Charged particles in strong magnetic fields

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PLAN OF THE LECTURES

- Lecture 1: Order of magnitude of B in different contexts. Units of B in different unit systems. Charged particles in a magnetic field: Classical case and non- relativistic quantum case.

- Lecture 2: Charged particles in a magnetic field: Relativistic quantum case. Charged fields and propagators for spin 0, ½ and 1

- Lecture 3: Schwinger phase (SP). SP and charged particle propagators. Connection between different representations of charged particle propagators. An example: Leading Order Correction to the charged pion propagator.

Schwinger phase

Given a particle P with electric charge Q_{P}, we denote the associated Schwinger phase (SP) by Φ_{P}. Its explicit form is

$$\boxed{\Phi_{\mathrm{P}}(x,y) = Q_{\mathrm{P}} \int_x^y d\xi_\mu \left[\mathcal{A}^\mu(\xi) + \frac{1}{2}\, F^{\mu\nu}\,(\xi_\nu - y_\nu)\right]} \;,$$

where $F^{\mu\nu}$ is assumed to be constant, and the integration is performed along an arbitrary path that connects x with y.

In general, the SP is found to be not invariant under either translations, rotations or gauge transformations.

On the other hand, the integral in Φ_{P} is shown to be path independent; thus, it can be evaluated using a straight line path.

$$\boxed{\xi^\mu = \frac{x^\mu + y^\mu}{2} - \lambda\, \frac{x^\mu - y^\mu}{2}} \qquad \text{with} \qquad -1 \leq \lambda \leq 1$$

Using, in addition

$$\boxed{\mathcal{A}^\mu(x) = \frac{1}{2} x_\nu F^{\nu\mu} + \partial^\mu \Psi(x)}$$

one can obtain a closed expression for the SP associated to a static and uniform magnetic field in an arbitrary gauge. It reads

$$\Phi_{\rm P}(x,y) = \frac{Q_{\rm P}}{2}\, x_\mu\, F^{\mu\nu}\, y_\nu - Q_{\rm P}\left[\Psi(x) - \Psi(y)\right]$$

For the “standard” gauges introduced in previous lecture

$$\boxed{\begin{array}{ll} \text{SG:} & \Phi_{\rm P}(x,y) = -\dfrac{Q_{\rm P}B}{2}(x^1y^2 - y^1x^2)\ ; \\ \text{LG1:} & \Phi_{\rm P}(x,y) = -\dfrac{Q_{\rm P}B}{2}(x^2 + y^2)(x^1 - y^1)\ ; \\ \text{LG2:} & \Phi_{\rm P}(x,y) = \dfrac{Q_{\rm P}B}{2}(x^1 + y^1)(x^2 - y^2)\ . \end{array}}$$

Under gauge transformation the SP transforms as

$$\boxed{\Phi_{\rm P}(x,y) \ \to\ \tilde{\Phi}_{\rm P}(x,y) = \Phi_{\rm P}(x,y) - Q_{\rm P}\left[\Lambda(x) - \Lambda(y)\right]}$$

SP always includes products that mix the coordinates of the points x and y . There is no way in which these combinations could be expressed in terms of the difference between a scalar function Λ evaluated at x and the same function evaluated at y

If the SP does not vanish in a given gauge it will be nonvanishing *in any gauge*. The SP cannot be ``gauged away''.

Schwinger phase and particle propagators

In the previous lecture we have seen how to write the particles propagators in terms of the Ritus eigenfunctions. Now, we will see an alternative form to express them. For simplicity we consider the case of S=0 (the pion). For S=1/2 and 1 similar arguments apply.

We have seen that a gauge transformation the pion propagator $\Delta_{\pi Q}(x,y)$ transforms as

$$\boxed{\Delta_{\pi\mathcal{Q}}(x,y) \;\rightarrow\; \tilde{\Delta}_{\pi\mathcal{Q}}(x,y) = e^{-iQ_\pi \Lambda(x)}\; \Delta_{\pi\mathcal{Q}}(x,y)\; e^{iQ_\pi \Lambda(y)}}$$

Given the way in which the SP transforms under the same transformation it is clear that we can always write

$$\boxed{\Delta_{\pi\mathcal{Q}}(x,y) = e^{i\Phi_{\pi\mathcal{Q}}(x,y)}\;\bar{\Delta}_{\pi\mathcal{Q}}(x,y)}$$

where $\bar{\Delta}_{\pi Q}(x,y)$ is a gauge invariant function; the gauge dependence of the propagator is carried by the SP, which has a well defined expression.

Since we are dealing with a system subject to a static and uniform magnetic field, the invariance under translations in time and space, under rotations around any axis parallel to the magnetic field, and under boosts indirections parallel to the magnetic field, is expected to be preserved.

Translations in time, as well as translations and boosts in the direction of $\vec{B}$, can be treated in the same way as in the case of a free particle, since they do not involve the axes 1 or 2.

We focus on the translations in the plane perpendicular to $\vec{B}$ and in the rotations around the $\vec{B}$ direction. The expected invariance seems to be at odds with the fact that the charged pion propagator is known to be not invariant under these transformations. For example, in previous lecture we have seen that it is expressed in terms of $\mathcal{F}_{\mathcal{Q}}(x,\bar{q})$ that when written in the SG are not invariant under translations

In what follows we clarify this point and see how the invariance implies further constraints on the form of the propagator.

Let us first consider space translations in the perpendicular plane, i.e. a general transformation of the form $x^\mu \to x'^\mu = x^\mu + b_\perp^\mu$. From $\mathcal{A}^\mu(x) = \frac{1}{2} x_\nu F^{\nu\mu} + \partial^\mu \Psi(x)$ under this transformation one has

$$\boxed{\mathcal{A}^\mu(x) \;\to\; \mathcal{A}_t^\mu(x) \equiv \mathcal{A}^\mu(x') = \mathcal{A}^\mu(x) - \frac{1}{2} F^{\mu\nu} b_{\perp\nu} + \partial^\mu \Psi(x') - \partial^\mu \Psi(x)}$$

It is rather easy to see that (as in the classical case discussed in 1st lecture) this is fully equivalent to a gauge transformation

$$\boxed{\mathcal{A}_\mu(x) \to \tilde{\mathcal{A}}^\mu(x) = \mathcal{A}^\mu(x) + \partial^\mu \Lambda_t(x; b_\perp)}$$

where

$$\boxed{\Lambda_t(x; b_\perp) = \Psi(x') - \Psi(x) - \frac{1}{2} x_\mu F^{\mu\nu} b_{\perp\nu}}$$

A similar relation between the translation and the gauge transformation can be obtained for the Schwinger phase. Under the translation, the SP transforms as

$$\boxed{\Phi_{\pi\mathcal{Q}}(x,y) \;\to\; \Phi_{\pi\mathcal{Q},t}(x,y) \equiv \Phi_{\pi\mathcal{Q}}(x',y') = \frac{Q_\pi}{2} x'_\mu F^{\mu\nu} y'_\nu - Q_\pi [\Psi(x') - \Psi(y')]}$$

whereas performing the corresponding gauge transformation one gets

$$\boxed{\Phi_{\pi\mathcal{Q}}(x,y) \;\to\; \tilde{\Phi}_{\pi\mathcal{Q}}(x,y) = \Phi_{\pi\mathcal{Q}}(x,y) - Q_\pi [\Lambda_t(x; b_\perp) - \Lambda_t(y; b_\perp)]}$$

Taking into account the form of $\Lambda_t(x; b_\perp)$ we observe that $\Phi_{\pi\mathcal{Q},t}(x,y) = \tilde{\Phi}_{\pi\mathcal{Q}}(x,y)$

From the above equations it is seen that under the considered translation the operator $\mathcal{D}^{\mu}\ \mathcal{D}_{\mu}+m_{\pi}^{2}$ and the factor $\exp[\Phi_{\pi Q}(x,y)]$ transform in the same way as under the gauge transformation $\Lambda_t(x;b_\perp)$.

Together with the requirement that the equation

$$\boxed{\left(\mathcal{D}^{\mu}\,\mathcal{D}_{\mu}+m_{\pi}^{2}\right)\Delta_{\pi\mathcal{Q}}(x,y)=-\,\delta^{(4)}(x-y)}$$

must be translational invariant, this implies that the gauge invariant factor $\bar{\Delta}_{\pi Q}(x,y)$ has to be also translational invariant and can be Fourier transformed

$$\boxed{\bar{\Delta}_{\pi\mathcal{Q}}(x,y)=\bar{\Delta}_{\pi\mathcal{Q}}(x-y)=\int\frac{d^4v}{(2\pi)^4}\;e^{-i\,v\,(x-y)}\;\bar{\Delta}_{\pi\mathcal{Q}}(v_\parallel,v_\perp)\,.}$$

where we have explicit that one can have different dependences on $v_\parallel$ and $v_\perp$.

Similar arguments hold for the rotations around the $\vec{B}$ direction (i.e. the 3-axis). This fact together with the invariance under boost along the 3-axis implies

$$\boxed{\Delta_{\pi\mathcal{Q}}(x,y)=e^{i\Phi_{\pi\mathcal{Q}}(x,y)}\;\int\frac{d^4v}{(2\pi)^4}\;e^{-i\,v\,(x-y)}\;\bar{\Delta}_{\pi\mathcal{Q}}(v_\parallel,v_\perp)\;.}$$

where $\bar{\Delta}_{\pi Q}(v_\parallel,v_\perp)$ can only depend on $v_\parallel^2$ and $v_\perp^2$

An entirely similar analysis can be performed for the case of spin 1/2 and spin 1 particles.

For the spin 1/2 fermion propagator one has

$$\boxed{S_f(x,y) = e^{i\Phi_f(x,y)} \int \frac{d^4v}{(2\pi)^4}\, e^{-i\,v(x-y)}\, \bar{S}_f(v_\parallel, v_\perp)}$$

Here, $\bar{S}_f(v_\parallel, v_\perp)$ is a matrix in Dirac space combination of $\gamma_\parallel . v_\parallel$ and $\gamma_\perp . v_\perp$ with coefficients depending $v_\parallel^2$, $v_\perp^2$,

For the spin 1 boson propagator one has

$$\boxed{D^{\nu\gamma}_{\rho\mathcal{Q}}(x,y) = e^{i\Phi_{\rho\mathcal{Q}}(x,y)} \int \frac{d^4v}{(2\pi)^4}\, e^{-i\,v(x-y)}\, \bar{D}^{\nu\gamma}_{\rho\mathcal{Q}}(v_\parallel, v_\perp)}$$

Similarly to the previous cases, invariance under rotations around the3-axis and under boosts in that direction implies that $D^{\nu\gamma}_{\rho Q}(v_\parallel, v_\perp)$ will be given by a linear combination of tensors of order two built from the tensors $g_\parallel^{\mu\nu}, g_\perp^{\mu\nu}, F^{\mu\nu}$ and the vectors $v_\parallel^\mu$ and $v_\perp^\mu$ with coefficients given by functions that depend only on $v_\parallel^2$ and $v_\perp^2$

Connection between different representations of propagators

In the previous lecture we have seen that $S=0$ propagator is

$$\Delta_{\pi^{\mathcal{Q}}}(x,y) \quad = \quad \sum\!\!\!\!\!\!\!\int_{\bar{q}} \mathbb{F}^{\mathcal{Q}}(x,\bar{q})\,\hat{\Delta}_{\pi^{\mathcal{Q}}}(k,q_{\parallel})\,\mathbb{F}^{\mathcal{Q}}(y,\bar{q})^{*}$$

where

$$\hat{\Delta}_{\pi^{\mathcal{Q}}}(k,q_{\parallel}) = 1/\left[q_{\parallel}^{2} - m_{\pi}^{2} - (2k+1)B_{\pi} + i\epsilon\right]$$

Now, we have learnt that for gauge and symmetry arguments it can be also written as

$$\Delta_{\pi^{\mathcal{Q}}}(x,y) = e^{i\Phi_{\pi^{\mathcal{Q}}}(x,y)} \int \frac{d^4 v}{(2\pi)^4}\; e^{-i\,v\,(x-y)}\;\bar{\Delta}_{\pi^{\mathcal{Q}}}(v_{\parallel},v_{\perp})$$

Which is the expression for $\bar{\Delta}_{\pi Q}(v_{\parallel}\ ,v_{\perp}\ )$?

Equating the two expressions we have

$$\int \frac{d^4v}{(2\pi)^4}\; e^{-i\,v\,(x-y)}\,\bar{\Delta}_{\pi\mathcal{Q}}(v_{\parallel}, v_{\perp}) \;=\; e^{-i\Phi_{\pi\mathcal{Q}}(x,y)} \sum\!\!\!\!\!\!\!\int_{\bar{q}} \mathbb{F}^{\mathcal{Q}}(x,\bar{q})\,\hat{\Delta}_{\pi\mathcal{Q}}(k,q_{\parallel})\,\mathbb{F}^{\mathcal{Q}}(y,\bar{q})^*$$

Making the change of variables $x = t + z/2$, and $y = t - z/2$ and recalling $\mathbb{F}^{\mathcal{Q}}(x,\bar{q}) = \mathcal{F}_{\mathcal{Q}}(x,\bar{q})$

$$\int \frac{d^4v}{(2\pi)^4}\; e^{-i\,v\,z}\,\bar{\Delta}_{\pi\mathcal{Q}}(v_{\parallel}, v_{\perp}) \;=\; e^{-i\Phi_{\pi\mathcal{Q}}\left(t+\frac{z}{2},t-\frac{z}{2}\right)} \sum\!\!\!\!\!\!\!\int_{\bar{q}} \mathcal{F}_Q\left(t+\frac{z}{2},\bar{q}\right)\,\hat{\Delta}_{\pi\mathcal{Q}}(k,q_{\parallel})\,\mathcal{F}_Q\left(t-\frac{z}{2},\bar{q}\right)^*$$

Now, we multiply both sides by $e^{i\,z\,p}$ and integrate over z

$$\begin{aligned}\int d^4z\; \int \frac{d^4v}{(2\pi)^4}\; e^{-i\,(v-p)\,z}\,\bar{\Delta}_{\pi\mathcal{Q}}(v_{\parallel}, v_{\perp}) &= \bar{\Delta}_{\pi\mathcal{Q}}(p_{\parallel}, p_{\perp}) \\ &= \int d^4z\; e^{i\,p\,z} e^{-i\Phi_{\pi\mathcal{Q}}\left(t+\frac{z}{2},t-\frac{z}{2}\right)} \sum\!\!\!\!\!\!\!\int_{\bar{q}} \mathcal{F}_Q\left(t+\frac{z}{2},\bar{q}\right)\,\hat{\Delta}_{\pi\mathcal{Q}}(k,q_{\parallel})\,\mathcal{F}_Q\left(t-\frac{z}{2},\bar{q}\right)^*\end{aligned}$$

Recalling

$$\sum\!\!\!\!\!\!\!\int_{\bar{q}} \equiv \frac{1}{2\pi}\sum_{k=0}^{\infty}\int_{q_\parallel}\sum\!\!\!\!\!\!\!\int_{\chi}$$

where

$$\int_{q_\parallel} = \int \frac{dq^0}{2\pi}\frac{dq^3}{2\pi}$$

$$\sum\!\!\!\!\!\!\!\int_{\chi} = \begin{cases} \frac{1}{2\pi}\sum_{\imath} & SG \\ \int \frac{dq^i}{2\pi} & LGi \end{cases}$$

We can write

$$\bar{\Delta}_{\pi\mathcal{Q}}(p_\parallel^2, p_\perp^2) = \frac{1}{2\pi}\sum_k \int_{q_\parallel} \hat{\Delta}_{\pi\mathcal{Q}}(k, q_\parallel)\ I(k, k, q_\parallel, t, p)$$

where

$$I(k, k', q_\parallel, t, p) = \sum\!\!\!\!\!\!\!\int_{\chi} \int d^4z\ e^{i\,p\,z}\ \mathcal{F}_Q\left(t - \frac{z}{2}, \bar{q}_{k'}\right)^*\ e^{-i\Phi_{\pi\mathcal{Q}}\left(t+\frac{z}{2}, t-\frac{z}{2}\right)}\ \mathcal{F}_Q\left(t + \frac{z}{2}, \bar{q}_k\right)$$

with $\bar{q}_k = (q^0, k, \chi, q^3)$.

I is gauge invariant. It can be calculated in any gauge.

Using e.g. LG2 one can show

$$\boxed{I(k,k',q_\parallel,t,p) = (2\pi)^2\delta^{(2)}(q_\parallel - p_\parallel)\ B_Q\ f_{kk'}(\vec{p}_\perp)}$$

Where $\vec{p}_\perp = (p^1,p^2) = |\vec{p}_\perp|\ (\cos\phi_\perp, \sin\phi_\perp)$ and

$$\boxed{f_{kk'}(\vec{p}_\perp) = \frac{4\pi(-i)^{k+k'}}{B_Q}\sqrt{\frac{k!}{k'!}}\left(\frac{2\vec{p}_\perp^{\,2}}{B_Q}\right)^{\frac{k'-k}{2}} L_k^{k'-k}\left(2\vec{p}_\perp^{\,2}/B_Q\right)\ e^{-\vec{p}_\perp^{\,2}/B_Q}\ e^{is(k-k')\phi_\perp}}$$

For the case we are interested in (i.e. $k = k'$) we have

$$\boxed{I(k,k,q_\parallel,t,p) = (2\pi)^2\delta^{(2)}(q_\parallel - p_\parallel)4\pi\ (-1)^k\ e^{-\vec{p}_\perp^{\,2}/B_Q}\ L_k\left(2\vec{p}_\perp^{\,2}/B_Q\right)}$$

Replacing in $\bar{\Delta}_{\pi\mathcal{Q}}(p_\parallel,p_\perp) = \frac{1}{2\pi}\sum_k\int_{q_\parallel}\hat{\Delta}_{\pi\mathcal{Q}}(k,q_\parallel)\ I(k,k,q_\parallel,t,p)$ we finally get

$$\boxed{\bar{\Delta}_{\pi\mathcal{Q}}(p_\parallel,p_\perp) = 2e^{-\vec{p}_\perp^{\,2}/B_Q}\ \sum_k \frac{(-1)^k\ L_k\left(2\vec{p}_\perp^{\,2}/B_Q\right)}{p_\parallel^2 - m_\pi^2 - (2k+1)B_\pi + i\epsilon}}$$

which is a function of $p_\parallel^2$ and $p_\perp^2$ as expected. I call this the LL representation of $\overline{\Delta}_{\pi\mathcal{Q}}$

There is still an alternative way to express $\bar{\Delta}_{\pi Q}$.

Using the relation

$$\frac{1}{x+i\epsilon} = -i\int_0^\infty d\sigma\ e^{i\sigma(x+i\epsilon)}$$

We have

$$\begin{aligned}\bar{\Delta}_{\pi Q}(p_\parallel, p_\perp) &= -2ie^{-\vec{p}_\perp^2/B_Q}\int_0^\infty d\sigma \sum_k (-1)^k\ e^{i\sigma\left[p_\parallel^2 - m_\pi^2 - (2k+1)B_\pi + i\epsilon\right]} L_k\left(2\vec{p}_\perp^2/B_Q\right) \\ &= -2ie^{-\vec{p}_\perp^2/B_Q}\int_0^\infty d\sigma\ e^{i\sigma\left[p_\parallel^2 - m_\pi^2 - B_\pi + i\epsilon\right]} \sum_k (-1)^k\ e^{-2i\sigma B_\pi k} L_k\left(2\vec{p}_\perp^2/B_Q\right)\end{aligned}$$

Using the relation

$$\sum_{n=0} z^n L_n^\alpha(x) = \frac{e^{xz/(z-1)}}{(1-z)^{\alpha+1}}$$

With $\alpha = 0,\ x = 2\,\vec{p}^2/B_\pi$, $z = -\exp(-2i\sigma B_\pi)$ we get

$$\sum_k (-1)^k\ e^{-2i\sigma B_\pi k}\ L_k\left(2\vec{p}_\perp^2/B_Q\right) = e^{-i\sigma B_\pi} e^{\vec{p}^2/B_\pi \tan\sigma B_\pi}/(2\cos\sigma B_\pi)$$

Replacing in previous expression we finally have

$$\bar{\Delta}_{\pi Q}(p_\parallel, p_\perp) = -i\int_0^\infty \frac{d\sigma}{\cos\sigma B_\pi}\ \exp\left[-i\sigma\left(m_\pi^2 - p_\parallel^2 + \vec{p}_\perp^2\,\frac{\tan\sigma B_\pi}{\sigma B_\pi} - i\epsilon\right)\right]$$

We call this the Proper-Time (PT) representation of $\bar{\Delta}_{\pi Q}$

We conclude from this that there are 3 alternative ways to express $\Delta_{\pi\mathcal{Q}}(x,y)$

$$\begin{aligned}
\Delta_{\pi\mathcal{Q}}(x,y) &= \sum\!\!\!\!\!\!\int_{\bar{q}} \mathbb{F}^{\mathcal{Q}}(x,\bar{q})\, \hat{\Delta}_{\pi\mathcal{Q}}(k,q_{\parallel})\, \mathbb{F}^{\mathcal{Q}}(y,\bar{q})^{*}\ , \\
&= e^{i\Phi_{\pi\mathcal{Q}}(x,y)} \int \frac{d^4p}{(2\pi)^4}\, e^{-i\,p\,(x-y)}\, \bar{\Delta}_{\pi\mathcal{Q}}(p_{\parallel},p_{\perp})
\end{aligned}$$

where

$$\hat{\Delta}_{\pi\mathcal{Q}}(k,q_{\parallel}) = 1/\left[q_{\parallel}^2 - m_\pi^2 - (2k+1)B_\pi + i\epsilon\right]$$

Ritus space

$$\bar{\Delta}_{\pi\mathcal{Q}}(p_{\parallel},p_{\perp}) = 2e^{-\vec{p}_{\perp}^{\,2}/B_\pi} \sum_k \frac{(-1)^k\ L_k\left(2\vec{p}_{\perp}^{\,2}/B_\pi\right)}{p_{\parallel}^2 - m_\pi^2 - (2k+1)B_\pi + i\epsilon}$$

LL representation

$$\bar{\Delta}_{\pi\mathcal{Q}}(p_{\parallel},p_{\perp}) = -i\int_0^\infty \frac{d\sigma}{\cos\sigma B_\pi}\ \exp\left[-i\sigma\left(m_\pi^2 - p_{\parallel}^2 + \vec{p}_{\perp}^{\,2}\frac{\tan\sigma B_\pi}{\sigma B_\pi} - i\epsilon\right)\right]$$

PT representation

Similarly, for spin ½ we have

$$
\begin{aligned}
S_f(x,y) &= \sum\!\!\!\!\!\!\int_{\bar q} \mathbb{E}^{\mathcal{Q}_f}(x,\bar q)\, \hat S_f(k,q_\parallel)\, \bar{\mathbb{E}}^{\mathcal{Q}_f}(y,\bar q) \\
&= e^{i\Phi_f(x,y)} \int \frac{d^4 v}{(2\pi)^4}\, e^{-i\,v(x-y)}\, \bar S_f(v_\parallel, v_\perp)
\end{aligned}
$$

where

$$
\hat S_f(k,q_\parallel) = \frac{\hat{\not{\Pi}}_s + m_f}{q_\parallel^2 - m_f^2 - 2kB_f + i\epsilon}
$$

Ritus space

$$
\begin{aligned}
\bar S_f(v_\parallel, v_\perp) &= 2e^{-\vec v_\perp^{\,2}/B_Q} \sum_k \frac{(-1)^k}{v_\parallel^2 - m_f^2 - 2kB_f + i\epsilon} \\
&\quad\times \left[\left(v_\parallel\cdot\gamma_\parallel + m_f\right) \sum_\lambda \Gamma^\lambda L_{k_{s\lambda}}\!\left(2\vec v_\perp^{\,2}/B_f\right) + 2\,\vec v_\perp\cdot\vec\gamma_\perp\; L^1_{k-1}\!\left(2\vec v_\perp^{\,2}/B_f\right)\right]
\end{aligned}
$$

LL representation

$$
\begin{aligned}
\bar S_f(v_\parallel, v_\perp) &= -i\int_0^\infty d\sigma\;\exp\!\left[-i\sigma\left(m_f^2 - v_\parallel^2 + \vec v_\perp^{\,2}\,\frac{\tan(\sigma B_f)}{\sigma B_f} - i\epsilon\right)\right] \\
&\quad\times \left[\left(v_\parallel\cdot\gamma_\parallel + m_f\right)\left(1 - s\,\gamma^1\gamma^2\tan(\sigma B_f)\right) - \frac{\vec v_\perp\cdot\vec\gamma_\perp}{\cos^2(\sigma B_f)}\right]
\end{aligned}
$$

PT representation

For definitions of $\widehat{\Pi}_s, \Gamma^\lambda$ and $k_{s\lambda}$ see 2nd lecture

The same happens for spin 1 (see Dumm et al, PRD 108 (2023) 1, 016012)

An example: Leading Order Correction to the charged pion propagator

Let us consider the quark-pion interaction Lagrangian

$$\boxed{\mathcal{L}_{\rm int}^{(\pi q)} = g_s\,\bar\psi(x)\,i\gamma_5\,\vec\tau\cdot\vec\pi(x)\,\psi(x)} \quad \text{where} \quad \psi(x) = \begin{pmatrix} \psi_u(x) \\ \psi_d(x) \end{pmatrix} \; ,$$

We analyze now the leading order correction (LOC) to the two-point π^+correlator. One has

$$\boxed{i\Delta_{\pi^+}^{(\rm LOC)}(y,y') = \frac{i^2}{2}\int d^4x\; d^4x'\;\langle 0|\,T\big[\pi^+(y)\,\pi^+(y')^\dagger\,\mathcal{L}_{\rm int}^{(\pi q)}(x)\,\mathcal{L}_{\rm int}^{(\pi q)}(x')\big]\,|0\rangle}$$

Considering the relevant terms in $\mathcal{L}_{\rm int}^{(\pi q)}$ we have

$$\boxed{\Delta_{\pi^+}^{(\rm LOC)}(y,y') = -i\,g_s^2\int d^4x\; d^4x'\;\Delta_{\pi^+}(y,x)\;J_{\pi^+}(x,x')\;\Delta_{\pi^+}(x',y')\;,}$$

where

$$\boxed{J_{\pi^+}(x,x') = -2N_c\;{\rm tr}_D\big[iS_u(x,x')\,i\gamma_5\,iS_d(x',x)\,i\gamma_5\big]}$$

We introduce the π^+polarization function in $\bar{q}$ (or Ritus) space, $J_{\pi^+}(\bar{q},\bar{q}')$

$$\boxed{J_{\pi^+}(\bar{q},\bar{q}') = \int d^4x\, d^4x'\; \mathbb{F}^+(x,\bar{q})^*\, J_{\pi^+}(x,x')\, \mathbb{F}^+(x',\bar{q}')}$$

Using the completeness relations of $\mathcal{F}_{\mathcal{Q}}(x,\bar{q})$ this can be inverted to give

$$\boxed{J_{\pi^+}(x,x') = \sum\!\!\!\!\!\!\!\int_{\bar{q},\,\bar{q}'} \mathbb{F}^+(x,\bar{q})\, J_{\pi^+}(\bar{q},\bar{q}')\, \mathbb{F}^+(x',\bar{q}')^*}$$

Replacing this relation into

$$\boxed{\Delta^{(\mathrm{LOC})}_{\pi^+}(y,y') = -i\, g_s^2 \int d^4x\; d^4x'\; \Delta_{\pi^+}(y,x)\; J_{\pi^+}(x,x')\; \Delta_{\pi^+}(x',y')\ ,}$$

and using the Ritus form of the π^+ propagator and the orthogonality of $\mathbb{F}^+$'s , the LOC can be written as

$$\boxed{\Delta^{(\mathrm{LOC})}_{\pi^+}(y,y') = -\, i\, g_s^2 \sum\!\!\!\!\!\!\!\int_{\bar{q},\,\bar{q}'} \mathbb{F}^+(y,\bar{q})\; \hat{\Delta}_{\pi^+}(k,q_{\|})\; J_{\pi^+}(\bar{q},\bar{q}')\; \hat{\Delta}_{\pi^+}(k',q'_{\|})\; \mathbb{F}^+(y',\bar{q}')^*}$$

We need to obtain $J_{\pi^+}(\bar{q},\bar{q}')$ given by

$$J_{\pi^+}(\bar{q},\bar{q}') = \int d^4x\, d^4x'\; \mathbb{F}^+(x,\bar{q})^*\, J_{\pi^+}(x,x')\, \mathbb{F}^+(x',\bar{q}')$$

where

$$J_{\pi^+}(x,x') = -2N_c\ \mathrm{tr}_D\left[iS_u(x,x')\, i\gamma_5\, iS_d(x',x)\, i\gamma_5\right]$$

Using form of the quark propagators in terms of the SP

$$S_f(x,x') = e^{i\Phi_f(x,x')} \int \frac{d^4v}{(2\pi)^4}\; e^{-i\,v(x-x')}\, \bar{S}_f(v_\parallel, v_\perp)$$

$J_{\pi^+}(x,x')$ can be written as

$$J_{\pi^+}(x,x') = e^{i\Phi_{\pi^+}(x,x')} \int \frac{d^4v}{(2\pi)^4}\; e^{-iv(x-x')}\, \bar{J}_{\pi^+}(v_\parallel, v_\perp)$$

where

$$\Phi^{Q_u}(x,x') + \Phi^{Q_d}(x',x) = \Phi^{Q_u-Q_d}(x,x') = \Phi^{Q_{\pi^+}}(x,x')$$

and

$$\bar{J}_{\pi^+}(v_\parallel, v_\perp) = -2N_c \int \frac{d^4p}{(2\pi)^4}\, \mathrm{tr}_D\left[i\bar{S}^u(p_\parallel^+, p_\perp^+)\, i\gamma_5\, i\bar{S}^d(p_\parallel^-, p_\perp^-)\, i\gamma_5\right]$$

with

$$p^{\pm,\mu} = p^\mu \pm v^\mu/2$$

Replacing this form of $J_{\pi^+}(x,x')$ into

$$J_{\pi^+}(\bar{q},\bar{q}') = \int d^4x\, d^4x'\; \mathbb{F}^+(x,\bar{q})^*\; J_{\pi^+}(x,x')\, \mathbb{F}^+(x',\bar{q}')\ .$$

we have

$$J_{\pi^+}(\bar{q},\bar{q}') = \int \frac{d^4v}{(2\pi)^4}\, h_{\pi^+}(\bar{q},\bar{q}',v_\parallel,v_\perp)\, \bar{J}_{\pi^+}(v_\parallel,v_\perp)\ ,$$

where

$$h_{\rm P}(\bar{q},\bar{q}',v_\parallel,v_\perp) = \int d^4x\, d^4x'\, \mathcal{F}_{Q_{\rm P}}(x,\bar{q})^*\, e^{i\Phi_{\rm P}(x,x')}\, \mathcal{F}_{Q_{\rm P}}(x',\bar{q}')\, e^{-iv(x-x')}$$

$h_{\rm P}$ is gauge invariant. It can be calculated in any gauge. One gets

$$h_{\rm P}(\bar{q},\bar{q}',v_\parallel,v_\perp) = \delta_{\chi\chi'}\ (2\pi)^4\ \delta^{(2)}(q_\parallel - q'_\parallel)\ (2\pi)^2\ \delta^{(2)}(q_\parallel - v_\parallel)\, f_{kk'}(\vec{v}_\perp)\ ,$$

where

$$f_{kk'}(\vec{v}_\perp) = \frac{4\pi(-i)^{k+k'}}{B_Q}\sqrt{\frac{k!}{k'!}}\left(\frac{2\vec{v}_\perp^{\,2}}{B_Q}\right)^{\frac{k'-k}{2}} L_k^{k'-k}\Big(2\vec{v}_\perp^{\,2}/B_Q\Big)\, e^{-\vec{v}_\perp^{\,2}/B_Q}\ e^{is(k-k')\phi_\perp}$$

Replacing h_P in

$$\boxed{J_{\pi^+}(\bar{q},\bar{q}') = \int \frac{d^4v}{(2\pi)^4}\, h_{\pi^+}(\bar{q},\bar{q}',v_\parallel,v_\perp)\,\bar{J}_{\pi^+}(v_\parallel,v_\perp)\ ,}$$

and performing the integral over $\phi_\perp$ (note $\bar{J}_{\pi^+}(v_\parallel,v_\perp)$ only depends on $\vec{v}_\perp{}^2$ as shown below), we get

$$\boxed{J_{\pi^+}(\bar{q},\bar{q}') = \hat{\delta}_{\bar{q}\bar{q}'}\,\hat{J}_{\pi^+}(k,q_\parallel)}$$

Diagonal in Ritus space

where

$$\boxed{\hat{J}_{\pi^+}(k,q_\parallel) = \int_0^\infty d|\vec{v}_\perp|^2\,\bar{J}_{\pi^+}\left(q_\parallel,v_\perp\right)\,\rho_k(\vec{v}_\perp^{\,2})}$$

with

$$\boxed{\rho_k(\vec{v}_\perp^{\,2}) = \frac{(-1)^k}{B_\pi}\,e^{-\vec{v}_\perp^{\,2}/B_\pi}\,L_k\left(\frac{2\vec{v}_\perp^{\,2}}{B_\pi}\right)}$$

Even for $k=0$ we have to integrate over $|\vec{v}_\perp|^2$! We cannot set $v_\perp = 0$. Like in the harmonic oscillator we have zero point motion in the g.s.

Given that $J_{\pi^+}(\bar{q},\bar{q}')$ is diagonal in $\bar{q}$-space, the LOC to the propagator can be written as

$$\boxed{\Delta_{\pi^+}^{(\mathrm{LOC})}(y,y') = \sum\!\!\!\!\!\!\int_{\bar{q}} \mathbb{F}^+(y,\bar{q})\,\hat{\Delta}_{\pi^+}^{(\mathrm{LOC})}(k,q_{\|})\,\mathbb{F}^+(y',\bar{q})^*}$$

where

$$\boxed{\hat{\Delta}_{\pi^+}^{(\mathrm{LOC})}(k,q_{\|}) = \hat{\Delta}_{\pi^+}(k,q_{\|})\,\hat{\Sigma}_{\pi^+}(k,q_{\|})\,\hat{\Delta}_{\pi^+}(k,q_{\|})}$$

with

$$\boxed{\hat{\Sigma}_{\pi^+}(k,q_{\|}) = -i\,g_s^2\,\hat{J}_{\pi^+}(k,q_{\|})}$$

Selfenergy diagonal in Ritus space

To get the final form of $\hat{J}_{\pi^+}(k,q_{\|})$ we need the explicit expression of $\bar{J}_{\pi^+}(v_{\|},v_{\perp})$.

The explicit expression of $\bar{J}_{\pi^+}(v_\parallel, v_\perp)$ can be readily obtained from the invariant part of the quark propagators given in the previous slides. One has

$$\boxed{\begin{aligned}\bar{J}_{\pi^+}(v_\parallel, v_\perp) \quad = \quad & -\frac{iN_c}{4\pi^2}\int_{-1}^{1} dx \int_0^\infty \frac{dz}{t_+}\, e^{-z\,\phi(x,v_\parallel^2)}\; e^{-(t_+^2 - t_-^2)\,\vec{v}_\perp^{\,2}/(4t_+)} \\ & \times \left\{ \left[m_u m_d + \frac{1}{z} + (1-x^2)\frac{v_\parallel^2}{4} \right](1 - t_u\, t_d) \right. \\ & \qquad \left. + \left[\frac{1}{t_+} - \left(1 - \frac{t_-^2}{t_+^2}\right)\frac{\vec{v}_\perp^{\,2}}{4} \right](1-t_u^2)(1-t_d^2) \right\} ,\end{aligned}}$$

where we have used the definition

$$\phi(x, v_\parallel^2) = (m_u^2 + m_d^2)/2 - x(m_u^2 - m_d^2)/2 - (1-x^2)\, v_\parallel^2/4 \ ,$$

as well as

$$t_u = \tanh\left[(1-x)zB_u/2\right] \ , \qquad t_d = \tanh\left[(1+x)zB_d/2\right] \ , \qquad t_\pm = t_u/B_u \pm t_d/B_d \ .$$

Replacing this expression in

$$\boxed{\hat{J}_{\pi^+}(k, q_{\|}) = \int_0^\infty d|\vec{v}_\perp|^2 \, \bar{J}_{\pi^+}\left(q_{\|}, v_\perp\right) \, \rho_k(\vec{v}_\perp^{\,2}) \; ,}$$

and performing the integral over $|\vec{v}_\perp|$ we finally obtain

$$\boxed{\begin{aligned} \hat{J}_{\pi^+}(k, q_{\|}) \quad = \quad & -\frac{iN_c}{4\pi^2} \int_0^\infty dz \, \int_{-1}^{1} dx \; e^{-z\phi(x, q_{\|}^2)} \, \frac{1}{\alpha_+} \left(\frac{\alpha_-}{\alpha_+}\right)^k \\ & \times \left\{ \left[m_u m_d + \frac{1}{z} + (1 - x^2) \frac{q_{\|}^2}{4} \right] (1 - t_u \, t_d) \right. \\ & \left. \quad + \frac{\alpha_- + k(\alpha_- - \alpha_+)}{\alpha_+ \alpha_-} \, (1 - t_u^2)(1 - t_d^2) \right\} \; , \end{aligned}}$$

where we have defined $\alpha_\pm$ as

$$\alpha_\pm = \frac{t_u}{B_u} + \frac{t_d}{B_d} \pm B_\pi \frac{t_u}{B_u} \frac{t_d}{B_d} = t_+ \pm B_\pi \frac{t_u}{B_u} \frac{t_d}{B_d} \; .$$

This expression is divergent. Therefore, it has to be properly regularized by i.e. subtracting the B=0 contribution.

Summary III

We have introduced the Schwinger phase. It carries all the gauge and non-rotational and translational invariance of the charged particle propagators.

The charged particle propagators can be written as a product of a SP term and a factor which is translational invariant. The latter can therefore be Fourier transformed. It also invariant under rotations and the boosts along the magnetic field direction.

The invariant part can be expressed in two alternative ways: as a sum over LL or as a PT integral. In addition, as seen in the second lecture, the full propagators can be expressed in terms of Ritus functions.

We have obtained a closed expression for the LOC correction to charged pion propagator which turns to be diagonal in Ritus space.

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