Thermodynamical analysis of a strongly interacting system with isospin imbalance using LSMq $N_f = 2$. The cold case. 1st part.

Non-trivial one Loop correction performing techniques

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I. The system

The system

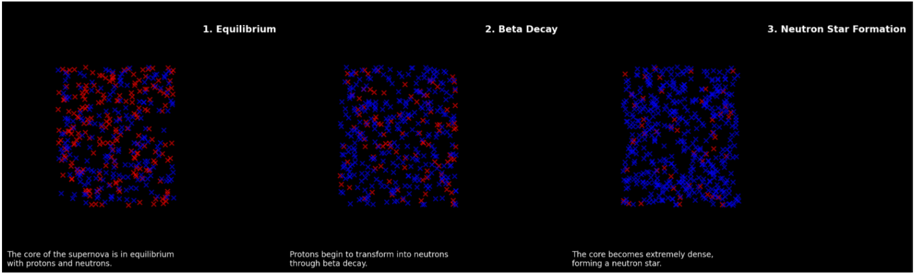


Figura: We are analysing a system with isospin imbalance, a natural example could be the temporal evolution from the super-nova explotion to a neutron star. At the begining after the explotion the star core is exposed, initialy at isospin equilibrium (aprox. the same number of protons and neutrons). When the was confined underneat the star it maintained the existence of charges into it, but now it need to minimize the energy to the lowest level, it is the neutral one. The beta decay is now is encoraged. 30 seconds (more or less) after the explotion later the core reaches the chemical equilibrium and the isospin density becomes locally constant.

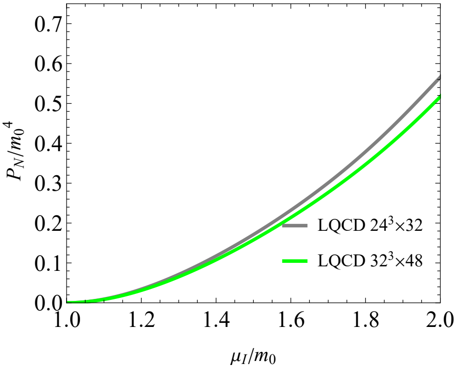
The system

Even thought, the direct observations of this process is practically impossible due the ejection of coronary matter, the luminosity of the event and the briefly of the evolution. We are able to simulate it the numerical model **Lattice QDCq**, nevertheless it has a limitation about the Barionic density due **the sign problem** or also known as the complex-action problem which is partially solved but it demands a huge computational power impossible right now (maybe with quantum computers...).
With this limitations to only possibility is to take diluted matter, which has no the problme. We are dealing to refine the theory in this regime same has really interesting points.

II. The data

The data

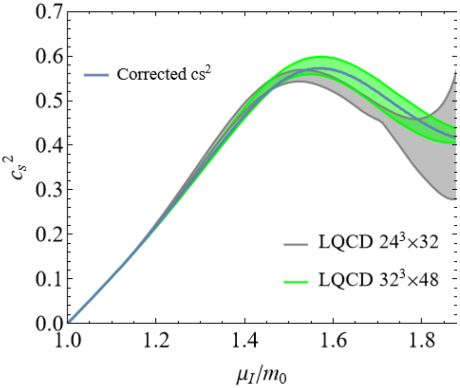
This escenario has been observed since beginings of the milenium, recently with a more precise grid (2022)



B. B. Brandt, F. Cuteri, and G. Endrodi, Equation of state and speed of sound of isospin-asymmetric QCD on the lattice, JHEP 07, 055, [hep-lat].arXiv:2212.14016

The data

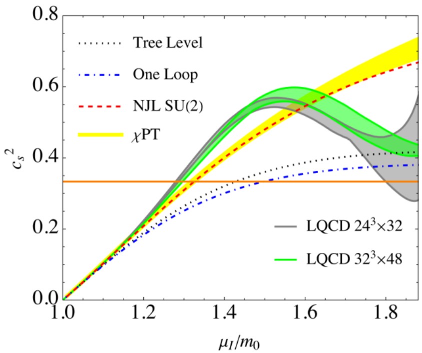
They give us some others thermodynamical variables, but the most interesting is the speed of sound.



Why exists a peak in the speed of sound? The most of time this manifests a phase transition but, if it is true... What states of matter are involved?

The data

Many people have work on this problem trying to answer this question, but the results has been, a little bit not satisfactory.



Therefore, working on this regime we can aim to two goals. To 'calibrate' our model and to try to describe the peak of speed of sound.

III. The ideal world

The ideal world

Imaging a really symetric, perfect (and boring) world.

$$\mathcal{L}_{BW} = \frac{1}{2}\left[(\partial_\mu\sigma)^2 + (\partial_\mu\vec{\pi})^2\right] - \frac{\mu^2}{2}(\sigma^2 + \vec{\pi}) - \frac{\lambda}{4}(\sigma^2 + \vec{\pi}^2)^2$$
$$+ i\bar{\psi}\partial_\mu\psi - ig\bar{\psi}\gamma^5\vec{\tau}\cdot\vec{\pi}\psi - g\bar{\psi}\psi\sigma$$

Then we can see everything is in the ground state, but it is the vacuum.
TOO BORING!!
Let's broke it a little bit :).

The ideal world

Let's broke it a little bit :).

We take $\mu^2 \to -a^2$ with $a^2 > 0$, then the new potential is the famous any-mexican-have-never-use mexican hat potential

IV. The sigma expectation value

The sigma expectation value

To determinate the shift of the solutions for sigma we define

$$\sigma \to \sigma + v$$

Where σ is the pure quantum funtion, while v is the classical solution, then we have

$$\begin{aligned}
\mathcal{L}_{LSMq} =& \frac{1}{2}\left[(\partial_\mu \sigma)^2 + (\partial_\mu \vec{\pi})^2\right] + \frac{a^2}{2}((\sigma+v)^2 + \vec{\pi}) - \frac{\lambda}{4}((\sigma+v)^2 + \vec{\pi}^2)^2 \\
&+ i\bar{\psi}\partial_\mu \psi - ig\bar{\psi}\gamma^5 \vec{\tau}\cdot\vec{\pi}\psi - g\bar{\psi}\psi(\sigma+v) \\
=& \frac{1}{2}\left[(\partial_\mu \sigma)^2 + (\partial_\mu \vec{\pi})^2\right] - \frac{3\lambda v^2 - a^2}{2}\sigma^2 - \frac{\lambda v^2 - a^2}{2}\vec{\pi} + i\bar{\psi}\left(\partial\!\!\!/ - gv\right)\psi \\
&- \frac{\lambda}{4}(\sigma^2 + \vec{\pi}^2)^2 \\
&- ig\bar{\psi}\gamma^5\vec{\tau}\cdot\vec{\pi}\psi - g\bar{\psi}\psi\sigma - \lambda v(\sigma^3 + \sigma\vec{\pi}^2) \\
&- v(\lambda v^2 - a^2)\sigma \\
&+ \frac{a^2}{2}v^2 - \frac{\lambda}{4}v^4
\end{aligned}$$

The sigma expectation value

Therefore, we have now a classical (tree level) potential as

$$-\frac{a^2}{2}v^2 + \frac{\lambda}{4}v^4$$

which we can found their local maximum and minimum derivating and equalling to zero

$$a^2 v - \lambda v^3 = 0$$

Then it has 3 roots,

$$v = 0 \quad ; \quad v = \pm\sqrt{\frac{a^2}{\lambda}}$$

Now, notice that including this the tadpole Lagrangian term

$$\mathcal{L}_{tadpole} = -v(\lambda v^2 - a^2)\sigma = 0$$

V. The explicitly symmetry breaking

The explicitly symmetry breaking

We found we cannot let free the model to fall into a spontaneous expectation value of sigma, then we propose to add an explicitly symmetry breaking term as

$$\mathcal{L} \to \mathcal{L} + h(\sigma + v)$$

therefore $h = v(\lambda v^2 - a^2)$, the tadpoles equation becomes

$$\mathcal{L}_{tadpole} = (h - v(\lambda v^2 - a^2))\sigma = 0$$

and the tree level potential becomes

$$-\frac{a^2}{2}v^2 + \frac{\lambda}{4}v^4 - hv$$

The explicitly symmetry breaking

later, the new roots of the tree level potential does not contain any more the zero expectation value

$$-a^2v + \lambda v^3 - h = 0$$

VI. The charged pion basis

The charged pion basis

Now, we have an stable state to start in matter, but still we need to change from the pion vector basis to the real ones, I mean the charged ones, defined as

$$\pi_1 = \frac{1}{\sqrt{2}}(\pi_+ + \pi_-) \quad ; \quad \pi_2 = \frac{i}{\sqrt{2}}(\pi_+ - \pi_-) \quad ; \quad \pi_3 = \pi_0$$

then the Lagrangian becomes

$$\begin{aligned}
\mathcal{L}_{LSMq} =& \frac{1}{2}\left[(\partial_\mu \sigma)^2 + (\partial_\mu \pi_0)^2\right] + \partial_\mu \pi_- \partial^\mu \pi_+ - \frac{m_\sigma^2}{2}\sigma^2 - \frac{m_{\pi_0}^2}{2}\pi_0^2 - m_{\pi_0}^2 \pi_- \pi_+ \\
& + i\bar{\psi}\left(\partial\!\!\!/ - m_f\right)\psi - \frac{\lambda}{4}(\sigma^2 + \pi_0^2 + 2\pi_-\pi_+)^2 - g\bar{\psi}\psi\sigma \\
& - ig\bar{\psi}\gamma^5(\tau_-\pi_- + \tau_+\pi_+ + \tau_3\pi_0)\psi - \lambda v\sigma(\sigma^2 + \pi_0^2 + 2\pi_-\pi_+) \\
& (h - vm_{\pi_0}^2)\sigma + \frac{a^2}{2}v^2 - \frac{\lambda}{4}v^4
\end{aligned}$$

The charged pion basis

Where the τ's are the Pauli matrices in the charged basis defined as

$$\tau_{\pm} = \frac{1}{\sqrt{2}}(\tau_1 \pm i\tau_2)$$

and where the masses are the dinamical masses defined as

$$m_f^2 = g^2 v^2 \quad ; \quad m_{\pi_0}^2 = \lambda v^2 - a^2 \quad ; \quad m_\sigma^2 = 3\lambda v^2 - a^2$$

Notice that the triplet of pions are degenerated in mass.

VII. Isospin chemical potential

Isospin chemical potential

The way to introduce the chemical potential is via the Covariant derivative for the charged pions as

$$\partial_\mu \pi_\pm \to \partial_\mu \pi_\pm \pm i\mu_I \delta^0_\mu \pi_\pm$$

similarlly as to change the energies of the states. For fermions the procedure is very similar

$$\partial_\mu \psi \to (\partial_\mu - i(\mu_B + \tau_3 \mu_I)\gamma^0 \delta^0_\mu)\psi$$

Including this covariant derivative, the new terms in the Lagrangian (it begins to be pretty large) are

$$\mathcal{L} \to \mathcal{L} + \mu_I^2 \pi_- \pi_+ + i\mu_I(\pi_+ \partial_0 \pi_- - \pi_- \partial_0 \pi_+) + \bar{\psi}\mu_B \gamma^0 \psi + \bar{\psi}\mu_I \tau_3 \gamma^0 \psi$$

VIII. Pion condensates

Pion condesates

Let's focus on the bosonic part. Once the chemical potential reaches a critical value over passing the pion vacuum mass, the condensation of pions minimizes the configuration energy, then $< \pi_\pm > \neq 0$, therefore we can include another condensate similar to v, with the following definition

$$\pi_\pm \to \pi_\pm + \frac{\Delta}{\sqrt{2}} \exp \pm i\theta$$

where θ is the symmetry of $U(1)_{I_3}$. In general every physical quatity is phase independent, keeping it help us to identify the non physical terms. Once substituded the condensation of pions the Lagragian can be written as

$$\mathcal{L}_{LSMq} = \mathcal{L}_{tree} + \mathcal{L}_1 + \mathcal{L}_2 + \mathcal{L}_3 + \mathcal{L}_4$$

where $\mathcal{L}_i$ is i-th order in the fields.

Pion condesates

$$\mathcal{L}_{tree} = \frac{a^2}{2}(v^2 + \Delta^2) - \frac{\lambda}{4}(v^2 + \Delta^2)^2 + \frac{1}{2}\mu_I^2\Delta^2 + hv$$

$$\mathcal{L}_1 = \frac{\Delta}{\sqrt{2}}\left((\mu_I^2 - m_{\pi_0}^2 - i\mu_I\partial^0)e^{-i\theta}\pi_+ + (\mu_I^2 - m_{\pi_0}^2 + -i\mu_I\partial^0)e^{i\theta}\pi_-\right) + (h - vm_{\pi_0}^2)\sigma$$

$$\mathcal{L}_2 = \frac{1}{2}[(\partial_\mu\sigma)^2 + (\partial_\mu\pi_0)^2] + \partial_\mu\pi_-\partial^\mu\pi_+ + i\mu_I(\pi_+\partial_0\pi_- - \pi_-\partial_0\pi_+) - \frac{m_\sigma^2}{2}\sigma - \frac{m_{\pi_0}^2}{2}\pi_0 - (m_{ch}^2 - \mu_I^2)\pi_-\pi_+ - \frac{\lambda\Delta^2}{2}(e^{-2i\theta}\pi_+^2 + e^{2i\theta}\pi_-^2) - \sqrt{2}\lambda\Delta v\sigma(e^{-i\theta}\pi_+ + e^{i\theta}\pi_-)$$

$$\mathcal{L}_3 = -\lambda\sigma(\sigma^2 + \pi_0^2 + 2\pi_-\pi_+) - \frac{\lambda\Delta}{\sqrt{2}}(\sigma^2 + \pi_0^2 + 2\pi_-\pi_+)(e^{-i\theta}\pi_+ + e^{i\theta}\pi_-)$$

$$\mathcal{L}_4 = \frac{\lambda}{4}(\sigma^2 + \pi_0^2 + 2\pi_-\pi_+)^2$$

Pion condesates

where the new dynamical masses are

$$m_f^2 = g^2 v^2 \quad ; \quad m_\sigma^2 = \lambda(3v^2 + \Delta^2) - a^2 \quad ; \quad m_{\pi_0}^2 = \lambda(v^2 + \Delta^2) - a^2$$

$$m_{ch}^2 = \lambda(v^2 + 2\Delta^2) - a^2$$

Notice that the neutral pion mass is now non-degenerated.

IX. Bosonic Propagators

Bosonic Propagators

From the second-order-field Lagrangian, we can compute the inverse propagator in the momentum space $D^{-1}(k) =$ as

$$\begin{pmatrix} k^2 - m_\sigma^2 & -\sqrt{2}\lambda v \Delta e^{-i\theta} & -\sqrt{2}\lambda v \Delta e^{i\theta} & 0 \\ -\sqrt{2}\lambda v e^{i\theta} & k^2 - m_{ch}^2 + \mu_I^2 + i\mu_I k_0 & -\lambda \Delta^2 e^{2i\theta} & 0 \\ -\sqrt{2}\lambda v e^{-i\theta} & -\lambda \Delta^2 e^{-2i\theta} & k^2 - m_{ch}^2 + \mu_I^2 - i\mu_I k_0 & 0 \\ 0 & 0 & 0 & k^2 - m_{\pi_0}^2 \end{pmatrix}$$

therefore, we can observe only the neutral pion is fully diagonal, then

$$D_{\pi_0}^{-1}(k) = k^2 - m_{\pi_0}^2$$

To found the other three propagators we could take the determinant and try to find the eigenvalues of it for k_0 (spoiler, the solutions are disgusting) nevertheless see has no sense some of that matrix elements we have some terms as $\lambda v\Delta\sigma\pi_{\pm}e^{\mp i\theta}$ (nevertheless v has an intrisic phase since it can be seen only as an exitation of the neutral pion and the way it is choosen is completly arbitraty, then taking a global average every $v\Delta$ terms cancel).

Then, the effective inverse porpagator is

$$\begin{pmatrix} k^2-m_\sigma^2 & 0 & 0 & 0 \\ 0 & k^2-m_{ch}^2+\mu_I^2+i\mu_I k_0 & -\lambda\Delta^2 e^{2i\theta} & 0 \\ 0 & -\lambda\Delta^2 e^{-2i\theta} & k^2-m_{ch}^2+\mu_I^2-i\mu_I k_0 & 0 \\ 0 & 0 & 0 & k^2-m_{\pi_0}^2 \end{pmatrix}$$

therefore, we have as inverse propagator of sigma as

$$D_\sigma^{-1}(k)=k^2-m_\sigma^2$$

And the charged pions propagators are given by the inner matrix determinant

Bosonic Propagators

$$\begin{vmatrix} k^2 - m_{ch}^2 + \mu_I^2 + i\mu_I k_0 & -\lambda\Delta^2 e^{2i\theta} \\ -\lambda\Delta^2 e^{-2i\theta} & k^2 - m_{ch}^2 + \mu_I^2 - i\mu_I k_0 \end{vmatrix} = 0$$

$$(k^2 - m_{ch}^2 + \mu_I^2)^2 + \mu_I^2 k_0^2 - \lambda^2\Delta^4 = 0$$

$$(k_0^2 - E_+^2(\vec{k})(k_0^2 - E_-^2(\vec{k}) = 0$$

where

$$E_\pm(\vec{k}) = \sqrt{\vec{k}^2 + m_{ch}^2 + \mu_I \mp \sqrt{4\mu_I^2(\vec{k}^2 + m_{ch}^2) + \lambda^2\Delta^4}}$$

X. Fermionic Propagators

Fermionic Propagators

For fermions we have the following cuadratic lagrangian

$$\mathcal{L}_2 = i\bar{\psi}\partial\!\!\!/\psi - m_f^2\bar{\psi}\psi + \bar{\psi}\mu_I\tau_3\gamma^0\psi - \frac{ig}{\sqrt{2}}\Delta\bar{\psi}\gamma^5(\tau_+ e^{i\theta} + \tau_- e^{-i\theta})\psi$$

performing an analogous procedure we can find that the propagators are

$$D_{u,d}^{-1} = k_0^2 - E_{u,d}^2(\vec{k})$$

where

$$E_u(\vec{k}) = \sqrt{\vec{k}^2 + g(v^2 + \Delta^2) + \mu_I^2 + 2\mu_I\sqrt{\vec{k}^2 + g^2v^2}}$$

$$E_d(\vec{k}) = \sqrt{\vec{k}^2 + g(v^2 + \Delta^2) + \mu_I^2 - 2\mu_I\sqrt{\vec{k}^2 + g^2v^2}}$$

XI. One Loop Corrections

One Loop Corrections

En this model wi have six particles:

- Quarks
- Triplet of Pions
- Sigma

Everyone of them has the following expression for the one Loop Corrections. For quarks

$$\sum_{f=u,d} V_f^1 = -2N_c \int \frac{d^3k}{(2\pi)^3} \left[E_u + E_d\right],$$

For charged Pions

$$\sum_{f=\pi^+,\pi^-} V_{\pi^\pm}^1 = \frac{1}{2} \int \frac{d^3k}{(2\pi)^3} \left[E_+ + E_-\right],$$

One Loop Corrections

For neutral pion

$$V^1_{\pi^0} = \frac{1}{2}\int \frac{d^3k}{(2\pi)^3}\left\{k^2 + m^2_{\pi_0}\right\}^{1/2},$$

and for the sigma

$$V^1_{\sigma} = \frac{1}{2}\int \frac{d^3k}{(2\pi)^3}\left\{k^2 + m^2_{\sigma}\right\}^{1/2},$$

Notice that the calculation for the neutral pion and the sigma are trivial using dimensional regularization and they are

$$V^1_{\pi_0} = -\frac{\left(\lambda\left(\Delta^2 + v^2\right) - a^2\right)^2}{64\pi^2}\left(\ln\left(\frac{\Lambda^2}{\lambda\left(\Delta^2 + v^2\right) - a^2}\right) + \frac{1}{\epsilon} + \frac{3}{2}\right)$$

$$V^1_{\sigma} = -\frac{\left(\lambda\left(\Delta^2 + 3v^2\right) - a^2\right)^2}{64\pi^2}\left(\ln\left(\frac{\Lambda^2}{\lambda\left(\Delta^2 + 3v^2\right) - a^2}\right) + \frac{1}{\epsilon} + \frac{3}{2}\right)$$

One Loop Corrections

For the charged pions and quarks we have non trivial expressions and the most of time we need to try to found a small parameter of expansion and hope it helps to implement the dimensional regularization formulas.
Now, I would like to show us another general way to solve this commom problem without aproximations and always effective.

XII. Asymptotic Renormalization Formalism (ARF)

The following five-steps technique let us to isolate the divergences of any integral, giving us a remaining numerical finite part, the most of time, small.
To introduce the technique we are going to isolate the divergences of the sum of energies of the charged pions

$$E_+ + E_- = \sqrt{\vec{k}^2 + m_{ch}^2 + \mu_I^2 - \sqrt{4\mu_I^2(\vec{k}^2 + m_{ch}^2) + \lambda^2\Delta^4}} + \sqrt{\vec{k}^2 + m_{ch}^2 + \mu_I^2 + \sqrt{4\mu_I^2(\vec{k}^2 + m_{ch}^2) + \lambda^2\Delta^4}}$$

ARF, Step 1

We perform an Poincaré expansion until d order where d is the dimension of the integral. (substitute $k \to 1/z$ and compute a Taylor serie around $z = 0$ to d order).

$$E_+ + E_- = 2k + \frac{m_{ch}^2}{k} - \frac{m_{ch}^4 + \lambda^2\Delta^4}{4k^3} + \mathcal{O}(k^5)$$

ARF, Step 2

To propouse a analitically allowed structure, in general

$$\sum_i a_i\sqrt{k^2+m_i^2}+\sum_i \frac{b_i}{\sqrt{k^2+m_i^2}}+\sum_i \frac{c_i}{(k^2+m_i^2)^{3/2}}+\dots$$

Note: The estructure choose does not affect the renormalization
In our case we are going to take 4 free parameters

$$A\sqrt{k^2+m_1^2}+B\sqrt{k^2+m_2^2}$$

ARF, Step 3

To repite the first step to your propousal

$$A\sqrt{k^2+m_1^2}+B\sqrt{k^2+m_2^2}=(A+B)k+\frac{Am_1^2+Bm_2^2}{2k}-\frac{Am_1^4+Bm_2^4}{8k^3}$$
$$+\mathcal{O}(1/k^5)$$

ARF, Step 4

To solve matching both expansion, in our case we decide to set A=B, then

$$A = B = 1$$
$$m_1^2 = m_{ch}^2 - \lambda\Delta^2$$
$$m_2^2 = m_{ch}^2 + \lambda\Delta^2$$

ARF, Step 5

The remaining finite part is given by the substraction of the full sum of energies and the solved propousal, for example

$$E_+ + E_- - \sqrt{k^2 + m_{ch}^2 + \lambda\Delta^2} - \sqrt{k^2 + m_{ch}^2 - \lambda\Delta^2}$$

Implementing this procedure we can found always an exact renormalization of any correction.
Note: Another simple posibility for renormalization could be taking

$$E_+ + E_- - \sqrt{k^2 + m_{ch}^2} + \frac{\lambda^2\Delta^4}{(k^2 + m_{ch}^2)^{3/2}}$$

Summary

Summary

- We have derived an everything-to-go Lagrangian to describe our system.
- We have found the inverse propagators for every single particle in the model.
- We have isolated all the divergences of the One Loop Correction

Thriller

Thriller

On monday we are to...

- Identify an ambiguity in in the election of Counter-Terms which does not allow us to have a unique renormalization.
- compute the Ward-Takahashi identities.
- Renormalize the Correction
- and ...

Thanks!!