# Second Order Fermions

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# Outline

# Space-time symmetries

The Poincaré Group is the group of transformations $x^\mu \to x'^\mu$ leaving invariant the interval $(ds)^2 = \eta_{\mu\nu} dx^\mu dx^\nu$, and can be split in two distinct pieces:

- Spacetime Translations $x'^\mu = x^\mu + a^\mu$, with constant parameters $a^\mu$.
- The group of transformations $x'^\mu = L^\mu{}_\nu x^\nu$ defined by the condition

$$L^\mu{}_\rho \eta_{\mu\nu} L^\nu{}_\sigma = \eta_{\rho\sigma}. \quad (1)$$

This group is known as $O(1,3)$ or Lorentz Group

# Lorentz Group

In turn, Lorentz Group $O(1,3)$ can be split into four disconnected pieces, namely:

- The continuous subgroup $SO(1,3)^+$ or Restricted Lorentz Group (RLG), with elements $\Lambda^\mu{}_\nu$. This subgroup contains proper orthochronous transformations (those continuously connected with the identity $\det\Lambda = 1$ and preserving the direction of time $\Lambda^0{}_0 > 0$).

- Improper orthochronous transformations, described by $[\mathcal{P}\Lambda]^{\mu}{}_{\nu}$, where the transformation

$$\mathcal{P} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} \quad (2)$$

can be identified as space inversion, or Parity, since $x'^{\mu} = \mathcal{P}^{\mu}{}_{\nu} x^{\nu}$ implies $t' = t$ and $\mathbf{x}' = -\mathbf{x}$. This sector is characterized by $\det \mathcal{P}\Lambda = -1$ and $[\mathcal{P}\Lambda]^{0}{}_{0} > 0$.

- Improper heterochronous transformations, described by $[\mathcal{T}\Lambda]^{\mu}{}_{\nu}$, with

$$\mathcal{T} = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \tag{3}$$

which implements Time Reversal, in the sense $x'^{\mu} = \mathcal{T}^{\mu}{}_{\nu} x^{\nu}$ implies $t' = -t$ and $\mathbf{x}' = \mathbf{x}$. In this case $\det \mathcal{T}\Lambda = -1$ and $[\mathcal{T}\Lambda]^{0}{}_{0} < 0$.

- Proper heterochronous transformations, described by $[\mathcal{P}\mathcal{T}\Lambda]^{\mu}{}_{\nu}$, with $\det \mathcal{P}\mathcal{T}\Lambda = 1$ and $[\mathcal{P}\mathcal{T}\Lambda]^{0}{}_{0} < 0$.

Thus the isometries (for vectors with fixed origin) in Minkowski space-time are described by the continuous RLG $SO(1,3)^+$ together with the discrete transformations $\mathcal{P}$ and $\mathcal{T}$ that belong to the quotient group

$$O(1,3)/SO(1,3)^+ \simeq Z_2 \otimes Z_2 = \{\mathbb{I}, \mathcal{P}, \mathcal{T}, \mathcal{PT}\}\,. \tag{4}$$

# Lorentz Algebra

The continuous $SO(1,3)^+$ group contains 6 parameters, describing

- 3 rotations, parameterized in cartesian coordinates by

$$\Lambda(\theta_1,0) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \cos\theta_1 & -\sin\theta_1 \\ 0 & 0 & \sin\theta_1 & \cos\theta_1 \end{pmatrix}, \tag{5}$$

$$\Lambda(\theta_2,0) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos\theta_2 & 0 & \sin\theta_2 \\ 0 & 0 & 1 & 0 \\ 0 & -\sin\theta_2 & 0 & \cos\theta_2 \end{pmatrix}, \tag{6}$$

$$\Lambda(\theta_3,0) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos\theta_3 & -\sin\theta_3 & 0 \\ 0 & \sin\theta_3 & \cos\theta_3 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \tag{7}$$

- 3 boosts, that can be parameterized in terms of 3 rapidities $\varphi_i$, defined by $v_i = \tanh \varphi_i$, as

$$\Lambda(0,\varphi_1) = \begin{pmatrix} \cosh\varphi_1 & \sinh\varphi_1 & 0 & 0 \\ \sinh\varphi_1 & \cosh\varphi_1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \tag{8}$$

$$\Lambda(0,\varphi_2) = \begin{pmatrix} \cosh\varphi_2 & 0 & \sinh\varphi_2 & 0 \\ 0 & 1 & 0 & 0 \\ \sinh\varphi_2 & 0 & \cosh\varphi_2 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \tag{9}$$

$$\Lambda(0,\varphi_3) = \begin{pmatrix} \cosh\varphi_3 & 0 & 0 & \sinh\varphi_3 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ \sinh\varphi_3 & 0 & 0 & \cosh\varphi_3 \end{pmatrix}. \tag{10}$$

Thus any element of $SO(1,3)^+$ in its defining representation can be written as

$$\Lambda(\boldsymbol{\theta}, \boldsymbol{\varphi}) = e^{-i(\boldsymbol{\theta}\cdot\mathbf{J}+\boldsymbol{\varphi}\cdot\mathbf{K})}, \tag{11}$$

where the generators are given by

$$J_i = i\,\frac{\partial\Lambda(\boldsymbol{\theta}, \boldsymbol{\varphi})}{\partial\theta_i}\bigg|_{\boldsymbol{\theta},\boldsymbol{\varphi}\to\mathbf{0}}, \qquad K_i = i\,\frac{\partial\Lambda(\boldsymbol{\theta}, \boldsymbol{\varphi})}{\partial\varphi_i}\bigg|_{\boldsymbol{\theta},\boldsymbol{\varphi}\to\mathbf{0}}. \tag{12}$$

Explicitly,

$$J_1 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -i \\ 0 & 0 & i & 0 \end{pmatrix}, \qquad K_1 = \begin{pmatrix} 0 & i & 0 & 0 \\ i & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix},$$
$$J_2 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & i \\ 0 & 0 & 0 & 0 \\ 0 & -i & 0 & 0 \end{pmatrix}, \qquad K_2 = \begin{pmatrix} 0 & 0 & i & 0 \\ 0 & 0 & 0 & 0 \\ i & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix},$$
$$J_3 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & -i & 0 \\ 0 & i & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \qquad K_3 = \begin{pmatrix} 0 & 0 & 0 & i \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ i & 0 & 0 & 0 \end{pmatrix}. \tag{13}$$

The algebra satisfied by these generators is

$$[J_i, J_j] = i\epsilon_{ijk}J_k\,, \quad [J_i, K_j] = i\epsilon_{ijk}K_k\,, \quad [K_i, K_j] = -i\epsilon_{ijk}J_k\,. \tag{14}$$

In terms of an antisymmetric tensor $\mathcal{J}^{\mu\nu} = -\mathcal{J}^{\nu\mu}$ defined as

$$J_i \equiv \tfrac{1}{2}\epsilon_{ijk}\mathcal{J}^{jk}, \qquad K_i \equiv \mathcal{J}^{0i}, \tag{15}$$

the Lorentz algebra takes the form

$$[\mathcal{J}^{\mu\nu}, \mathcal{J}^{\rho\sigma}] = i(\eta^{\mu\sigma}\mathcal{J}^{\nu\rho} + \eta^{\nu\rho}\mathcal{J}^{\mu\sigma} - \eta^{\mu\rho}\mathcal{J}^{\nu\sigma} - \eta^{\nu\sigma}\mathcal{J}^{\mu\rho}), \tag{16}$$

and the defining elements of the Lorentz group can be written as

$$\Lambda = e^{-\frac{i}{2}\Omega_{\mu\nu}\mathcal{J}^{\mu\nu}} = e^{-i(\boldsymbol{\theta}\cdot\mathbf{J}+\boldsymbol{\varphi}\cdot\mathbf{K})}, \tag{17}$$

with $\Omega_{\mu\nu} = -\Omega_{\nu\mu}$, and $\theta_i \equiv \frac{1}{2}\epsilon_{ijk}\Omega_{jk}$, $\varphi_i \equiv \Omega_{0i}$.

Another useful representation of the Lorentz generators can be found by defining the linear combinations

$$A_i = \frac{1}{2}(J_i + iK_i), \quad B_i = \frac{1}{2}(J_i - iK_i). \tag{18}$$

In terms of $\mathbf{A}$ and $\mathbf{B}$, the Lorentz algebra becomes

$$[A_i, A_j] = i\epsilon_{ijk}A_k, \quad [B_i, B_j] = i\epsilon_{ijk}B_k, \quad [A_i, B_j] = 0, \tag{19}$$

which is locally isomorphic to two copies of $\mathfrak{su}(2)$. We often quote that result as

$$SO(1,3)^+ \simeq SU(2)_A \otimes SU(2)_B. \tag{20}$$

We can further classify the irreps of the RLG in terms of $SU(2)_A$ and $SU(2)_B$, by noticing that

- Irreps of each $SU(2)_A \otimes SU(2)_B$ can described by the states $\{|a, m_a\rangle\, |b, m_b\rangle\}$, with

$$\mathbf{A}^2 |a, m_a\rangle = a(a+1)\, |a, m_a\rangle\,,\; A_3\, |a, m_a\rangle = m_a\, |a, m_a\rangle\,, \tag{21}$$
$$\mathbf{B}^2 |b, m_b\rangle = b(b+1)\, |b, m_b\rangle\,,\; B_3\, |b, m_b\rangle = m_b\, |b, m_b\rangle\,. \tag{22}$$

- RLG irreps have dimension $(2a+1)(2b+1)$ and can be labeled by the two half integers $(a, b)$.

RLG irreps form the following tower of states

$$
\begin{array}{c}
(0,0) \\
(\tfrac{1}{2},0) \quad (0,\tfrac{1}{2}) \\
(1,0) \quad (\tfrac{1}{2},\tfrac{1}{2}) \quad (0,1) \\
(\tfrac{3}{2},0) \quad (1,\tfrac{1}{2}) \quad (\tfrac{1}{2},1) \quad (0,\tfrac{3}{2}) \\
(2,0) \quad (\tfrac{3}{2},\tfrac{1}{2}) \quad (1,1) \quad (\tfrac{1}{2},\tfrac{3}{2}) \quad (0,2)
\end{array}
$$

Remarkably, in the Standard Model only four of them are included: Scalars, Spin 1/2 fermions and Gauge bosons. We can explicitly identify the form of the Lorentz group representation $S(\Lambda)$ and the Lorentz generators $\mathcal{J}^{\mu\nu}$ for the fields transforming under irreps of the RLG.

If we denote a general field transforming under some irreducible representation of the HGL as $\phi_a$, we can write a finite Lorentz transformation as

$$\phi'_a(x') = [S(\Lambda)]_a{}^b \phi_b(x) = [e^{-\frac{i}{2}\Omega_{\mu\nu}\mathcal{J}^{\mu\nu}}]_a{}^b \phi_b(x), \tag{23}$$

with $x'^\mu = \Lambda^\mu{}_\nu x^\nu$.

• Scalar Field $(0, 0)$

Defined by the transformation rule $\phi'(x') = \phi(x)$, we can easily conclude that $S_0(\Lambda) = 1$, and $\mathcal{J}_0^{\mu\nu} = 0$.

• Left-handed Weyl Spinor $(\frac{1}{2}, 0)$

This field transforms according to

$$\mathbf{B} = \frac{1}{2}(\mathbf{J} - i\mathbf{K}) = 0 \Rightarrow \mathbf{J} = i\mathbf{K} \qquad \Rightarrow \mathbf{A} = \frac{1}{2}(\mathbf{J} + i\mathbf{K}) = \mathbf{J} = \frac{\boldsymbol{\sigma}}{2}$$

$$S_L(\Lambda) = e^{-i\frac{\boldsymbol{\sigma}}{2}\cdot\boldsymbol{\theta} - \frac{\boldsymbol{\sigma}}{2}\cdot\boldsymbol{\varphi}},$$

We can identify its generators as $\mathcal{J}_L^{ij} = \epsilon_{ijk}\frac{\sigma^k}{2}, \quad \mathcal{J}_L^{0i} = -i\frac{\sigma^i}{2}$. Defining $\sigma^\mu = (1, \boldsymbol{\sigma})$ and $\overline{\sigma}^\mu = (1, -\boldsymbol{\sigma})$, we have

$$\mathcal{J}_L^{\mu\nu} = \frac{i}{4}(\sigma^\mu\overline{\sigma}^\nu - \sigma^\nu\overline{\sigma}^\mu), \tag{24}$$

and the transformation rule for the left-handed fermion becomes

$$\phi'_a(x') = [S_L(\Lambda)]_a{}^b\phi_b(x) = [e^{-\frac{i}{2}\Omega_{\mu\nu}\mathcal{J}_L^{\mu\nu}}]_a{}^b\phi_b(x). \tag{25}$$

- Right-handed Weyl Spinor $(0, \frac{1}{2})$

$$\mathbf{A} = \frac{1}{2}(\mathbf{J} + i\mathbf{K}) = 0 \Rightarrow \mathbf{J} = -i\mathbf{K} \qquad \Rightarrow \mathbf{B} = \frac{1}{2}(\mathbf{J} - i\mathbf{K}) = \mathbf{J} = \frac{\boldsymbol{\sigma}}{2}$$

$$S_R(\Lambda) = e^{-i\frac{\boldsymbol{\sigma}}{2}\cdot\boldsymbol{\theta} + \frac{\boldsymbol{\sigma}}{2}\cdot\boldsymbol{\varphi}},$$

and therefore $\mathcal{J}_R^{ij} = \epsilon_{ijk}\frac{\sigma^k}{2}$, $\mathcal{J}_R^{0i} = i\frac{\sigma^i}{2}$, or equivalently

$$\mathcal{J}_R^{\mu\nu} = \tfrac{i}{4}(\overline{\sigma}^\mu \sigma^\nu - \overline{\sigma}^\nu \sigma^\mu). \tag{26}$$

The Lorentz transformation of a right-handed spinor is written as

$$\psi'^{\dot{a}}(x') = [S_R(\Lambda)]^{\dot{a}}{}_{\dot{b}}\psi^{\dot{b}}(x) = [e^{-\frac{i}{2}\Omega_{\mu\nu}\mathcal{J}_R^{\mu\nu}}]^{\dot{a}}{}_{\dot{b}}\psi^{\dot{b}}(x). \tag{27}$$

Notice that

$$S_R^{-1}(\Lambda) = S_L^{\dagger}(\Lambda), \quad S_L^{-1}(\Lambda) = S_R^{\dagger}(\Lambda). \tag{28}$$

• Dirac spinor

The Dirac spinor field $\chi_a(x)$, $a = 1, \ldots, 4$ transforms in the $(\frac{1}{2}, 0) \oplus (0, \frac{1}{2})$ representation of the RLG

$$\chi_a(x) \to S(\Lambda)_{ab}\chi_b(\Lambda^{-1}x), \qquad S(\Lambda) = e^{-\frac{i}{2}\Omega_{\rho\sigma}\mathcal{J}^{\rho\sigma}_{1/2}}. \tag{29}$$

In the chiral basis, its RLG rep is

$$S(\Lambda) = \begin{pmatrix} S_L(\Lambda) & 0 \\ 0 & S_R(\Lambda) \end{pmatrix} = \begin{pmatrix} e^{-i\frac{\boldsymbol{\sigma}}{2}\cdot\boldsymbol{\theta} - \frac{\boldsymbol{\sigma}}{2}\cdot\boldsymbol{\varphi}} & 0 \\ 0 & e^{-i\frac{\boldsymbol{\sigma}}{2}\cdot\boldsymbol{\theta} + \frac{\boldsymbol{\sigma}}{2}\cdot\boldsymbol{\varphi}} \end{pmatrix}, \tag{30}$$

with $\mathcal{J}^{\mu\nu}_{1/2} = i[\gamma^\mu, \gamma^\nu]/4$. In this basis, the Dirac matrices are given by

$$\gamma^\mu = \begin{pmatrix} 0 & \sigma^\mu \\ \overline{\sigma}^\mu & 0 \end{pmatrix}. \tag{31}$$

The Gamma matrices $\gamma^\mu$, satisfy the *Clifford algebra*

$$\{\gamma^\mu, \gamma^\nu\} \equiv \gamma^\mu\gamma^\nu + \gamma^\nu\gamma^\mu = 2\eta^{\mu\nu}. \tag{32}$$

The relation between Clifford algebra and the Lorentz group emerges from the observation that the Dirac Lorentz generator can be written as

$$\left(1 + \tfrac{i}{2}\omega_{\rho\sigma}\mathcal{J}^{\rho\sigma}_{1/2}\right)\gamma^\mu\left(1 - \tfrac{i}{2}\omega_{\lambda\tau}\mathcal{J}^{\lambda\tau}_{1/2}\right) = \left(1 - \tfrac{i}{2}\omega_{\rho\sigma}\mathcal{J}^{\rho\sigma}_{1}\right)^\mu{}_\nu\gamma^\nu, \tag{33}$$

with the generators of the vector representation

$$[\mathcal{J}^{\rho\sigma}_1]^\mu{}_\nu = i[\eta^{\rho\mu}\delta^\sigma_\nu - \eta^{\sigma\mu}\delta^\rho_\nu]. \tag{34}$$

This equation is just the infinitesimal form of

$$S(\Lambda)^{-1}\gamma^\mu S(\Lambda) = \Lambda^\mu{}_\nu\gamma^\nu. \tag{35}$$

# Dirac Equation

A Dirac spinor field satisfies a Lorentz covariant first order equation, the Dirac equation

$$(i\gamma^\mu\partial_\mu - m)\chi = 0. \quad (36)$$

If $\chi$ satisfies the Dirac equation, then it automatically satisfies the Klein-Gordon equation. We can show this by acting with $(-i\gamma^\mu\partial_\mu - m)$ the above equation

$$\begin{aligned}
&(-i\gamma^\mu\partial_\mu - m)(i\gamma^\nu\partial_\nu - m)\chi = (\gamma^\mu\gamma^\nu\partial_\mu\partial_\nu + m^2)\chi \\
&= (\tfrac{1}{2}\{\gamma^\mu, \gamma^\nu\}\partial_\mu\partial_\nu + m^2)\chi = (\eta^{\mu\nu}\partial_\mu\partial_\nu + m^2)\chi(x) \\
&= (\partial_\mu\partial^\mu + m^2)\chi = 0.
\end{aligned} \quad (37)$$

There is an operator that distinguishes the different chiral spinors inside a Dirac one, the chirality operator

$$\gamma^5 \equiv i\gamma^0\gamma^1\gamma^2\gamma^3 = -\frac{i}{4!}\epsilon^{\mu\nu\rho\sigma}\gamma_\mu\gamma_\nu\gamma_\rho\gamma_\sigma. \tag{38}$$

with $\epsilon^{0123} = -\epsilon_{0123} = 1$. In the chiral representation $\gamma^5$ is diagonal

$$\gamma^5 = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} \tag{39}$$

and in general satisfies

$$\begin{aligned} (\gamma^5)^\dagger &= \gamma^5, \\ (\gamma^5)^2 &= \mathbf{1}, \\ \{\gamma^5, \gamma^\mu\} &= 0. \end{aligned} \tag{40}$$

If $\chi$ satisfies the Dirac equation, then

$$(-i\gamma^\mu\partial_\mu - m)\gamma^5\chi = \gamma^5(i\gamma^\mu\partial_\mu - m)\chi = 0, \tag{41}$$

and therefore, $\gamma^5\chi$ also satisfies the Klein-Gordon Equation. It can be shown that the most general solution to the Klein Gordon equation for spin $1/2$ fields can be written in terms of two solutions to the Dirac equation $\chi_1$ and $\chi_2$ as (Cufaro Petroni et al, 1985)

$$\psi = \frac{1}{\sqrt{2m}}(\chi_1 + \gamma^5\chi_2). \tag{42}$$

# Second Order Fermions

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Naive Theory

The simplest second order theory for free spin 1/2 fermions is described by the Lagrangian

$$\mathcal{L}_0 = \partial^\mu \bar{\psi} \partial_\mu \psi - m^2 \bar{\psi} \psi \tag{43}$$

where the Dirac dual is defined as $\bar{\psi} = \psi^\dagger \gamma^0$, rendering the theory Lorentz invariant. The corresponding equations of motion coincide with the Klein-Gordon equation for each component of the field and its dual.

Observations

- The conjugate-momenta are given by $\dot{\bar{\psi}}$ and $\dot{\psi}$ Without imposing further constraints, this second order formalism is described by a spinor with 8 degrees of freedom, the double compared to Dirac.
- The mass dimension of the field is one, in sharp contrast with Dirac spinors which have mass dimension $3/2$.

Unfortunately this simple theory has an undesirable feature. It can be shown that if the following canonical anti-commutation relations are imposed at equal times

$$\left\{\psi_\alpha(\mathbf{x},t),\dot{\bar{\psi}}_\beta(\mathbf{x}',t)\right\}=-\left\{\bar{\psi}_\alpha(\mathbf{x},t),\dot{\psi}_\beta(\mathbf{x}',t)\right\}=i\delta_{\alpha\beta}\delta^{(3)}(\mathbf{x}-\mathbf{x}'), \quad (44)$$

an indefinite metric problem emerges, leading to the presence of negative norm states in the spectrum (Kibble and Polkinghorne,1958).

In this talk, we show that the canonical quantization of second order fermions is possible by the adoption of a pseudo hermitian quantum field theory through an adequate redefinition of the dual field and the identification of an operator $\eta$ such that

$$\mathcal{L}^{\#} \equiv \eta^{-1}\mathcal{L}^{\dagger}\eta = \mathcal{L}. \tag{45}$$

## Pseudo-Hermitian Quantum Mechanics

The dynamics of the pseudo-hermitian quantum theory with $H^{\#} = \eta^{-1} H^{\dagger} \eta = H$ are guaranteed to be unitary if the inner product between two states is defined as

$$\langle a|b\rangle_{\eta} \equiv \langle a| \eta |b\rangle . \tag{46}$$

In this way, probability amplitudes are preserved under time evolution:

$$\langle a(t)|b(t)\rangle_{\eta} = \langle a| e^{iH^{\dagger}t} \eta e^{-iHt} |b\rangle = \langle a| \eta e^{iHt} e^{-iHt} |b\rangle = \langle a|b\rangle_{\eta} . \tag{47}$$

Similarly, the new inner product renders the energy spectrum real

$$(E - E^{*}) \langle a_E|a_E\rangle_{\eta} = \langle a_E| (\eta H - H^{\dagger}\eta) |a_E\rangle = 0, \tag{48}$$

where the state $|a_E\rangle$ is an energy eigenstate $H |a_E\rangle = E |a_E\rangle$.

# Pseudo-Hermitian Second Order Fermions

The Lagrangian for the spin $1/2$ field is given by

$$\mathcal{L} = \partial^\mu \widehat{\psi} \partial_\mu \psi - m^2 \widehat{\psi} \psi, \tag{49}$$

where $\widehat{\psi}$ is not the Dirac adjoint of $\psi$, but instead a redefinition of its dual that renders the theory pseudo-hermitian.

$$\widehat{\psi} = \eta^{-1} \bar{\psi} \eta = \eta^{-1} \psi^\dagger \eta \gamma^0. \tag{50}$$

It can be shown that if $\psi$ is the most general solution to the Klein-Gordon equation for spin $1/2$ fields

$$\psi = \frac{1}{\sqrt{2m}}(\chi_1 + \gamma^5\chi_2), \tag{51}$$

then, the dual $\widehat{\psi}$ that renders the theory pseudo-hermitian is

$$\widehat{\psi} = \eta^{-1}\bar{\psi}\eta = \frac{1}{\sqrt{2m}}\eta^{-1}(\overline{\chi}_1 - \overline{\chi}_2\gamma^5)\eta = \frac{1}{\sqrt{2m}}(\overline{\chi}_1 + \overline{\chi}_2\gamma^5). \tag{52}$$

Expanding $\psi$ and $\widehat{\psi}$ into plane waves, we have

$$\psi(x) = \int \frac{d^3\mathbf{p}}{(2\pi)^3 2\sqrt{m\omega_\mathbf{p}}} \sum_s \Big\{ \left[u^s_\mathbf{p} a^{1s}_\mathbf{p} + \gamma^5 u^s_\mathbf{p} a^{2s}_\mathbf{p}\right] e^{-ip\cdot x} + \left[v^s_\mathbf{p} b^{1s\dagger}_\mathbf{p} + \gamma^5 v^s_\mathbf{p} b^{2s\dagger}_\mathbf{p}\right] e^{ip\cdot x} \Big\}, \tag{53}$$

$$\widehat{\psi}(x) = \int \frac{d^3\mathbf{p}}{(2\pi)^3 2\sqrt{m\omega_\mathbf{p}}} \sum_s \Big\{ \left[\bar{u}^s_\mathbf{p} a^{1s\dagger}_\mathbf{p} + \bar{u}^s_\mathbf{p} \gamma^5 a^{2s\dagger}_\mathbf{p}\right] e^{ip\cdot x} + \left[\bar{v}^s_\mathbf{p} b^{1s}_\mathbf{p} + \bar{v}^s_\mathbf{p} \gamma^5 b^{2s}_\mathbf{p}\right] e^{-ip\cdot x} \Big\}, \tag{54}$$

with $\omega_\mathbf{p} = +\sqrt{|\mathbf{p}|^2 + m^2}$, $p^\mu = (\omega_\mathbf{p}, \mathbf{p})$, $u^s_\mathbf{p}$, $v^s_\mathbf{p}$ as the positive and negative energy solutions of the Dirac free equation, and $s = \pm\frac{1}{2}$.

Thus, the action of the operator $\eta$ is defined by

$$\eta^{-1} a_{\mathbf{p}}^{js} \eta = (-1)^{j-1} a_{\mathbf{p}}^{js}, \qquad \eta^{-1} b_{\mathbf{p}}^{js\dagger} \eta = (-1)^{j-1} b_{\mathbf{p}}^{js\dagger}, \tag{55}$$

where $j = 1, 2$. Assuming equal time canonical anti-commutation relations

$$\left\{ \psi_\alpha(\mathbf{x}, t), \dot{\widehat{\psi}}_\beta(\mathbf{x}', t) \right\} = -\left\{ \widehat{\psi}_\alpha(\mathbf{x}, t), \dot{\psi}_\beta(\mathbf{x}', t) \right\} = i\delta_{\alpha\beta}\delta^{(3)}(\mathbf{x} - \mathbf{x}'), \tag{56}$$

we have

$$\begin{aligned} \left\{ a_{\mathbf{p}}^{js}, a_{\mathbf{p}'}^{kr\dagger} \right\} &= (2\pi)^3 \delta^{jk} \delta^{rs} \delta^{(3)}(\mathbf{p} - \mathbf{p}'), \\ \left\{ b_{\mathbf{p}}^{js}, b_{\mathbf{p}'}^{kr\dagger} \right\} &= (2\pi)^3 \delta^{jk} \delta^{rs} \delta^{(3)}(\mathbf{p} - \mathbf{p}'), \end{aligned} \tag{57}$$

with all other anti-commutators vanishing.

An explicit solution for the operator $\eta$ is given by

$$\eta = \exp\left[i\pi \int \frac{d^3\mathbf{p}}{(2\pi)^3} \sum_s \left(a_{\mathbf{p}}^{2s\dagger} a_{\mathbf{p}}^{2s} + b_{\mathbf{p}}^{2s\dagger} b_{\mathbf{p}}^{2s}\right)\right]. \tag{58}$$

This operator satisfies $\eta = \eta^\dagger$ and $\eta\eta^\dagger = 1$, meaning that $\eta^2 = 1$, and thus their eigenvalues are $\pm 1$.

# Microcausality

Due to the redefinition of the dual implemented by the $\eta$ operator, the fields display the correct properties under microcausality. In particular, we have

$$\{\psi_\alpha(x), \psi_\beta(y)\} = \left\{\widehat{\psi}_\alpha(x), \widehat{\psi}_\beta(y)\right\} = 0, \tag{59}$$

and

$$\begin{aligned}\left\{\psi_\alpha(x), \widehat{\psi}_\beta(y)\right\} &\equiv \Delta(x-y)\delta_{\alpha\beta} \\ &= \int \frac{d^3\mathbf{p}}{(2\pi)^3 2\omega_{\mathbf{p}}}\left\{e^{-ip\cdot(x-y)} - e^{ip\cdot(x-y)}\right\}\delta_{\alpha\beta},\end{aligned} \tag{60}$$

where $\Delta(x-y)$ is the well known Lorentz invariant and causal Schwinger's Green function.

# Hamiltonian and Momentum

The Hamiltonian and the momentum operator are given by

$$\begin{aligned} H &=: \int d^3\mathbf{x} \left\{ \dot{\widehat{\psi}}\dot{\psi} + \nabla\widehat{\psi}\cdot\nabla\psi + m^2\widehat{\psi}\psi \right\} :, \\ \mathbf{P} &=: -\int d^3\mathbf{x} \left\{ \dot{\widehat{\psi}}\nabla\psi + \nabla\widehat{\psi}\dot{\psi} \right\} :, \end{aligned} \tag{61}$$

where : : stands for normal ordering. In terms of the momentum space operators, the generators of space-time translations $P^\mu = (H, \mathbf{P})$ read

$$P^\mu = \int \frac{d^3\mathbf{p}}{(2\pi)^3} p^\mu \sum_{j,s} \left\{ a^{js\dagger}_{\mathbf{p}} a^{js}_{\mathbf{p}} + b^{js\dagger}_{\mathbf{p}} b^{js}_{\mathbf{p}} \right\}. \tag{62}$$

# Spin

The spin of the field is

$$\mathbf{S} =: -i \int d^3\mathbf{x} \left\{ \dot{\widehat{\psi}} \mathbf{J} \psi - \widehat{\psi} \mathbf{J} \dot{\psi} \right\} :, \tag{63}$$

where the components of $\mathbf{J}$ are given by $J^k = \frac{1}{2}\epsilon_{ijk}\mathcal{J}^{ij}_{1/2}$. One can explicitly show that the field contains particles of spin $1/2$:

$$S^3 a^{js\dagger}_{\mathbf{0}} \left|0\right\rangle = s a^{js\dagger}_{\mathbf{0}} \left|0\right\rangle , \quad S^3 b^{js\dagger}_{\mathbf{0}} \left|0\right\rangle = s b^{js\dagger}_{\mathbf{0}} \left|0\right\rangle . \tag{64}$$

# Abelian Global Symmetry

The free theory of second order fermions is invariant under the phase transformation

$$\psi \to \psi' = e^{i\theta}\psi, \qquad \widehat{\psi} \to \widehat{\psi}' = \widehat{\psi} e^{-i\theta}, \tag{65}$$

where $\theta$ is a constant real parameter. The conserved charge associated to this $\mathrm{U}(1)$ global symmetry is

$$\begin{aligned} Q &= :i\int d^3\mathbf{x}\left\{\widehat{\psi}\dot{\psi} - \dot{\widehat{\psi}}\psi\right\}: \\ &= \int \frac{d^3\mathbf{p}}{(2\pi)^3}\sum_{j,s}\left\{a^{jr\dagger}_{\mathbf{p}} a^{js}_{\mathbf{p}} - b^{jr\dagger}_{\mathbf{p}} b^{js}_{\mathbf{p}}\right\}. \end{aligned} \tag{66}$$

From the commutation relation of this operator with the creation and annihilation operators, one can conclude that $a_{\mathbf{p}}^{js\dagger}$ and $b_{\mathbf{p}}^{js}$ have charge $+1$, whereas $a_{\mathbf{p}}^{js}$ and $b_{\mathbf{p}}^{js\dagger}$ have charge $-1$. Labeling the one-particle states with this eigenvalue, one can show that they are eight-fold degenerate

$$\begin{aligned} H a_{\mathbf{p}}^{js\dagger} \left|0\right\rangle &\propto H \left|\mathbf{p}, +, j, s\right\rangle = \omega_{\mathbf{p}} \left|\mathbf{p}, +, j, s\right\rangle , \\ H b_{\mathbf{p}}^{js\dagger} \left|0\right\rangle &\propto H \left|\mathbf{p}, -, j, s\right\rangle = \omega_{\mathbf{p}} \left|\mathbf{p}, -, j, s\right\rangle . \end{aligned} \tag{67}$$

# Symplectic Symmetry

Since the field $\psi$ and its dual anticommute $\{\psi_\alpha(x), \widehat{\psi}_\beta(x)\} = 0$, we can write the free lagrangian as

$$\mathcal{L} = \frac{1}{2}\partial^\mu \Psi^T \Omega \partial_\mu \Psi - \frac{m^2}{2}\Psi^T \Omega \Psi, \tag{68}$$

where $\Psi$ is a column matrix defined as

$$\Psi(x) = \begin{pmatrix} \widehat{\psi}^T(x) \\ \psi(x) \end{pmatrix}, \tag{69}$$

and $\Omega$ is the $8 \times 8$ symplectic matrix, written in $4 \times 4$ blocks as

$$\Omega = \begin{pmatrix} 0_{4\times4} & 1_{4\times4} \\ -1_{4\times4} & 0_{4\times4} \end{pmatrix}. \tag{70}$$

Thus, the free theory is symmetric under the global transformations $\Psi \to \Psi' = S\Psi$ with $S^T \Omega S = \Omega$. This is the defining relation for an element of the symplectic group $\mathrm{Sp}(8, \mathbb{C})$, whose algebra has 36 generators.

# Discrete Symmetries

The second order theory is invariant under Parity ($\mathrm{P}$), Charge conjugation ($\mathrm{C}$) and Time reversal ($\mathrm{T}$), and therefore under $\mathrm{CPT}$. We define the discrete transformations of the $\psi$ field through their action on the creation operators, as follows:

$$\begin{aligned} \mathrm{P}a_{\mathbf{p}}^{js\dagger}\mathrm{P}^{-1} &= -i(-1)^{j-1}a_{-\mathbf{p}}^{js\dagger}, & \mathrm{P}b_{\mathbf{p}}^{js\dagger}\mathrm{P}^{-1} &= -i(-1)^{j-1}b_{-\mathbf{p}}^{js\dagger}, \\ \mathrm{C}a_{\mathbf{p}}^{js\dagger}\mathrm{C}^{-1} &= b_{\mathbf{p}}^{js\dagger}, & \mathrm{C}b_{\mathbf{p}}^{js\dagger}\mathrm{C}^{-1} &= a_{\mathbf{p}}^{js\dagger}, \\ \mathrm{T}a_{\mathbf{p}}^{js\dagger}\mathrm{T}^{-1} &= 2sa_{-\mathbf{p}}^{j(-s)\dagger}, & \mathrm{T}b_{\mathbf{p}}^{is\dagger}\mathrm{T}^{-1} &= 2sb_{-\mathbf{p}}^{i(-s)\dagger}. \end{aligned} \tag{71}$$

With this choice, the discrete transformations have the familiar representations

$$\begin{aligned} \mathrm{P}\psi(x)\mathrm{P}^{-1} &= i\gamma^0\psi(\mathcal{P}x), & \mathrm{P}\widehat{\psi}(x)\mathrm{P}^{-1} &= -i\widehat{\psi}(\mathcal{P}x)\gamma^0, \\ \mathrm{C}\psi(x)\mathrm{C}^{-1} &= \mathcal{C}\widehat{\psi}^T, & \mathrm{C}\widehat{\psi}\mathrm{C}^{-1} &= \psi^T\mathcal{C} \\ \mathrm{T}\psi(x)\mathrm{T}^{-1} &= \mathcal{C}\gamma^5\psi(\mathcal{T}x), & \mathrm{T}\widehat{\psi}\mathrm{T}^{-1} &= -\widehat{\psi}(\mathcal{T}x)\gamma^5\mathcal{C}, \end{aligned} \quad (72)$$

where we have defined $\mathcal{P} = \mathrm{diag}(1,-1,-1,-1)$, $\mathcal{T} = \mathrm{diag}(-1,1,1,1)$, and $\mathcal{C} = -i\gamma^2\gamma^0$ in the chiral representation.

# Interactions

The simplest $\mathrm{C}$, $\mathrm{P}$ and $\mathrm{T}$ invariant pseudo-hermitian interactions that can be introduced in this framework are 4-fermion self-interactions represented by a dimension-four renormalizable operator

$$\begin{aligned}\mathcal{L}_{\text{int}}^{\text{self}} =& \frac{\lambda_1}{2}\left(\widehat{\psi}\psi\right)^2 + \frac{\lambda_2}{2}\left(\widehat{\psi}\gamma^5\psi\right)\left(\widehat{\psi}\gamma^5\psi\right) \\ &+ \frac{\lambda_3}{2}\left(\widehat{\psi}M^{\mu\nu}\psi\right)\left(\widehat{\psi}M_{\mu\nu}\psi\right),\end{aligned} \tag{73}$$

with $M^{\mu\nu} = \mathcal{J}_{1/2}^{\mu\nu}$.

The spin $1/2$ field can also be coupled with an abelian gauge field according to

$$\mathcal{L}_{\text{gauge}} = -\frac{1}{4}F^{\mu\nu}F_{\mu\nu} + D^{\dagger\mu}\widehat{\psi}G_{\mu\nu}D^{\nu}\psi - m^2\widehat{\psi}\psi \tag{74}$$

where $D_\mu = \partial_\mu + iqA_\mu$ is the covariant derivative, and we have introduced the space-time tensor

$$G^{\mu\nu} \equiv g^{\mu\nu} - igM^{\mu\nu}, \tag{75}$$

to produce a theory analogous to Scalar QED, supplemented with a Pauli interaction characterized by the coupling constant $g$, that can be identified as an arbitrary gyromagnetic factor. The resulting pseudo-hermitian interaction Lagrangian is

$$\mathcal{L}^{\text{gauge}}_{\text{int}} = iq[\partial^\mu\widehat{\psi}\psi - \widehat{\psi}\partial^\mu\psi]A_\mu + q^2\widehat{\psi}\psi A^\mu A_\mu - \frac{qg}{2}\widehat{\psi}M_{\mu\nu}\psi F^{\mu\nu} \tag{76}$$

The one loop renormalization properties of this theory have been studied in arXiv:1205.1557 [hep-ph]. The beta functions and anomalous dimensions of the theory at one-loop are

$$\beta_q = \frac{q^3}{12\pi^2}\left(\frac{3}{4}g^2 - 1\right),$$

$$\beta_g = \frac{g}{32\pi^2}\left[q^2\left(g^2-4\right) - 4(\lambda_1+\lambda_2-3\lambda_3)\right],$$

$$\beta_{\lambda_1} = -\frac{1}{16\pi^2}\Bigg\{\frac{3}{4}q^4\left(g^2-4\right)^2 + 3q^2\left[\left(4+g^2\right)\lambda_1 + g^2\lambda_3\right]$$

$$+ 4\lambda_2\left(\lambda_1+\lambda_2\right) + 6\lambda_3\left(2\lambda_1+\lambda_3\right)\Bigg\},$$

$$\beta_{\lambda_2} = -\frac{1}{16\pi^2}\left\{3q^2\left[\left(4+g^2\right)\lambda_2 + g^2\lambda_3\right] + 4\lambda_2\left(3\lambda_1-\lambda_2\right) + 6\lambda_3\left(2\lambda_2+\lambda_3\right)\right\},$$

$$\beta_{\lambda_3} = -\frac{1}{16\pi^2}\left\{q^2\left[\left(12-g^2\right)\lambda_3 + 2g^2\left(\lambda_1+\lambda_2\right)\right] + 4\lambda_3[3(\lambda_1+\lambda_2)-2\lambda_3]\right\}.$$

Fixed points for $\beta_g$

- Dirac-like configuration $\{g = \pm 2,\ \lambda_1 = \lambda_2 = \lambda_3 = 0\}$, the renormalization group equations (RGE) coincide with those obtained in the Dirac framework, except for the number of degrees of freedom propagating inside the fermion loops, the double compared with respect to Dirac.
- Scalar-like solution $g = 0$ with arbitrary self interactions $\lambda_i$. In this scenario the fermi statistics drive $\beta_q$ negative. Thus, at one loop level, the solution $g = 0$ with vanishing self-interactions displays asymptotic freedom for the coupling $q$.

Conversely, in the limit of vanishing gauge interactions $q = 0$, $g = 0$, we have a renormalizable theory of self-interacting fermions with rich phenomenology. Taking, for example, $\lambda_1 = \lambda,\ \lambda_2 = \lambda_3 = 0$, we obtain the simplest model of self-interacting fermions

$$\mathcal{L} = \partial^\mu \widehat{\psi} \partial_\mu \psi - m^2 \widehat{\psi} \psi + \frac{\lambda}{2} \left( \widehat{\psi} \psi \right)^2 , \tag{77}$$

with vanishing one-loop beta function $\beta_\lambda = 0$.

# Wimp Dark Matter

The second order fermion can couple with the Higgs field $H$ through a quartic dimensionless coupling

$$\mathcal{L}_{\psi H} = \frac{\lambda_H}{2}\left(\widehat{\psi}\psi\right) H^{\dagger} H. \tag{78}$$

Furthermore, since $\psi$ has nothing to decay in, it is stable and could play the role of Wimp dark matter through a Higgs portal.

# Conclusions

- Spin $1/2$ fermions can be successfully described by a second order formalism, provided the underlying Quantum Field Theory is pseudo-hermitian.
- Second Order Fermions are causal, with real spectrum, Hamiltonian bounded from below, unitary evolution and CPT invariance.
- The field contains 8 degrees of freedom
- Since the field has mass dimension one, the theory contains renormalizable self-interactions.
- The field is a Wimp Dark Matter candidate.

Thanks