Graphene in magnetic fields generated by supersymmetry

Juan D. García-Muñoz

Instituto de Física y Matemáticas (IFM)
Universidad Michoacana de San Nicolás de Hidalgo (UMSNH)

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Content

SUSY-QM

In the SUSY-QM framework, we have two Schrödinger-like hamiltonians:

$$H^{\pm} = -\frac{d^2}{dx^2} + V^{\pm}(x), \tag{1}$$

and they are intertwined through the operational relation

$$H^{+}L^{-} = L^{-}H^{-}, \tag{2}$$

where the operator L^{-} is known as intertwining operator and it is given by

$$L^{-} = \frac{d}{dx} + w(x), \tag{3}$$

with $w(x)$ being a real function called superpotential.

SUSY-QM

Substituting Eqs. (1) and (3) into the intertwining relation (2), we arrive at the following system of equations[1]

$$V^{+}(x) = V^{-}(x) + 2w'(x), \tag{4}$$

$$w(x)V^{+}(x) - w''(x) = w(x)V^{-}(x) + (V^{-}(x))'. \tag{5}$$

If we introduce Eq. (4) into Eq. (5), we have that

$$2w(x)w'(x) - w''(x) = (V^{-}(x))'. \tag{6}$$

Integrating, we onbtain that

$$w^{2}(x) - w'(x) = V^{-}(x) - \epsilon, \tag{7}$$

where ϵ is a constant called factorization energy. The previous equation is a particular case of the Ricatti equation.

[1] $w'(x) \equiv dw(x)/dx$.

SUSY-QM

If we porpose that

$$w(x) = -\frac{u'(x)}{u(x)}, \tag{8}$$

substituting into Eq. (7), it turns out to be that

$$-u''(x) + V^{-}(x)u(x) = \epsilon u(x), \tag{9}$$

i.e., the function $u(x)$, called seed solution, is an eigenfunction of the Schrödinger equation for the Hamiltonian H^{-}, associated to the eigenvalue ϵ.
On the other hand, the Hamiltonians $H^{\pm}$ are factorized as follows

$$L^{+}L^{-} = H^{-} - \epsilon, \quad L^{-}L^{+} = H^{+} - \epsilon, \tag{10}$$

with $L^{+} = (L^{-})^{\dagger}$.

SUSY-QM

If we know the solutions of the Hamiltonian H^-, i.e., their eigenfunctions $\psi_n^-(x)$ and their eigenvalues E_n, $n = 0, 1, 2, \ldots$ Then, from Eqs. (2) y (10), we have that

$$\psi_n^+(x) = \frac{L^-\psi_n^-(x)}{\sqrt{E_n - \epsilon}}, \quad \psi_n^-(x) = \frac{L^+\psi_n^+(x)}{\sqrt{E_n - \epsilon}}, \tag{11}$$

where $\psi_n^+(x)$ is a eigenfucntion of H^+ with eigenvalue E_n. However, the spectra of H^- and H^+ are not necessarily the same. Since $H^-u(x) - \epsilon u = 0$, given the Eq. (10), the seed solution lies in the *kernel* of the operator L^-, using the Eq. (3), we arrive at

$$\frac{du(x)}{dx} + w(x)u(x) = 0 \Rightarrow u(x) \propto e^{-\int w(y)dy}. \tag{12}$$

Depending on the square-integrability of the seed solution $u(x)$, the factorization energy can belong to the spectrum of H^-. Thus, defining two kinds of supersymmetric transformations, isospectral (broken) and non-isospectral (unbroken).

SUSY-QM

The eigenfunction $\psi_\epsilon^+(x)$ of H^+ associated to the eigenvalue ϵ is given by

$$\psi_\epsilon^+(x) \propto e^{\int w(y)dy} = \frac{1}{u(x)}. \tag{13}$$

Thus the seed solution $u(x)$ must be a nodeless function. Moreover, if we caculate the expedtation value for the operator L^+L^- given in the Eq. (10), onto a eigenfucntion $\psi_n^-(x)$, we obtain that

$$0 \leq |L^-\psi_n^-(x)|^2 = E_n - \epsilon \Rightarrow E_0 \geq \epsilon. \tag{14}$$

Supersymmetric Algebra

The Supersymmetric Algebra is defined by means of the operators $Q^{\pm}$ called supercharges and the supersymmetric Hamiltonian H_{SS}, which follow the commutation rules

$$\{Q^+, Q^-\} = H_{SS}, \quad [Q^{\pm}, H_{SS}] = 0. \tag{15}$$

For the fisrt-order SUSY-QM, we have that

$$Q^+ = \begin{pmatrix} 0 & L^+ \\ 0 & 0 \end{pmatrix}, \quad Q^- = \begin{pmatrix} 0 & 0 \\ L^- & 0 \end{pmatrix}, \tag{16}$$

thus, the supersymmteric Hamiltonian turns out to be

$$H_{SS} = \begin{pmatrix} L^+L^- & 0 \\ 0 & L^-L^+ \end{pmatrix} = \begin{pmatrix} H^- - \epsilon & 0 \\ 0 & H^+ - \epsilon \end{pmatrix} \tag{17}$$

Álgebra Supersimétrica

Since the intertwining operators $L^{\pm}$ are hermitian conjugates, the spectrum of H_{SS} is non-negative and the ground state is have a corresponding eigenfucntion $\Psi_0(x)$, which has the following form

$$\Psi_0(x) = \begin{pmatrix} u(x) \\ \psi_\epsilon^+(x) \end{pmatrix} . \tag{18}$$

If neither $u(x)$ nor $\psi_\epsilon^+(x)$ are square-integrable, the energy eigenvalue of the zero mode does not belong to the spectrum of H_{SS}, when this happen we have the case of broken SUSY, i.e. an isospectral transformation. In the opposite case, if one of the function $u(x)$ or ψ_ϵ^+ is square-integrable, the energy eigenvalue of the zero mode belong to the spectrum of H_{SS} and we have that SUSY is unbronken, corresponding to the case of a non-isospectral transformation.

Harmonic Oscillator Solutions

Let us consider a harmonic oscillator potential $V^{-}(x) = x^2$. Its solutions are well-known and they are given by

$$\psi_n^{-}(x) = c_n e^{-\frac{x^2}{2}} \mathcal{H}_n(x), \quad E_n = 2n + 1, \quad n = 0, 1, 2, ..., \tag{19}$$

where $\mathcal{H}_n(x)$ are the Hermite polynomials and c_n is the corresponding normalization constant. Moreover, in general, the solutions for the eigenvalue problem of H^{-} are

$$u(x) = e^{-\frac{x^2}{2}} \left[{}_1F_1\left(\frac{1-\varepsilon}{4}, \frac{1}{2}; x^2\right) + 2x\lambda \frac{\Gamma\left(\frac{3-\varepsilon}{4}\right)}{\Gamma\left(\frac{1-\varepsilon}{4}\right)} \, {}_1F_1\left(\frac{3-\varepsilon}{4}, \frac{3}{2}; x^2\right) \right], \tag{20}$$

with ${}_1F_1(a, b; x)$ being the confluent hypergeometric function, ε is the corresponding energy eigenvalue and λ is a constant such that for $|\lambda| \leq 1$ and $\varepsilon \leq 1$ the function $u(x)$ is nodeless.

Unbroken SUSY

Taking the ground state eigenfunction of the harmonic oscillator as seed solution, $u(x) = \psi_0^-(x) = c_0 e^{-x^2/2}$, and thus, the factorization energy is $\epsilon = 1$. From Eq. (8), the superpotential is

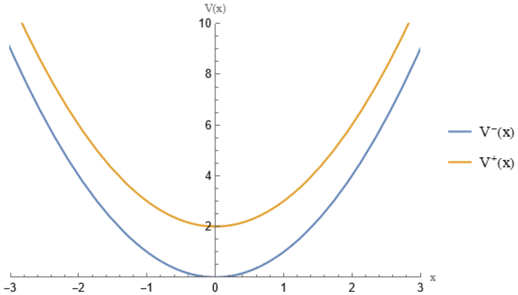
$$w(x) = x. \tag{21}$$

Substituting Eqs. (7) and (4), we have that the SUSY partner potentials are

$$V^-(x) = x^2, \quad V^+(x) = x^2 + 2. \tag{22}$$

These potentials are known as shape-invariant potentials. Furtnermore, the solutions of these potentials are such that $\psi_n^-(x) = \psi_{n-1}^+(x),\ n = 1, 2, 3, ...$

Non-isospectral Transformation



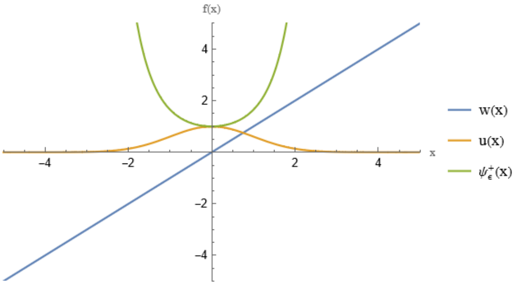


Figura: (Left) Shape-invariant potentials. (Right) The superpotential, the seed function and the function $\psi_{\epsilon}^{+}(x)$ with energy $\epsilon = 1$.

Isospectral Transformation

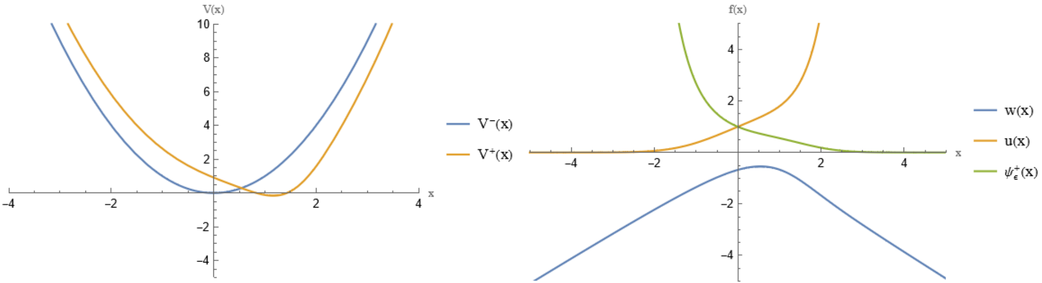


Figura: (Left) Non-shape-invariant potentials. (Right) The corresponding superpotential, the seed function and the function $\psi_{\epsilon}^{+}(x)$ with energy $\epsilon = 0$ and $\lambda = 1$.

Effective Hamiltonian

The charge carriers behave as massless Dirac particles. This behavior is described by the effective Hamiltonian:

$$H = v_F(\boldsymbol{\sigma} \cdot \mathbf{p}), \tag{23}$$

where $v_F \approx c/300$ is the Fermi velocity, $\boldsymbol{\sigma} = (\sigma_x, \sigma_y)$ are the Pauli matrices and $\mathbf{p} = -i\hbar(\partial_x, \partial_y)$ is the quantum momentum operator.

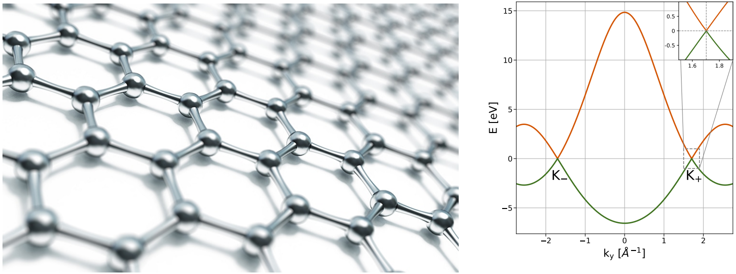


Figura: (Left) A skecth of the graphene. (Right) The energy band structure of graphene.

The Dirac-Weyl Equation

Then, the DIrac-Weyl Equation is:

$$v_F(\boldsymbol{\sigma} \cdot \mathbf{p})\Psi(t,x,y) = i\hbar \frac{\partial \Psi(t,x,y)}{\partial t}, \tag{24}$$

with $\Psi(t,x,y)$ is a two-component *spinor*. We suppose that our system evolve in a standard way, i.e.,

$$\Psi(t,x,y) = e^{-i\frac{E}{\hbar}t}\psi(x,y), \tag{25}$$

the stationary Dirac-Weyl Equation can be written as

$$v_F(\boldsymbol{\sigma} \cdot \mathbf{p})\psi(x,y) = E\psi(x,y). \tag{26}$$

Dirac-Weyl Equation in a magnetic field

Let us consider a magnetic field perpendicular to the graphene surface $\mathbf{B} = (0, 0, B(x))$. In the Landau gauge, the vector potential that generates this magnetic field can be chosen such that

$$\mathbf{A} = (0, A(x), 0), \quad B(x) = \frac{dA(x)}{dx}. \tag{27}$$

Using the minimal coupling rule, $\mathbf{p} \to \mathbf{p} + \frac{e}{c}\mathbf{A}$, with $(-e)$ being the charge of the electron. Thus, the stationary Dirac-Weyl Euqation is given by

$$\left[p_x\sigma_x + \left(p_y + \frac{e}{c}A(x)\right)\sigma_y\right]\psi(x, y) = \frac{E}{v_F}\psi(x, y) \tag{28}$$

The equivalent problem

The system have translational symmetry in the y-direction, thus, we can propose the *spinor* $\psi(x, y)$ has the following form

$$\psi(x, y) = e^{iky} \begin{pmatrix} \psi^{+}(x) \\ i\psi^{-}(x) \end{pmatrix}. \tag{29}$$

Substituting the Eq. (26), we arrive at the system of equations

$$\left[\frac{d}{dx} + k + \frac{e}{c\hbar}A(x)\right]\psi^{-}(x) = \varepsilon\psi^{+}(x), \tag{30}$$

$$\left[-\frac{d}{dx} + k + \frac{e}{c\hbar}A(x)\right]\psi^{+}(x) = \varepsilon\psi^{-}(x), \tag{31}$$

where $\varepsilon = E/v_F\hbar$.

Decoupled System

We can decouple the system of equations (30) and (31), we have that

$$\left[-\frac{d^2}{dx^2}+\left(k+\frac{eA(x)}{c\hbar}\right)^2+\frac{e}{c\hbar}\frac{dA(x)}{dx}\right]\psi^+=\varepsilon^2\psi^+(x), \tag{32}$$

$$\left[-\frac{d^2}{dx^2}+\left(k+\frac{eA(x)}{c\hbar}\right)^2-\frac{e}{c\hbar}\frac{dA(x)}{dx}\right]\psi^-=\varepsilon^2\psi^-(x). \tag{33}$$

We can associated these operators to two Schrödinger-like Hamiltonians $H^{\pm}$ with potentials

$$V^{\pm}(x)=\left(k+\frac{eA(x)}{c\hbar}\right)^2\pm\frac{e}{c\hbar}\frac{dA(x)}{dx} \tag{34}$$

The SUSY transformation

Comparing with the solutions of the previous solutions, we can observe that the Hamiltonians $H^{\pm}$ are SUSY partners, while the intertwining operators are

$$L^{\pm} = \mp \frac{d}{dx} + w(x), \quad w(x) = k + \frac{eA(x)}{c\hbar}. \tag{35}$$

From the Eq. (34), we can take the factorization energy $\epsilon = 0$. And, from Eq. (14) it follows that the Hamiltonians $H^{\pm}$ are positives, thus, the seed solution (if it is square-integrable) is the ground state eigenfucntion of H^{-}.
Moreover, taking the derivative of the superpotential, we arrive at

$$B(x) = \frac{dw(x)}{dx} = \frac{e}{c\hbar}\frac{dA(x)}{dx}. \tag{36}$$

Dirac-Weyl Equation solutions

If the Hamiltonians $H^{\pm}$ have solutions, with eigenfunctions $\psi_n^{\pm}(x)$ and eigenvalues ε_n, the *spinor* $\psi(x, y)$ satisfying the Dirac-Weyl Eq. (26) has the following form

$$\psi_n(x, y) = e^{iky} \begin{pmatrix} \psi_{n-1}^{+}(x) \\ i\psi_n^{-}(x) \end{pmatrix}, \quad E_n = \hbar v_F \sqrt{\varepsilon_n}, \quad n = 1, 2, 3, \ldots \tag{37}$$

In particular, the zero-mode turns out to be

$$\psi_0(x, y) = e^{iky} \begin{pmatrix} 0 \\ \psi_0^{-}(x) \end{pmatrix}, \quad E_0 = 0. \tag{38}$$

Constant magnetic field

Taking a constant magnetic field:

$$\mathbf{B} = (0, 0, B_0) \Rightarrow \mathbf{A} = (0, B_0 x, 0). \tag{39}$$

Using the Eq. (35), we have the superpotential is

$$w(x) = k + \frac{eB_0}{c\hbar} x. \tag{40}$$

And substituting into the Eq. (34), the SUSY partner potentials are

$$V^{\pm}(x) = \frac{\omega^2}{4} \left(x + \frac{2k}{\omega} \right)^2 \pm \frac{\omega}{2}, \tag{41}$$

where $\omega = 2B_0/c\hbar$.

Constant magnetic field

The eigenfunctions are given in terms of the Hermite polynomials:

$$\psi_n^{\pm}(x) = c_n e^{\frac{\omega}{4}\left(x+\frac{2k}{\omega}\right)^2} \mathcal{H}\left(\sqrt{\frac{\omega}{2}}\left(x + \frac{2k}{\omega}\right)\right), \quad \varepsilon_n = \omega n, \quad n = 0, 1, 2, \ldots \tag{42}$$

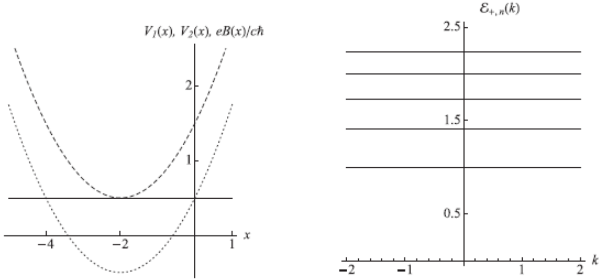


Figura: (Left) Shape-invariant potentials for a constant magnetic field. (Right) Energies for the Dirac electron. The parameter values taken are $\omega = k = 1$. Picture taken of[2].

[2] Ş. Kuru/J. Negro y L. M. Nieto: Exact analytic solutions for a Dirac electron moving in graphene under magnetic fields, en: J. Phys.: Condens. Matter 21 (2009), pág. 455305.

Trigonomtric singular well

In this case, we consider a magnetic field of the form

$$\mathbf{B} = (0, 0, B_0 \csc^2 \mu x) \Rightarrow \mathbf{A} = \left(0, -\frac{B_0}{\mu} \cot \mu x, 0\right), \quad 0 \leq \mu x \leq \pi. \tag{43}$$

The corresponding superpotential is

$$w(x) = k - D \cot \mu x, \quad D = \frac{eB_0}{\mu c \hbar}. \tag{44}$$

While the SUSY partner potentials tun out to be

$$V^{\pm}(x) = k^2 - D^2 + D(D \pm \mu) \csc^2 \mu x - 2kD \cot \mu x. \tag{45}$$

Trigonometric singular well

The eigenfunctions are in terms of the pseudo-Jacobi polynomials:

$$\psi_n^{\pm}(z) = (z^2-1)^{-(s_{\pm}+n)/2} e^{a_{\pm}\mu x} \mathcal{P}_n^{(-s_{\pm}-n+ia_{\pm}, -s_{\pm}-n-ia_{\pm})}(z), \quad \varepsilon_n = k^2 - D^2 + (D+n\mu)^2 - \frac{k^2 D^2}{(D+n\mu)^2}, \tag{46}$$

where $s_- = D/\mu$, $s_+ = S_- + 1$, $a_- = -kD/\mu(D+n\mu)$, $a_+ = -kD/\mu(D+n\mu+\mu)$, $z(x) = i\cot\mu x$ and $n = 0, 1, 2, ...$

The square-integrable conditions of the eigenfunctions limit the parameter range, in other words, $D, \mu > 0$.

Trigonometric singular well

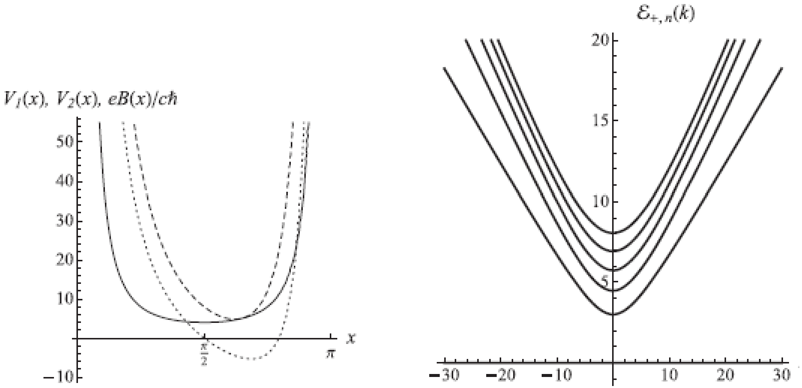


Figura: (Left) Shape-invariant potentials for the trigonometric singular well. (Right) Energies for the Dirac electron. The parameter values taken are $D = 4$, $\mu = 1$, $k = -2$. Picture taken of[3].

[3] Kuru/Negro y Nieto: Exact analytic solutions for a Dirac electron moving in graphene under magnetic fields (ver n. 2).

Exponential decaying field

As last example, let us take an exponential decaying field:

$$\mathbf{B} = (0, 0, B_0 e^{-\mu x}) \Rightarrow \mathbf{A} = \left(0, -\frac{B_0}{\mu} e^{-\mu x}, 0\right). \tag{47}$$

The corresponding superpotential:

$$w(x) = k - De^{-\mu x}, \quad D = \frac{eB_0}{\mu c\hbar}. \tag{48}$$

Thus, the SUSY partner potential are

$$V^{\pm}(x) = k^2 + D^2 e^{-2\mu x} - 2D\left(k \mp \frac{\mu}{2}\right) e^{-\mu x}. \tag{49}$$

Exponential decaying field

The eigenfunctions are given in terms of the associated Laguerre polynomials:

$$\psi_n^{\pm}(x) = z^{s_\pm - n} e^{-z/2} \mathcal{L}_n^{2(s_\pm - n)}(z), \quad \varepsilon_n = k^2 - (k - n\mu)^2, \tag{50}$$

with $s_- = k/\mu$, $s_+ = s_- - 1$ y $z(x) = (2D/\mu)e^{-\mu x}$. In this case, the square-integrable conditions limit the parameter values $D, k, \mu > 0$ and the number of bound states by means of the following inequality $k > \mu n$.

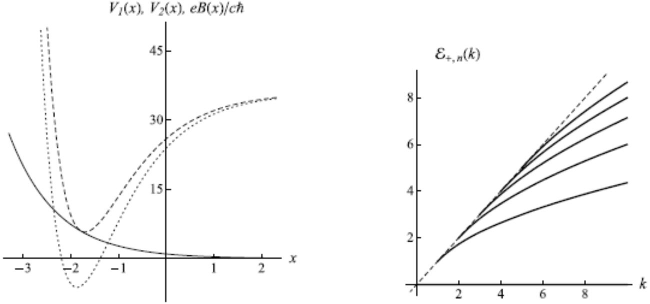


Figura: (Left) Shape-invariant potentials for the exponential decaying field. (Right) Energies for the Dirac electron. The parameter values taken are $D = 1$, $\mu = 1$, $k = 6$. Picture taken of[4].

[4] Kuru/Negro y Nieto: Exact analytic solutions for a Dirac electron moving in graphene under magnetic fields (ver n. 2).

SUSY-QM and complex superpotentials

If we have intertwining superpotentials with complex superpotentials, i.e.,

$$L^{\pm} = \mp \frac{d}{dx} + w(x), \quad w(x) \in \mathbb{C}. \tag{51}$$

In this case, the operator L^{+} is not the hermitian conjugate of L^{-}. thus, both hamiltonians $H^{\pm}$ are not hermitian, and in general their eigenvalues ε_n will be complex.

However, ¡the supersymmetric algorithm works similar to the real case!

Graphene in complex magnetic fields

Let us consider a magnetic field perpendicular to the graphene surface of the form

$$\mathbf{B} = B(x)\ \hat{\mathbf{z}}, \quad B(x) \in \mathbb{C}. \tag{52}$$

In the Landau Gauge, the vector potential that generates **B** can be written as

$$\mathbf{A} = A(x)\ \hat{\mathbf{y}}, \quad B(x) = \frac{dA(x)}{dx}. \tag{53}$$

And, using the minimal coupling rule, we can obtain a similar system of equations

$$\left[\frac{d}{dx} + k + \frac{e}{c\hbar}A(x)\right]\psi^-(x) = \varepsilon\psi^+(x), \tag{54}$$

$$\left[-\frac{d}{dx} + k + \frac{e}{c\hbar}A(x)\right]\psi^+(x) = \varepsilon\psi^-(x). \tag{55}$$

Constant magnetic field

Our first example is a constant magnetic field $\mathbf{B} = B\,\hat{\mathbf{z}}, B \in \mathbb{C}$. In the Landau gauge $\mathbf{A} = Bx\,\hat{\mathbf{y}}$ and the superpotential turns out to be

$$w(x) = k + \frac{\omega}{2}x, \quad \omega = \frac{2eB}{c\hbar}. \tag{56}$$

While the SUSY partner potentials are

$$V^{\pm}(x) = \frac{\omega^2}{4}\left(x + \frac{2k}{\omega}\right)^2 \pm \frac{\omega}{2}. \tag{57}$$

Their corresponding eigenfunctions and eigenvalues are given by

$$\psi_n^{\pm}(x) = \begin{cases} c_n e^{-\frac{\zeta^2}{2}} \mathcal{H}(\zeta), & -\frac{\pi}{2} < \theta < \frac{\pi}{2}, \\ c_n e^{-\frac{\xi^2}{2}} \mathcal{H}(\xi), & \frac{\pi}{2} < \theta < \frac{3\pi}{2}, \end{cases} \qquad \varepsilon_n = n\omega, \tag{58}$$

where $\zeta = \sqrt{\omega/2}(x + 2k/\omega)$, $\xi = \sqrt{-\omega/2}(x - 2k/\omega)$ and $\omega = |\omega|e^{i\theta}$.

Constant magnetic field

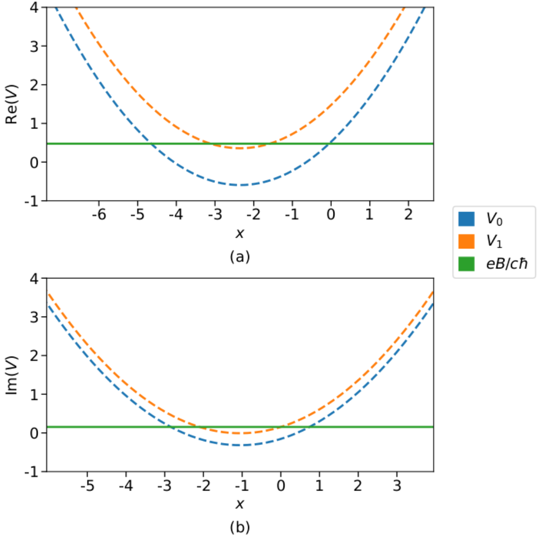


Figura: Real part (a) and imaginary part (b) of the potentials $V^{\pm}(x)$, in the case of constant magnetic field with $|\omega| = k = 1$ and $\theta = \pi/10$.

Constant magnetic field

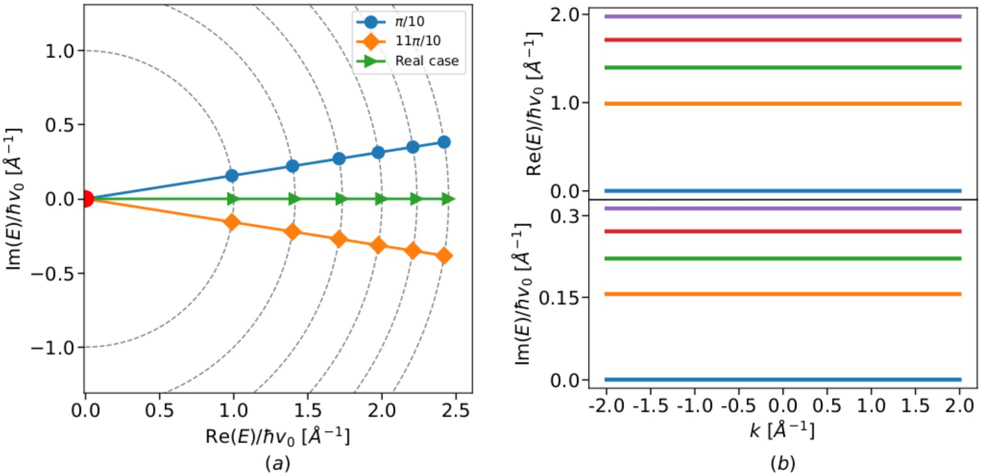


Figura: (a) First energy eigenvalues in the complex plane for a constant magnetic field. (b) Real and imaginary part of the first energy eigenvaluesas function of k for $|\omega| = 1$ y $\theta = \pi/10$.

Trigonometric singular well

Now, let us consider a magnetic field of the form $\mathbf{B} = B\csc^2\mu x\ \hat{\mathbf{z}}, B \in \mathbb{C}, \mu \in \mathbb{R}^+$. Then, the vector potential is $\mathbf{A} = -B/\mu \cot\mu x\ \hat{\mathbf{y}}$, and the superpotential is

$$w(x) = k - D\cot\mu x, \quad D = \frac{eB}{\mu c\hbar}. \tag{59}$$

The SUSY partener potential turns out to be

$$V^{\pm}(x) = D(D \pm \mu)\csc^2\mu x - 2Dk\cot\mu x + k^2 - D^2, \tag{60}$$

whose solutions are given by

$$\begin{aligned} \psi^{\pm}(\zeta) &= c_n(-1)^{-(s_\pm+n)/2}(1+\zeta^2)^{-(s_\pm+n)/2}e^{r_\pm \operatorname{arccot}(\zeta)}\mathcal{P}_n^{(-s_\pm-n-ir_\pm,-s_\pm-n+ir_\pm)}(i\zeta), \\ \varepsilon_n &= k^2 - D^2 + (D+n\mu)^2 - \frac{k^2D^2}{(D+n\mu)^2}, \quad -\frac{\pi}{2} < \theta < \frac{\pi}{2}, \end{aligned} \tag{61}$$

with $s_- = D/\mu$, $s_+ = s_- + 1$, $r_- = -kD/\mu(D+n\mu)$, $r_+ = -kD/\mu(D+n\mu+\mu)$, $D = |D|e^{i\theta}$ and $\zeta = \cot\mu x$.

Trigonometric singular well

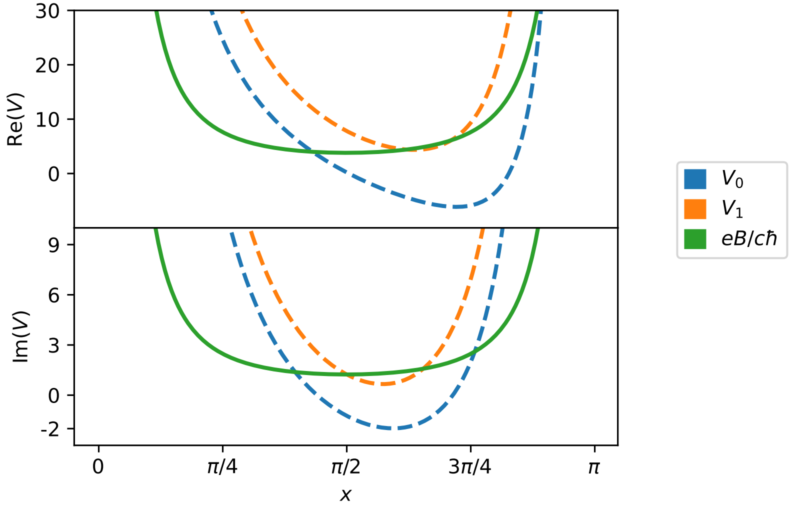


Figura: Real part (a) and imaginary part (b) of the potentials $V^{\pm}(x)$ in the case of a trigonometric singular well with $|D| = 4$, $k = -2$, $\theta = \pi/10$ and $\mu = 1$.

Trigonometric singular well

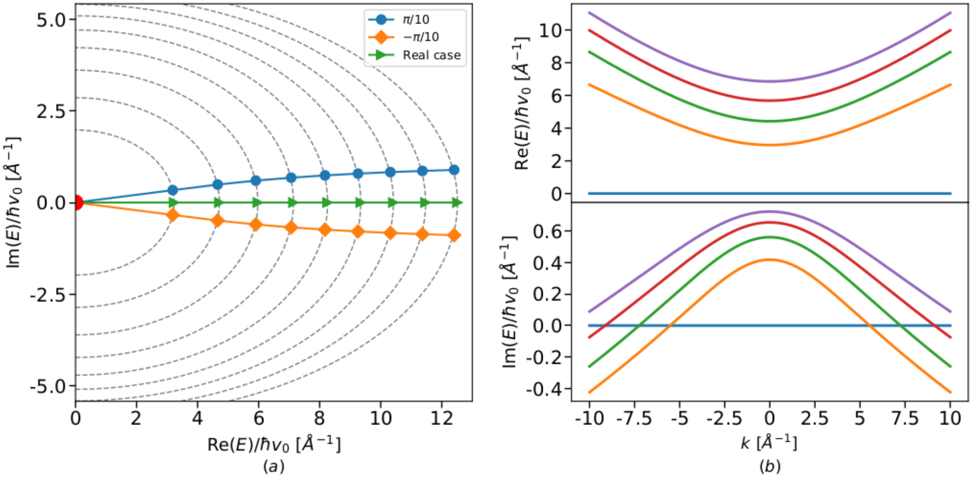


Figura: (a) First energy eigenvalues in the complex plane for the trigonometric singular well. (b) Real and imaginary part of the first energy eigenvalues as functions of k for $|D| = 4$, $k = -2$, $\mu = 1$ and $\theta = \pi/10$.

Exponenetial decaying field

Our last example is an exponential decaying field $\mathbf{B} = Be^{-\mu x}\ \hat{\mathbf{z}}, B \in \mathbb{C}, \mu \in \mathbb{R}^{+}$. The corresponding vector potential is $\mathbf{A} = -B/\mu e^{-\mu x}\ \hat{\mathbf{y}}$. Thus, the superpotential is given by

$$w(x) = k - De^{-\mu x}, \quad D = \frac{eB}{\mu c\hbar}. \tag{62}$$

While the SUSY partner potentials are

$$V^{\pm}(x) = k^2 + D^2 e^{-2\mu x} - 2D\left(k \pm \frac{\mu}{2}\right) e^{-\mu x}, \tag{63}$$

whose solutions are

$$\psi_n^{\pm}(\zeta) = c_n \zeta^{s_{\pm}-n} e^{-\frac{\zeta}{2}} \mathcal{L}_n^{2(s_{\pm}-n)}(\zeta), \quad \varepsilon_n = k^2 - (k - n\mu)^2, \tag{64}$$

where $s_- = k/\mu$, $s_+ = s_- - 1$, $D = |D|e^{i\theta}$, $-\pi/2 < \theta < \pi/2$ and $\zeta = 2D/\mu e^{-\mu x}$, $k > n\mu$.

Exponential decaying field

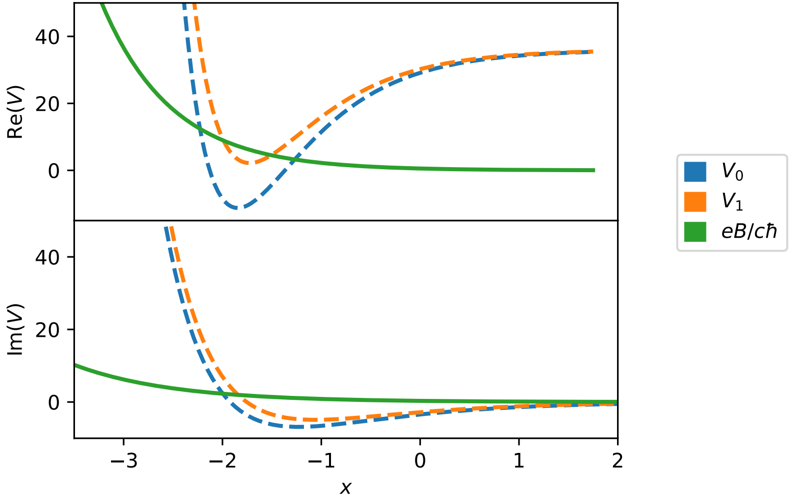


Figura: Real part (a) and imaginary part (b) of the potentials $V^{\pm}(x)$ in the case of the exponenetial decaying field with $|D| = 1$, $k = 6$, $\theta = \pi/10$ and $\mu = 1$.

Exponenetial decaying field

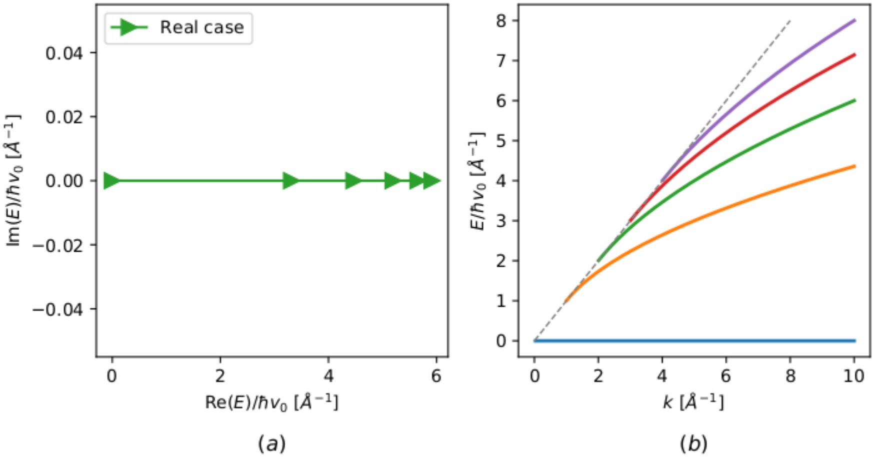


Figura: (a) First energy eigenvalues in the complex plane for the exponential decaying field. (b) Real and imaginary part of the first energy eigenvalues as functions of k for $k = 6$, $\mu = 1$.

A possible physical interpretation

The total probability density of a bound state of the effective Hamiltonian of the graphene in the presence of a complex magnetic field evolves as follows

$$P_T(t) = \langle \psi_n(t) | \Psi_n(t) \rangle = e^{2\frac{\mathrm{Im}[E_n]}{\hbar}} \langle \Psi_n(0) | \Psi_n(0) \rangle. \tag{65}$$

For the case of a constant magnetic field, if the argument $\theta \ll 1$, the imaginary part of the energy eigenvalue is approximately $\theta/2$, and the maximum time for which the probability density is conserved is limit by

$$T \ll \frac{1}{V_F \theta \sqrt{|\omega| n}}, \tag{66}$$

for the first exited state and a magnetic strength of 5 T and $\theta = \pi/36$, $T \ll 95$ fs.

In summary

- The supersymmetric transformation is a useful tool to find the solutions to the effective Hamiltonian describing the graphene monolayer under the presence of a magnetic field perpendicular to it.
- It is also possible to find the solutions for the case of complex magnetic field profiles, which makes the effective Hamiltonian of the monolayer non-hermitian. However, this system decays rapidly to the real case if we consider that the argument of the field amplitude is small.
- The use of SUSY-QM to solve Dirac-like equations describing 2D materials still has a long way to go. We have worked Dirac materials in $(3+1)$-dimensions and determined their zero energy mode, see[5]. We are currently working on graphene under uniaxial stresses and have found that there are a finite number of Landau levels.

[5] Julio Cesar Pérez-Pedraza/Juan D. García-Muñoz y A. Raya: Dirac materials in parallel non-uniform electromagnetic fields generated by SUSY: A new class of chiral Planar Hall Effect?, 2023.

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thank
you!