The effective QCD Phase diagram: Parameter fixing, fluctuations and C_s

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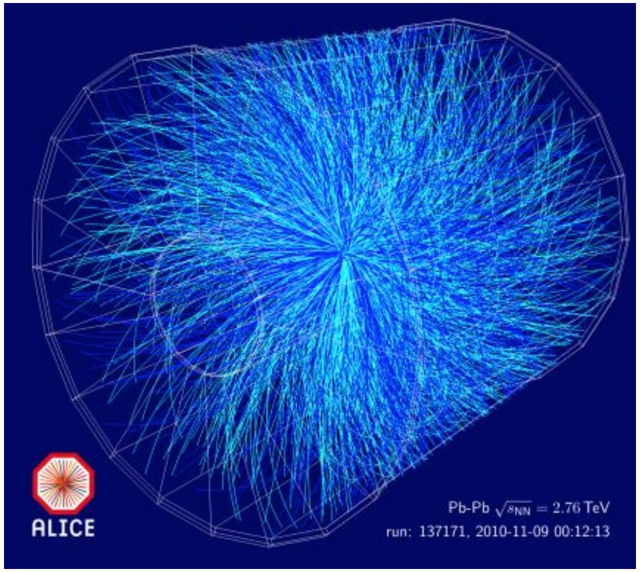
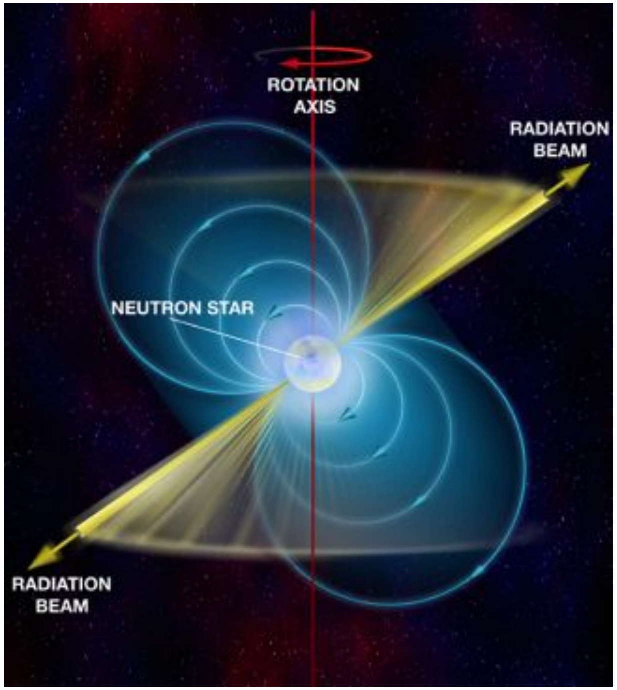
Outline

- Motivation
- Chiral Symmetry
- The Linear Sigma model
- Results and comments so far...
- Study of fluctuations as a signature of CEP
- Results again
- What's next???

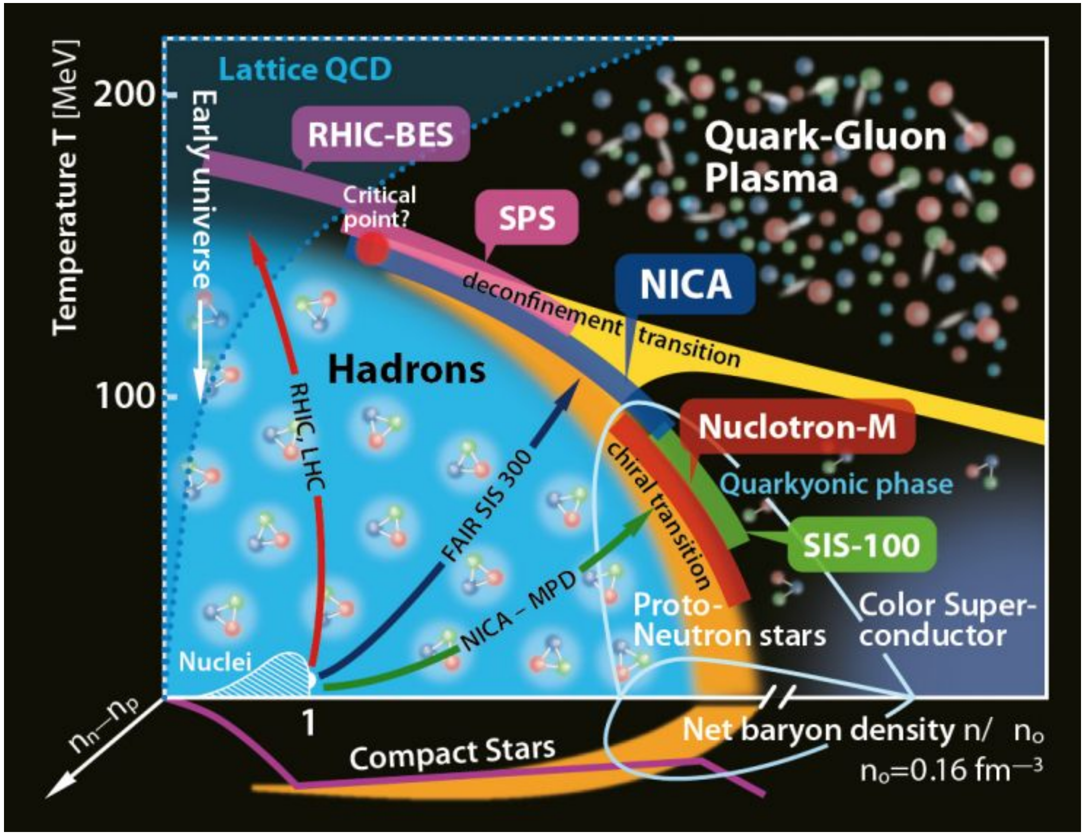
Motivation

- QCD under extreme conditions (temperature and finite quark density) play an important role in understanding the transitions that took place in the early universe.





Motivation

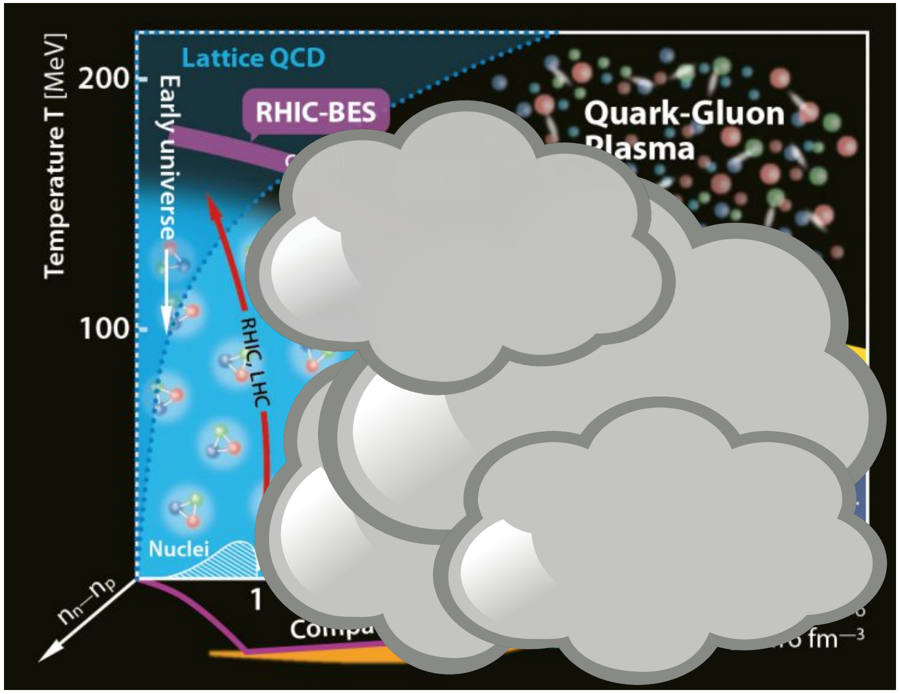


Phase Transition ⇔ Restore/Broke Symmetry

Confinement y/o Chiral Symmetry restauration

Motivation

- There is only reliable information at low densities.
- There are experimental efforts to dissipate doubts at higher densities.
  - NICA
  - RHIC(BESII)
  - JPARC
  - HADES



Chiral Symmetry

- Lagrangian QCD with massless quarks.

$$\mathcal{L}^0_{QCD} = \overline{\psi}(x) i\gamma_\mu \partial^\mu \psi + \mathcal{L}_{quark-gluon} + \mathcal{L}_{glue}$$

- Chirality.

- Chiral Charges.

$$Q^a_L = \frac{1}{2}(Q^a + Q^a_5)\,; \quad Q^a_R = \frac{1}{2}(Q^a - Q^a_5)$$

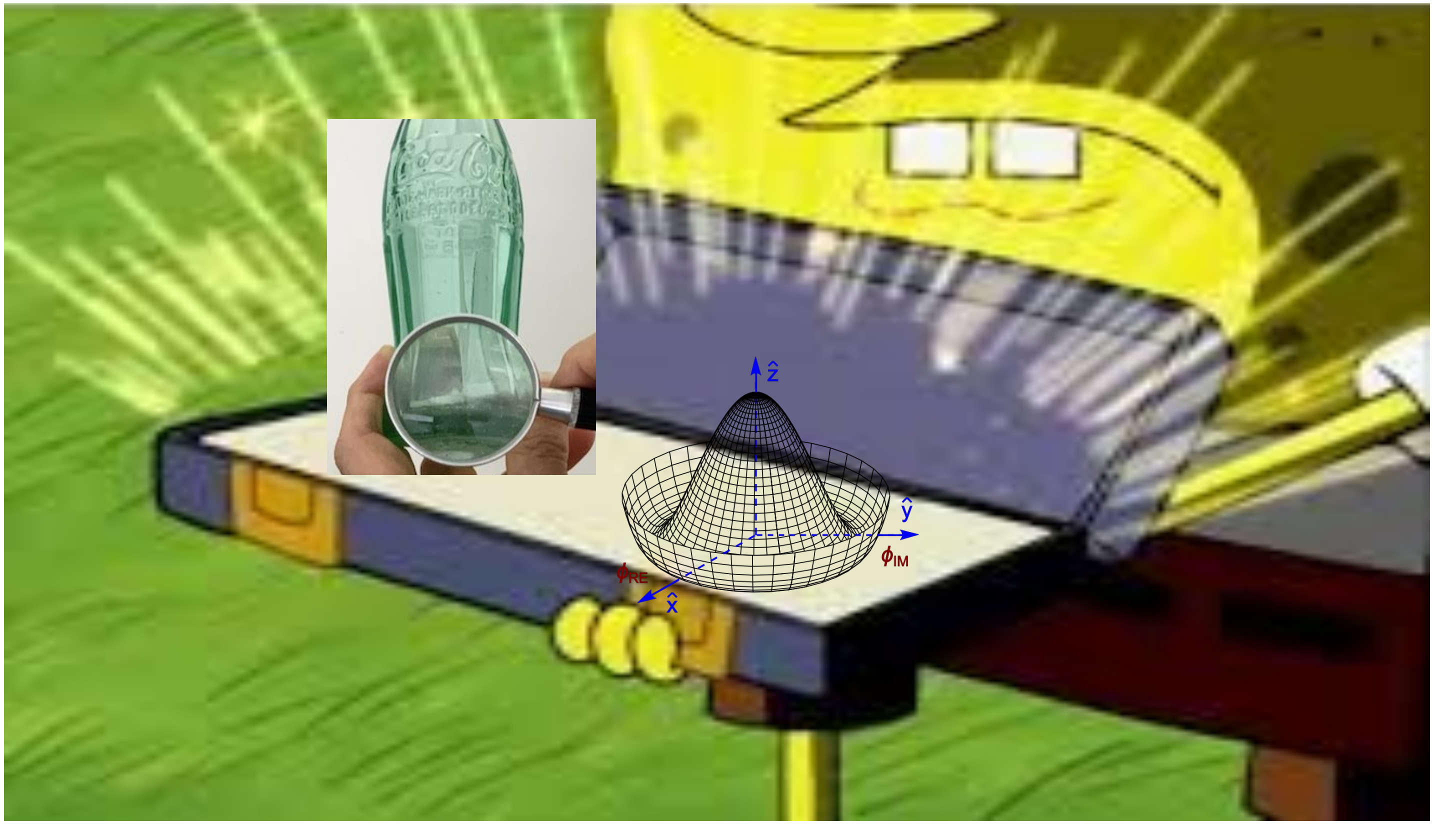
Chiral Symmetry

- If we have a mass:

$$\Lambda_V : m\left(\bar{\varphi}\varphi\right) \rightarrow m\left(\bar{\varphi}\varphi\right)$$

$$\Lambda_A : m\left(\bar{\varphi}\varphi\right) \rightarrow m\left(\bar{\varphi}\varphi\right) - im\Theta\left(\bar{\varphi}\gamma_5\tau\varphi\right)$$

- The Lagrangian is not symmetric!!!
- There must be a way to focus only on chiral symmetry but describe approximately the behavior of QCD.



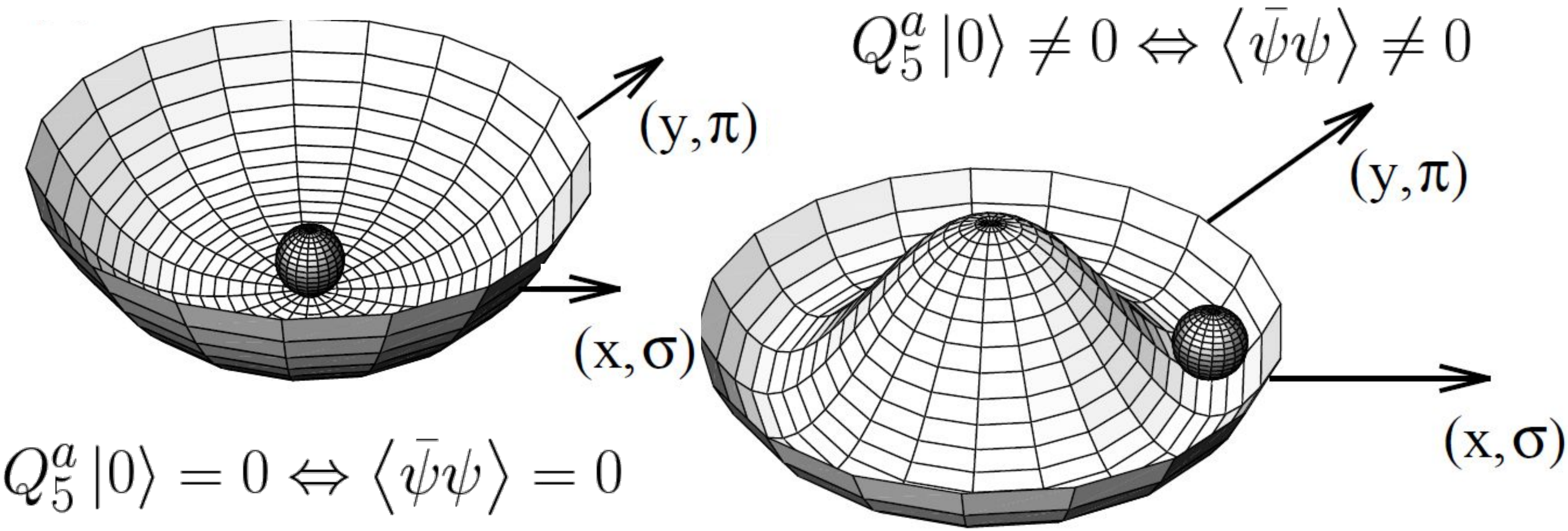
Old Reliable!!!

Linear Sigma Model

- Effective model for low-energy QCD.
- Effects of quarks and mesons on the chiral phase transition.
- Implement ideas of chiral symmetry and spontaneous symmetry breaking

Linear Sigma Model

- Spontaneous breaking of the chiral symmetry.



Linear Sigma Model

- Lagrangian

$$\mathcal{L} = \frac{1}{2}(\partial_\mu\sigma)^2 + \frac{1}{2}(\partial_\mu\vec{\pi})^2 + \frac{a^2}{2}(\sigma^2 + \vec{\pi}^2) - \frac{\lambda}{4}(\sigma^2 + \vec{\pi}^2)^2$$
$$+ i\bar{\psi}\gamma_\mu\partial^\mu\psi - g\bar{\psi}(\sigma + i\gamma_5\vec{\tau}\cdot\vec{\pi})\psi,$$

- To allow for spontaneous symmetry breaking

$$\sigma \to \sigma + v$$

$$\langle\sigma\rangle = v; \qquad \langle\pi\rangle = 0.$$

- where v is identified as the order parameter

Linear Sigma Model

- After the shift

$$\mathcal{L} = -\frac{1}{2}[\sigma(\partial_\mu + iqA_\mu)^2\sigma] \underline{-\frac{1}{2}\left(3\lambda v^2 - a^2\right)\sigma^2}$$
$$-\frac{1}{2}[\vec{\pi}(\partial_\mu + iqA_\mu)^2\vec{\pi}] \underline{-\frac{1}{2}\left(\lambda v^2 - a^2\right)\vec{\pi}^2}$$
$$+ i\bar{\psi}\gamma^\mu D_\mu\psi \underline{-gv\bar{\psi}\psi} + \frac{a^2}{2}v^2 - \frac{\lambda}{4}v^4$$
$$-\frac{\lambda}{4}[(\sigma^2 + \pi_0^2)^2 + 4\pi^+\pi^-(\sigma^2 + \pi_0^2 + \pi^+\pi^-)]$$
$$-g\hat{\psi}(\sigma + i\gamma_5\vec{\tau}\cdot\vec{\pi})\psi$$

with masses

$$m_\sigma^2 = 3\lambda v^2 - a^2$$
$$m_\pi^2 = \lambda v^2 - a^2$$
$$m_f = gv$$

→ which increases with the order parameter

Linear Sigma Model

- We calculate the effective potential for fermions and bosons at temperature and finite chemical potential

$$V_b = s_b T \sum_n \int \frac{d^3k}{(2\pi)^3} \ln\left(D^{-1}\right)^{1/2}, \qquad V_f = s_f T \sum_n \int \frac{d^3k}{(2\pi)^3} \ln\left(S^{-1}\right)$$

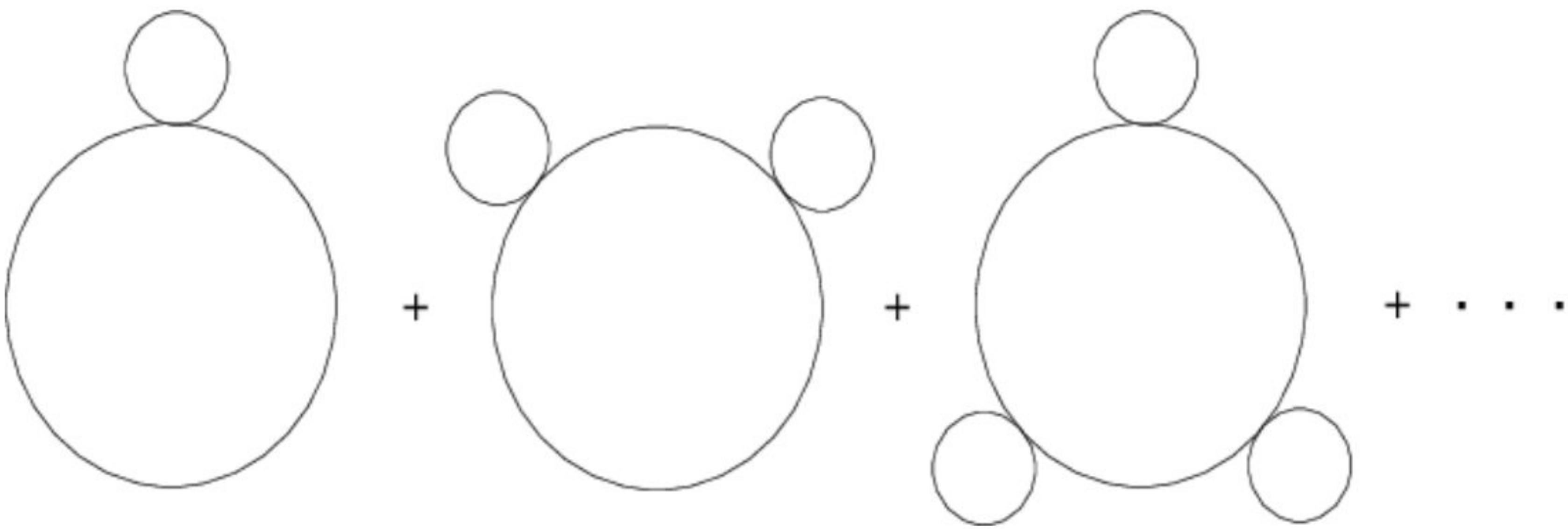
.

where the thermal boson and fermion propagators are given by

$$D = \frac{1}{k^2 + m_b^2 + \omega_n^2}, \qquad S = \frac{\not{k} + m_f}{k^2 + m_f^2 + (\omega_n - i\mu)^2}.$$

Linear Sigma Model

- In order to include media effects on the mesons we need to go beyond mean field and include the Ring diagrams to the boson contribution

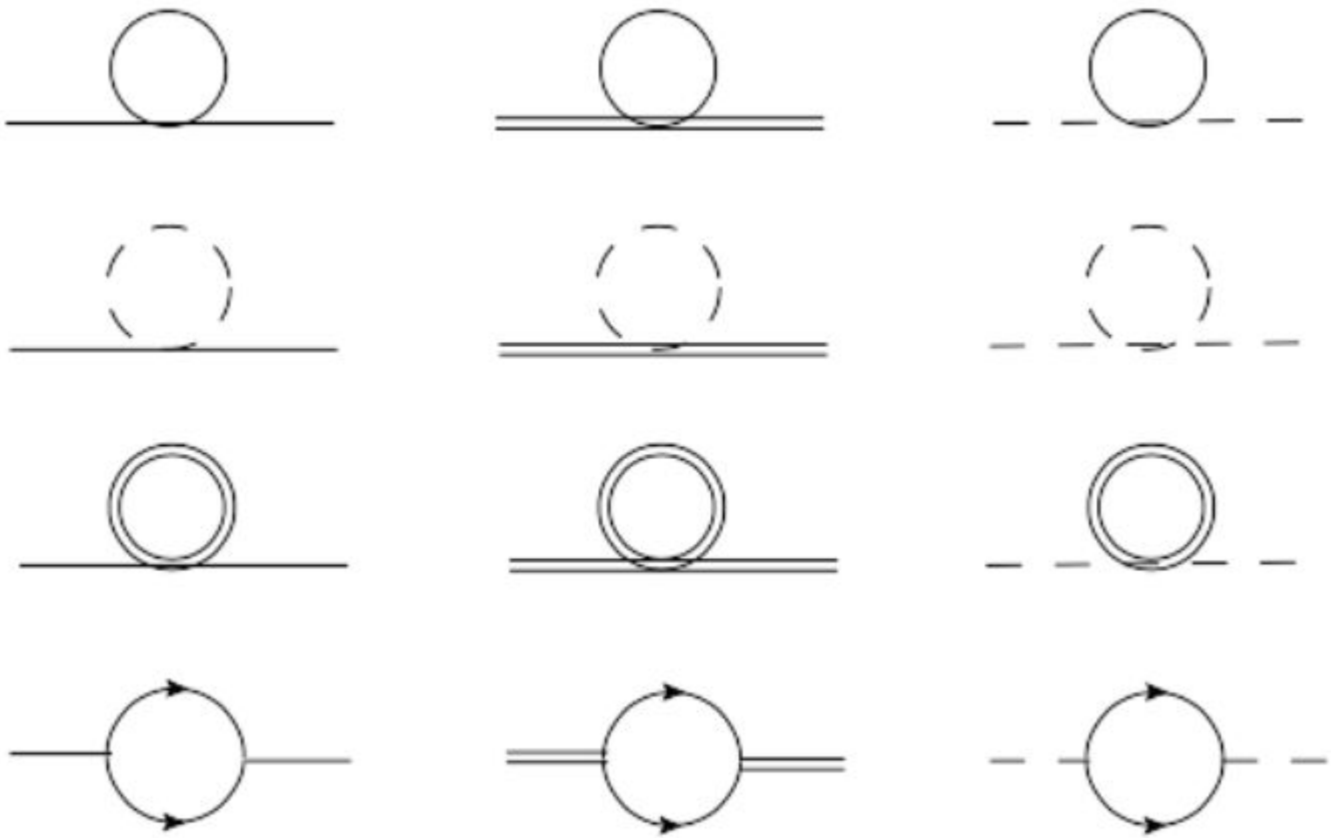


Then the full effective potential is:

$$
\begin{aligned}
V^{\mathrm{eff}} &= V^{\mathrm{tree}} + V^{\mathrm{b}} + V^{\mathrm{f}} + V^{\mathrm{Ring}} \\
V^{\mathrm{tree}}(v) &= -\frac{a^2}{2}v^2 + \frac{\lambda}{4}v^4 \\
V^{\mathrm{b}}(v,T) &= T\sum_n \int \frac{d^3k}{(2\pi)^3} \ln D_{\mathrm{b}}(\omega_n,\vec{k})^{1/2} \\
V^{\mathrm{f}}(v,T,\mu) &= -T\sum_n \int \frac{d^3k}{(2\pi)^3} \mathrm{Tr}[\ln S_{\mathrm{f}}(\tilde{\omega}_n,\vec{k})^{-1}] \\
V^{\mathrm{Ring}}(v,T,\mu) &= \frac{T}{2}\sum_n \int \frac{d^3k}{(2\pi)^3} \ln[1+\Pi_{\mathrm{b}} D(\omega_n,\vec{k})]
\end{aligned}
$$

with Π the self energy

$$\Pi = \lambda\frac{T^2}{2} - N_f N_c g^2 \frac{T^2}{\pi^2}[Li_2(-e^{\mu/T}) + Li_2(-e^{-\mu/T})]$$



High Temperature

$$V^{(eff)} = \boxed{-\frac{a^2}{2}v^2 + \frac{\lambda}{4}v^4} + \sum_{i=\sigma,\pi^0} \left\{ \frac{m_i^4}{64\pi^2}\left[\ln\left(\frac{(4\pi T)^2}{2a^2}\right) - 2\gamma_E + 1\right] - \frac{\pi^2 T^4}{90} + \frac{m_i^2 T^2}{24} - \frac{T}{12\pi}\boxed{(m_i^2 + \Pi)^{3/2}} \right\}$$

$$- N_c \sum_{f=u,d}\left[\frac{m_f^4}{16\pi^2}\left[\ln\left(\frac{(4\pi T)^2}{2a^2}\right) + \psi^0\left(\frac{1}{2} + \frac{i\mu}{2\pi T}\right) + \psi^0\left(\frac{1}{2} - \frac{i\mu}{2\pi T}\right)\right] + 8m_f^2 T^2 [Li_2(-e^{\mu/T}) + Li_2(-e^{-\mu/T})] - 32T^4 [Li_4(-e^{\mu/T}) + Li_4(-e^{-\mu/T})]\right]$$

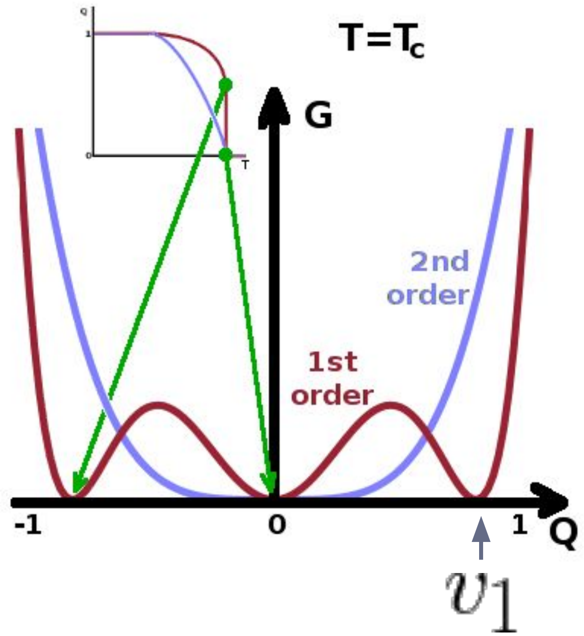
Criticality

- Now the criterion to find the temperature and the chemical potential where the chiral symmetry is restored, is the following.

  - Second Order

$$\left.\frac{\partial^2}{\partial v^2} V^{eff}\right|_{v=0} = 0$$

  - First Order



$$V^{eff}(0) = V^{eff}(v_1); \quad \left.\frac{\partial}{v\partial v} V^{eff}\right|_{v=0} = \left.\frac{\partial}{v\partial v} V^{eff}\right|_{v=v_1} = 0$$

Model Parameters

- The parameter space consists of the λ and g coupling constants and the mass parameter a, which can be fix by LQCD data (PRL 125, 052001 (2020)).

Fixing a with:

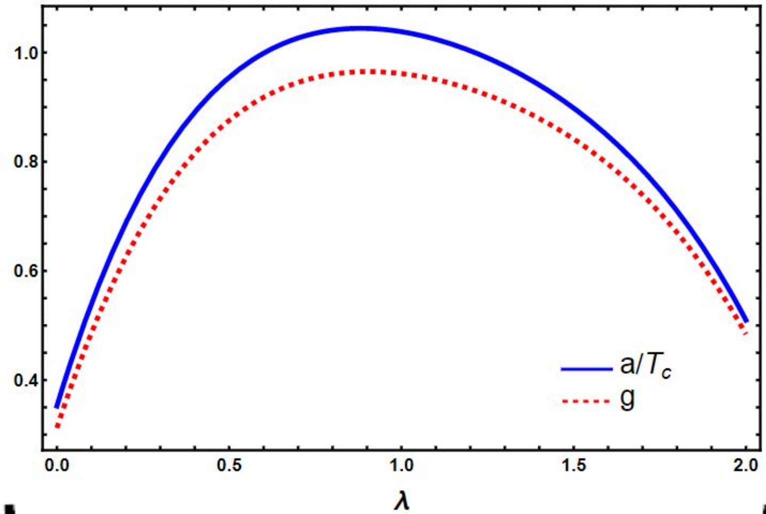
$$6\lambda\left(\frac{T_c^2}{12}-\frac{T_c}{4\pi}\left(\Pi_b(T_c,\mu_B=0)-a^2\right)^{1/2}+\frac{a^2}{16\pi^2}\left[\ln\left(\frac{\tilde{\mu}^2}{T_c^2}\right)\right]\right)$$
$$+g^2T_c^2-a^2=0.$$

Fixing λ and g with the collection of curves that obey this relation:

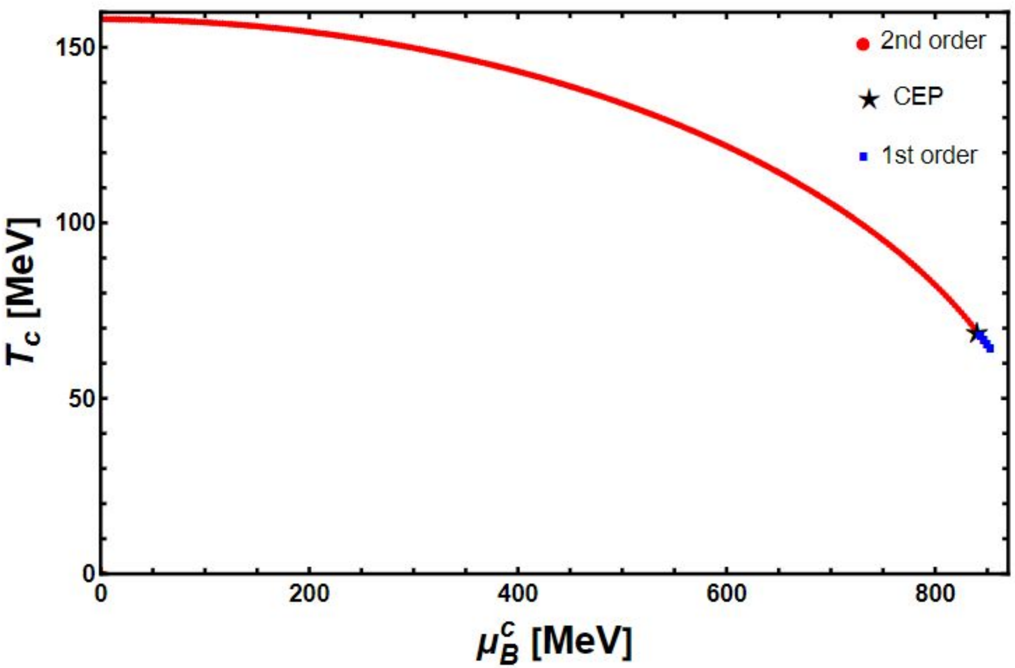
$$\frac{T_c(\mu_B)}{T_c^0}=1-\kappa_2\left(\frac{\mu_B}{T_c^0}\right)^2+\kappa_4\left(\frac{\mu_B}{T_c^0}\right)^4$$

$\kappa_2 = 0.0153$ and $\kappa_4 = 0.00032$

Results



$$\lambda = 0.4,\;\; g = 0.88 \;\;\text{and}\;\; a = 141.38\ \text{MeV}$$



$$768\ \text{MeV} < \mu_B^{CEP} < 849\ \text{MeV}$$

$$69\ \text{MeV} < T^{CEP} < 70.3\ \text{MeV}$$

Ayala, A, et al, Eur. Phys. J. A 58, 87 (2022).

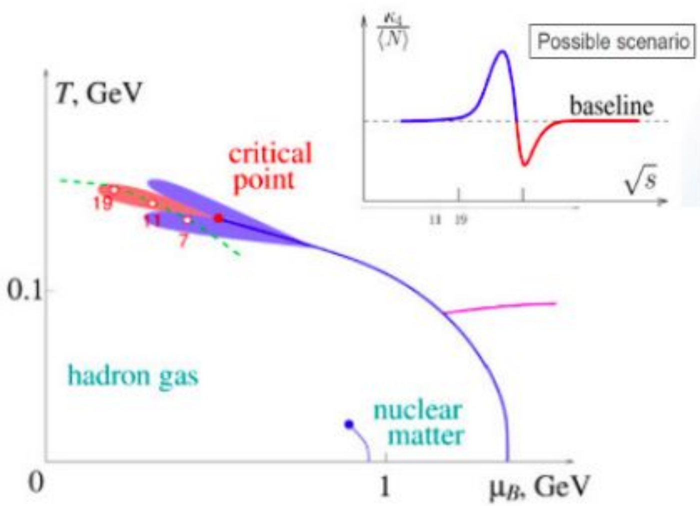
Comments so far..

- The linear sigma model is an effective tool that allows the analytical analysis of properties of great interest in QCD.
- Up to this point we have learned that the parameter space with physical relevance is neither large nor arbitrary.
- There is still room for improvement and plenty of things to learn from this model and its possible extensions.

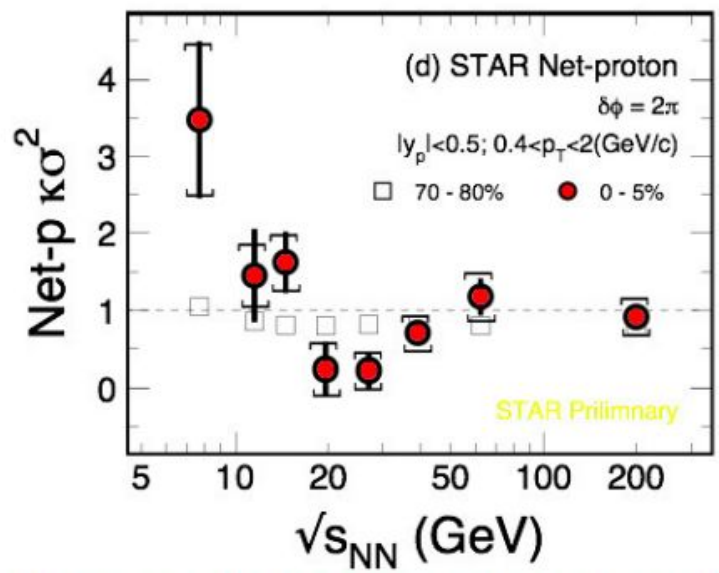
What is next?

$\kappa\sigma^2 = C_4/C_2$

Model



STAR BES Data



M.A. Stephanov, PRL107, 052301 (2011).
Schaefer&Wanger,PRD 85, 034027 (2012)
Vovchenko et al., PRC92, 054901 (2015)
JW Chen et al., PRD93, 034037 (2016)
arXiv: 1603.05198.

Non-monotonic energy dependence is observed for 4th order net-proton fluctuations in most central Au+Au collisions.

Xiaofeng Luo ISMD 2016, Jeju, Korea, Aug. 29-Sept. 2, 2016 11 / 18

Experimental Observables

- A powerful tool to experimentally locate the CEP is the study of event-by-event fluctuations in relativistic heavy-ion collisions.

- The relation with thermodynamics comes through the partition function Z, which is the fundamental object.

- The partition function is also the moment generating function and therefore the cumulant generating function is given by ln Z

- The main point can be illustrated using a description of the fluctuations based on the probability distribution of an order parameter field

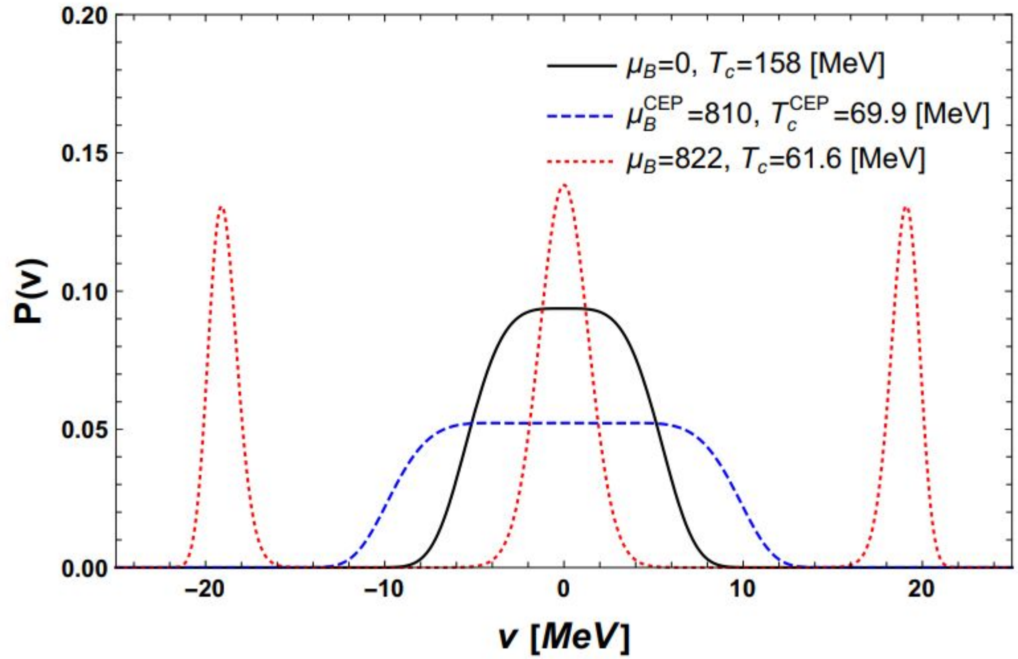
$$\mathcal{Z}(T,\mu,v) = exp[-\Omega V^{eff}(v)/T]$$

Moments generating function

$$G(\theta) = \int_{-\infty}^{\infty} dv \; e^{v\theta} \mathcal{Z}(T,\mu,v)$$

Cumulants generating function

$$K(\theta) = ln \; G(\theta)$$



Cumulants

- Therefore the cumulants are given by:

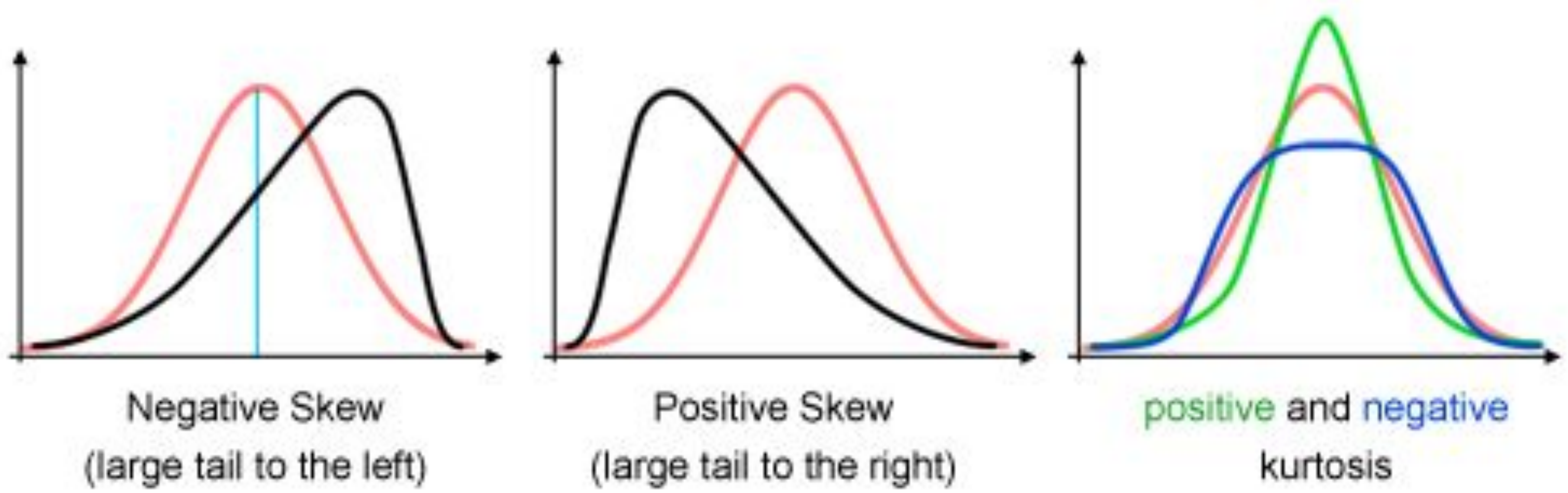
$$\mu'_n = \frac{\int_{-\infty}^{\infty} v^n \mathcal{Z}(T,\mu,v)}{\int_{-\infty}^{\infty} \mathcal{Z}(T,\mu,v)}$$

$$\begin{aligned}
\kappa_1 &= \mu'_1 \\
\kappa_2 &= \mu'_2 - \mu'^2_1 \\
\kappa_3 &= \mu'_3 - 3\mu'_2\mu'_1 + 2\mu'^3_1 \\
\kappa_4 &= \mu'_4 - 4\mu'_3\mu'_1 - 3\mu'^2_2 + 12\mu'_2\mu'^2_1 - 6\mu'^4_1
\end{aligned}$$

$$\begin{aligned}
S &\equiv \kappa_3/\sigma^3 \\
\kappa &\equiv \kappa_4/\kappa_2^2 \\
S\sigma &= \kappa_3/\kappa_2 \\
\kappa\sigma^2 &= \kappa_4/\kappa_2
\end{aligned}$$

Cumulants

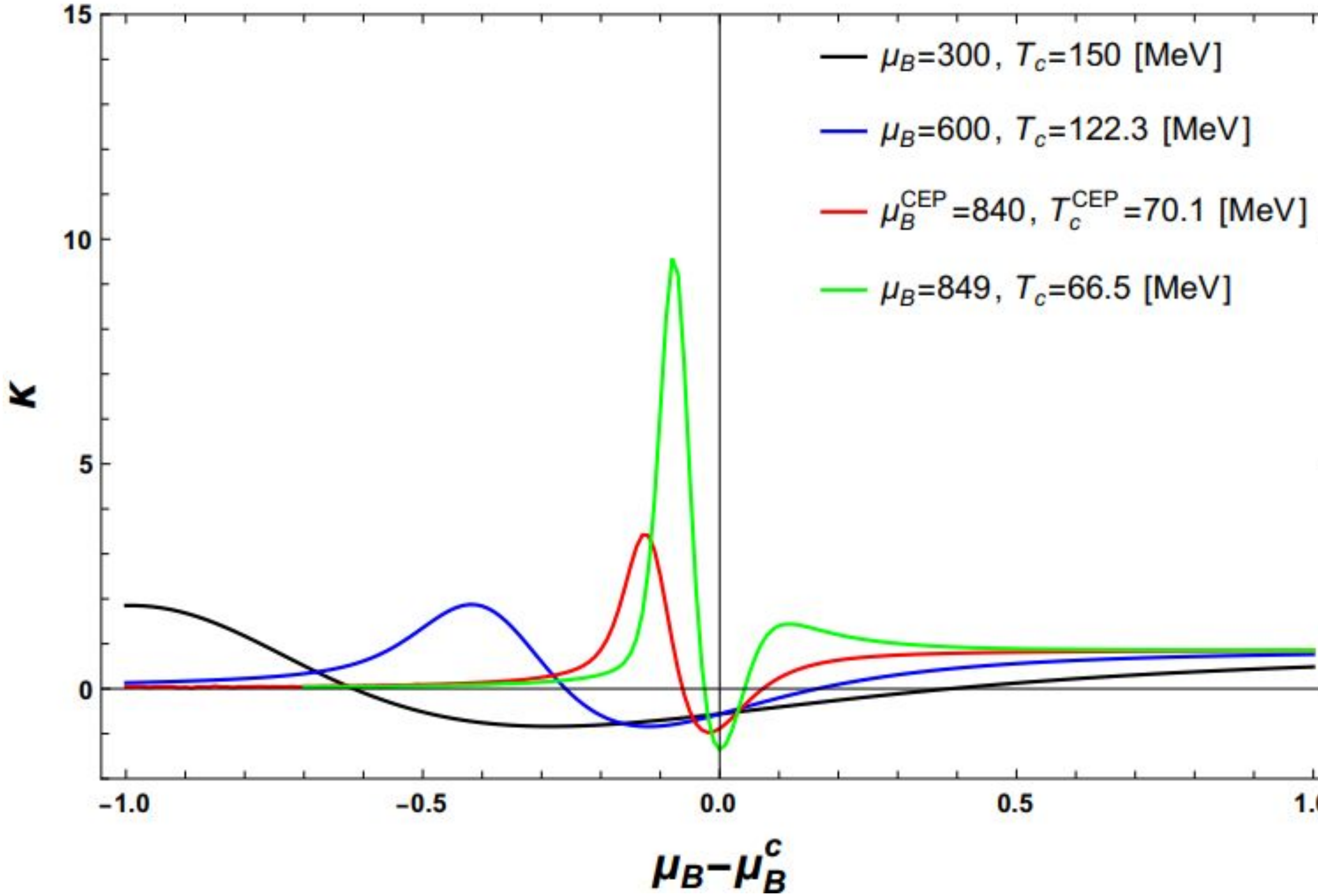
- The first cumulant is the mean, the second cumulant is the variance, the third cumulant is the same as the third central moment or Skewness and the fourth cumulant is call kurtosis.



Comments about the CEP

- The main idea is to establish that some observables present fluctuations around the critical point that can be directly related to these cumulants

- The fact that non-Gaussian moments have stronger dependence on correlation length than, e.g., quadratic moments makes those higher moments more sensitive signatures of the critical point.

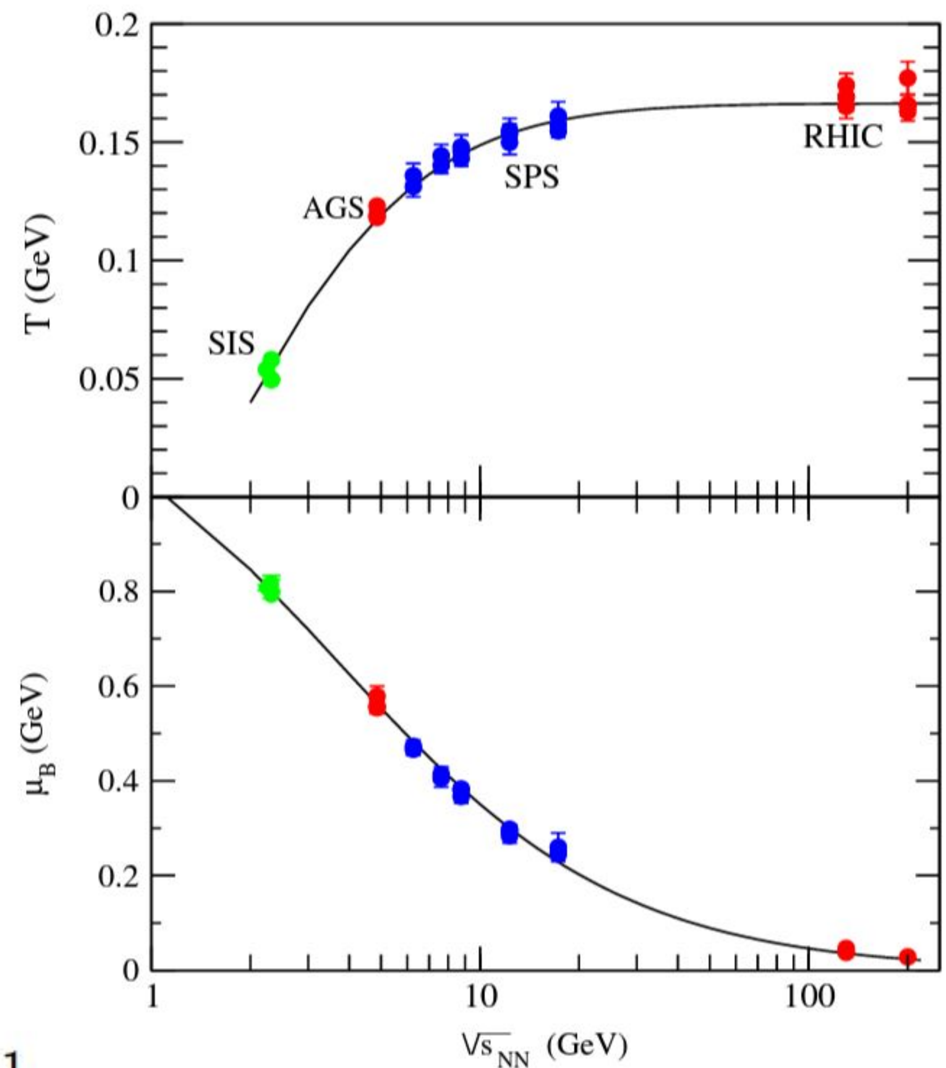


15
10
5
0
κ
−1.0
−0.5
0.0
0.5
1.0
$\mu_B - \mu_B^c$
μ_B=300, T_c=150 [MeV]
μ_B=600, T_c=122.3 [MeV]
μ_B^{CEP}=840, T_c^{CEP}=70.1 [MeV]
μ_B=849, T_c=66.5 [MeV]

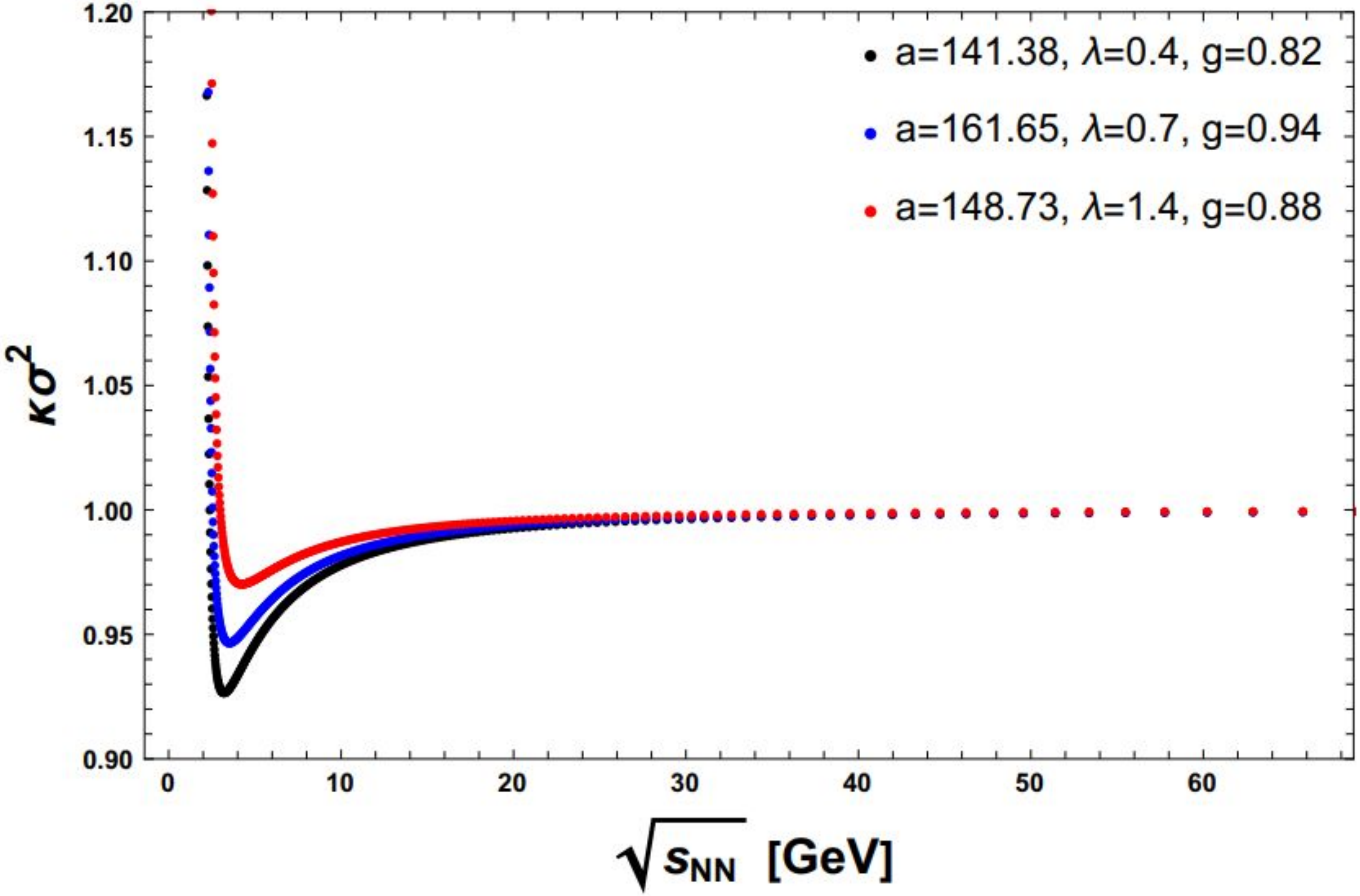
- Finally, we can parameterize the baryon chemical potential in terms of the collision energy to compare the fluctuations of the cumulants and their dependence on it.

$$\mu_B(\sqrt{s_{NN}}) = \frac{d}{1 + e\sqrt{s_{NN}}}$$

$$d = 1.308\text{GeV}, \;\; e = 0.273\text{GeV}^{-1}$$



Phys. Rev. C **73**, 034905



a=141.38, λ=0.4, g=0.82
a=161.65, λ=0.7, g=0.94
a=148.73, λ=1.4, g=0.88
1.20
1.15
1.10
1.05
1.00
0.95
0.90
κσ²
0
10
20
30
40
50
60
√s_NN [GeV]

Comments so far…

- Ring diagrams inclusion is equivalent to introducing screening effects at finite T and μB .

- CEP signaled by "divergence" of $\kappa\sigma^2$

- CEP found at low T and high μB

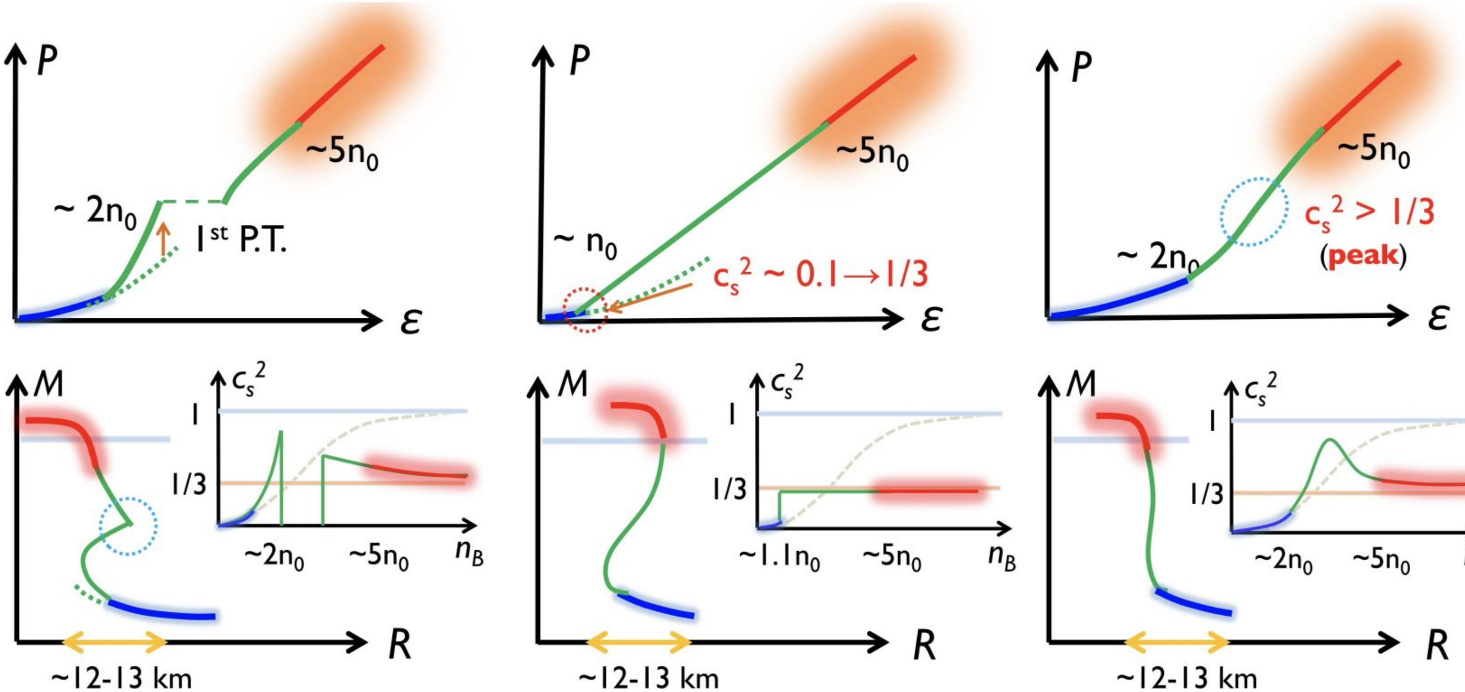
- the CEP can be located for collision energies , $\sqrt{s_{NN}} \sim 2$ GeV, namely, in the lowest NICA or within the HADES energy domain.

What is next?

- Speed of sound is closely related with the thermodynamics properties of any system, including the EoS.
- For example, in neutron start researches, the c_s behavior as a function of baryon number density influences the mass-radius relationship.
- In HIC, c_s also conveys relevant information; for example, it displays a local minimum at a crossover transition.

What is next?

- TOV and EoS can give some insight about the transition quark-nucleon matter. Directly related with the speed of sound behavior.



T. Kojo arXiv:2011.10940v2 (2020)

Speed of sound

- The square of the speed of sound is usually defined as

$$c_\chi^2 = \left(\frac{\partial p}{\partial \epsilon}\right)_\chi$$

where χ denotes the parameter fixed in the calculation of the speed of sound.

- According to the properties on the propagation medium, it may be more useful to keep one quantity fixed rather than another.

Speed of sound

- For this work, we will focus on

$$c_{\rho_B}^2 = \frac{\partial(p,\rho_B)}{\partial(\epsilon,\rho_B)} = \frac{s\chi_{\mu\mu} - \rho_B\chi_{\mu T}}{T(\chi_{TT}\chi_{\mu\mu} - \chi_{\mu T}^2)},$$

$$c_s^2 = \frac{\partial(p,s)}{\partial(\epsilon,s)} = \frac{\rho_B\chi_{TT} - s\chi_{\mu T}}{\mu_B(\chi_{TT}\chi_{\mu\mu} - \chi_{\mu T}^2)},$$

$$c_{s/\rho_B}^2 = \frac{\partial(p,s/\rho_B)}{\partial(\epsilon,s/\rho_B)} = \frac{c_{\rho_B}^2 Ts + c_s^2\mu_B\rho_B}{Ts + \mu_B\rho_B}.$$

Speed of sound

- The pressure, entropy and baryon number densities can be derived using the thermodynamics relations in the grand canonical ensemble as

$$p = -\Omega, \qquad \epsilon = -p + Ts + \mu_B \rho_B$$

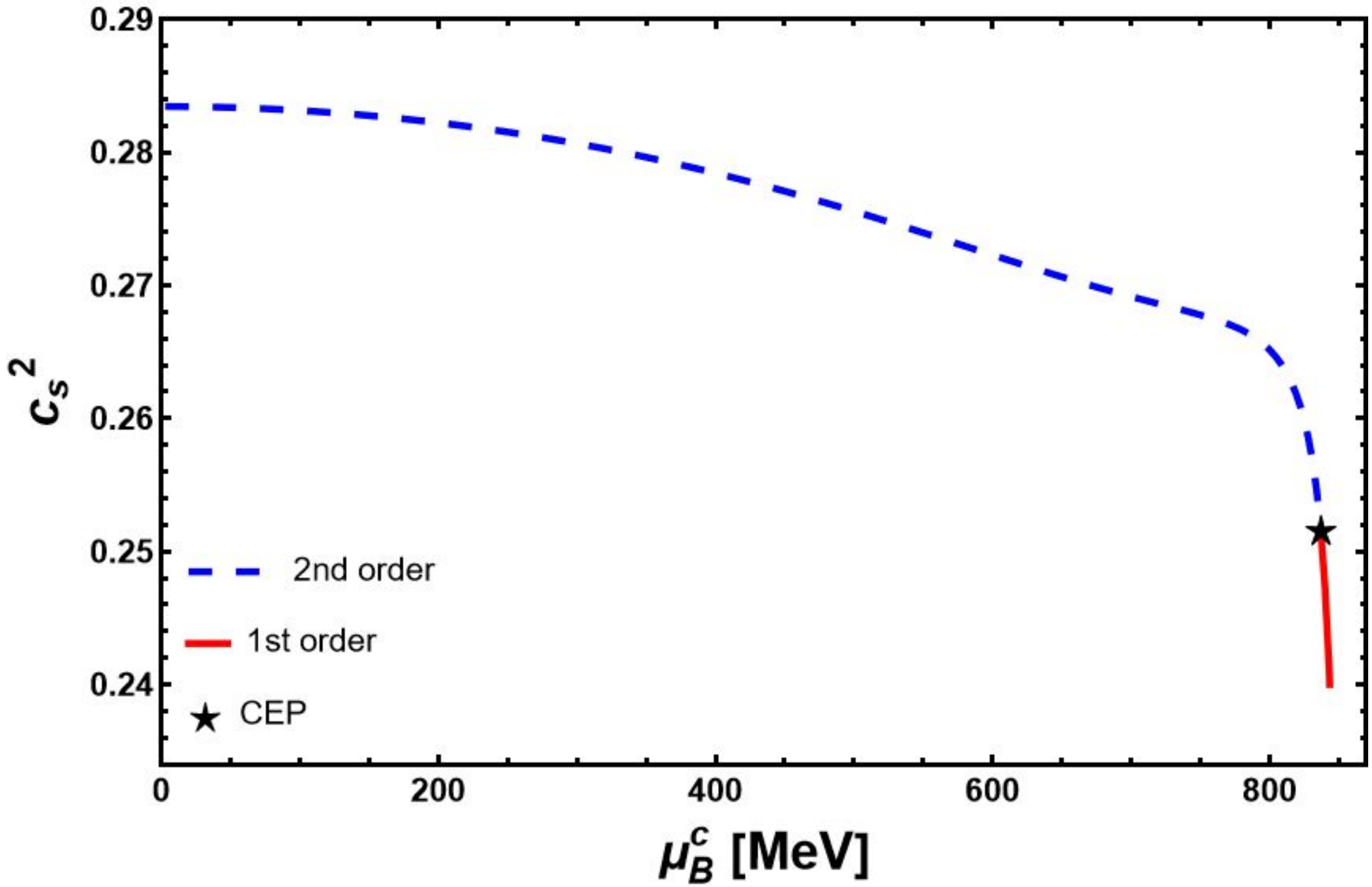
$$s = \left(\frac{\partial p}{\partial T}\right)_{\mu_B} \quad \text{and} \quad \rho_B = \left(\frac{\partial p}{\partial \mu_B}\right)_T .$$

where

$$\Omega(T, \mu) = V^{(eff)}(v = 0, T, \mu)$$

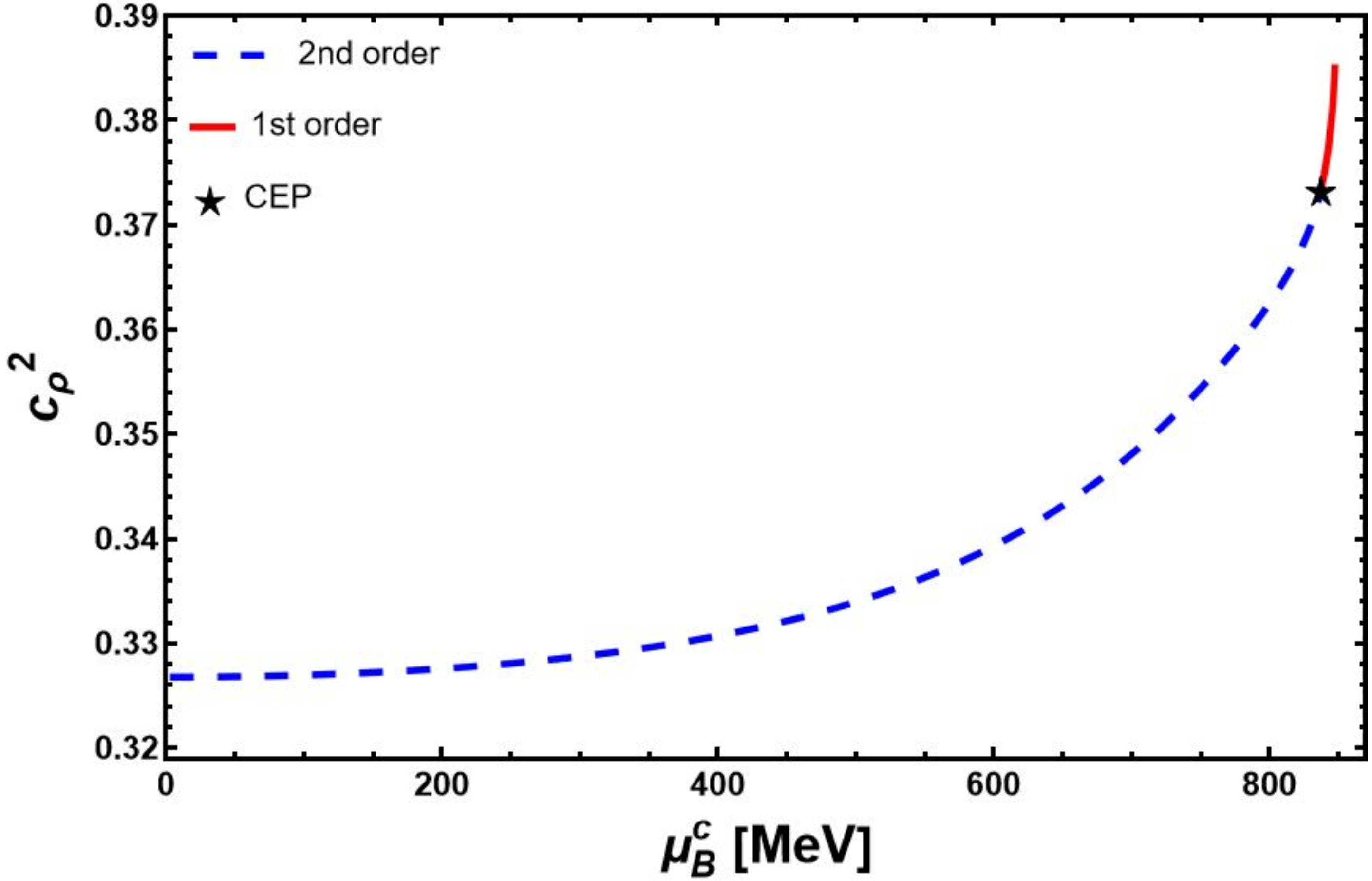
Results

$$\lambda = 0.4, \;\; g = 0.88 \;\; \text{and} \;\; a = 141.38 \text{ MeV}$$



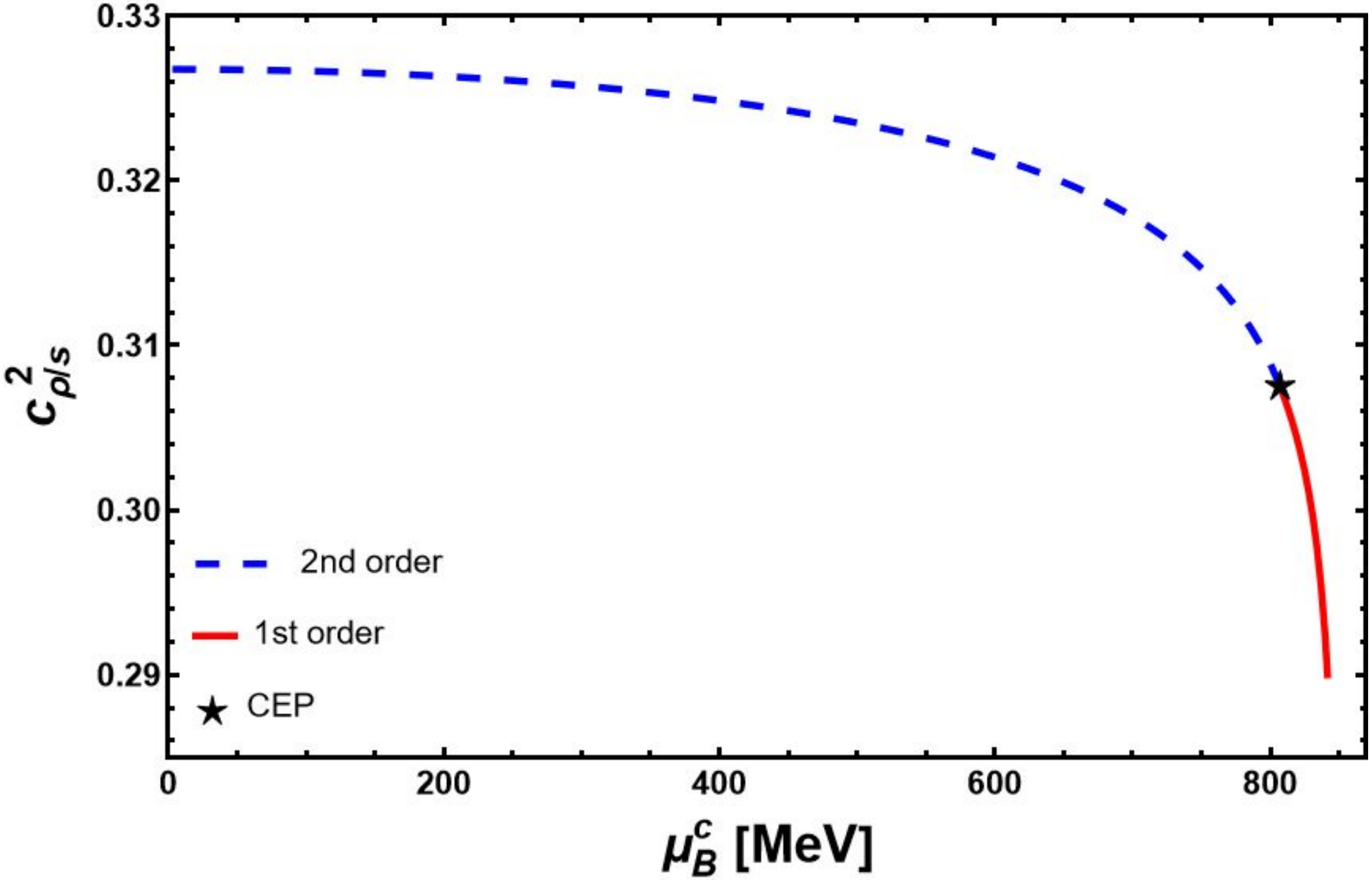
Results

$\lambda = 0.4,\ g = 0.88$ and $a = 141.38$ MeV

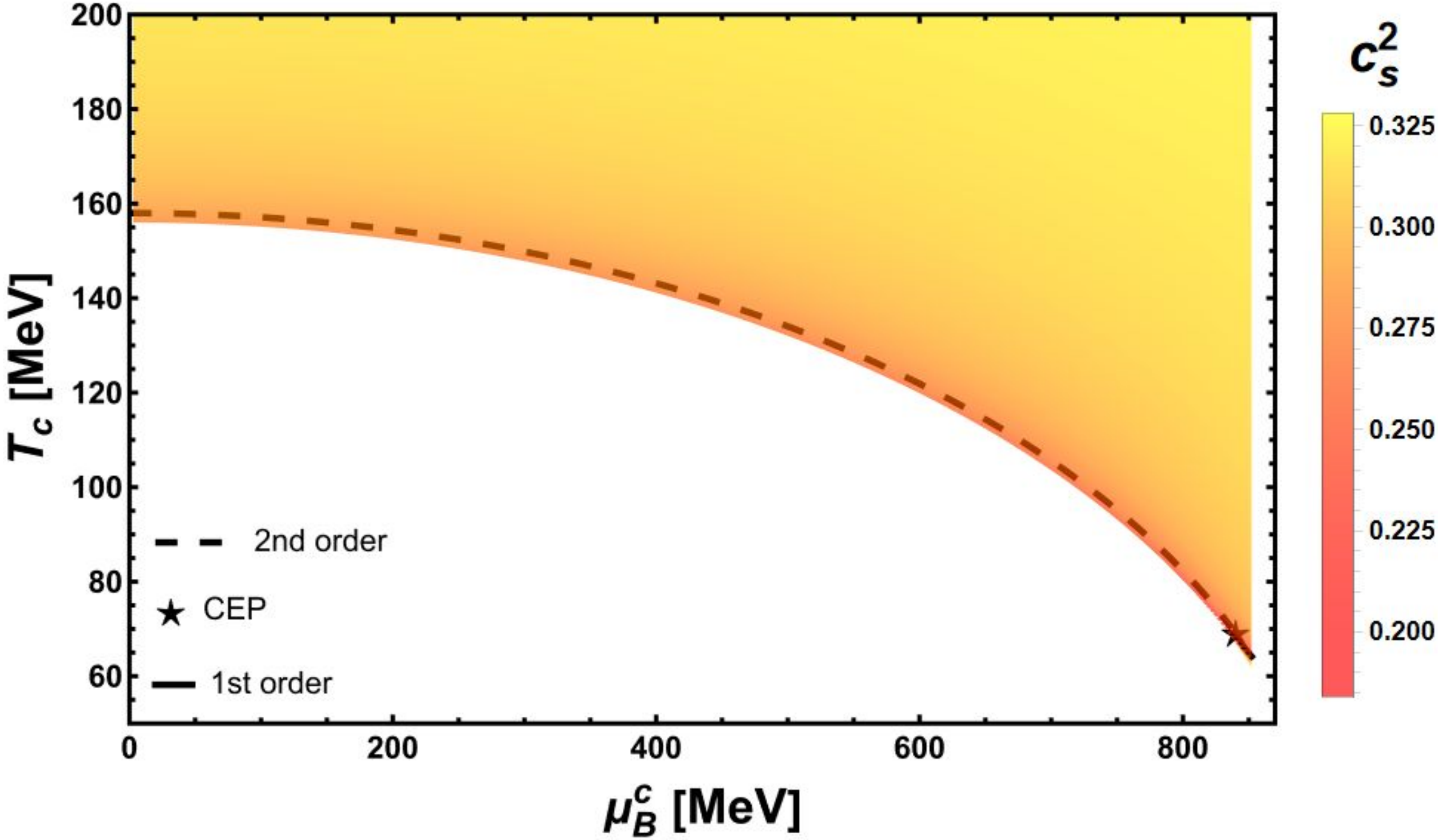


Results

$\lambda = 0.4, \;\; g = 0.88 \;\;$ and $\;\; a = 141.38 \; MeV$

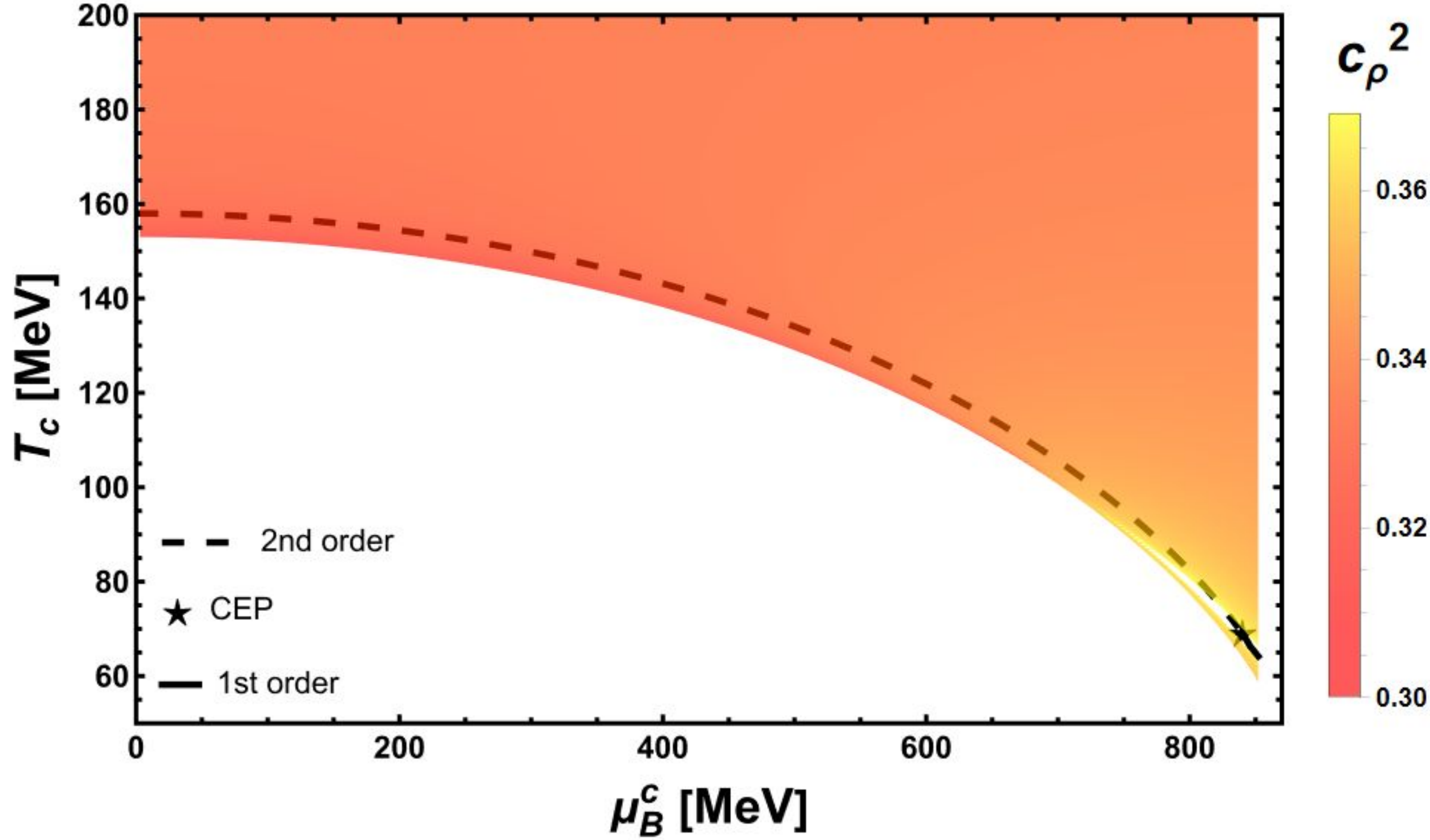


Results



c_s^2
0.325
0.300
0.275
0.250
0.225
0.200
200
180
160
140
120
100
80
60
T_c [MeV]
2nd order
CEP
1st order
0
200
400
600
800
μ_B^c [MeV]

Results



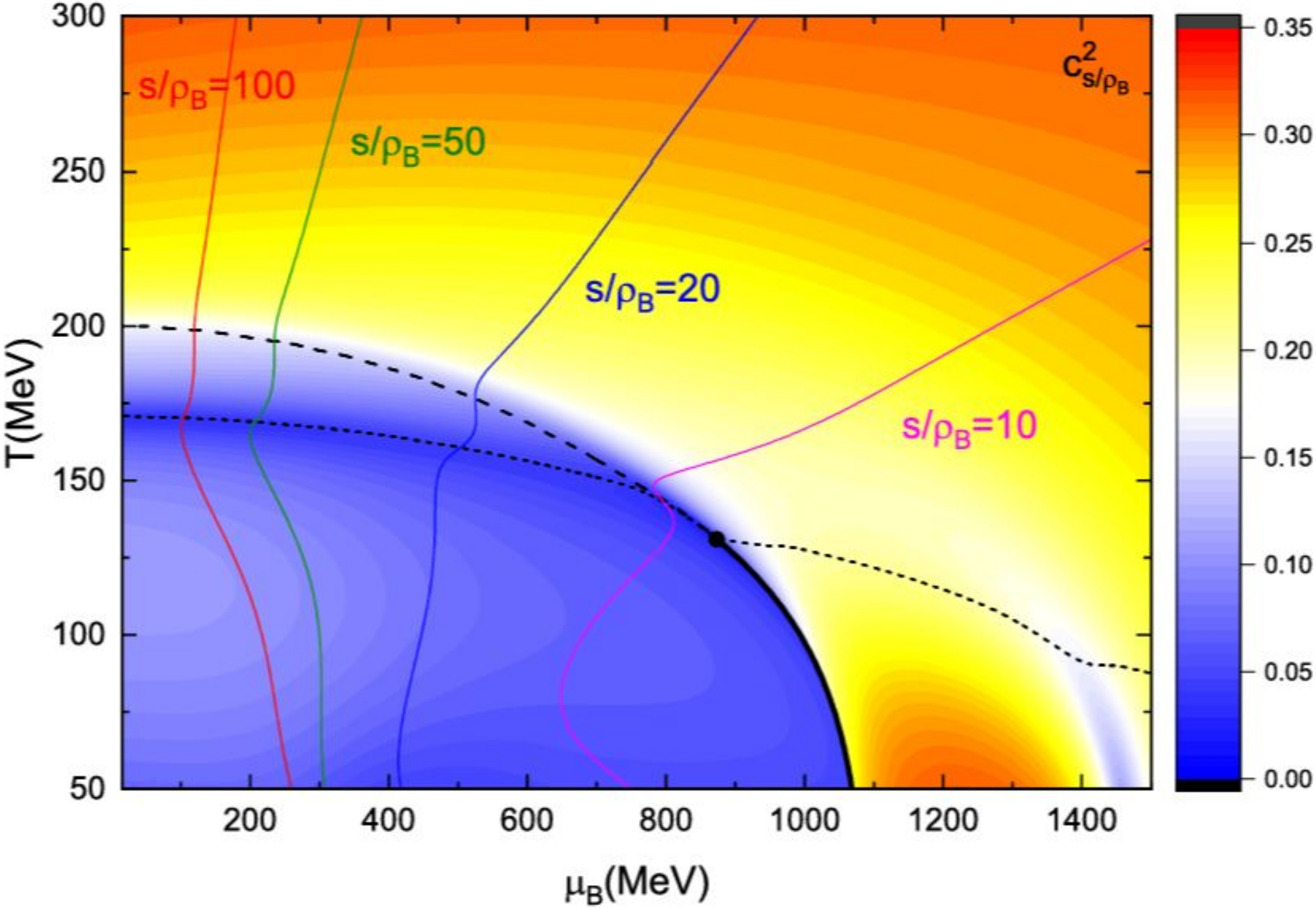
200
180
160
140
120
100
80
60
0
200
400
600
800
T_c [MeV]
μ_B^c [MeV]
c_ρ^2
0.36
0.34
0.32
0.30
2nd order
CEP
1st order

Speed of sound so far…

- The main idea is to establish that the speed of sound exhibits different behaviors around the critical point that can be directly related to drastic changes in the medium.

- For example, a drastic change in the behavior of the speed of sound at fix density, it's a clear sign that we are approaching the CEP.

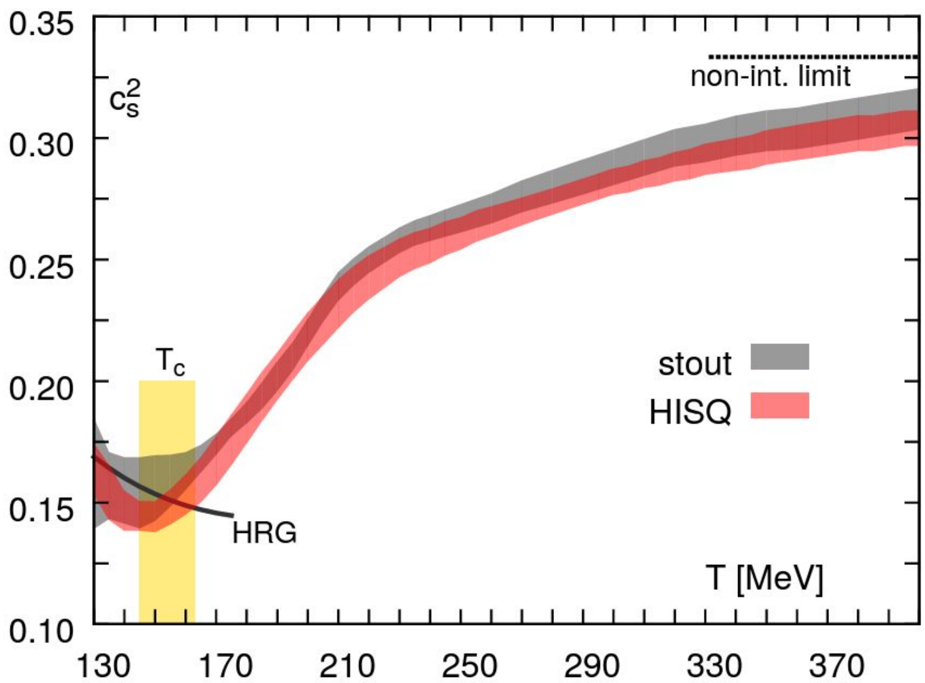
- This criterion can be complemented with the other criteria that have been shown in previous works of the collaboration.



https://doi.org/10.1103/PhysRevD.105.094024

Is this useful?

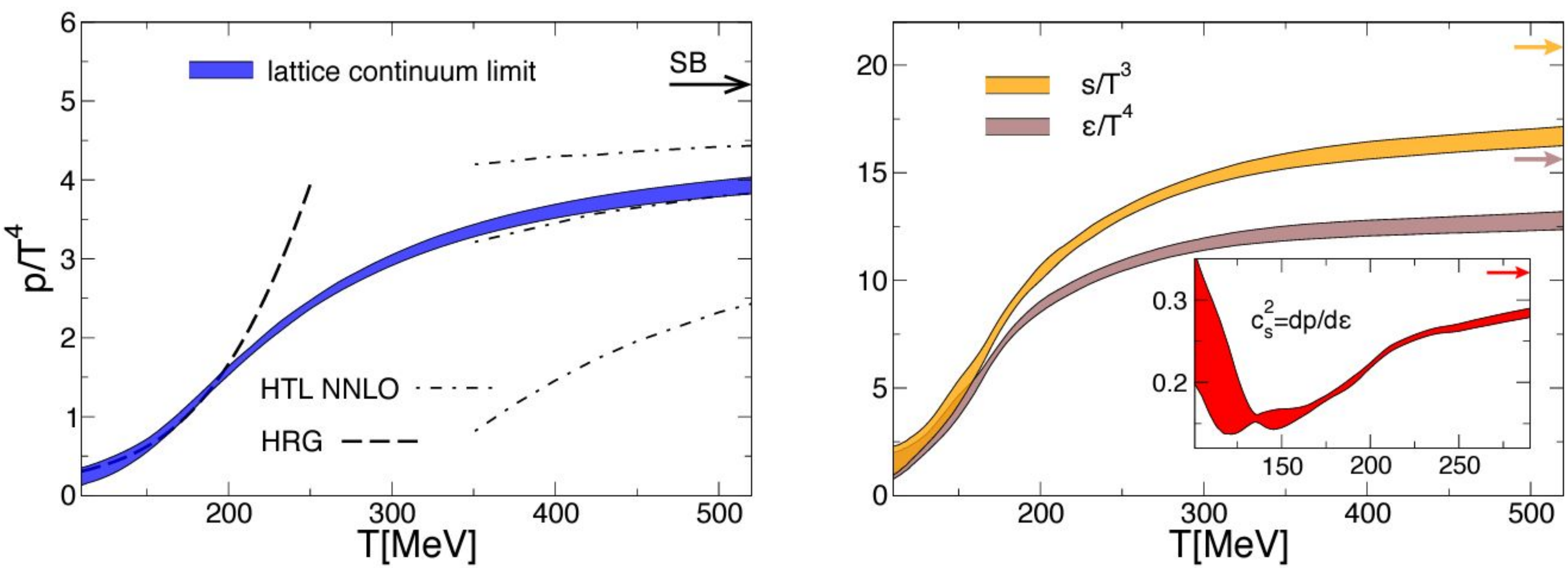
This show the speed of sound squared from lattice QCD and the HRG model versus temperature, the yellow band shows the value of the critical temperature at zero chemical potential. $T_C = 154 \pm 9$



https://doi.org/10.1103/PhysRevD.90.094503

Is this useful?

This show the speed of sound squared from lattice QCD and the HRG model versus temperature, the yellow band shows the value of the critical temperature at zero chemical potential. $T_C = 154 \pm 9$



https://doi.org/10.1016/j.physletb.2014.01.007

Model Parameters

- The parameter space consists of the λ and g coupling constants and the mass parameter a, which can be fix by LQCD data (PRD 102, 034027).

$N_f = 2$; T_C = 166 MeV

$N_f = 2 + 1$; T_C = 158 MeV

Fixing a with:

$$6\lambda\left(\frac{T_c^2}{12} - \frac{T_c}{4\pi}\left(\Pi_b(T_c, \mu_B = 0) - a^2\right)^{1/2} + \frac{a^2}{16\pi^2}\left[\ln\left(\frac{\tilde{\mu}^2}{T_c^2}\right)\right]\right) + g^2 T_c^2 - a^2 = 0.$$

Fixing λ and g with the collection of curves that obey this relation:

$$\frac{T_c(\mu_B)}{T_c^0} = 1 - \kappa_2\left(\frac{\mu_B}{T_c^0}\right)^2 + \kappa_4\left(\frac{\mu_B}{T_c^0}\right)^4$$

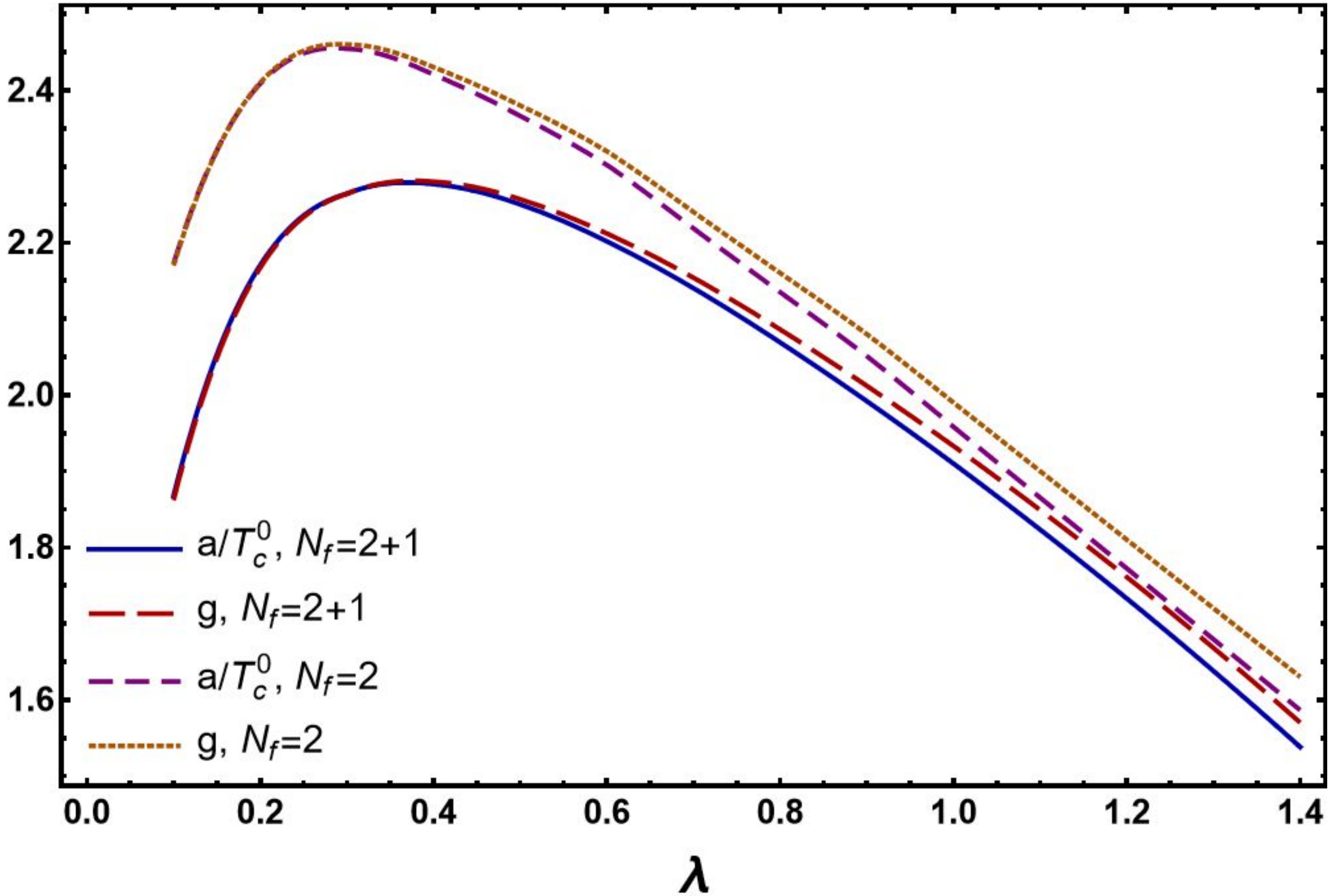
$N_f = 2$

κ_2 = 0.0176, and κ_4 = 0.0

$N_f = 2 + 1$

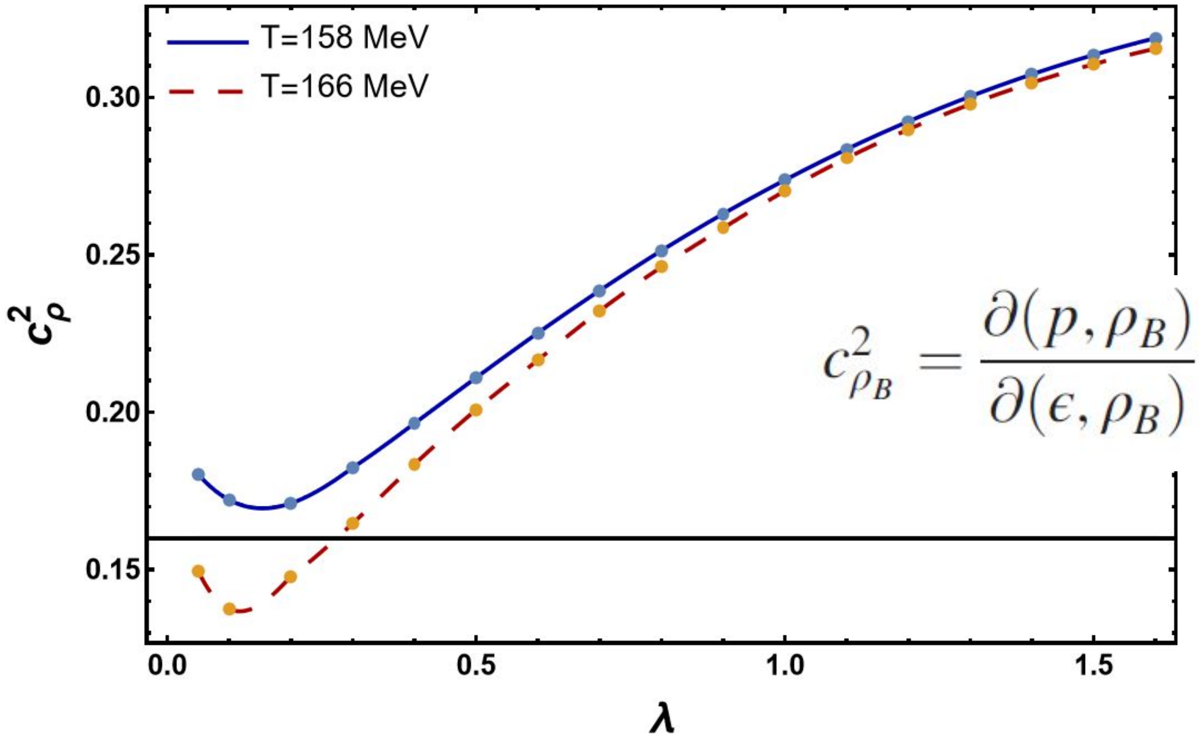
κ_2 = 0.0153, and κ_4 = 0.0

Is this useful?



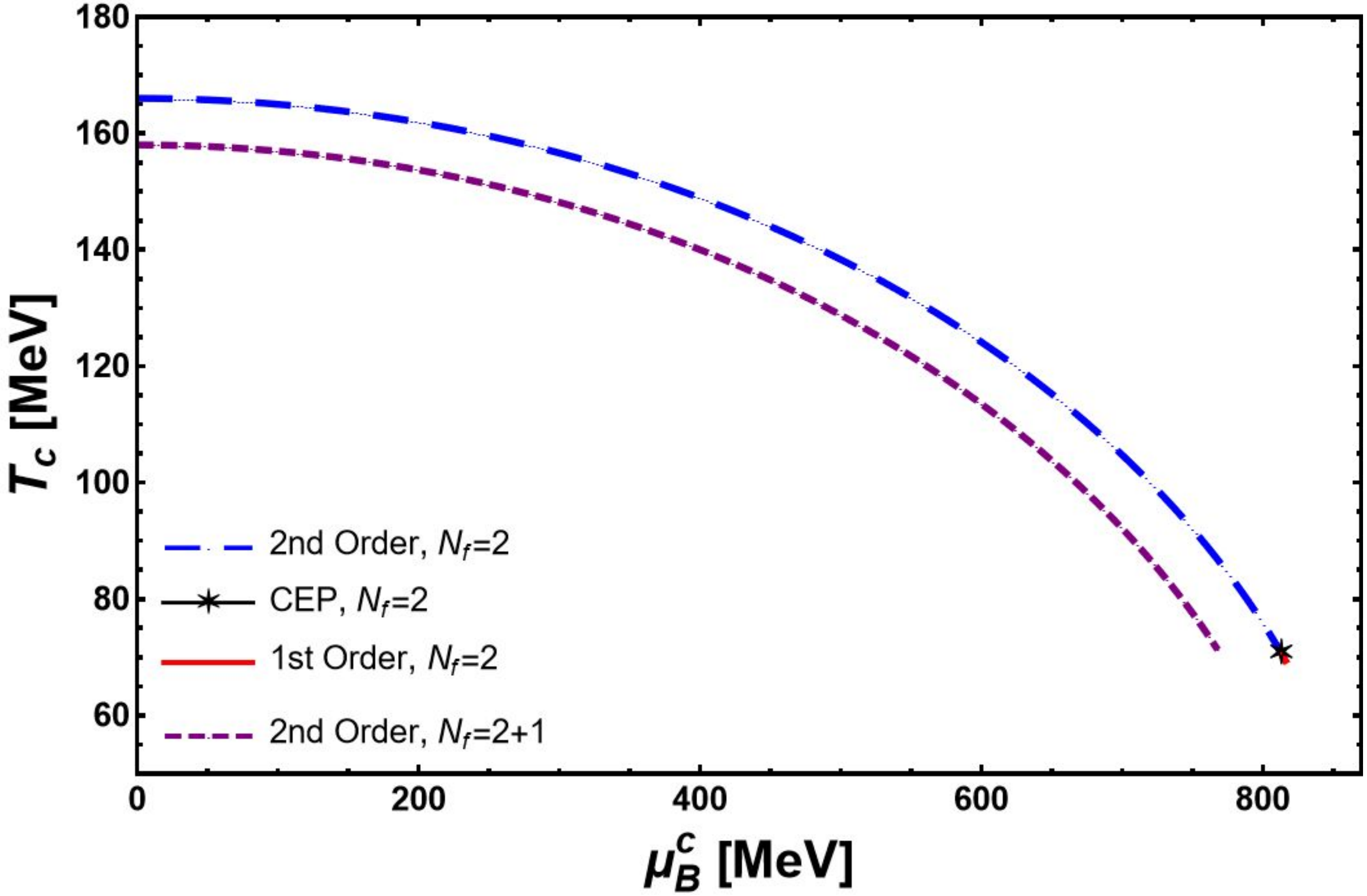
Model Parameters

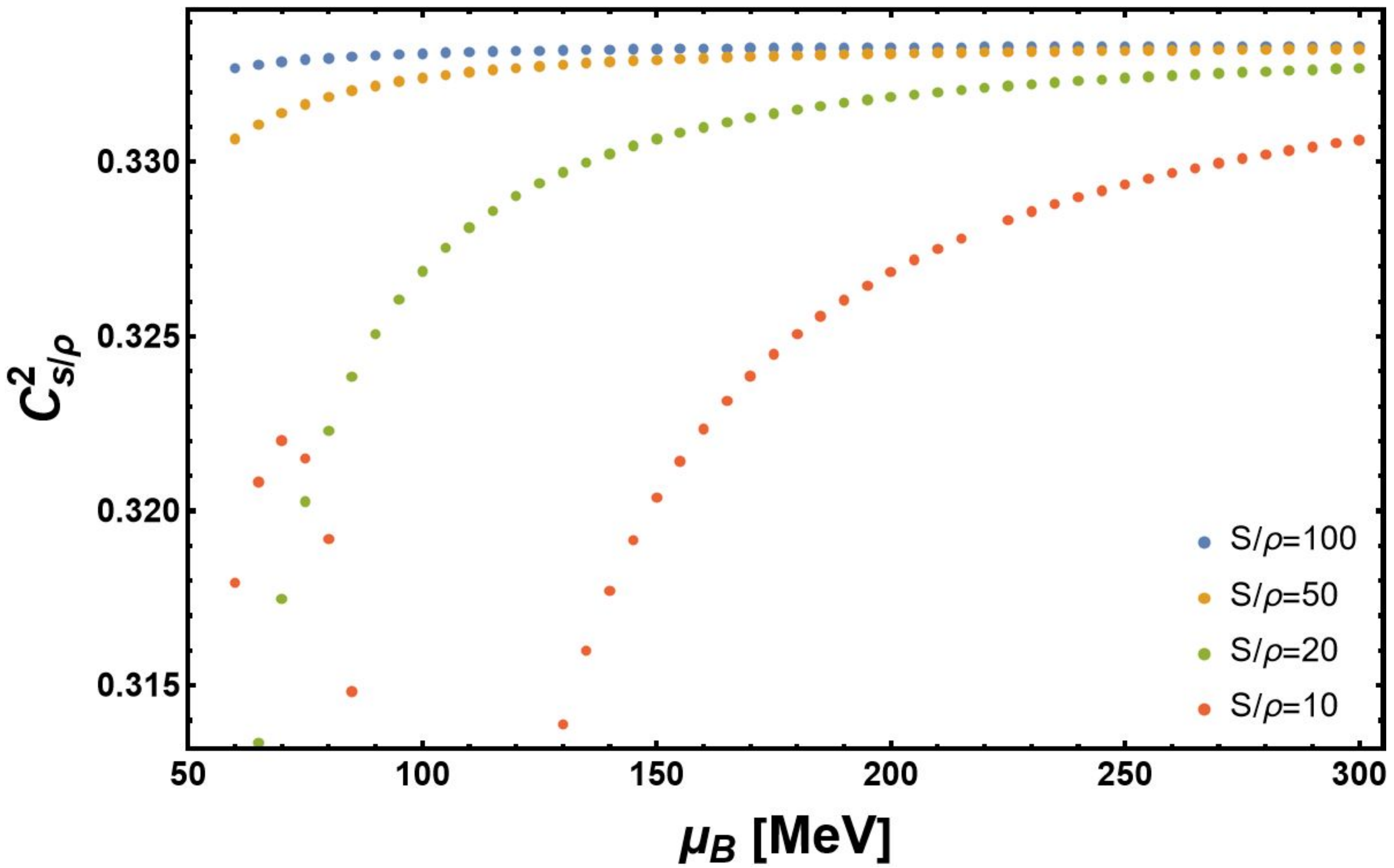
- Now, let us use the value of the speed of sound at $\mu = 0$, meaning $C_S \approxeq 0.16$ and try to fix lambda.

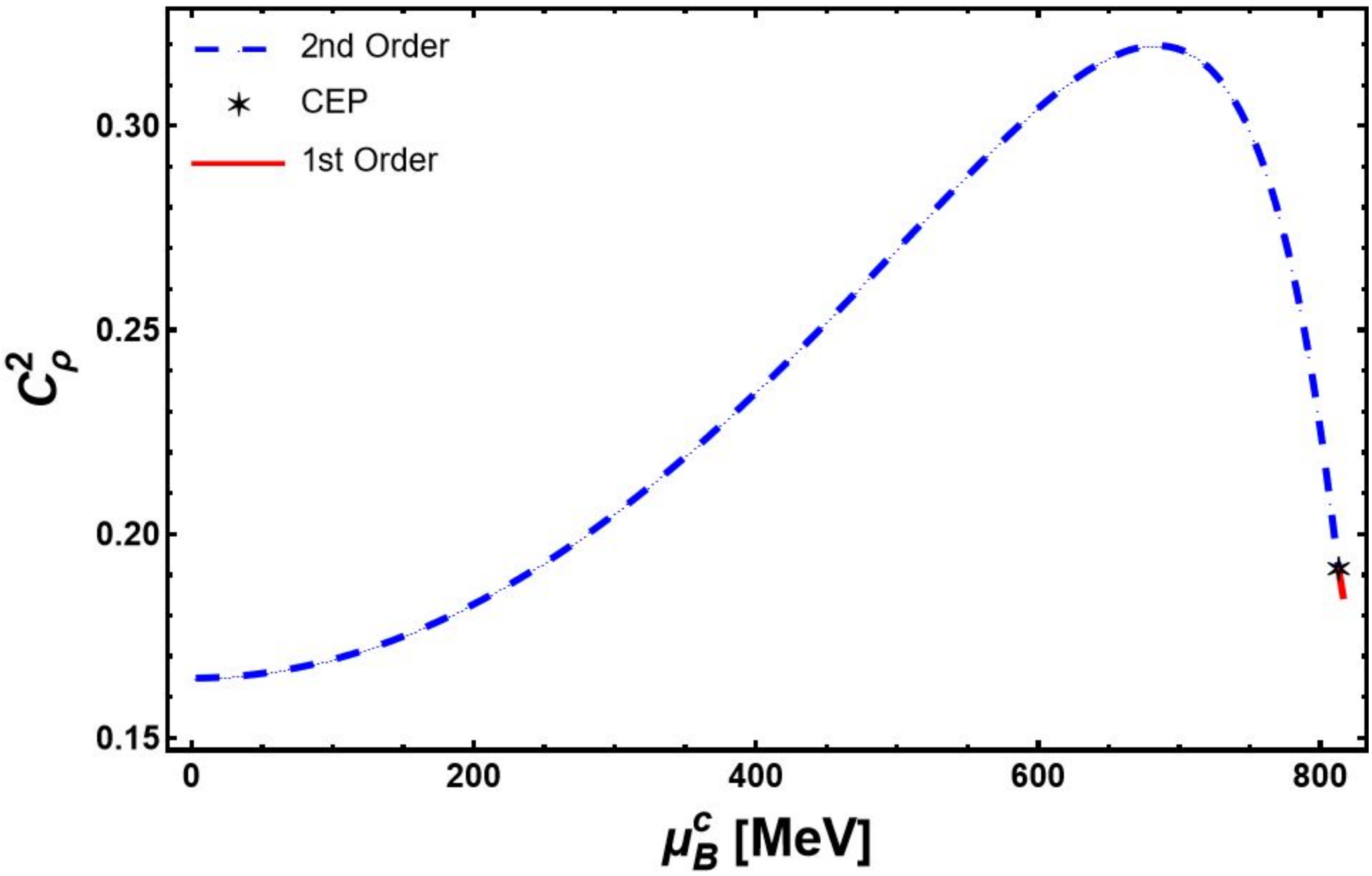


$$c^2_{\rho_B} = \frac{\partial(p, \rho_B)}{\partial(\epsilon, \rho_B)} = \frac{s\chi_{\mu\mu} - \rho_B \chi_{\mu T}}{T(\chi_{TT}\chi_{\mu\mu} - \chi^2_{\mu T})},$$

Nf = 2+1: a = 325.744, $\boldsymbol{\lambda}$ = 0.15, g = 2.057
Nf = 2 f: a = 407.441, $\boldsymbol{\lambda}$ = 0.30, g = 2.461




0.330
0.325
0.320
0.315
$C^2_{s/\rho}$
50
100
150
200
250
300
μ_B [MeV]
S/ρ=100
S/ρ=50
S/ρ=20
S/ρ=10



2nd Order
CEP
1st Order
0.30
0.25
0.20
0.15
C_ρ^2
0
200
400
600
800
μ_B^c [MeV]

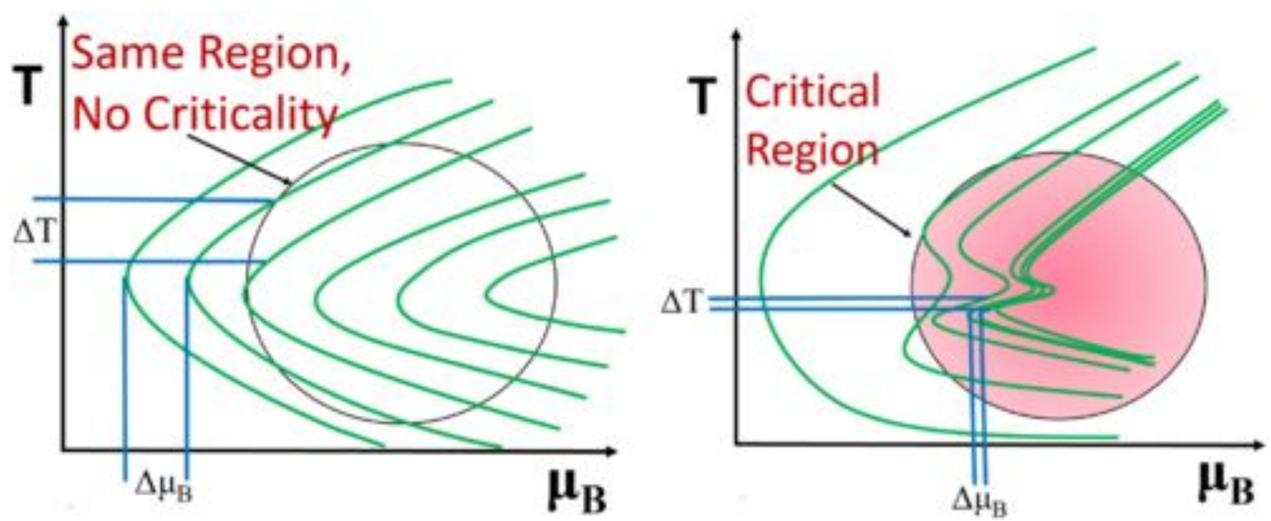
Final Comments

- We were able to use the speed of sound at $\mu = 0$ as an additional criterion to set the parameters.
- Using the observables from LQCD for Nf = 2+1, We were not able to observe the CEP, given the limitations of the approximation.
- Still, using the observables for two flavors we were able to appreciate the CEP, and to note that there is some behavior in the speed of sound that can serve as a criterion for the location of the CEP.

For the future…

- Critical Lensing: the critical point is an atractor of isentropic trajectories.

How do the size and shape of the critical region affected the isentropes trajectories???



THANKS FOR WATCHING!