Thermodynamical analysis of a strongly interacting system with isospin imbalance using LSMq $N_f = 2$. The cold case. 2nd Part.

Non-trivial one Loop correction performing techniques

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0. The review

The review

The last presentation we derived the full Lagrangian:

$$\mathcal{L}_{tree} = \frac{a^2}{2}(v^2 + \Delta^2) - \frac{\lambda}{4}(v^2 + \Delta^2)^2 + \frac{1}{2}\mu_I^2\Delta^2 + hv$$

$$\mathcal{L}_1 = \frac{\Delta}{\sqrt{2}}\left((\mu_I^2 - m_{\pi_0}^2 - i\mu_I\partial^0)e^{-i\theta}\pi_+ + (\mu_I^2 - m_{\pi_0}^2 + -i\mu_I\partial^0)e^{i\theta}\pi_-\right) + (h - vm_{\pi_0}^2)\sigma$$

$$\mathcal{L}_2 = \frac{1}{2}[(\partial_\mu\sigma)^2 + (\partial_\mu\pi_0)^2] + \partial_\mu\pi_-\partial^\mu\pi_+ + i\mu_I(\pi_+\partial_0\pi_- - \pi_-\partial_0\pi_+) - \frac{m_\sigma^2}{2}\sigma - \frac{m_{\pi_0}^2}{2}\pi_0 - (m_{ch}^2 - \mu_I^2)\pi_-\pi_+ - \frac{\lambda\Delta^2}{2}(e^{-2i\theta}\pi_+^2 + e^{2i\theta}\pi_-^2) - \sqrt{2}\lambda\Delta v\sigma(e^{-i\theta}\pi_+ + e^{i\theta}\pi_-)$$

$$\mathcal{L}_3 = -\lambda\sigma(\sigma^2 + \pi_0^2 + 2\pi_-\pi_+) - \frac{\lambda\Delta}{\sqrt{2}}(\sigma^2 + \pi_0^2 + 2\pi_-\pi_+)(e^{-i\theta}\pi_+ + e^{i\theta}\pi_-)$$

$$\mathcal{L}_4 = \frac{\lambda}{4}(\sigma^2 + \pi_0^2 + 2\pi_-\pi_+)^2$$

The review

We identified the dynamical masses as

$$m_f^2 = g^2 v^2 \quad ; \quad m_\sigma^2 = \lambda(3v^2 + \Delta^2) - a^2 \quad ; \quad m_{\pi_0}^2 = \lambda(v^2 + \Delta^2) - a^2$$

$$m_{ch}^2 = \lambda(v^2 + 2\Delta^2) - a^2$$

and we discussed that the inverse propagator is given by

$$\begin{pmatrix} k^2 - m_\sigma^2 & -\sqrt{2}\lambda v \Delta e^{-i\theta} & -\sqrt{2}\lambda v \Delta e^{i\theta} & 0 \\ -\sqrt{2}\lambda v e^{i\theta} & k^2 - m_{ch}^2 + \mu_I^2 + i\mu_I k_0 & -\lambda \Delta^2 e^{2i\theta} & 0 \\ -\sqrt{2}\lambda v e^{-i\theta} & -\lambda \Delta^2 e^{-2i\theta} & k^2 - m_{ch}^2 + \mu_I^2 - i\mu_I k_0 & 0 \\ 0 & 0 & 0 & k^2 - m_{\pi_0}^2 \end{pmatrix}$$

The review

We took an aproximation to have as effective propagators

$$D_{\pi_0}^{-1}(k) = k^2 - m_{\pi_0}^2$$

$$D_{\sigma}^{-1}(k) = k^2 - m_{\sigma}^2$$

And the charged pions propagators are given by the inner matrix determinant as

$$D_{\pi_\pm}^{-1}(k) = k0^2 - E_{\pi_\pm}^2$$

with

$$E_\pm(\vec{k}) = \sqrt{\vec{k}^2 + m_{ch}^2 + \mu_I \mp \sqrt{4\mu_I^2(\vec{k}^2 + m_{ch}^2) + \lambda^2\Delta^4}}$$

And we performed the ARF to extract the divergences from the charged pions

I. The Scoccola's solution

The Scoccola's solution

In the last talk, Norberto convinced me I should use the full determinant of the inverse propagator

$$\begin{vmatrix} k^2 - m_\sigma^2 & -\sqrt{2}\lambda v\Delta e^{-i\theta} & -\sqrt{2}\lambda v\Delta e^{i\theta} \\ -\sqrt{2}\lambda v e^{i\theta} & k^2 - m_{ch}^2 + \mu_I^2 + i\mu_I k_0 & -\lambda\Delta^2 e^{2i\theta} \\ -\sqrt{2}\lambda v e^{-i\theta} & -\lambda\Delta^2 e^{-2i\theta} & k^2 - m_{ch}^2 + \mu_I^2 - i\mu_I k_0 \end{vmatrix}$$

The determinant can be carried to the form

$$(k_0^2 - E_1^2)(k_0^2 - E_2^2)(k_0^2 - E_3^2) = 0$$

where every E_i are the dispertion relations of the three envolved particles, the sigma and the charged pions.

The Scoccola's solution

The dispertion relations has a very similar estructure between them, for example, the first one is

$$
\begin{aligned}
= \Big\{\Big\{x \to{}& \frac{1}{3}\left(3k^2+3m^2+2v^2\lambda+2\Delta^2\lambda+2\mu I^2\right) - \\
&\left(2^{1/3}\left(-4v^4\lambda^2-8v^2\Delta^2\lambda^2-4\Delta^4\lambda^2-12k^2\mu I^2-12m^2\mu I^2+4v^2\lambda\mu I^2-14\Delta^2\lambda\mu I^2-\mu I^4\right)\right) \Big/ \\
&\Big(3\Big(16v^6\lambda^3+48v^4\Delta^2\lambda^3+48v^2\Delta^4\lambda^3+16\Delta^6\lambda^3-144k^2v^2\lambda\mu I^2-144m^2v^2\lambda\mu I^2+72k^2\Delta^2\lambda\mu I^2+72m^2\Delta^2\lambda\mu I^2- \\
&\quad 24v^4\lambda^2\mu I^2+60v^2\Delta^2\lambda^2\mu I^2+84\Delta^4\lambda^2\mu I^2+72k^2\mu I^4+72m^2\mu I^4+12v^2\lambda\mu I^4+66\Delta^2\lambda\mu I^4- \\
&\quad 2\mu I^6+\sqrt{}\Big(4\left(-4v^4\lambda^2-8v^2\Delta^2\lambda^2-4\Delta^4\lambda^2-12k^2\mu I^2-12m^2\mu I^2+4v^2\lambda\mu I^2-14\Delta^2\lambda\mu I^2-\mu I^4\right)^3+ \\
&\qquad \big(16v^6\lambda^3+48v^4\Delta^2\lambda^3+48v^2\Delta^4\lambda^3+16\Delta^6\lambda^3-144k^2v^2\lambda\mu I^2-144m^2v^2\lambda\mu I^2+72k^2\Delta^2\lambda\mu I^2+72m^2\Delta^2\lambda\mu I^2- \\
&\qquad\quad 24v^4\lambda^2\mu I^2+60v^2\Delta^2\lambda^2\mu I^2+84\Delta^4\lambda^2\mu I^2+72k^2\mu I^4+72m^2\mu I^4+12v^2\lambda\mu I^4+66\Delta^2\lambda\mu I^4-2\mu I^6\big)^2\Big)\Big)^{1/3}\Big) + \\
&\frac{1}{3\times 2^{1/3}}\Big(16v^6\lambda^3+48v^4\Delta^2\lambda^3+48v^2\Delta^4\lambda^3+16\Delta^6\lambda^3-144k^2v^2\lambda\mu I^2-144m^2v^2\lambda\mu I^2+72k^2\Delta^2\lambda\mu I^2+ \\
&\quad 72m^2\Delta^2\lambda\mu I^2-24v^4\lambda^2\mu I^2+60v^2\Delta^2\lambda^2\mu I^2+84\Delta^4\lambda^2\mu I^2+72k^2\mu I^4+72m^2\mu I^4+12v^2\lambda\mu I^4+66\Delta^2\lambda\mu I^4- \\
&\quad 2\mu I^6+\sqrt{}\Big(4\left(-4v^4\lambda^2-8v^2\Delta^2\lambda^2-4\Delta^4\lambda^2-12k^2\mu I^2-12m^2\mu I^2+4v^2\lambda\mu I^2-14\Delta^2\lambda\mu I^2-\mu I^4\right)^3+ \\
&\qquad \big(16v^6\lambda^3+48v^4\Delta^2\lambda^3+48v^2\Delta^4\lambda^3+16\Delta^6\lambda^3-144k^2v^2\lambda\mu I^2-144m^2v^2\lambda\mu I^2+72k^2\Delta^2\lambda\mu I^2+72m^2\Delta^2\lambda\mu I^2- \\
&\qquad\quad 24v^4\lambda^2\mu I^2+60v^2\Delta^2\lambda^2\mu I^2+84\Delta^4\lambda^2\mu I^2+72k^2\mu I^4+72m^2\mu I^4+12v^2\lambda\mu I^4+66\Delta^2\lambda\mu I^4-2\mu I^6\big)^2\Big)\Big)^{1/3}\Big\},
\end{aligned}
$$

The Scoccola's solution

All of them has the same structure

$$E_1^2 = C^2 + \frac{B}{\sqrt[3]{A + \sqrt{A^2 - B^3}}} + \sqrt[3]{A + \sqrt{A^2 - B^3}}$$

In order to identify, we can to take $k = 0$, which gives us up its correspondent mass, this is identically as when it is as simple as $k^2 + m^2$. Doing this, we can identify that the Energy E_1^2 is the sigma propagator and the other two corresponds to the charged pions

$$E_1^2(k = 0, \Delta = 0, \mu_I = 0) = 3\lambda v^2 - a^2$$
$$E_2^2(k = 0, \Delta = 0, \mu_I = 0) = \lambda v^2 - a^2$$
$$E_2^3(k = 0, \Delta = 0, \mu_I = 0) = \lambda v^2 - a^2$$

This is just a verification, since it is not relevant, because the One Loop Correction is given by

II. The One Loop Corrections

One Loop Corrections

For neutral pion

$$V^1_{\pi^0} = \frac{1}{2}\int \frac{d^3k}{(2\pi)^3}\left\{k^2 + m^2_{\pi_0}\right\}^{1/2},$$

and for the others

$$V^1_{comb} = \frac{1}{2}\int \frac{d^3k}{(2\pi)^3}\left\{E_1 + E_2 + E_3\right\},$$

In fact, we need the whole sum of Eigenvalues since each one cannot be idependently asymptotically renormalized. Using ARF we have

$$E_1 + E_2 + E_3 \approx \sqrt{k^2 + m^2_\sigma} + 2\sqrt{k^2 + m^2_{ch}} - \frac{\lambda^2\left(\Delta^4 + 4v^2\Delta^2\right)}{4\left(k^2 + m^2_{ch}\right)^{3/2}}$$

One Loop Corrections

Applying the dimensional regularization formulas, we have that the divergences of the One Loop correction are given by

$$
\begin{aligned}
V^{(1)}_{b,vac} = &-\frac{\left(\Delta^2\lambda^2\left(\Delta^2+4v^2\right)\right)\left(\log\left(\frac{\Lambda^2}{\lambda(2\Delta^2+v^2)-a^2}\right)+\frac{1}{\epsilon}\right)}{32\pi^2}\\
&-\frac{\left(\lambda\left(\Delta^2+v^2\right)-a^2\right)^2\left(\log\left(\frac{\Lambda^2}{\lambda(\Delta^2+v^2)-a^2}\right)+\frac{1}{\epsilon}+\frac{3}{2}\right)}{64\pi^2}\\
&-\frac{\left(\lambda\left(\Delta^2+3v^2\right)-a^2\right)^2\left(\log\left(\frac{\Lambda^2}{\lambda(\Delta^2+3v^2)-a^2}\right)+\frac{1}{\epsilon}+\frac{3}{2}\right)}{64\pi^2}\\
&-\frac{\left(\lambda\left(2\Delta^2+v^2\right)-a^2\right)^2\left(\log\left(\frac{\Lambda^2}{\lambda(2\Delta^2+v^2)-a^2}\right)+\frac{1}{\epsilon}+\frac{3}{2}\right)}{32\pi^2}
\end{aligned}
$$

The method to compute the fermionic correction is completly analogous, and it gives us

$$V_{f,vac}^{(1)} = \frac{N_c g^4 \left(\Delta^2 + v^2\right)^2}{8\pi^2} \left(\log\left(\frac{\Lambda^2}{g^2\left(\Delta^2 + v^2\right)}\right) + \frac{1}{\epsilon} + \frac{3}{2}\right) - \frac{N_c \Delta^2 g^2 \mu_I^2}{2\pi^2} \left(\log\left(\frac{\Lambda^2}{g^2\left(\Delta^2 + v^2\right)}\right) + \frac{1}{\epsilon}\right)$$

III. The Ambiguity

The Ambiguity

Notice that if we sum terms which goes as $1/\epsilon$, we have

$$-\frac{a^4}{16\pi^2}+\frac{3a^2\Delta^2\lambda}{16\pi^2}+\frac{3a^2\lambda v^2}{16\pi^2}-\frac{3\Delta^4\lambda^2}{16\pi^2}+\frac{\Delta^4 g^4 \mathrm{Nc}}{8\pi^2}+\frac{g^4\mathrm{Nc}v^4}{8\pi^2}$$
$$+\frac{\Delta^2 g^4\mathrm{Nc}v^2}{4\pi^2}-\frac{\Delta^2 g^2\mu \mathrm{I}^2\mathrm{Nc}}{2\pi^2}-\frac{3\lambda^2 v^4}{16\pi^2}-\frac{3\Delta^2\lambda^2 v^2}{8\pi^2}$$

The first one is just a spurious divergence, but we have six condensate-medium dependent, which must be renormalized. To do it we can propouse the most general form of Counter-Terms as

$$\delta_{2,0}\frac{a^2}{2}v^2+\delta_{0,2}\frac{a^2}{2}\Delta^2$$
$$-\delta_{4,0}\frac{\lambda}{4}v^4-\delta_{2,2}\frac{\lambda}{4}v^2\Delta^2-\delta_{0,4}\frac{\lambda}{4}\Delta^4$$
$$+\delta_\mu\frac{\mu_I^2\Delta^2}{2}$$

The Ambiguity

In principle, we could apply the *n*-th derivative of the potential to fit the Cts, nevertheless, what set should we use?

$$\frac{\partial^{n+m} V^{(1)}}{\partial v^n \partial^m}$$

In particular it is confusing to set $v^2\Delta^2$, we should use $n = 1, m = 2$ or $n = 2, m = 2$? they are unequivalent!!
This problem led us to analyze the system from the point of view of the Ward Identities.

IV. The Renormalization

The Renormalization

If we consider, we started from the unrenormalized Lagrangian, with divergent fields, masses and coupling constants, then, we can define the renormalization parameters

$$\pi_0^0 = Z_M^{1/2} \pi_0$$
$$\pi_\pm^0 = Z_{ch}^{1/2} \pi_\pm$$
$$\sigma^0 = Z_M^{1/2} \sigma$$

Where is natural to introduce a new parameter Z_{ch} since, the charged pions has an independent behaviour than the neutral ones.

The Renormalization

This led us to define the renormalization parameters of the coupling constants as

$$\lambda^0 Z_M^2 = \lambda Z_{4,0}$$
$$\lambda^0 Z_M Z_{ch} = \lambda Z_{2,2}$$
$$\lambda^0 Z_{ch}^2 = \lambda Z_{0,4}$$
$$(a^2)^0 Z_M = a^2 Z_{2,0}$$
$$(a^2)^0 Z_{ch} = a^2 Z_{0,2}$$
$$Z_{ch} = Z_\mu$$

then, the most general renormalization is only allowed if the renormalization parameter for charged and neutral mesons are different.

The Renormalization

Nevertheless, if the Counter terms are condesates-medium independent, then, for $\Delta = 0$ they must be unchaged, however, in this case one more time the mass is degenerated, then $Z_M = Z_{ch}$ then we cannot have independent v, Δ renormalization parametres and the most general allowed CTs structure is

$$\delta_2 \frac{a^2}{2}(v^2 + \Delta^2)$$
$$-\delta_4 \frac{\lambda}{4}(v^2 + \Delta^2)^2$$
$$+\delta_\mu \frac{\mu_I^2 \Delta^2}{2}$$

This has restricted the number of CTs from six to three, but we can still apply the 'no conservation' of the Axial current

V. The Ward Identity

The Ward Identity

To be capable to find analytical a Ward Identity we need to go back to $\Delta = 0, \mu_I = 0$, in this regime, all the propagators are diagonal, and trivial

$$D_\sigma^{-1} = k^2 + m_\sigma^2$$
$$D_\pi^{-1} = k^2 + m_\pi^2$$

And the neutral particles related Ward Identity (the derivation is not complicated) is

$$-2\lambda v^2 = D_\sigma^{-1} - D_\pi^{-1}$$

Notice that this is already satisfied

The Ward Identity

Usign now the Renormalizating parameter, the Ward identity is

$$-2\lambda v^2 Z_M^{-2} = D_\sigma^{-1} Z_M^{-1} - D_\pi^{-1} Z_M^{-1}$$
$$-2\lambda v^2 Z_4^{-1} = D_\sigma^{-1} Z_2^{-1} - D_\pi^{-1} Z_2^{-1}$$
$$\rightarrow Z_4 = Z_2$$

This means the CTs structure is at the end of the day

$$\delta\frac{a^2}{2}(v^2+\Delta^2) - \delta\frac{\lambda}{4}(v^2+\Delta^2)^2$$
$$+\delta_\mu\frac{\mu_I^2\Delta^2}{2}$$

Then it can be computed using only

$$\frac{\partial V}{\partial v} = 0 \quad ; \quad \frac{\partial^2 V}{\partial \Delta^2} = 0$$

VI. The Coupling Constant Relation

The Coupling Constant Relation

Let's go back to the sum terms which goes as $1/\epsilon$

$$-\frac{a^4}{16\pi^2} + \frac{a^2}{2}\left(\Delta^2 + v^2\right)\frac{3\lambda}{8\pi^2} - \frac{\lambda}{4}\left(\Delta^2 + v^2\right)^2 \frac{\left(3\lambda - 2g^4 N_c/\lambda\right)}{4\pi^2} - \frac{\Delta^2 g^2 \mu \mathrm{l}^2 \mathrm{Nc}}{2\pi^2}$$

notice the second and third terms must be canceled by only one CT

$$\delta\frac{a^2}{2}(v^2 + \Delta^2) - \delta\frac{\lambda}{4}(v^2 + \Delta^2)^2$$

by one side we have the term

$$\delta = \frac{3\lambda}{8\pi^2\epsilon}$$

but, by the other side

$$\delta = \frac{\left(3\lambda - 2g^4 N_c/\lambda\right)}{4\pi^2\epsilon}$$

this means that the coupling constant cannot be arbitrary, the need to satisfy

The Coupling Constant Relation

$$\frac{3\lambda}{8\pi^2\epsilon} = \frac{\left(3\lambda - 2g^4 N_c/\lambda\right)}{4\pi^2\epsilon}$$
$$3\lambda^2 = 6\lambda^2 - 4g^4 N_c$$
$$4g^4 N_c = 3\lambda^2$$

if we take $N_c = 3$

$$4g^4 = \lambda^2$$
$$2g^2 = \lambda$$

e.i, $\lambda = 20$, then $g = 3{,}33$

The Coupling Constant Relation

moreover, using the definitions of the masses at vaccum, we can derive that

$$\lambda = \frac{m_\sigma^2 - m_{\pi_0}}{2f_\pi^2}$$
$$g = \frac{m_f}{f_\pi}$$

then, substituing

$$m_\sigma^2 = 4m_f^2 + m_{\pi_0}^2$$

This two equations restricts strongly the space of parameter and let us to identify, the full One Loop Correction is proportional to λ

VII. The matter part

The matter part

The numerical finite part of the One Loop Contribution given by

$$V^1_{b,mat} = \frac{1}{2}\int \frac{d^3k}{(2\pi)^3}\Bigg\{E_1 + E_2 + E_3 \\ -\sqrt{k^2+m_\sigma^2} - 2\sqrt{k^2+m_{ch}^2} + \frac{\lambda^2\left(\Delta^4 + 4v^2\Delta^2\right)}{4\left(k^2+m_{ch}^2\right)^{3/2}}\Bigg\}$$

(disclaimer anouncement)
If well, the integration is numerically calculated, to find the solutions of the system is mandatory to compute multiples derivatives over that integral what is computational heavy, I didn't have time to optimize it and could compute it. I hope along the week could I share you the plots with matter.

VIII. The Termo

The Termo

The system is going to be compared by four termodynamical variables

$$\textit{Pressure} \quad P(v, \Delta, \mu_I) = -V(v, \Delta, \mu_I) + V(f_\pi, 0, m_{\pi_0})$$

$$\textit{Isospin density} \quad n_I = \left.\frac{\partial P}{\partial \mu_I}\right|_T$$

$$\textit{Energy density} \quad \varepsilon = -P + Ts + \mu_I n_I$$

$$\textit{squared speed of sound} \quad c_s^2 = \left.\frac{\partial P}{\partial \varepsilon}\right|_{T=0}$$

nevertheless notice that the derivatives must be implemented as finite diferences since we have no an analitycal structure for the condensates medium dependent

IX. The Condensates

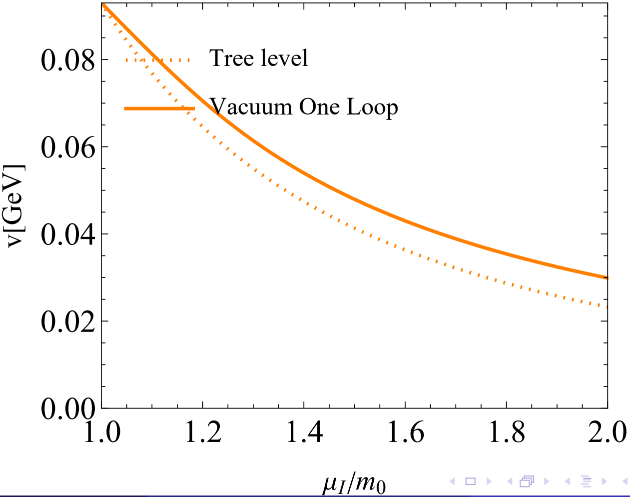
The Condensates

The main procedure to find the values of condensates is looking for every single μ_I point the values of the condensates which minimize the potential, in step by step scheme it is

1. $i = n$;
2. for $\mu_I = m + i\delta$, take $vs = v0[\mu_I]$ and $\Delta s = \Delta 0[\mu_I]$;
3. are they the roots?
4. if true, then save them and go to 6
5. else, vary them and repite from 3.
6. n++;
7. repeat

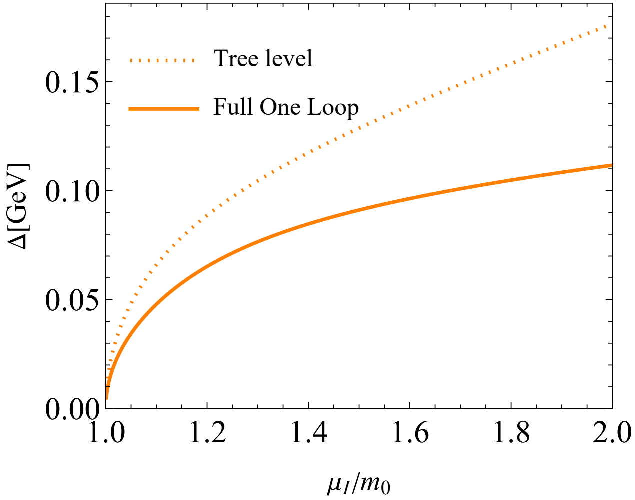
The Condensates

Doing the previus algorith we can find the following solutions for the condesates for $f_\pi = 0{,}093\,GeV$, $m_\pi = 0{,}140\,GeV$, $m_f = 0{,}105\,GeV$, $m_\sigma = 0{,}252389\,GeV$



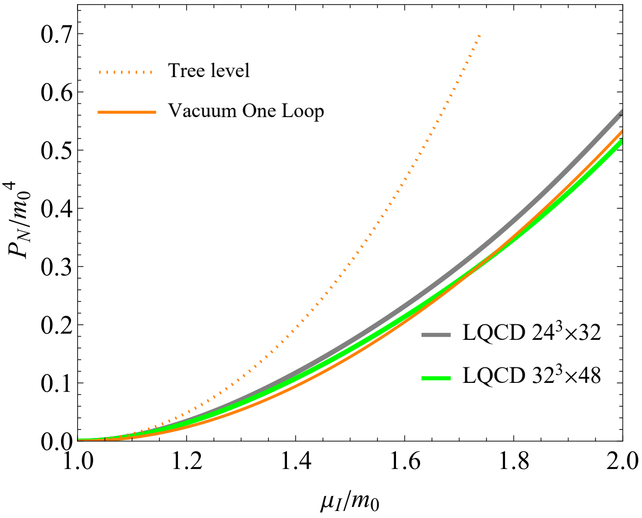
The Condensates

and for Δ

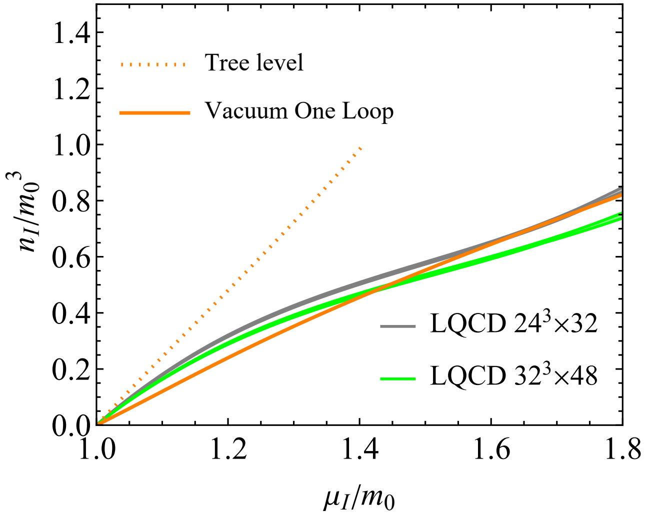


X. The Comparison

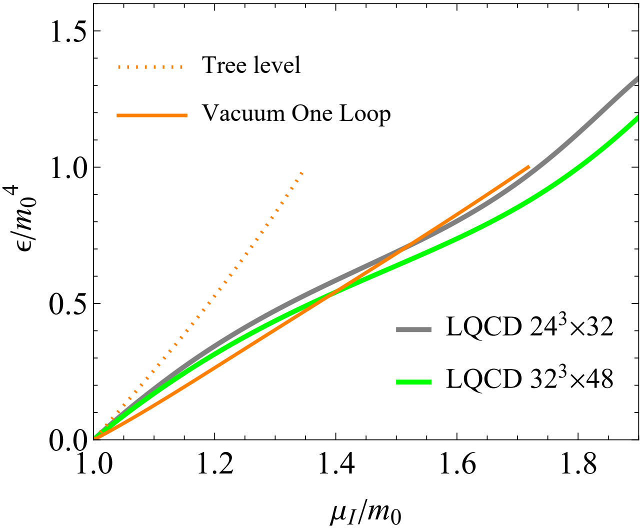
The Comparison, Pressure



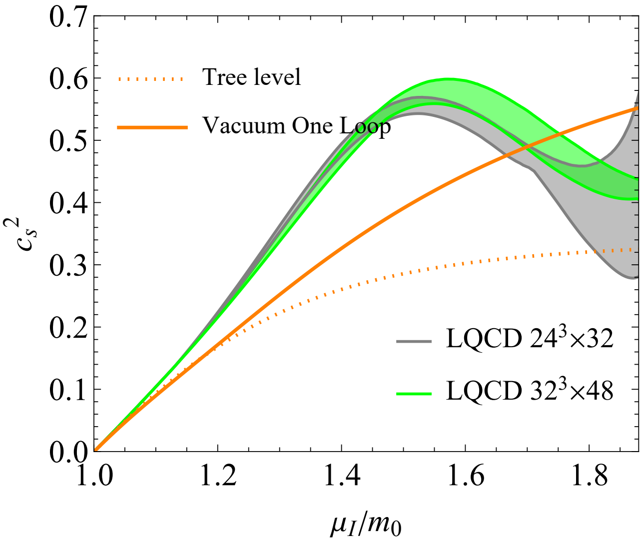
The Comparison, Isospin Density



The Comparison, Energy Density



The Comparison, Speed of Sound



Summary

Summary

- We identify an ambiguity in in the election of Counter-Terms which does not allow us to have a unique renormalization.
- We use the the Ward-Takahashi identities and the fields renormalization to reduce the free parameters from six to two.
- We found exists a non trivial relation between the coupling constants and masses
- We found a very well agreement on three of four T.V.
- We can expect the inclusion of the matter part, we can release the value of the fermion mass and modify the curvature to describe the speed of sound since the vacuum bosonic part is μ_I independent, therefore is not still captured the charged pion condensation.

Thanks!!