Minkowski space solution of the fermion Dyson-Schwinger equation (and a little bit more)

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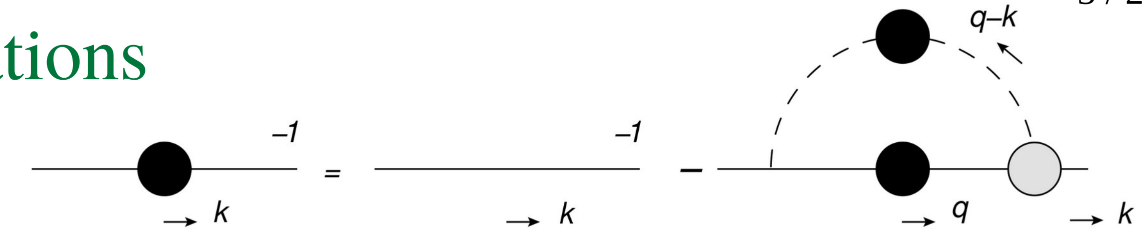
Outline

- Motivation;
- The Nakanishi Integral Representation;
- Solving Dyson-Schwinger equation in Minkowski space;
- Recent developments;
- Summary, perspectives and questions.

The Dyson-Schwinger equations



Equation of motion: tower of infinite coupled functional differential equations.

F. Dyson, "The S Matrix in Quantum Electrodynamics", Phys. Rev. 75 (11) 1736 (1949).

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Naturally sum infinitely many diagrams and therefore automatically contain nonperturbative information.

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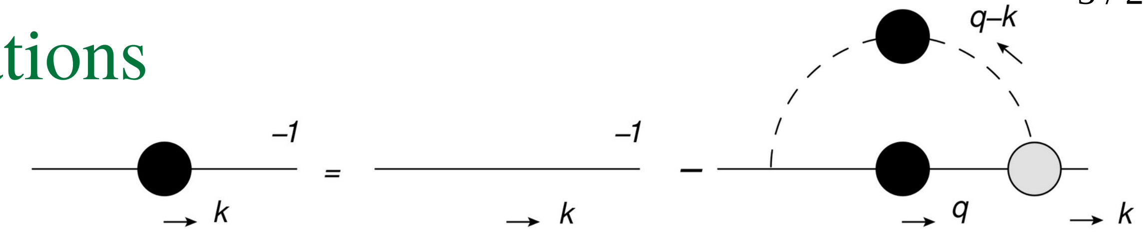

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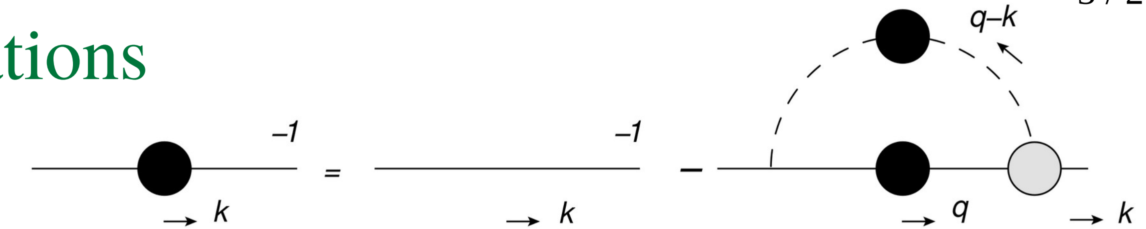

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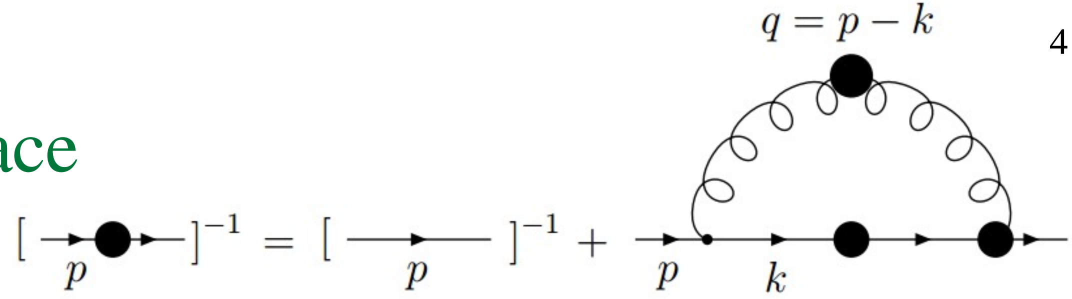
Wide range of applications, from solid state physics to strong interaction problems.

DSE in Minkowski space



Usually defined and solved in Euclidean space:

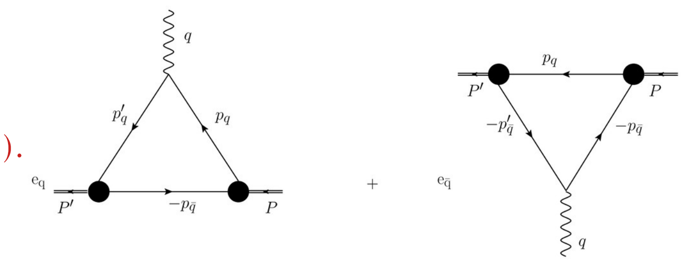
- Lattice gauge theory simulations and its numerical solutions;
- QCD perturbation theory are strictly valid only at spacelike-momenta, the only possibility for Euclidean formulation.

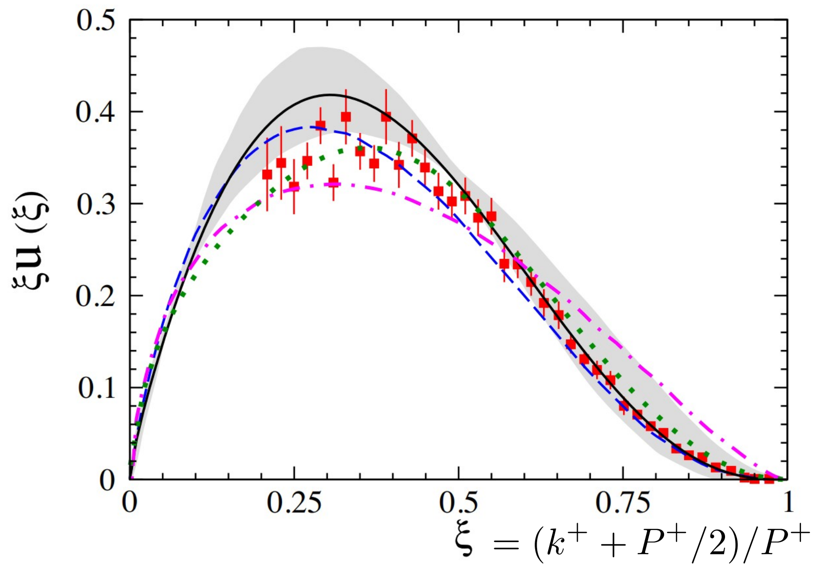
Why Minkowski? Difficulties to deal with singular behavior of physical quantities...

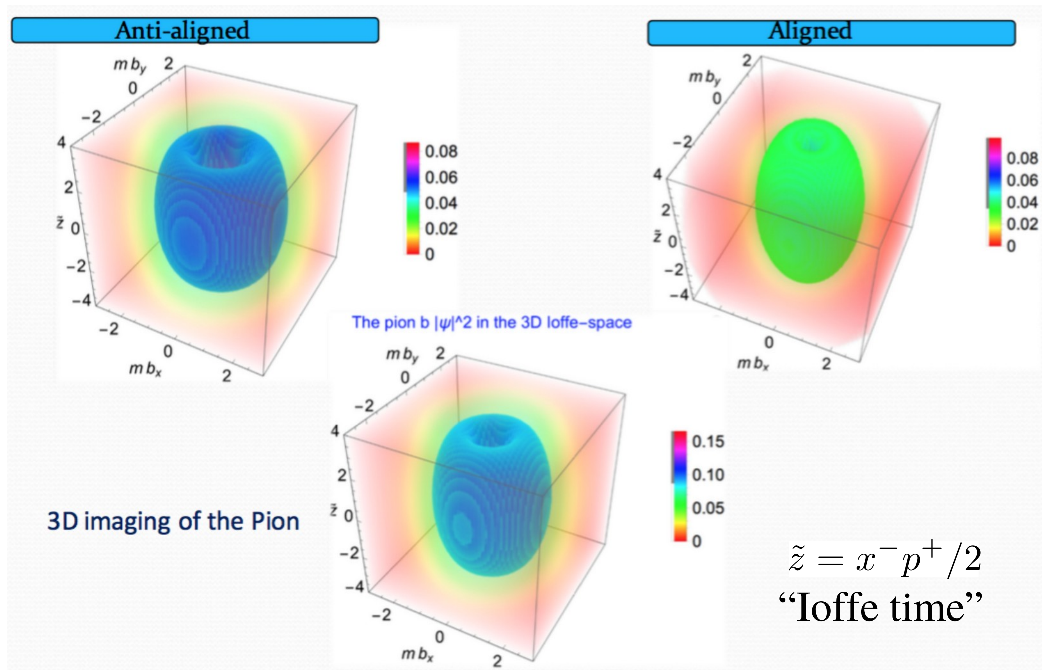
- Dynamical observables defined in the light-front;
- Electromagnetic form-factors (singularities!);
- 3d imaging that may clarify the hadron content (EIC facility in the future);
- ...
- QCD at finite density?
- Finite magnetic field?

Developments in Minkowski space (ITA group)

- Two-fermion homogeneous BSE. W. de Paula et al., PRD 94, 071901 (2016).
- Solution of the 3-body BSE. E.Ydrefors et al., PLB 791, 276 (2019).
- Pion electromagnetic form-factor. E. Ydrefors et al., PLB 820 136494 (2021).
- Proton image. Ydrefors and Frederico, PRD 104, 114012 (2021)
- ....







de Paula,Ydrefors,Nogueira,Frederico,Salmè, PRD 105, L071505 (2022).

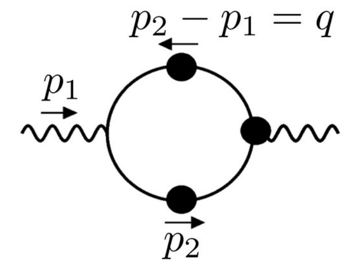
Main Tool: Nakanishi Integral Representation

- Generalization of the Källén-Lehmman integral representation of two point functions, for n-point functions. N. Nakanishi, Phys. Rev. 130, 1230 (1963) and Prog.Theor.Phys.Suppl. 43, 1 (1969).

- Goal: Construction of a compact representations of the transition amplitude for a generic scattering process involving N external particles.

- Let's consider a connected Feynman diagram G with N external legs, n inner propagators and k loops. If p_i are the external four-momenta:

$$\sum_{i=1}^{N} p_i = 0$$



- Using now:
l_j,m_j: the four-momenta and mass of particles propagating inside the loops (j = 1,2,…,n);
q_r: four-momenta to be integrated on the r-th loop (r = 1,2,…,k).

Momentum conservations in each vertex gives: $l_j = \sum_{r=1}^{k} b_{jr} q_r + \sum_{i=1}^{N} c_{ji} p_i$
(b_{jr},c_{ji} = 1-, 0 or 1)

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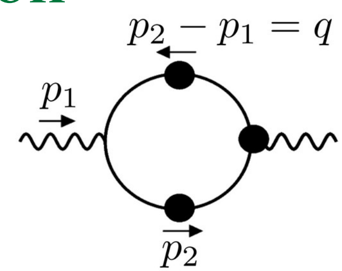

- Transition amplitude:

$$f_G(p_1, p_2, \cdots, p_N) = \sharp \prod_{r=1}^{k} \int d^4 q_r \frac{1}{(l_1^2 - m_1^2 + i\epsilon)(l_2^2 - m_2^2 + i\epsilon)\cdots(l_n^2 - m_n^2 + i\varepsilon)}$$

- With the Feynman parametric formula:

$$\frac{1}{A^\alpha B^\beta \cdots E^\epsilon} = \frac{\Gamma(\alpha+\beta+\cdots+\epsilon)}{\Gamma(\alpha)\Gamma(\beta)\cdots\Gamma(\epsilon)} \int_0^1 dx\, dy \cdots dz \delta(1 - x - y - \cdots - z) \frac{x^{\alpha-1} y^{\beta-1} \cdots z^{\epsilon-1}}{(Ax + By + \cdots + Ez)^{\alpha+\beta+\cdots+\epsilon}}$$

we obtain

$$f_G(p_1, p_2, \cdots, p_N) = \sharp \prod_{r=1}^{k} \int d^4 q_r \prod_{i=1}^{n} \int d\alpha_i \frac{\delta\left(1 - \sum_{j=1}^{n} \alpha_j\right)}{\left[\sum_{j=1}^{n} (l_j^2 - m_j^2)\alpha_j\right]^n + i\varepsilon}$$

Information about the number of loops and the masses

Main Tool: Nakanishi Integral Representation

NIR idea: Remove all this dependence to the numerator, keeping the whole global analytic behavior in the denominator $\longrightarrow$ Possibility to create an integral representation of the infinite sum of Feynman diagram with N external legs!

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$$f_G(s) = \prod_h \int_0^1 dz_h \delta\left(1 - \sum_h z_h\right) \int_0^\infty d\chi \frac{\tilde{\phi}_G(z_h, \chi)}{\chi - \sum_h z_h s_h - i\varepsilon}\,, \qquad \tilde{\phi}_G(z_h, \chi) = \frac{1}{n-2k-1} \frac{\partial^{n-2k-1} \phi_G(z_h, \chi)}{\partial \chi^{n-2k-1}}$$

$\phi_G(z_h, \chi)$ Weight function that carries out all the information about the loops;

s Set $\{s_h\}$ of all the independent scalars that can be constructed from N external four-momenta p_i.

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Independent of the internal structure of the diagrams!

$$f_N(s) = \prod_h \int_0^1 dz_h \delta\left(1 - \sum_h z_h\right) \int_0^\infty d\chi \frac{\phi_N(z_h,\chi)}{\chi - \sum_h z_h s_h - i\varepsilon}, \text{with} \qquad \phi_N(z,\chi) = \sum_G \mathcal{N} \tilde{\phi}_G(z_h,\chi)$$

Nakanishi also proved that the weight function is unique in any perturbative order for a bosonic theory, and subsequently its validity was confirmed in the nonperturbative domain: Uniqueness of the integral representation!

Comparison with Un-Wick rotated results: Bethe-Salpeter vertex

Wick rotation is the exact analytical continuation of the Minkowski space Nakanishi representation: Explorations in the complex plane.

Un-Wick comparison with rotation in the ladder bosonic BSE

The Bethe-Salpeter approach to bound states: from Euclidean to Minkowski space

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[2]Università di Roma La Sapienza, INFN, Sezione di Roma, P.le A. Moro 5, 00187 Roma, Italy
[3]Department of Physics and Astronomy, Iowa State University, Ames, IA 50011, USA

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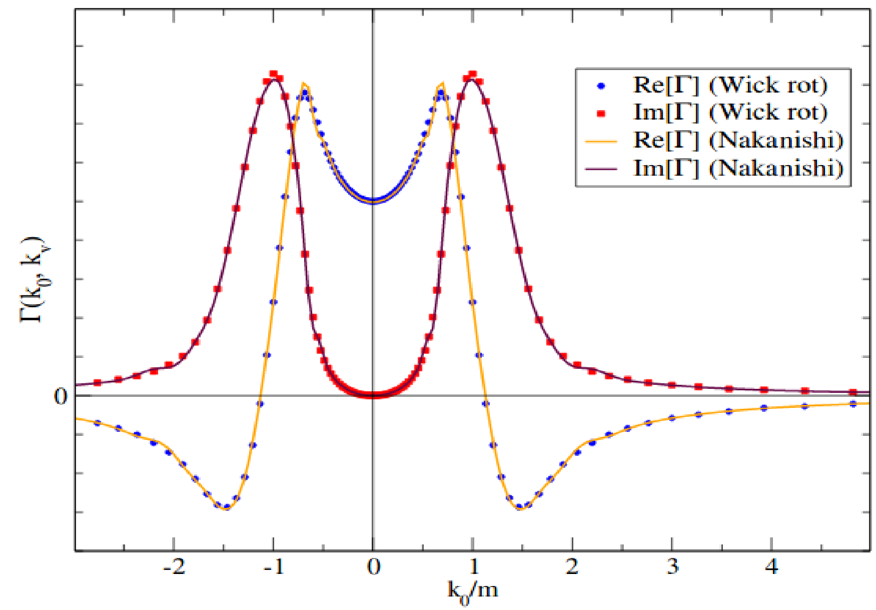
Abstract. The challenge to obtain from the Euclidean Bethe–Salpeter amplitude the amplitude in Minkowski is solved by resorting to un-Wick rotating the Euclidean homogeneous integral equation. The results obtained with this new practical method for the amputated Bethe–Salpeter amplitude for a two-boson bound state reveals a rich analytic structure of this amplitude, which can be traced back to the Minkowski space Bethe–Salpeter equation using the Nakanishi integral representation. The method can be extended to small rotation angles bringing the Euclidean solution closer to the Minkowski one and could allow in principle the extraction of the longitudinal parton density functions and momentum distribution amplitude, for example.



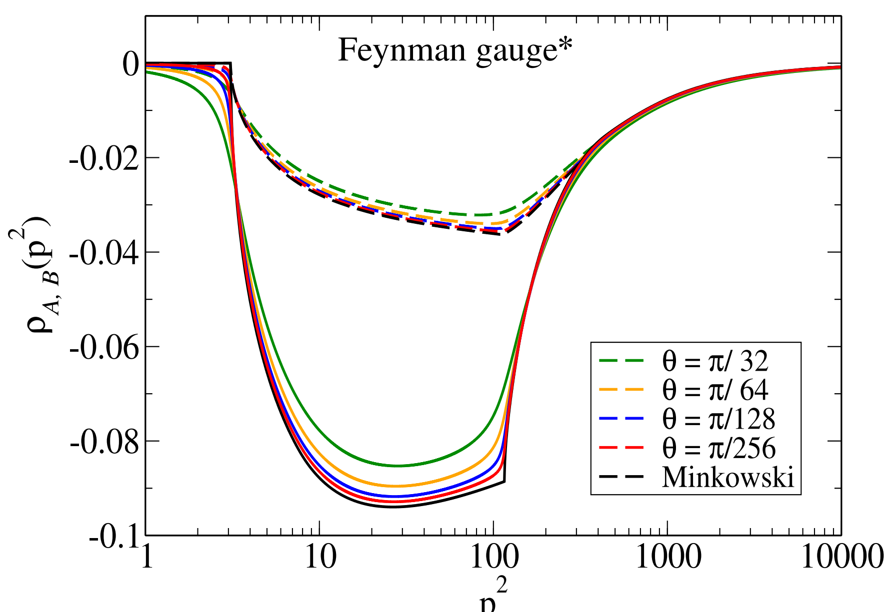
A. Castro et al., Journal of Physics: Conf. Series 1291 012006 (2019)

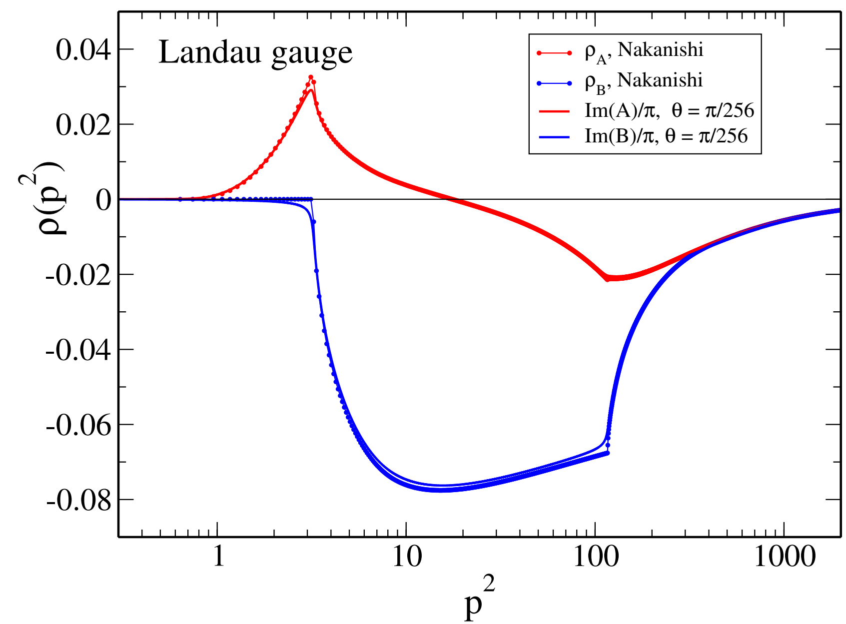
Comparison with Un-Wick rotated results:DSE in the Weak coupling limit

- From Euclidean space formulation, in increments of δ: $p_0 =\to \exp(-i\delta)p_0$ $k_0 =\to \exp(-i\delta)k_0$

- Minkowski space: $\delta = \pi/2$, or in a more conveninent notation $\Theta = \pi/2 - \delta$.

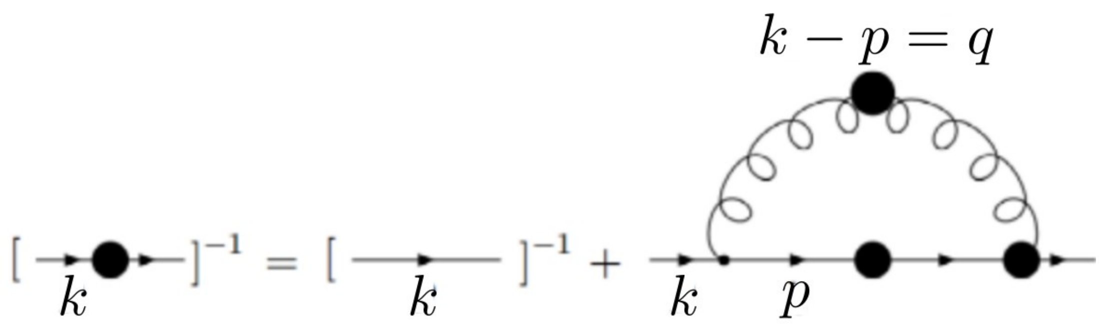
$$\theta = 0, \begin{cases} p_0^2 = 0, & \vec{p}^2 > 0 \quad \text{spacelike region} \\ p_0^2 > 0, & \vec{p}^2 = 0 \quad \text{timelike region} \end{cases}$$





*S. Jia et al., Proceedings of HADRON-2019, arXiv:1912.00063, T. Frederico et al., Proceedings of NTSE-2018, arXiv:1905.00703.

Fermion Dyson Schwinger Equation (Rainbow-Ladder)



- DSE for the above schematic representation:

$$S_f^{-1}(k) = \not{k} - m_B + ig^2 \int \frac{d^4q}{(2\pi)^4} \Gamma_\mu(q,k) S_f(k-q) \gamma_\nu D^{\mu\nu}(q)$$

Rainbow ladder approximation: $\Gamma_\mu(q,k) = \gamma_\mu$

Gluon propagator:

$$D^{\mu\nu}(q) = \frac{1}{q^2 - m_\sigma^2 + \imath\epsilon} \left[g^{\mu\nu} - \frac{(1-\xi)q^\mu q^\nu}{q^2 - \xi m_\sigma^2 + \imath\epsilon} \right]$$

Fermion Dyson Schwinger Equation (Rainbow-Ladder)

⟹ Dressed fermion propagator: $S_f\left(k\right) = \dfrac{1}{\not{k} - m_B + \not{k} A_f\left(k^2\right) - B_f\left(k^2\right) + i\epsilon}$

$$A_f(k^2) = \int_0^\infty ds \frac{\rho_A(s)}{k^2 - s + i\epsilon}, \qquad B_f(k^2) = \int_0^\infty ds \frac{\rho_B(s)}{k^2 - s + i\epsilon}$$

⟹ Integral representation of the fermion propagator:

$$S_f(k) = R \frac{\not{k} + \overline{m}_0}{k^2 - \overline{m}_0^2 + i\epsilon} + \not{k} \int_0^\infty ds \frac{\rho_v(s)}{k^2 - s + i\epsilon} + \int_0^\infty ds \frac{\rho_s(s)}{k^2 - s + i\epsilon}$$

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Vector and scalar Self-Energy densities

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Vector and scalar spectral densities

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Vector and scalar spectral densities

$$\not{k} A(k^2) - B(k^2) = i g^2 \int \frac{d^4 q}{(2\pi)^4} \frac{\gamma_\mu S_f(k-q) \gamma_\nu}{q^2 - m_g^2 + i\epsilon} \left[g^{\mu\nu} - \frac{(1-\xi)\, q^\mu q^\nu}{q^2 - \xi m_g^2 + i\epsilon} \right]$$

$$- i g^2 \int \frac{d^4 q}{(2\pi)^4} \frac{\gamma_\mu S_f(k-q) \gamma_\nu}{q^2 - \Lambda^2 + i\epsilon} \left[g^{\mu\nu} - \frac{(1-\xi) q^\mu q^\nu}{q^2 - \xi \Lambda^2 + i\epsilon} \right]$$

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Vector and scalar spectral densities

Gauge fixing

$$\not{k}A(k^2) - B(k^2) = ig^2 \int \frac{d^4q}{(2\pi)^4} \frac{\gamma_\mu S_f(k-q)\gamma_\nu}{q^2 - m_g^2 + i\epsilon}\left[g^{\mu\nu} - \frac{(1-\xi)\, q^\mu q^\nu}{q^2 - \xi m_g^2 + i\epsilon}\right]$$

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Pauli-Villars regulator

Fermion Dyson Schwinger Equation

- Parameters: $\alpha = \dfrac{g^2}{4\pi}$, Λ , m_g , $\overline{m}_0$.

- Spectral densities are obtained from the IR of the self-energy:

$$\rho_A(\gamma) = -\frac{1}{\pi}\mathrm{Im}\left[A(\gamma)\right]$$
$$\rho_B(\gamma) = -\frac{1}{\pi}\mathrm{Im}\left[B(\gamma)\right]$$

- Solutions of DSE obtained writing the trivial relation $S_f^{-1} S_f = 1$ in a suitable form:

$$\frac{R}{\gamma - \overline{m}_0^2 + i\epsilon} + \int_0^\infty ds \frac{\rho_v(s)}{\gamma - s + i\epsilon} = \frac{A(\gamma)}{\gamma A^2(\gamma) - B^2(\gamma) + i\epsilon}$$
$$\frac{R\,\overline{m}_0}{\gamma - \overline{m}_0^2 + i\epsilon} + \int_0^\infty ds \frac{\rho_s(s)}{\gamma - s + i\epsilon} = \frac{B(\gamma)}{\gamma A^2(\gamma) - B^2(\gamma) + i\epsilon}$$

Fermion DSE solution

$$\rho_A(\gamma) = R\mathcal{K}^{\xi}_{0A}(\gamma, \overline{m}_0^2, m_g^2) + \int_0^\infty ds\, \mathcal{K}^{\xi}_A(\gamma, s, m_g^2)\, \rho_v(s) - [m_g \to \Lambda]$$

$$\rho_B(\gamma) = R\,\overline{m}_0\, \mathcal{K}^{\xi}_{0B}(\gamma, \overline{m}_0^2, m_g^2) + \int_0^\infty ds\, \mathcal{K}^{\xi}_B(\gamma, s, m_g^2)\, \rho_s(s) - [m_g \to \Lambda]$$

- Driving term:

$$\mathcal{K}^{\xi}_{0A(0B)} = K_{A(B)} + m_g^{-2} \bar{K}^{\xi}_{A(B)}$$

- Kernel:

$$\mathcal{K}^{\xi}_A(\gamma, s, m_g^2) = K_A(\gamma, s, m_g^2)\, \Theta(s - (\overline{m}_0 + m_g)^2) + m_g^{-2} \bar{K}^{\xi}_A(\gamma, s, m_g^2)\, \Theta(s - (\overline{m}_0 + \sqrt{\xi} m_g)^2)$$

$$f_A(\gamma) = 1 + \int_0^\infty ds \frac{\rho_A(s)}{\gamma - s}$$

$$f_B(\gamma) = m_B + \int_0^\infty ds \frac{\rho_B(s)}{\gamma - s}$$

$$d(\gamma) = \left[\gamma f_A^2(\gamma) - \pi^2\, \gamma\, \rho_A^2(\gamma) - f_B^2(\gamma) + \pi^2\, \rho_B^2(\gamma)\right]^2 + 4\pi^2 \left[\gamma \rho_A(\gamma)\, f_A(\gamma) - \rho_B(\gamma)\, f_B(\gamma)\right]^2$$

Connection Formulas

$$\rho_v(\gamma) = -2\frac{f_A(\gamma)}{d(\gamma)} \left[\gamma \rho_A(\gamma)\, f_A(\gamma) - \rho_B(\gamma)\, f_B(\gamma)\right] + \frac{\rho_A(\gamma)}{d(\gamma)} \left[\gamma\, f_A^2(\gamma) - \pi^2 \gamma \rho_A^2(\gamma) - f_B^2(\gamma) + \pi^2 \rho_B^2(\gamma)\right]$$

$$\rho_s(\gamma) = -2\frac{f_B(\gamma)}{d(\gamma)} \left[\gamma \rho_A(\gamma)\, f_A(\gamma) - \rho_B(\gamma)\, f_B(\gamma)\right] + \frac{\rho_B(\gamma)}{d(\gamma)} \left[\gamma\, f_A^2(\gamma) - \pi^2 \gamma \rho_A^2(\gamma) - f_B^2(\gamma) + \pi^2 \rho_B^2(\gamma)\right]$$

Feynman gauge kernel ($\xi = 1$):

$$K_A\left(\gamma, s, m_g^2\right) = -\frac{\alpha}{4\pi}\frac{\gamma - m_g^2 + s}{\gamma^2}\sqrt{(\gamma - m_g^2 + s)^2 - 4\gamma\, s}\;\;\Theta\left[\gamma - (m_g + \sqrt{s})^2\right]$$

$$K_B\left(\gamma, s, m_g^2\right) = -\frac{\alpha}{4\pi}\frac{4}{\gamma}\sqrt{(\gamma - m_g^2 + s)^2 - 4\gamma s}\;\;\Theta\left[\gamma - (m_g + \sqrt{s})^2\right]$$

Remaining arbitrary ξ-gauge contribution:

$$\bar{K}_A^{\xi}(\gamma, s, m_g^2) = -\frac{\alpha}{4\pi}\frac{(\gamma - s)^2 - m_g^2(\gamma + s)}{2\gamma^2}\sqrt{(\gamma - m_g^2 + s)^2 - 4\gamma s}$$
$$\times\,\Theta\left[\gamma - (m_g + \sqrt{s})^2\right] - [m_g^2 \to \xi m_g^2]$$

$$\bar{K}_B^{\xi}(\gamma, s, m_g^2) = \frac{\alpha\, m_g^2}{4\pi}\frac{\sqrt{(\gamma - m_g^2 + s)^2 - 4\gamma s}}{\gamma}\Theta\left[\gamma - (m_g + \sqrt{s})^2\right] - [m_g^2 \to \xi m_g^2]$$

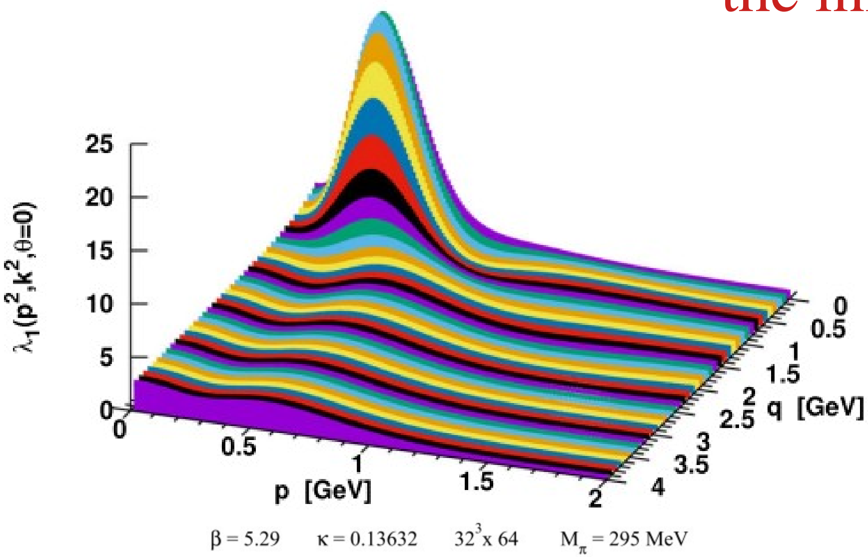
Bare mass:

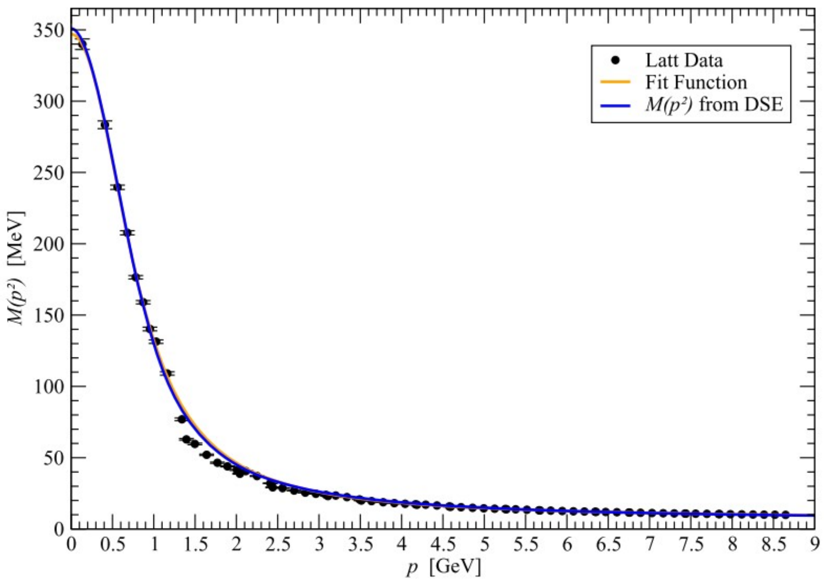
$$m_B = \overline{m}_0 + \overline{m}_0 \fint_0^{\infty} ds\, \frac{\rho_A(s)}{\overline{m}_0^2 - s} - \fint_0^{\infty} ds\, \frac{\rho_B(s)}{\overline{m}_0^2 - s}$$

Residue:

$$R^{-1} = 1 + \fint_0^{\infty} ds\, \frac{\rho_A(s)}{\overline{m}_0^2 - s} - 2\overline{m}_0^2 \fint_0^{\infty} ds\, \frac{\rho_A(s)}{(\overline{m}_0^2 - s)^2}$$
$$+ 2\overline{m}_0 \fint_0^{\infty} ds\, \frac{\rho_B(s)}{(\overline{m}_0^2 - s)^2}$$

DSE+Lattice QCD propagators: Enhancement of the quark-gluon vertex at the infrared region![†]





[†]O. Oliveira et al., EPJC 80:484 (2020).

Pauli-Villars regulator can also be effectively associated with the form factor of the γ^μ component of the quark-gluon vertex:

$$\lambda_1(q^2) = \frac{m_g^2 - \Lambda^2}{q^2 - \Lambda + i\epsilon}$$

[†]Rojas et al., JHEP 10 (2013) 193; O. Oliveira et al., EPJC 79, 116 (2019).

Large coupling regime: Phenomenological model

Calibration of the model: Possibility to explore the chiral symmetry breaking region!

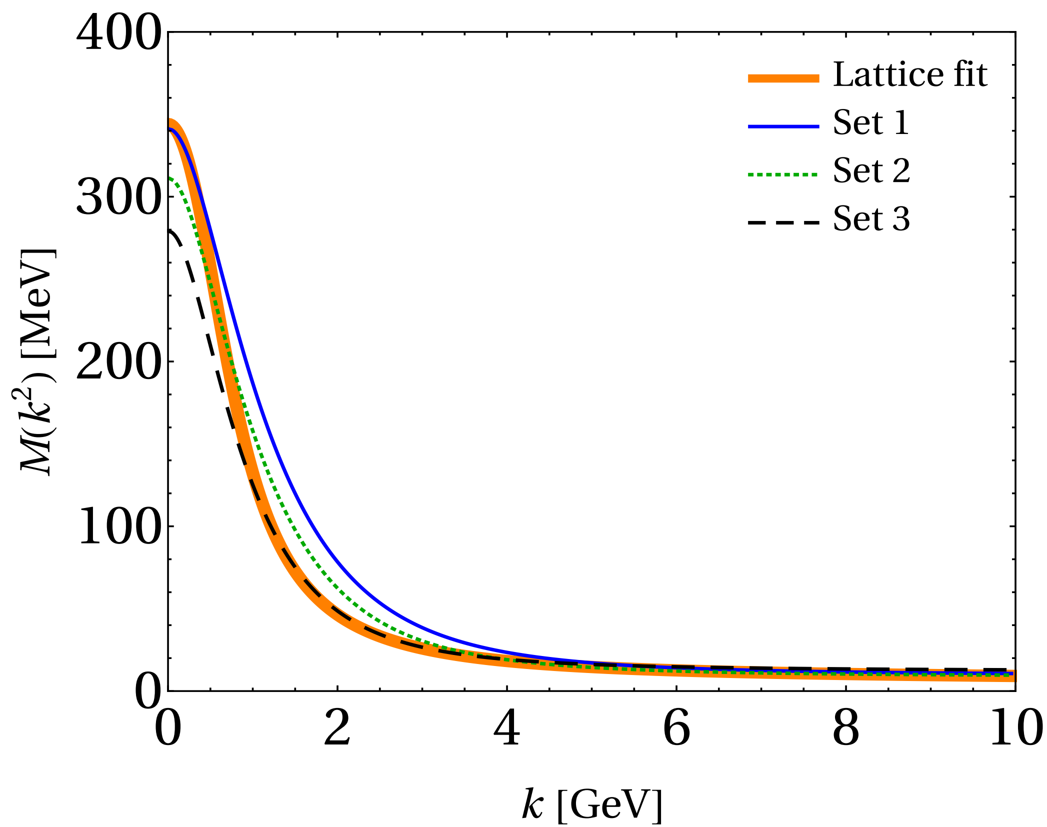
Set	$\bar{m}_0$ (GeV)	m_g (GeV)	Λ (GeV)	α
1	0.42	0.84	1.20	19.70
2	0.38	0.78	1.10	20.30
3	0.35	0.60	1.00	13.25

Set	(Outputs)	m_B (MeV)	R
1		9.06	2.22
2		8.53	2.09
3		12.25	2.64

Appropriate behavior in the infrared require a large enough Kernel

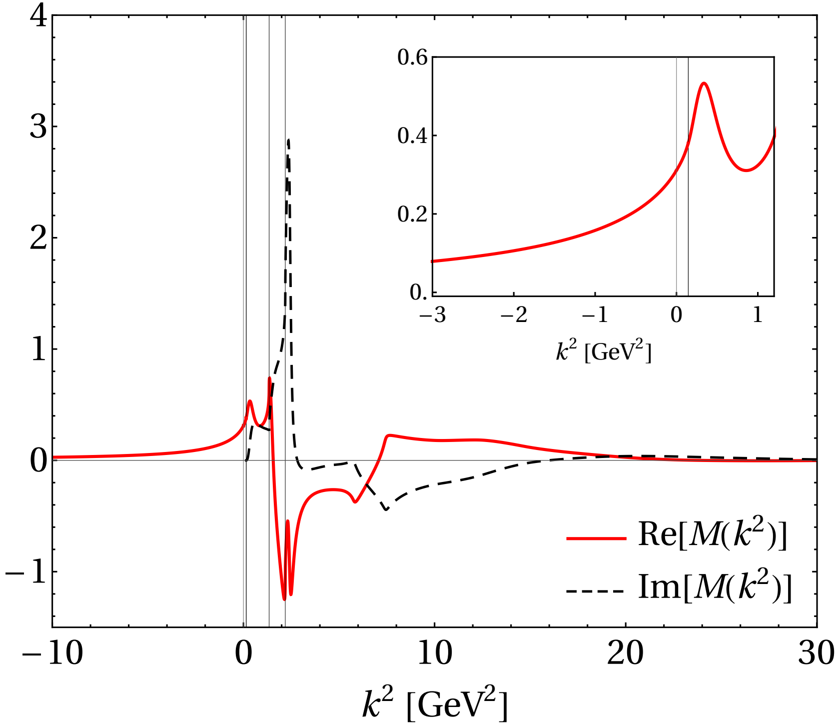
⇓

Λ cannot be large compared to m_σ, and as a consequence, α must increase!



DD, Frederico, de Paula, Ydrefors PRD 105, 114055 (2022). Fit of lattice data from O. Oliveira, et al., PRD 99, 094506 (2019).

Large coupling regime: Phenomenological model

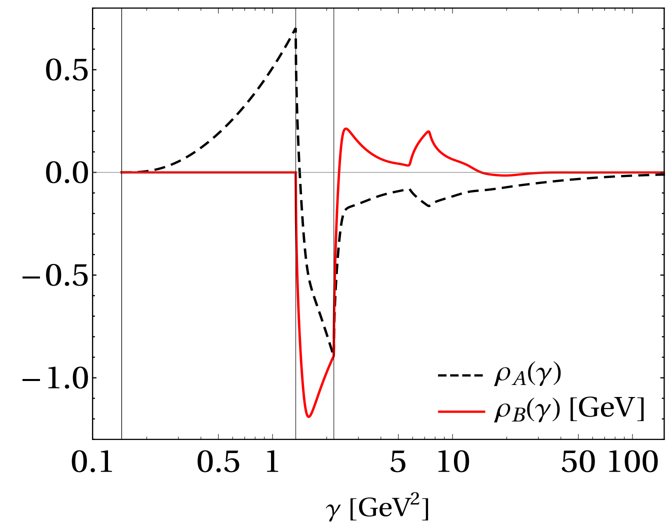


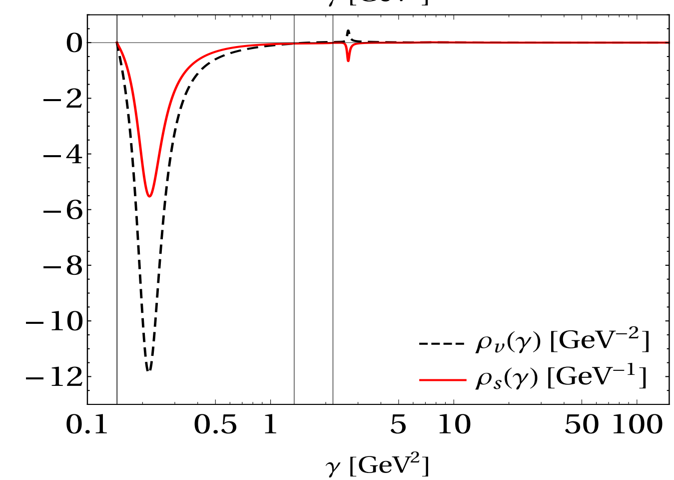
$\overline{m}_0^2$

$(\overline{m}_0 + m_g)^2$

$(\overline{m}_0 + \Lambda)^2$

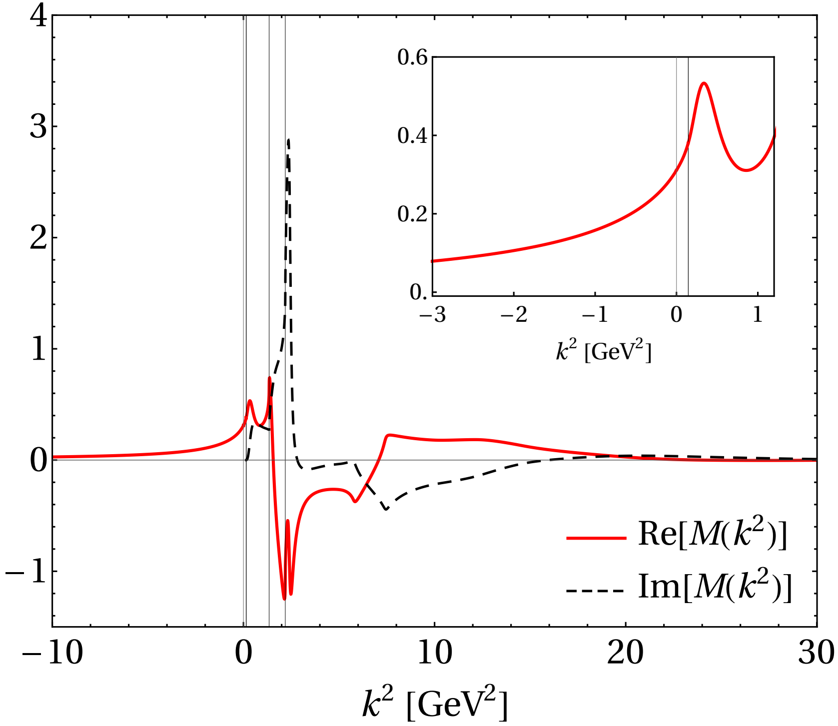
PRD 105, 114055 (2022)





DD, T. Frederico, W. de Paula, E. Ydrefors Phys. Rev. D 105, 114055 (2022).

Large coupling regime: Phenomenological model

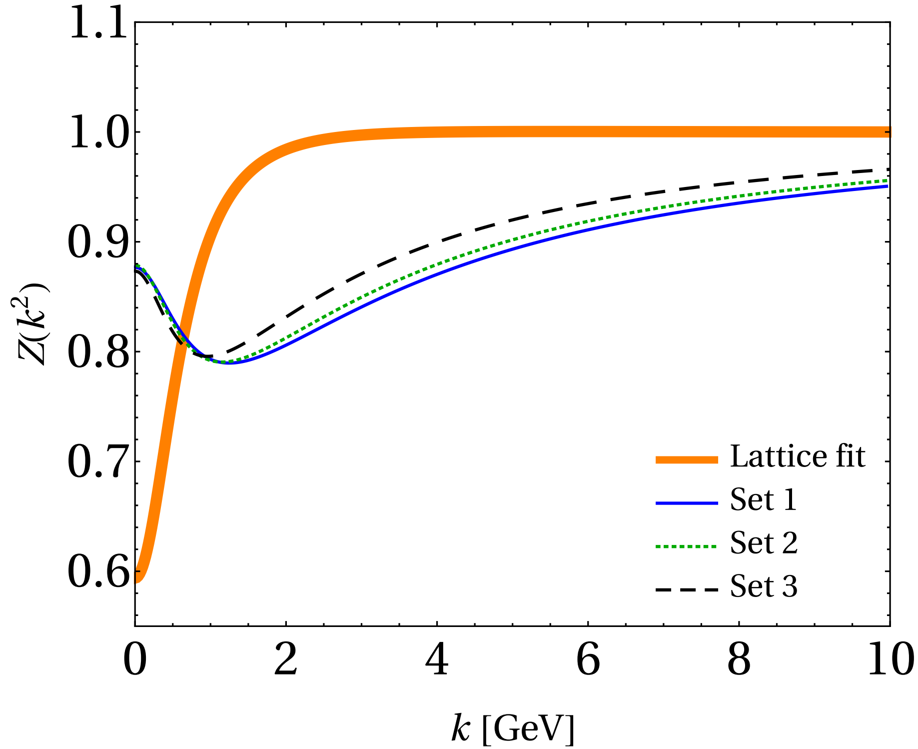


$\overline{m}_0^2$

$(\overline{m}_0 + m_g)^2$

$(\overline{m}_0 + \Lambda)^2$

Not enough to get a good fit for $Z(k^2)$

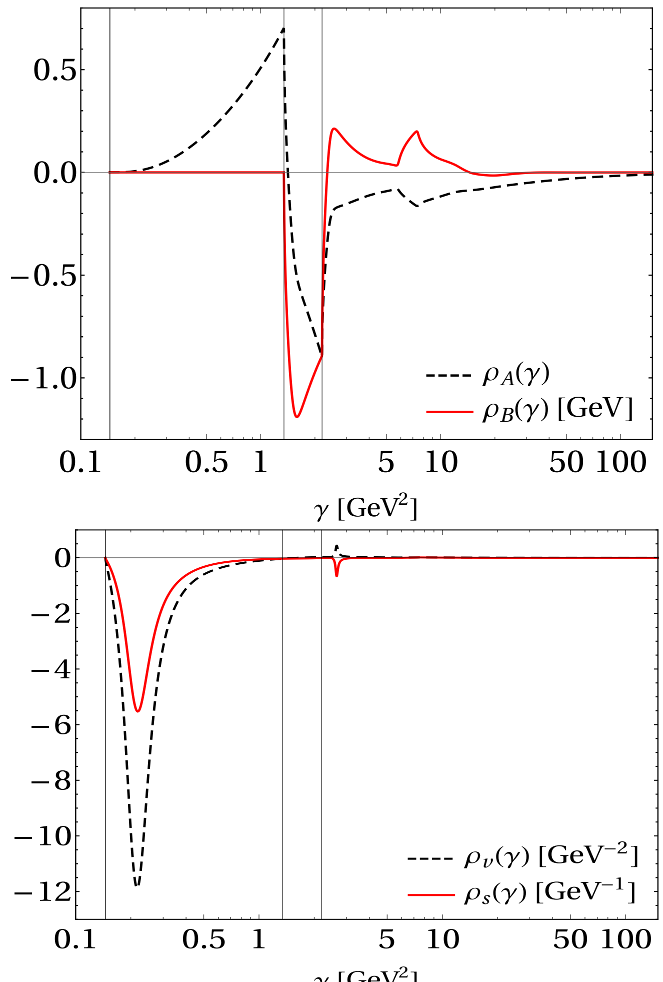


DD, T. Frederico, W. de Paula, E. Ydrefors Phys. Rev. D 105, 114055 (2022).

Recent developments and perspectives

Spectral densities evaluated by solving the DSE the method described previously as inputs for the pion Bethe-Salpeter equation.

PRD 105, 114055 (2022)



$$\Psi_\pi(k;p) = S_F(k_q)\Gamma_\pi(k;p)S_F(k_{\bar{q}})$$

$$S_F(k) = \frac{1}{A(k^2)\not{k} - B(k^2)}$$
$$= S_v(k^2)\not{k} + S_s(k^2)$$

$$S_v(k^2) = \frac{R}{k^2 - \overline{m}_0^2 + i\epsilon} + \int_0^\infty ds \frac{\rho_v(s)}{k^2 - s + \imath\epsilon}$$

$$S_s(k^2) = \frac{R\overline{m}_0}{k^2 - \overline{m}_0^2 + i\epsilon} + \int_0^\infty ds \frac{\rho_s(s)}{k^2 - s + i\epsilon}$$

$$\Gamma_\pi(k,p) = \gamma_5[\imath E_\pi(k,p) + F_\pi(k,p)$$
$$+ k^\mu p_\mu\, G_\pi(k,p) + \sigma_{\mu\nu}k^\mu p^\nu H_\pi(k,p)]$$

Recent developments and perspectives

- Bethe Salpeter equation: $\Psi_\pi(p,P) = S(q)\Gamma_\pi(p,P)S(\bar{q})$

$$\Gamma_\pi(p,P) = \gamma_5 \Big[iE_\pi(p,P) + \not{P}\, F_\pi(p,P) + p^\mu P_\mu \not{p} G_\pi(p,P) + \sigma_{\mu\nu} p^\mu P^\nu H_\pi(p,P) \Big]$$

Recent developments and perspectives

- Bethe Salpeter equation: $\Psi_\pi(p,P) = S(q)\Gamma_\pi(p,P)S(\bar{q})$

$$\Gamma_\pi(p,P) = \gamma_5 \left[iE_\pi(p,P) + \not{P}\, F_\pi(p,P) + p^\mu P_\mu \not{p} G_\pi(p,P) + \sigma_{\mu\nu} p^\mu P^\nu H_\pi(p,P) \right]$$

- First approximation: Chiral limit! In this case, the pion quark-antiquark vertex is given by**

$$E_\pi(k) = -\frac{1}{f_\pi^0} B(k^2)\,, \qquad B(k^2) = \int_0^\infty ds \frac{\rho_B(s)}{k^2 - s + i\epsilon}$$

- Calculation of observables, more rigorous study of chiral symmetry breaking, ingredients from LQCD...

Pion decay constant: $ip^\mu f_\pi = N_c \int \frac{d^4k}{(2\pi)^4} \mathrm{Tr}\left[\gamma^\mu \gamma^5 \Psi_\pi(k,p)\right]$

**C. S. Mello, et al., Phys. Lett. B, 766 86–93 (2017), L. Chang et al., PRL 110, 132001 (2013).

Recent developments and perspectives

Set	Pion decay constant (in MeV)	
Set	$m_\pi = 0$	$m_\pi = 140$ MeV
1	71.03	71.87
2	65.72	66.63
3	53.51	54.58
4	84.85	85.60

PRD 105, 114055 (Sets 1–3)

New! (Set 4)

$\overline{m}_0 = 420$ MeV

$m_g = 450$ MeV

$\Lambda = 800$ MeV

Recent developments and perspectives

	Set	Pion decay constant (in MeV)	
		$m_\pi = 0$	$m_\pi = 140$ MeV
PRD 105, 114055	1	71.03	71.87
	2	65.72	66.63
	3	53.51	54.58
New!	4	84.85	85.60

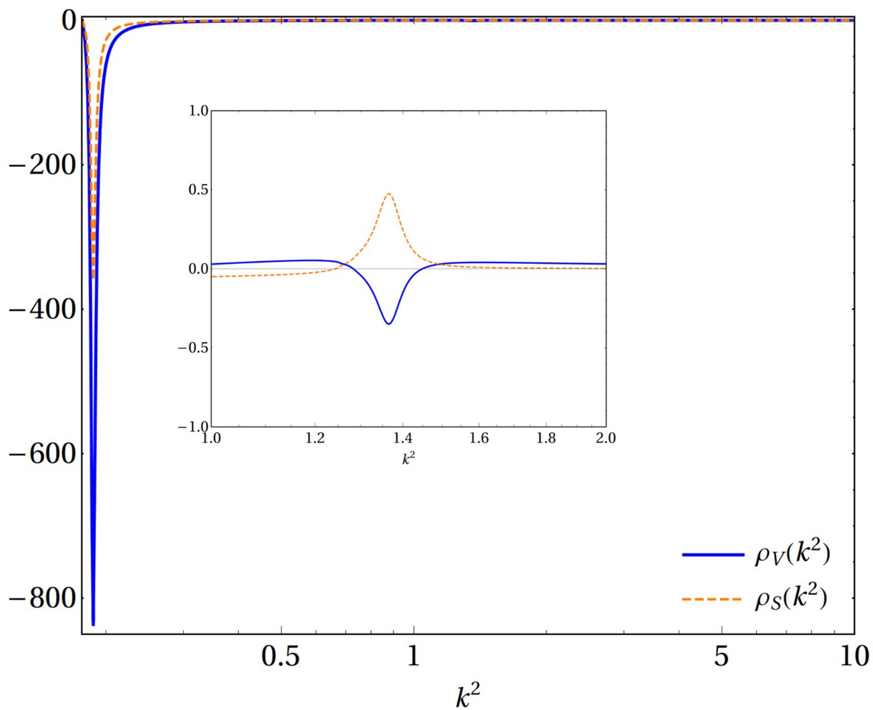
$\overline{m}_0 = 420$ MeV
$m_g = 450$ MeV
$\Lambda = 800$ MeV

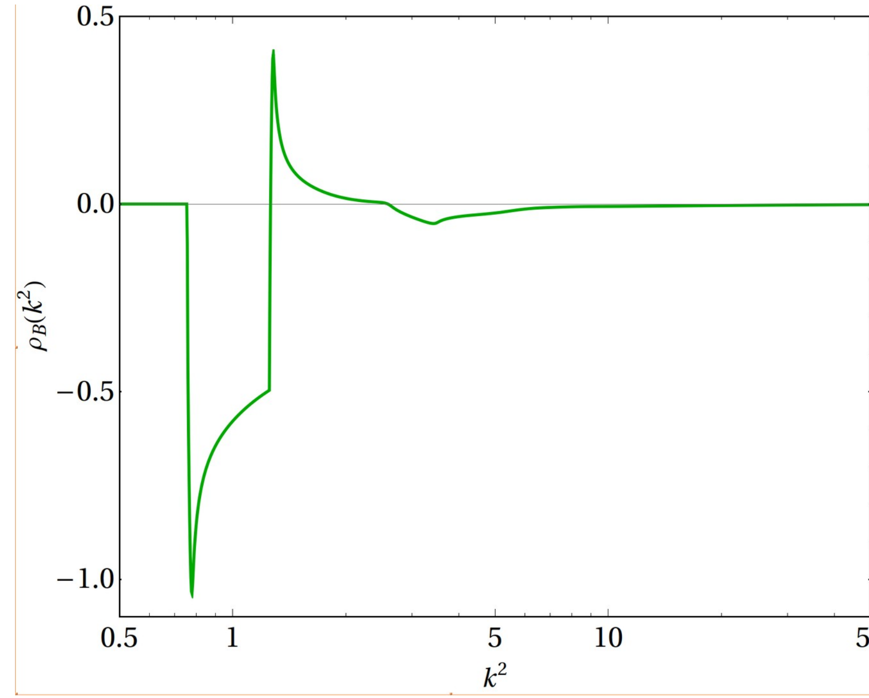
Next steps:

- Valence wave function and probability amplitude.
- BSA normalization:

$$\mathrm{Tr}\left\{\int \frac{d^4k}{(2\pi)^4}\frac{\partial}{\partial p'^{\mu}}\left[S^{-1}\left(k-\frac{p'}{2}\right)\bar{\Psi}_\pi(k,p)S^{-1}\left(k+\frac{p'}{2}\right)\Psi_\pi(k,p)\right]_{p'=p}\right\} = -2ip_\mu$$

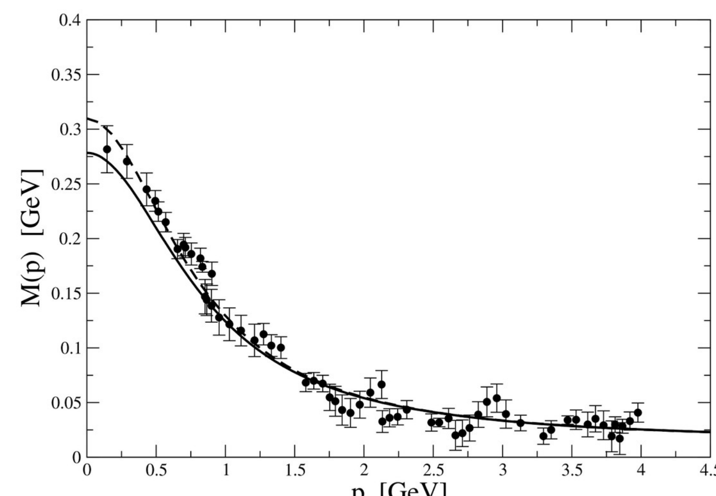
- Calculation of more observables, more rigorous study of chiral symmetry breaking, ingredients from LQCD...





- Better results for pion decay constant are obtained with solutions that converge very slow, due to the shape of the propagator spectral densities. However a similar discussion was already made in a phenomenological model.

$$S_F(k) = \imath \, \frac{\left(k^2-\lambda^2\right)^2 (\not k + m_0) - \left(k^2-\lambda^2\right) m^3}{\prod_{i=1,3}(k^2 - m_i^2 + \imath\epsilon)}$$



**C. S. Mello, et al., Phys. Lett. B, 766 86–93 (2017). Lattice data from M.B. Parappilly, et al., Phys. Rev. D 73 (2006) 054504.

Questions:

- How to construct the EoS at finite density and/or temperature in Minkowski space?

  Mallik, Sarkar: EPJC 61:489-494(2009): Real-time propagators at finite temperature and chemical potential.

  ⟶ Quark condensate (regularization): $\langle \bar{q}q \rangle = -\mathrm{Tr} \int \frac{d^4k}{(2\pi)^4} S_F(k)$

- Is it possible to retain the integral representation while including magnetic field effects?

$$
\begin{aligned}
S(k) &= -i \int_0^\infty \frac{ds}{\cos(qBs)} e^{is\left(k_\parallel^2 - k_\perp^2 \frac{\tan(qBs)}{qBs} - m^2\right)} \\
&\times \left\{ \left[\cos(qBs) + \gamma_1\gamma_2 \sin(qBs)\right](m + \not{k}_\parallel) \right. \\
&- \left. \frac{\not{k}_\perp}{\cos(qBs)} \right\},
\end{aligned}
$$

$$
\tilde{S}(\omega, p^3; \mathbf{p}_\perp) = 2ie^{-p_\perp^2 \ell^2} \sum_{n=0}^{\infty} \frac{(-1)^n D_n(\omega, p^3; \mathbf{p}_\perp)}{\omega^2 - 2n|eB| - (p^3)^2 - m^2}
$$

$$
\begin{aligned}
D_n(\omega, p^3; \mathbf{p}_\perp) = (\gamma^0\omega - \gamma^3 p^3 + m)\left[L_n(2p_\perp^2\ell^2)\mathscr{P}_+ - L_{n-1}(2p_\perp^2\ell^2)\mathscr{P}_-\right] \\
+ 2(\boldsymbol{\gamma}_\perp \cdot \mathbf{p}_\perp) L_{n-1}^1(2p_\perp^2\ell^2)
\end{aligned}
$$

⟶ Is there a way to write down the fermion propagator in such a way that overall the analytic structure is carried out in the denominator?

Summary

- Possibility of calculation of dynamical observables;
- The Integral Representation as a very important tool to solve DSE and BSE;
- Inclusion of more sophisticated ingredients, as quark-gluon vertex, Lattice QCD (self energy, vertex, ...) $\Rightarrow$ more realistic theories!
- Wide range of applications: Form factors, parton distribution functions, analytic structure of pion, kaon, nucleon, Nakanishi weight functions ...
- Perspective of an EoS in Minkowski space for the first time in the literature (??)

Thanks for your attention!