Little group contraction and The massless limit in QFT.

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Introduction

In the last decade we have had an explosion of progress about understanding scattering amplitudes in gauge theories and gravity [1]. The key question here is: What defines a particle? The answer is given by the concept of **Little group** which is a subgroup of Lorentz group.

Little Group tell us that the massless states of particles are uniquely defined by its *helicity* and described by $U(1) \times T(2) = E(2)$, the Euclidean group. The case of massive states are characterized by its *mass and spin* and described by $SO(3)$ group. It is possible to take the limit $m \to 0$ and go from SO(3) to E(2), such is the little group contraction (LGC) presented by Inonu and Wigner [3], and later Kim [4]. How to go from the massive little group to massless one is actually needed in order to identify the massless limit in QFT.

On the other hand, on-shell methods like the spinor helicity formalism, are useful for computing amplitudes of massless particles in four dimensional theories at tree level order. In the spinor helicity formalism, momentum and polarization four vectors are reparameterized as the product of two spinors of two components (Weyl spinors) through Pauli matrices. The extension of this formalism describe particles of any mass and spin known as Spin-Spinors formalism; making it possible, in principle, to apply these methods to any fundamental theory of particles [5].

1 The Group Contraction

It is possible construct new Lie groups from old ones by a process named **group contraction**. This process consist in a reparameterization of the Lie group´s parameter space such that the properties of the original Lie group have well-defined limits in the contracted Lie group [2]. Here, we review a simple class called Inonu-Wigner contractions.

1.0.1 Inonu-Wigner contractions

The best way to understand the Inonu-Wigner contractions or the contraction of $O(3)$ to $E(2)$, is to consider a sphere with large radius and a small area around the north pole. The generators of $O(3)$ in the matrix form are:

$$L_1 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -i \\ 0 & i & 0 \end{pmatrix}, L_2 = \begin{pmatrix} 0 & 0 & i \\ 0 & 0 & 0 \\ -i & 0 & 0 \end{pmatrix}, L_3 = \begin{pmatrix} 0 & -i & 0 \\ i & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}. \tag{1}$$

Here L_3 generates rotations around the north pole. These generators satisfy the commutation relations:

$$[L_i, L_j] = i\epsilon_{ijk}L_k. \tag{2}$$

The Euclidean group is generated by L_3, P_1, and P_2, where

$$P_1 = \begin{pmatrix} 0 & 0 & i \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad P_2 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & i \\ 0 & 0 & 0 \end{pmatrix}, \tag{3}$$

and they satisfy the commutation relations

$$[P_1, P_2] = 0, \quad [L_3, P_1] = iP_2, \quad [L_3, P_2] = -iP_1. \tag{4}$$

We can then write

$$\begin{pmatrix} x \\ y \\ 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \frac{1}{R} \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix}. \tag{5}$$

The column vectors on the left- and right-hand sides are, respectively, the coordinate vectors on which the $E(2)$ and $O(3)$ transformations are applicable, here the 3×3 matrix is A. In the limit of large R, we have:

$$L_3 = AL_3A^{-1}, \tag{6}$$

$$P_1 = \left(\frac{1}{R}\right) AL_2A^{-1}, \tag{7}$$

$$P_2 = -\left(\frac{1}{R}\right) AL_1A^{-1}. \tag{8}$$

Further, in terms of P_1, P_2 and L_3, the commutations relations for $O(3)$ given in (2) become

$$[L_3, P_1] = iP_2, [L_3, P_2] = -iP_1, [P_1, P_2] = i\left(\frac{1}{R}\right)^2 L_3. \tag{9}$$

In the large-R limit, the commutator $[P_1, P_2]$ vanishes, so the above set of commutators becomes the Lie algebra for $E(2)$ (4).

1.1 Little group contraction

Consider first $p^\mu = (m, 0, 0, 0)$ a massive particle at rest . In the case the LG is the rotation group $O(3)$, with generators J_i. Then we perform a boost along z axis $B(\eta)$, such that

$$p^\mu \longrightarrow p^\mu = (E, 0, 0, E). \tag{10}$$

In this case J_3 remmains unchanged but J_1 and J_2 set boosted with

$$J_i \longrightarrow J_i' = B(\eta)J_iB(-\eta), \tag{11}$$

expanding $B(\eta)$ we find

$$J_1' = \cosh\eta J_1 + \sinh\eta K_2, \quad J_2' = \cosh\eta J_2 - \sinh\eta K_1. \tag{12}$$

These operators satisfy the same algebra

$$[J_i', J_j'] = i\epsilon_{ijk}J_k'. \tag{13}$$

We define

$$N_1 = -\frac{J_2}{\cosh\eta}, \quad N_2 = \frac{J_1}{\cosh\eta}. \tag{14}$$

So, the massless limit is obtained when the boost parameter $\eta \longrightarrow \infty$, then in this limit one gets

$$N_1 = K_1 - J_2, \quad N_2 = K_2 + J_1, \tag{15}$$

and one can see that these generators satisfy the algebra:

$$[J_3, N_1] = iN_2, \quad [J_3, N_2] = -iN_1, \quad [N_1, N_2] = 0. \tag{16}$$

Thus, in the massless limit this algebra is the same as the $E(2)$ Euclidian group in $2 - dimensions$

2 On-shell methods

2.1 Spinor helicity formalism

2.1.1 Massless case

In the massless case, **helicity spinors** are defined as real or complex doublets transforming in the $(\frac{1}{2}, 0)$ or $(0, \frac{1}{2})$ representations of the Lorentz group. To represent momenta as bispinors, we use sigma Pauli matrices:

$$p^{\dot a a} = (\bar\sigma^\mu)^{\dot a a} p_\mu = \begin{pmatrix} p^0 - p^3 & -p^1 + ip^2 \\ -p^1 - ip^2 & p^0 + p^3 \end{pmatrix} = \lambda_a\tilde\lambda_{\dot a} \equiv |p\rangle[p| \tag{17}$$

Dirac spinors can be either left or right handed, in the Weyl basis we can write them as:

$$v_+(p) = \begin{pmatrix} |p]_a \\ 0 \end{pmatrix} \equiv |p], \quad v_-(p) = \begin{pmatrix} 0 \\ |p\rangle^{\dot a} \end{pmatrix} \equiv |p\rangle,$$

$$\bar u_-(p) = (0, \langle p|_{\dot a}) \equiv \langle p| \quad , \bar u_+(p) = ([p|^a, 0) \equiv [p|. \tag{18}$$

According to (17), in spherical coordinates we have

$$\lambda_a = \sqrt{2E}\begin{pmatrix} -e^{i\phi}\sin\frac{\theta}{2} \\ \cos\frac{\theta}{2} \end{pmatrix}, \quad \tilde\lambda^{\dot a} = \sqrt{2E}\begin{pmatrix} \cos\frac{\theta}{2} \\ \sin\frac{\theta}{2}e^{i\phi} \end{pmatrix}. \tag{19}$$

2.2 Spin-Spinors formalism

2.2.1 Massive case

On the other hand, in the massive case we adopt the convention of ref. [5], a massive momentum $p^2 = m^2$ can be decomposed in terms of chiral and antichiral spinors similarly to eq.(17), but two pairs of massles spinors are required,

$$\mathbf{p}_{a\dot a} = \epsilon_{IJ}\lambda_a^I\tilde\lambda_{I\dot a} = \lambda_a^I\tilde\lambda_{I\dot a} = |p^I\rangle[p_I|, \tag{20}$$

$$\bar{\mathbf{p}}^{\dot a a} = \epsilon^{IJ}\tilde\lambda_I^{\dot a}\lambda_J^a = -\tilde\lambda^{I\dot a}\lambda_I^a = -|p^I]\langle p_I|, \tag{21}$$

here $I = 1, 2$, and boldface is used to denote massive momenta and their corresponding spinors. It is possible rewrite massive spinors parametrization as

$$\lambda_a^I = \sqrt{E+p}\begin{pmatrix} -e^{i\phi}\sin\frac{\theta}{2} \\ \cos\frac{\theta}{2} \end{pmatrix}_a \otimes \begin{pmatrix} 0 \\ 1 \end{pmatrix}^I + \frac{m}{\sqrt{E+p}}\begin{pmatrix} \cos\frac{\theta}{2} \\ e^{i\phi}\sin\frac{\theta}{2} \end{pmatrix}_a \otimes \begin{pmatrix} 1 \\ 0 \end{pmatrix}^I, \tag{22}$$

$$\tilde\lambda^{\dot a I} = \sqrt{E+p}\begin{pmatrix} \cos\frac{\theta}{2} \\ e^{i\phi}\sin\frac{\theta}{2} \end{pmatrix}^{\dot a} \otimes \begin{pmatrix} 1 \\ 0 \end{pmatrix}^I + \frac{m}{\sqrt{E+p}}\begin{pmatrix} -e^{i\phi}\sin\frac{\theta}{2} \\ \cos\frac{\theta}{2} \end{pmatrix}^{\dot a} \otimes \begin{pmatrix} 0 \\ 1 \end{pmatrix}^I. \tag{23}$$

Taking the limit $m \to 0$, we have

$$\lambda_a^I \longrightarrow \lambda_a\zeta_-^I, \quad \tilde\lambda^{\dot a I} \longrightarrow \tilde\lambda^{\dot a}\zeta_+^I, \tag{24}$$

with $\zeta_-^I \equiv (0, 1)$ and $\zeta_+^I \equiv (1, 0)$. Then it reproduces above massless spinors (19).

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3 Expectations

In QFT we need wave functions for external particles that participate in a process, for instance, $Z^0 \longrightarrow f\bar f$. We want to unify the math concept of group contraction and the modern on-shell methods, then study scattering amplitudes at tree level that involve some spin cases as $1/2, 1, 3/2$ and 2.

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