Dirac materials in parallel electromagnetic fields generated by supersymmetry: A chiral Planar Hall Effect

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Contents

Dirac equation in electromagnetic fields

The Dirac equation, considering the minimal coupling rule, in the massless case, can be written as

$$\gamma^{\mu}\pi_{\mu}\Psi(\mathbf{x}), \quad \pi_{\mu} = i\partial_{\mu} - A_{\mu}, \tag{1}$$

where $A_{\mu} = (\Phi, \mathbf{A})$ is the electromagnetic potential of a external field. We are interested in solved the squared operator $(\gamma \cdot \pi)^2$, which takes the form

$$\begin{aligned} (\gamma \cdot \pi)^2 = \gamma^{\mu}\gamma^{\nu}\pi_{\mu}\pi_{\nu} = \pi^2 + \frac{\sigma^{\mu\nu}}{2}F_{\mu\nu}, \quad \sigma^{\mu\nu} = \frac{i}{2}[\gamma^{\mu}, \gamma^{\nu}], \\ F_{\mu\nu} = [\pi_{\mu}, \pi_{\nu}] = \begin{pmatrix} 0 & E_x & E_y & E_z \\ -E_x & 0 & -B_z & B_y \\ -E_y & B_z & 0 & -B_x \\ -E_z & -B_y & B_x & 0 \end{pmatrix} \end{aligned} \tag{2}$$

Dirac equation in electromagnetic fields

Developing each term separately, we obtain that

$$
\begin{aligned}
\pi^2 &= -\partial_t^2 - i\frac{\partial A_0}{\partial t} - 2iA_0\partial_t + A_0^2 + \partial_j^2 + i\frac{\partial A_j}{\partial x^j} + 2iA_j\partial_j - A_j^2, \\
\frac{\sigma^{\mu\nu}}{2}F_{\mu\nu} &= \begin{pmatrix} -\sigma_3F_{12}+\sigma_2F_{13}-\sigma_1F_{23} & i(\sigma_1F_{01}+\sigma_2F_{02}+\sigma_3F_{03}) \\ i(\sigma_1F_{01}+\sigma_2F_{02}+\sigma_3F_{03}) & -\sigma_3F_{12}+\sigma_2F_{13}-\sigma_1F_{23} \end{pmatrix},
\end{aligned} \tag{3}
$$

where we have used the relations $[\gamma^0,\gamma^i] = 2\alpha_i$, $[\gamma^i,\gamma^j] = i2\epsilon_{ijk}\sigma_k \otimes I_{2\times 2}$, $F_{\mu\nu} = -F_{\nu\mu}$ and $\sigma^{\mu\nu} = -\sigma^{\nu\mu}$, with the Dirac matrices being

$$
\begin{aligned}
\gamma^0 = \beta = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad \gamma^i = \beta\alpha_i = \begin{pmatrix} 0 & \sigma_i \\ -\sigma_i & 0 \end{pmatrix}, \\
\alpha_i = \begin{pmatrix} 0 & \sigma_i \\ \sigma_i & 0 \end{pmatrix}, \quad \gamma^5 = i\gamma^0\gamma^1\gamma^2\gamma^3 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix},
\end{aligned} \tag{4}
$$

and σ_i being the Pauli matrices.

Parallel electromagnetic fields

Let us consider a static parallel field configuration, namely, a magnetic field $\mathbf{B} = B(x)\,\hat{\mathbf{z}}$ and an electric field $\mathbf{E} = E(z)\,\hat{\mathbf{z}}$. By taking the Landau gauge the vector potential generating $\mathbf{B}$ can be chosen as $\mathbf{A} = A(x)\,\hat{\mathbf{y}}$. Moreover, since the rotational of the electric field is equal to zero, the electric potential ϕ is a function only depending on z. Thus, the field strengths turn out to be

$$B(x) = \frac{dA(x)}{dx}, \quad E(z) = -\frac{d\phi(z)}{dz}. \tag{5}$$

On the other hand, we assume the *spinor* $\Psi_D(t,x,y,z)$, such that $(\gamma\cdot\pi)^2\Psi_D(t,x,y,z) = 0$, has a standard temporal behavior and since the system exhibits translational symmetry along the y-direction, we propose $\Psi_D(t,x,y,z)$ can be written as follows

$$\Psi_D(t,x,y,z) = e^{i(\varepsilon t+ky)}\Psi(x,z) = e^{i(\varepsilon t+ky)}\begin{pmatrix}\psi_\uparrow^\alpha\\ \psi_\downarrow^\beta\\ \psi_\uparrow^\beta\\ \psi_\downarrow^\alpha\end{pmatrix}. \tag{6}$$

Parallel electromagnetic fields

The time-independent *spinor* $\Psi(x, z)$ fulfills an eigenvalue problem that is equivalent to the following coupled system of equations

$$\left\{\left[\partial_x^2 - (k + A(x))^2 \pm \frac{dA(x)}{dx}\right] + \left[\partial_z^2 + (\phi(z) + \varepsilon)^2\right]\right\}\psi_{\uparrow\downarrow}^{\alpha,\beta} \mp i\frac{d\phi(z)}{dz}\psi_{\uparrow\downarrow}^{\beta,\alpha} = 0. \tag{7}$$

This system can be decoupled defining the variables

$$\varphi_{\uparrow\downarrow}^{\pm} \equiv \psi_{\uparrow\downarrow}^{\alpha} \pm \psi_{\uparrow\downarrow}^{\beta}. \tag{8}$$

Then, the system of equations can be written as

$$\left[H_A^- + H_\phi^\pm\right]\varphi_\uparrow^\pm = 0, \quad \left[H_A^+ + H_\phi^\mp\right]\varphi_\downarrow^\pm = 0, \tag{9}$$

where

$$H_A^\pm = -\partial_x^2 + (k + A(x))^2 \pm \frac{dA(x)}{dx}, \quad H_\phi^\pm = -\partial_z^2 + [i(\phi(z) + \varepsilon)]^2 \pm i\frac{d\phi(z)}{dz}. \tag{10}$$

Parallel electromagnetic fields

The previous fact implying the functions $\varphi_\uparrow^\pm = \chi_\uparrow^-(x)\zeta_\uparrow^\pm(z)$, $\varphi_\downarrow^\pm = \chi_\downarrow^+(x)\zeta_\downarrow^\mp(z)$ satisfy that

$$H_A^\pm \chi_{\uparrow\downarrow}^\pm = \varepsilon_A \chi_{\uparrow\downarrow}^\pm, \quad H_\phi^\pm \zeta_{\uparrow\downarrow}^\pm = \varepsilon_\phi \zeta_{\uparrow\downarrow}^\pm. \tag{11}$$

Hence, we obtain a relation between the energies ε_A and ε_ϕ, given by

$$\varepsilon_A = -\varepsilon_\phi. \tag{12}$$

The SUSY transformations are carried out by means of the superpotentials

$$\mathrm{w}_A = k + A(x), \quad \mathrm{w}_\phi = i(\varepsilon + \phi(z)). \tag{13}$$

Confining case: Pöschl-Teller-like potentials

Let us consider the electromagnetic potentials of the form

$$A(x) = \frac{B_0}{\nu}\sec(\nu x), \; -\frac{\pi}{2} < \nu x < \frac{\pi}{2}; \quad \phi(z) = \frac{E_0}{\mu}\mathrm{sech}(\mu z), \tag{14}$$

With the definitions above, the SUSY partner potentials are

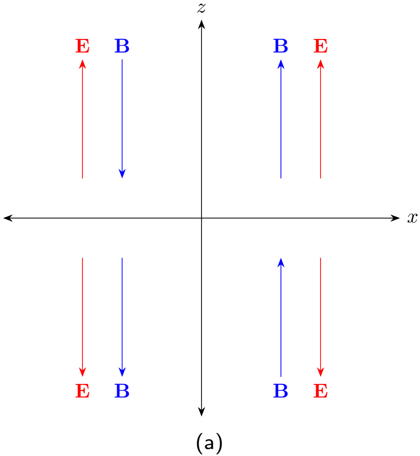
$$\begin{aligned} V_A^{\pm}(x) &= k^2 + 2kD_A\sec(\nu x) + D_A^2\sec^2(\nu x) \pm \nu D_A\sec(\nu x)\tan(\nu x), \\ V_\phi^{\pm}(z) &= -\varepsilon^2 - 2\varepsilon D_\phi\mathrm{sech}(\mu z) - D_\phi^2\mathrm{sech}^2(\mu z) \mp i\mu D_\phi\mathrm{sech}(\mu z)\tanh(\mu z), \end{aligned} \tag{15}$$

where $D_A = B_0/\nu$, $D_\phi = E_0/\mu$. In order to solve the eigenvalue equation of the Hamiltonians $H_\phi^{\pm}$, we apply the change of variable $u = i\sinh(\mu z)$, it is obtained that

$$\left[\mu^2(1-u^2)\frac{d^2}{du^2} - \mu^2 u\frac{d}{du} - \varepsilon^2 - 2\frac{\varepsilon D_\phi}{\sqrt{1-u^2}} - \frac{D_\phi^2}{1-u^2} \mp \mu D_\phi\frac{u}{1-u^2} - \varepsilon_\phi\right]\zeta^{\pm}(u) = 0. \tag{16}$$

This differential equation leads us to the Jacobi equation, if first we take $\varepsilon = 0$.

Confining case: Pöschl-Teller-like potentials



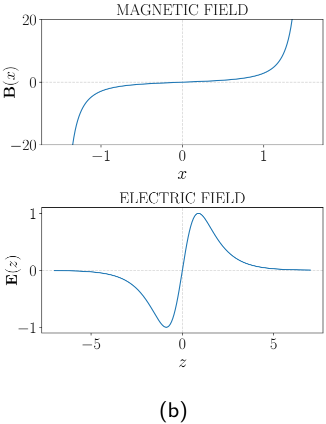


Figure: A sketch of the alignment of the fields in the plane $X - Z$ (a). The electromagnetic fields generated by the potentials in Eq. (14) (b). The scale of the graphs is set by parameters $E_0 = 2.0$, $B_0 = 1.0$, $\mu = 1.0$ and $\nu = 1.0$.

Confining case: Pöschl-Teller-like potentials

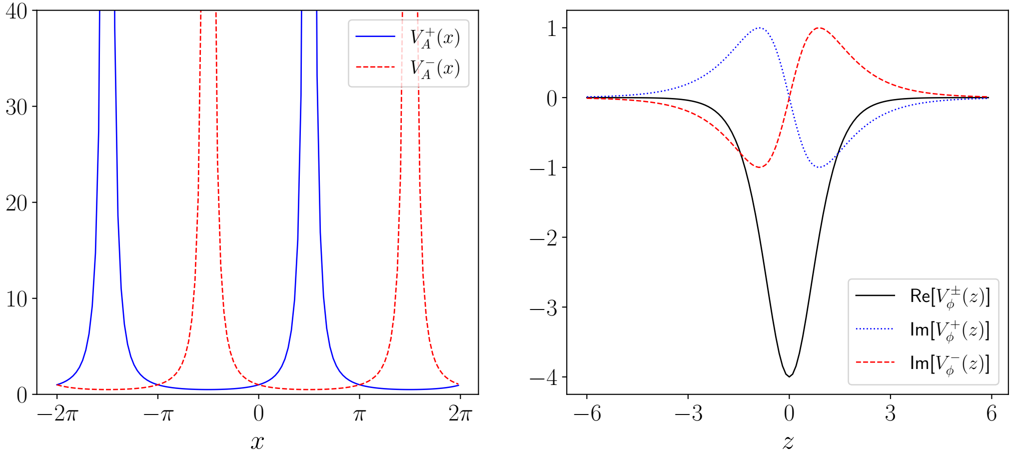


Figure: Plot of the Pöschl-Teller-like SUSY partner potentials $V_A^\pm(x)$ (left). Real and imaginary parts of the potentials $V_\phi^\pm(z)$ (right). The scale of the graphs is set by parameters $E_0 = 2.0$, $B_0 = 1.0$, $\mu = 1.0$, $\nu = 1.0$ and $\varepsilon = k = 0$.

Confining case: Pöschl-Teller-like potentials

The eigenfunctions of the Hamiltonians $H_\phi^\pm$ are given by

$$\begin{aligned}
\zeta_n^\pm(u) &= (1-u)^{\frac{1}{4}(1-r_\pm)}(1+u)^{\frac{1}{4}(1-r_\mp)}P_n^{\left(-\frac{1}{2}r_\pm,-\frac{1}{2}r_\mp\right)}(u),\\
\varepsilon_\phi &= -\mu^2\left[n(n-Q_\phi+1)+\frac{(1-Q_\phi)^2}{4}\right], \quad r_\pm = \sqrt{1+4S_\phi(S_\phi\pm1)},
\end{aligned} \tag{17}$$

where $S_\phi = D_\phi/\mu$, $Q_\phi = \frac{\sqrt{1+4S_\phi(S_\phi+1)}+\sqrt{1+4S_\phi(S_\phi-1)}}{2}$, $|S_\phi| > 1$ and $Q_\phi - 1 \geq n$. While, the solutions of the Hamiltonians $H_A^\pm$, when $k=0$, can be written as follows

$$\begin{aligned}
\chi_n^\pm(u) &= (1-u)^{\frac{1}{4}(1+s_\pm)}(1+u)^{\frac{1}{4}(1+s_\mp)}P_n^{\left(\frac{1}{2}s_\pm,\frac{1}{2}s_\mp\right)}(u),\\
\varepsilon_A &= \nu^2\left[n(n+Q_A+1)+\frac{(1+Q_A)^2}{4}\right], \quad s_\pm = \sqrt{1+4S_A(S_A\pm1)},
\end{aligned} \tag{18}$$

with $u=\sin(\nu x)$, $S_A = D_A/\nu$, $Q_A = \frac{\sqrt{1+4S_A(S_A+1)}+\sqrt{1+4S_A(S_A-1)}}{2}$ and $S_A(S_A\pm1) > -1/4$. The relation between the energies leads us to $|\nu(Q_A+1+2m)| = |\mu(Q_\phi-1-2n)|$.

Probability and current densities

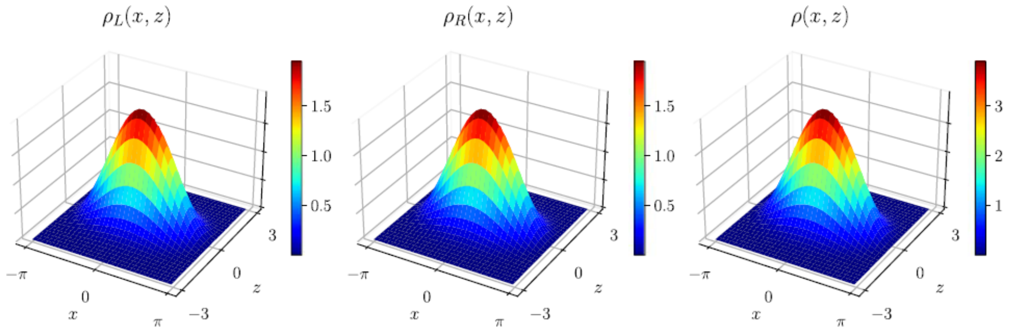
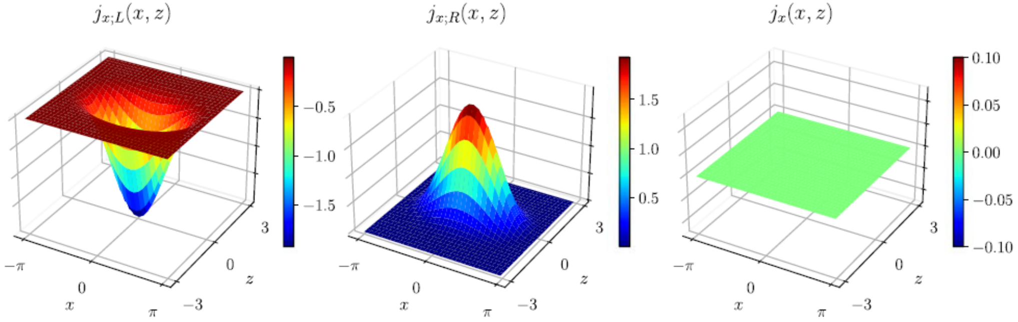
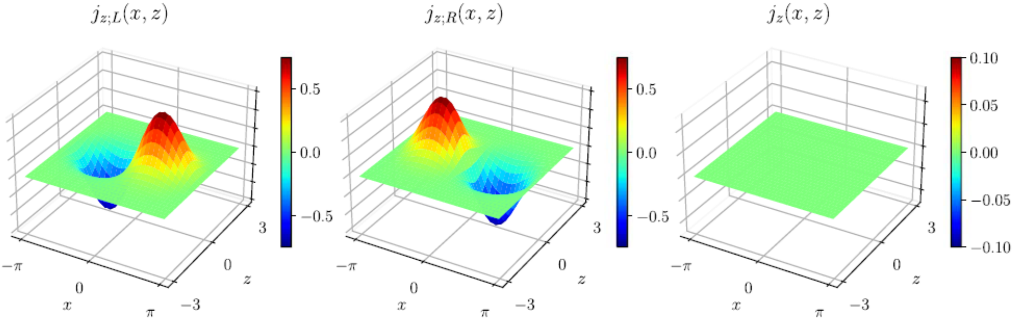


Figure 3. Probability densities for the L- and R-handed *spinors* in equation (39) (first and second column, respectively) and total probability density (third column), corresponding to the zero-mode *spinor* in equation (35). The scale of the plot is set by the parameters $E_0 = 2.0$, $B_0 = 1.0$, $\mu = 1.0$ and $\nu = 1.0$. Note that we are using natural units, $\hbar = c = e = 1$, where x, y, z variables possess units of $(\mathrm{MeV})^{-1}$, whereas the probability density has units of $(\mathrm{MeV})^3$.

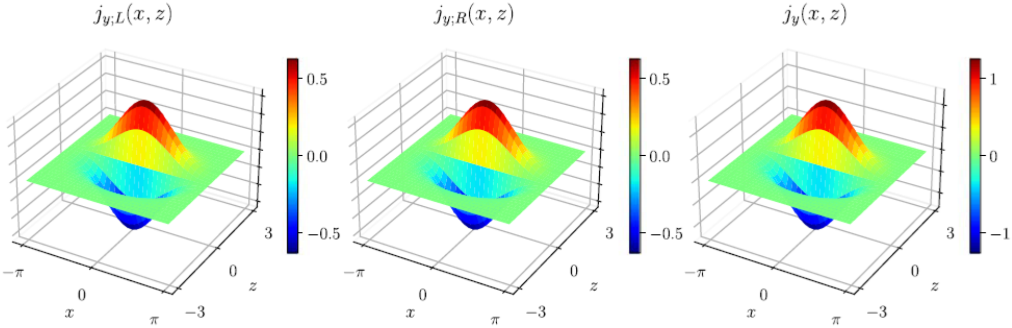
Probability and current densities



Probability and current densities



Probability and current densities



Probability and current densities

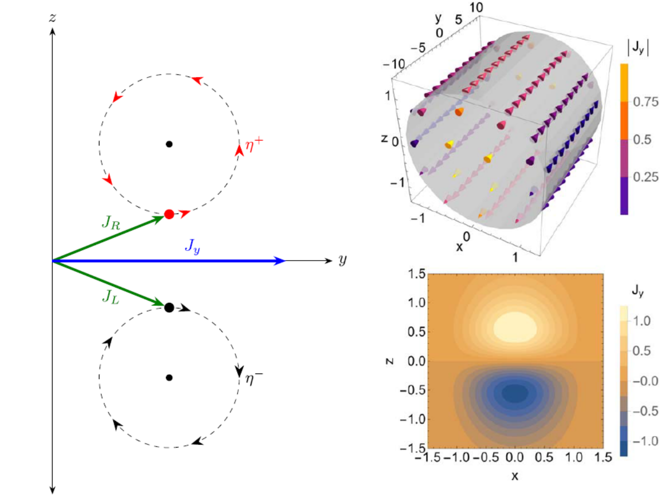


Figure 5. (Left) A representation of the Right and Left chiralities. The sum of the corresponding L- and R-handed currents is non-zero in y-direction. (Right-top) j_y current on the surface of a cylinder and (Right-bottom) its projection to the $x-z$ plane. It can be seen the direction switching through passing the $z=0$ plane. Note that we are using natural units, $\hbar=c=e=1$, where x, y, z variables possess units of $(\mathrm{MeV})^{-1}$, whereas the probability current densities has units of $(\mathrm{MeV})^3$.

Summary and outlook

- The (3+1) Dirac equation describing a Dirac material in the presence of static non-uniform parallel electromagnetic fields is solved within a SUSY-QM framework.
- The current densities vanish in all spatial directions, except for the current along the y-direction, which defines a plane in which it lies perpendicularly to the electromagnetic fields. Hence, it is appropriate to assume that a PHE develops in the system dealt here.
- **¡The Dirac material addressed here shows a new class of chiral PHE!**
- The electromagnetic profiles used in this work are tough to realize in the laboratory. Nevertheless, a configuration of pseudo-electromagnetic fields, associated to strains in the material, could become analogous to the system worked here. Such configuration could be feasible in the laboratory through modern strain techniques in Dirac materials, such as scanning tunneling spectroscopy
- Solving the SUSY partner potentials $V_\phi^\pm(z)$, $V_A^\pm(x)$ for $\varepsilon \neq 0$ is an interesting work for the future.

Reference

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thank
you!