Towards two-loop master integrals for ttH

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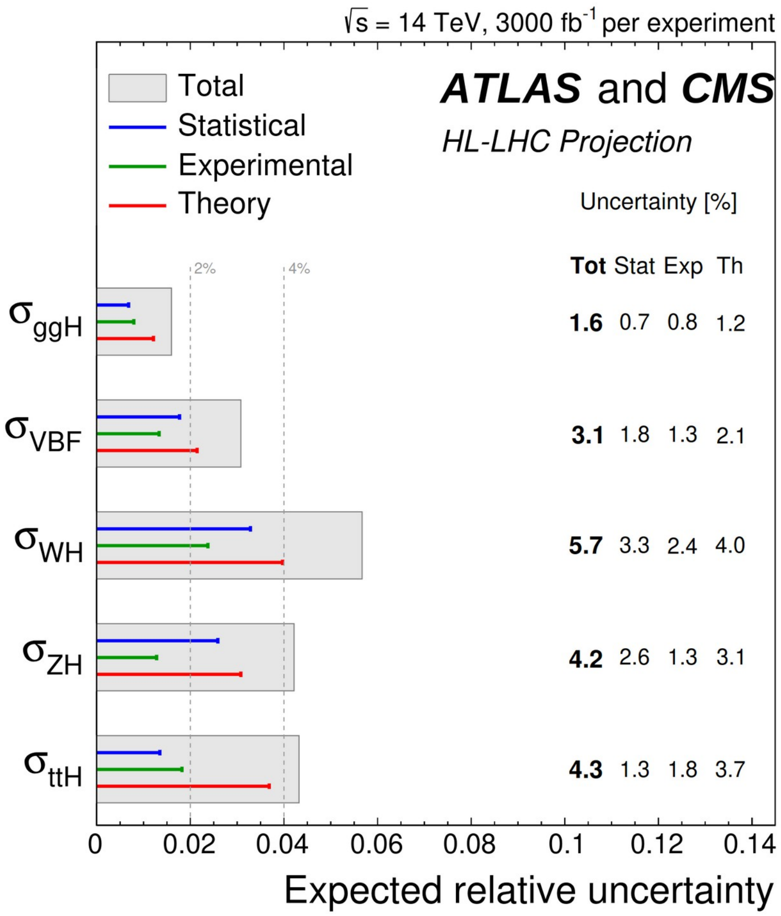
Motivation



[Yellow Report, CERN-2019-007]

$t\bar{t}H$ allows for a direct measurement of

$$y_t = \frac{\sqrt{2}m_t}{v}$$

Exp. Precision: $\mathcal{O}(20\%) \to \mathcal{O}(2\%)$

Approximate NNLO:

σ [pb]	$\sqrt{s} = 13\,\mathrm{TeV}$	$\sqrt{s} = 100\,\mathrm{TeV}$
σ_{LO}	$0.3910\,^{+31.3\%}_{-22.2\%}$	$25.38\,^{+21.1\%}_{-16.0\%}$
σ_{NLO}	$0.4875\,^{+5.6\%}_{-9.1\%}$	$36.43\,^{+9.4\%}_{-8.7\%}$
σ_{NNLO}	$0.5070\,(31)^{+0.9\%}_{-3.0\%}$	$37.20(25)\,^{+0.1\%}_{-2.2\%}$

[Catani et al., arXiv:2210.07846]

The missing ingredient:

Two-Loop Virtual Amplitudes

Status of NNLO 2 -> 3 with tops

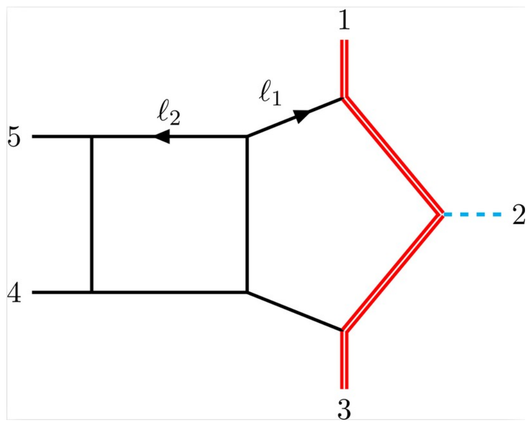
$pp \to t\bar{t}j$

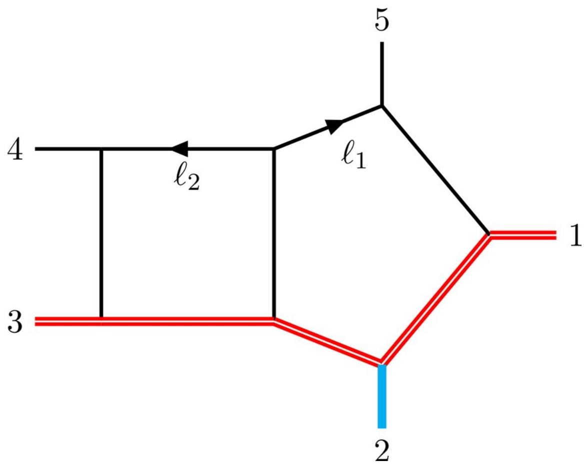
- One-loop Amplitudes at $@\ \mathcal{O}(\epsilon^2)$
  [Badger, Bechetti, Chaubey, Marzucca, Sarandrea, arXiv:2201.12188]
- A planar two-loop integral family
  [Badger, Bechetti, Chaubey, Marzucca, arXiv:2210.17477]
- Leading-color two-loop integrals
  [Badger, Bechetti, Giraudo, Zoia, arXiv:2404.12325]

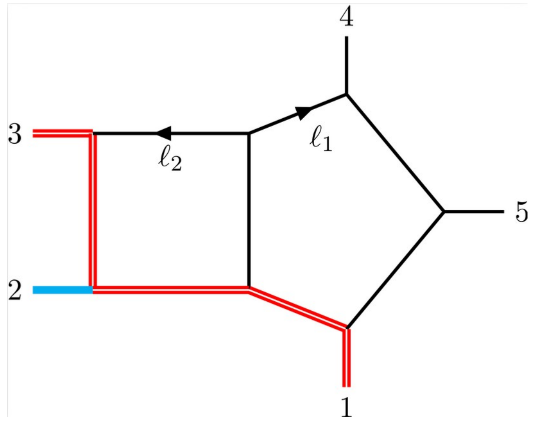
$pp \to t\bar{t}H$

- IR divergences of two-loop amplitudes
  [Chen, Ma, Wang, Yang, Ye, arXiv:2202.02913]
- Two-loop amplitudes in soft Higgs/boosted limit
  [Catani, Devoto, Grazzini, Kallweit, Mazzitelli, Savoini, arXiv:2210.07846] [Wang, Xia, Yang, Ye, arXiv:2401.07632]
- Two-loop planar master integrals for light-quark loop
  [Febres Cordero, Figueiredo, MK, Page, Reina, arXiv:2312.08131]
- One-loop $gg \to t\bar{t}H\ @\ \mathcal{O}(\epsilon^2)$
  [Buccioni, Kreer, Liu, Tancredi, arXiv:2312.10015]
- Two-loop $q\bar{q} \to t\bar{t}H$ amplitudes N_f contribution
  [Agarwal, Heinrich, Jones, Kerner, Klein, Lang, Magerya, Olsson, arXiv:2402.03301]

Two-loop ttH @ Leading Color







111 MIs

Nested root

156 MIs

More nested roots ...

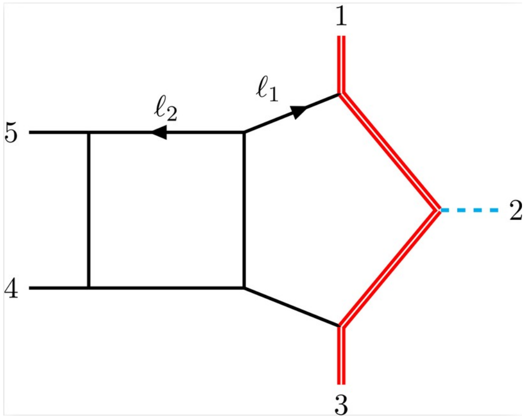
Elliptic sectors

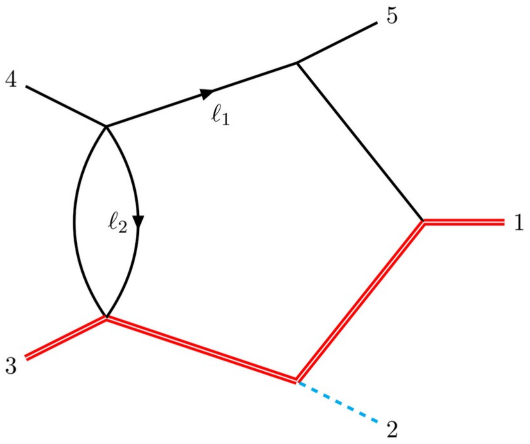
186 MIs

Even more nested roots ...

Even more elliptics ...

Two-loop ttH @ Leading Color





111 MIs

Nested root

19 MIs

No suprises :)

All Master Integrals for Leading-Color + closed light fermion for ttH

Scattering Kinematics

Channel: $g(p_4)g(p_5) \to t(p_1)H(p_2)\bar{t}(p_3)$

Five-point, two massless legs, two on-shell tops, generic Higgs

$$p_1^2 = p_3^2 = m_t^2\,, \qquad p_2^2 = q^2\,, \qquad p_4^2 = p_5^2 = 0$$

7 independent kinematic invariants

$$\vec{s} = \{v_{12}, v_{23}, v_{34}, v_{45}, v_{15}, m_t^2, q^2\}, \qquad v_{ij} = 2(p_i \cdot p_j)$$

11 square roots:

6 – Gram determinants

2 – mod. Cayleys

3 – Maximal cut

2 nested roots:

$$\sqrt{N_\pm} = \sqrt{q^2\left(N_b \pm \sqrt{N_b^2 - N_c}\right)}$$

N_b degree 3 polynomial in 5 variables

General Strategy

1. Setup differential equations for system of master integrals

$$\frac{d\vec{I}}{dx} = B_x \; \vec{I}$$

[Kotikov, ‘91; Remiddi ‘97; Gehrman, Remiddi ‘01]

2. Find a basis in ϵ-factorized form

$$\vec{J} = T\vec{I} \qquad \frac{d\vec{J}}{dx} = \epsilon A_x \vec{J}$$

[Henn ‘13]

3. Rewrite differential matrix in terms of basis of one-forms

$$A_x = \frac{d}{dx} \sum_\alpha M_\alpha \mathrm{d}\log(\omega_\alpha)$$

4. Numerical solutions via series expansions

[Moriello ‘19; Hidding ‘20, Liu; Ma ‘22]

Construction of ϵ-factorized Basis

Construction of a canonical Basis is non-trivial!

Our Approach:

1. Find initial choice of masters such that

$$B(\epsilon, \vec{s}) = B^{(0)}(\vec{s}) + \epsilon B^{(1)}(\vec{s})$$

2. Change basis $\vec{J} = T\vec{I}$, where T satisfies homogeneous DE

$$dT_{ij} = B^{(0)}_{ik} T_{kj}$$

3. If necessary compute subtraction terms for off-shell corrections by explicit integration

Computation in Finite-Fields – Analytics only where necessary!

The Easy Sectors

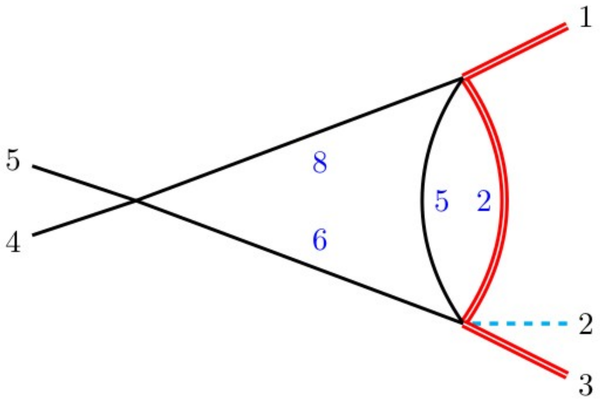
$$\mathrm{B}^{(0)} = \begin{pmatrix} \star & 0 & 0 \\ \star & \star & 0 \\ \star & \star & \star \end{pmatrix}$$

1. First remove the diagonal elements via

$$T_{ii} = \exp\left(-\int dx B_{ii}^{(0)}\right)$$

2. Then remove off-diagonal elements

$$T_{ij} = -\int dx B_{ij}^{(0)}$$

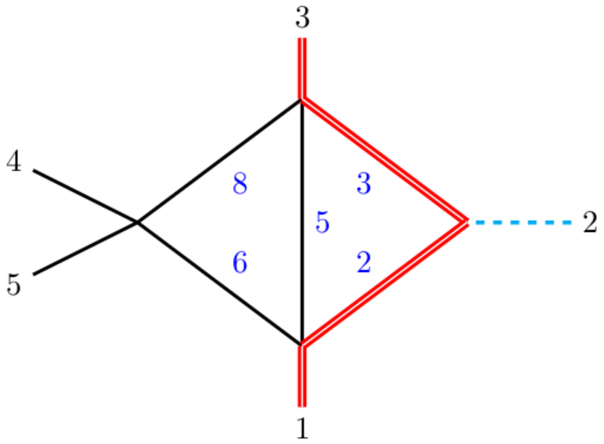


$$\mathcal{N}_{31}^{(1)} = \epsilon^3 \sqrt{\Delta_3^{(3)}}\,\frac{1}{\rho_2}\,,$$

$$\mathcal{N}_{32}^{(1)} = \epsilon^3 \sqrt{\Delta_3^{(3)}}\,\frac{1}{\rho_5}\,,$$

$$\mathcal{N}_{33}^{(1)} = \epsilon^2 \left[\frac{m_t^2 v_{45}}{\rho_2 \rho_6} + \epsilon(q^2 + v_{23} - v_{45})\left(\frac{1}{\rho_2} + \frac{1}{2\rho_5}\right)\right]$$

The fun-stuff: Kite



This sector has 7 master integrals

Using the previous methods we can reduce the problem to a coupled 3x3 system

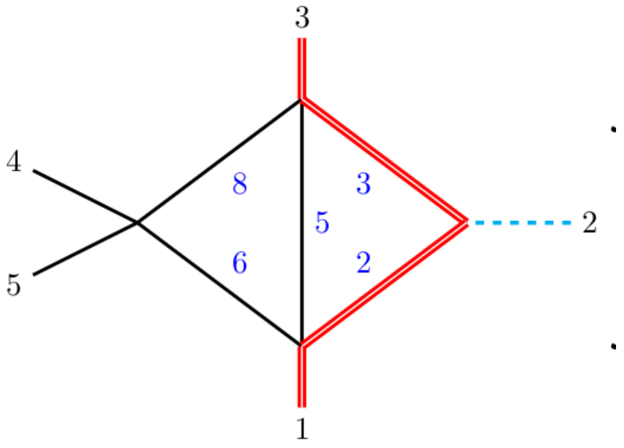
$$\mathrm{B}^{(0)} = \begin{pmatrix} 0 & \star & 0 \\ \star & 0 & 0 \\ \star & \star & 0 \end{pmatrix}$$

Needs Magnus expansion of entire 3x3 system!

$$\mathrm{dT} = \begin{pmatrix} 0 & \frac{1}{4} d\log(A) & 0 \\ \frac{1}{4} d\log(A) & 0 & 0 \\ x & y & 0 \end{pmatrix} \mathrm{T}$$

A - algebraic

The fun-stuff: Kite



$$\mathrm{dT} = \begin{pmatrix} 0 & \frac{1}{4}d\log(A) & 0 \\ \frac{1}{4}d\log(A) & 0 & 0 \\ x & y & 0 \end{pmatrix} \mathrm{T}$$

A - algebraic

Solution introduces a nested root in the integral basis

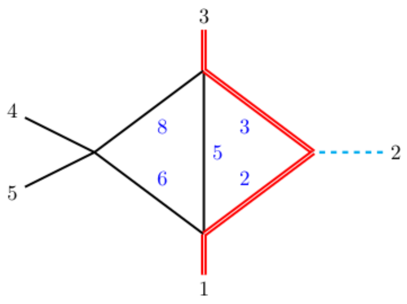
$$\sqrt{N_\pm} = \sqrt{q^2\left(N_b^2 \pm \sqrt{N_b^2 - N_c}\right)}$$

$$N_b = q^2\left[(v_{14}+v_{15})^2 + (v_{34}+v_{35})^2\right] - 2m_t^2(v_{24}+v_{25})^2$$

$$N_c = q^2(q^2-4m_t^2)(v_{12}-v_{23})^2(v_{24}+v_{25}+2v_{45})^2$$

New analytic structure!

The fun-stuff: Kite



$$
\begin{aligned}
\mathcal{N}_{60}^{(1)} &= \epsilon^4 \sqrt{r_1}\,, \\
\mathcal{N}_{61}^{(1)} &= \epsilon^3 v_{45} \sqrt{\Delta_3^{(2)}}\,\frac{1}{\rho_8}\,, \\
\mathcal{N}_{62}^{(1)} &= \epsilon^3 v_{45} \sqrt{\Delta_3^{(1)}}\,\frac{1}{\rho_6}\,, \\
\mathcal{N}_{63}^{(1)} &= \epsilon^3 \sqrt{\mathcal{C}_2}\,\frac{1}{\rho_5}\,, \\
\mathcal{N}_{64}^{(1)} &= \epsilon^3 \left[\frac{\sqrt{N_+}}{2}\left(\frac{1}{\rho_3}-\frac{1}{\rho_2}\right)+\frac{\sqrt{\mathcal{C}_1}\sqrt{N_-}}{2q^2}\left(\frac{1}{\rho_3}+\frac{1}{\rho_2}\right)\right], \\
\mathcal{N}_{65}^{(1)} &= \epsilon^3 \left[\frac{\sqrt{N_-}}{2}\left(\frac{1}{\rho_3}-\frac{1}{\rho_2}\right)+\frac{\sqrt{\mathcal{C}_1}\sqrt{N_+}}{2q^2}\left(\frac{1}{\rho_3}+\frac{1}{\rho_2}\right)\right], \\
\mathcal{N}_{66}^{(1)} &= \epsilon^2 \frac{m_t^2 v_{45}(q^2+v_{12})(q^2+v_{23})}{2q^2+v_{12}+v_{23}}\left(\frac{1}{\rho_2\rho_6}+\frac{1}{\rho_3\rho_8}\right) \\
&\quad + \epsilon^3\left(C_{66}^{(1)}\frac{1}{\rho_5}+C_{66}^{(2)}\frac{1}{\rho_6}+C_{66}^{(3)}\frac{1}{\rho_8}\right) \\
&\quad + \epsilon^3\left(C_{66}^{(4)}\left(\frac{1}{\rho_3}+\frac{1}{\rho_2}\right)+C_{66}^{(5)}\left(\frac{1}{\rho_3}-\frac{1}{\rho_2}\right)\right) \\
&\quad + C_{66}^{(6)}\left[\rho_2\,\mathcal{N}_{36}^{(1)}-\rho_3\,\mathcal{N}_{33}^{(1)}+\rho_2\rho_8\,\mathcal{N}_{3}^{(1)}\right]+C_{66}^{(7)}\left[\rho_5\,\mathcal{N}_{28}^{(1)}\right] \\
&\quad + C_{66}^{(8)}\left[\rho_6\,\mathcal{N}_{26}^{(1)}-\rho_8\,\mathcal{N}_{13}^{(1)}\right]+C_{66}^{(9)}\left[\rho_3\rho_6\,\mathcal{N}_{10}^{(1)}+\rho_2\rho_8\,\mathcal{N}_{2}^{(1)}\right] \\
&\quad + C_{66}^{(10)}\left[\rho_3\rho_6\,\mathcal{N}_{9}^{(1)}\right].
\end{aligned}
$$

Reconstruction of DEs

We follow the random direction DE approach: [Abreu, Ita, Moriello, BP, Tschernow, Zeng '20]

$$(\vec{r}\cdot\nabla_{\vec{s}})\vec{I} = C(\epsilon,\vec{s})\vec{I}\,, \qquad C(\epsilon,\vec{s}) = \epsilon\sum_{\alpha} M_{\alpha}(\vec{r}\cdot\nabla_{\vec{s}})\omega_{\alpha}$$

From which we obtain the number of linear independent basis functions:

$$\mathcal{C} = \{C(\epsilon,\vec{s}_1),\ldots,C(\epsilon,\vec{s}_N)\} \quad \to \quad \mathrm{Rank}[\mathcal{C}] = \min\{N,\dim(\omega)\} \to \dim(\omega) = 152$$

Now we can build an Ansatz Matrix:

$$W_{\alpha n} \equiv (\vec{r}\cdot\nabla_{\vec{s}})\omega_{\alpha}(\vec{s})\Big|_{\vec{s}=\vec{s}_n}$$

And fit only the coefficient matrices of the entire DE:

$$M_{\alpha} = \frac{1}{\epsilon}\sum_{n=1}^{152} W^{-1}_{\alpha n}\cdot C(\epsilon,\vec{s}_n)$$

Problem reduced to computation of basis functions

Alphabet

PentaBox Topology has 152 letters

- Much larger than single family for $pp \to Wjj$
- 9 Letters are irrelevant -> only arise at @ $\mathcal{O}(\epsilon^5)$

122 relevant letters without nested roots

Mass dimension	1	2	3	4	5	6	Σ
# Polynomial Letters	19	10	8	5	0	1	43
# Algebratic, single odd	14	12	15	0	0	1	42
# Algebratic, double odd	5	21	10	1	0	1	37

Remaining 21 relevant letters depend on nested roots!

Letters with nested roots

Many entries can be put into dLog form

$$d\log\left(\frac{q^2[v_{45}+s_{13}-q^2]-\sqrt{N_+}}{q^2[v_{45}+s_{13}-q^2]+\sqrt{N_+}}\right),\quad d\log\left(\frac{q^2(v_{12}-v_{23})(v_{45}+s_{13}-q^2)+\sqrt{r_1}\sqrt{N_+}}{q^2(v_{12}-v_{23})(v_{45}+s_{13}-q^2)-\sqrt{r_1}\sqrt{N_+}}\right)$$

We were unable to find dLog forms in 4 cases generated by

$$\omega^E = \frac{\Omega^E}{m_t^2(q^2-v_{23})\sqrt{G(p_2,p_3)}\sqrt{N_+}\sqrt{N_b^2-N_cW_{32}}}$$

Generates only single-poles when expanded around zeros of denominator

Not clear, if can be brought into dLog form. Lack of decision procedure

Numerical Solutions - I

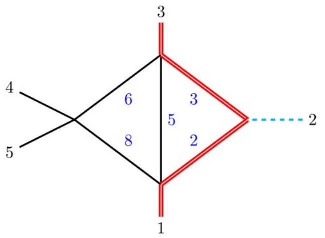
Use AMFlow to compute 100-digit physical region boundary constants

[Liu,Ma, arXiv:2201.11669]

Solve DE using generalized series expansions techniques

[Moriello, arXiv:1907.13234]

Public implementations: No nested roots -> use auxiliary basis



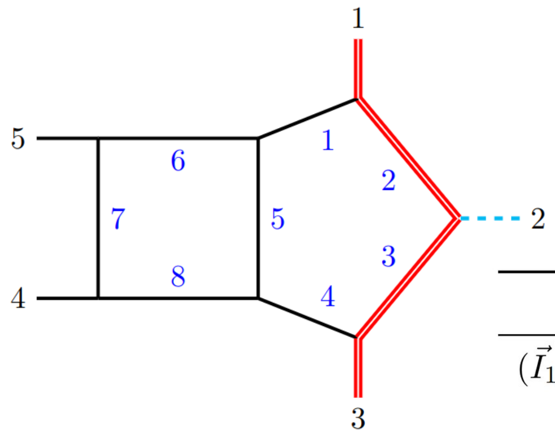
$$= \left\{ \epsilon^3 (q^2)^2 \left(\frac{1}{\rho_3} + \frac{1}{\rho_2} \right), \epsilon^3 (q^2)^2 \left(\frac{1}{\rho_3} - \frac{1}{\rho_2} \right) \right\}$$

Proof of concept DiffExp implementation in ancillary files

[Febres Cordero, Figueiredo, MK, Page, Reina, arXiv:2312.08131] [Hidding, arXiv:2006.05510]

Numerical Solutions - II

$$\vec{s}_1 = \left\{ \frac{19}{3}, \frac{46}{3}, -\frac{24}{7}, \frac{383}{5}, -\frac{61}{28}, \frac{25}{118}, \frac{97}{896} \right\},$$



	$\mathcal{O}(\epsilon^0)$	$\mathcal{O}(\epsilon^1)$	$\mathcal{O}(\epsilon^2)$	$\mathcal{O}(\epsilon^3)$	$\mathcal{O}(\epsilon^4)$
$(\vec{I}_1)_{109}$	0	0	0	-3.703380133 $+5.885655074\,i$	2.149576969 $-10.432322830\,i$
$(\vec{I}_1)_{110}$	0	0	0	0	0
$(\vec{I}_1)_{111}$	0	0	-1.306045093 $-12.647039669\,i$	2.05552771 $+25.35139955\,i$	-85.55528965 $-75.93834102\,i$

Solving DE is approximately 2 orders of magnitude faster than AMFlow

Summary & Outlook

Summary:

- We computed the first set of master integrals for

$$pp \to t\bar{t}H \quad @ \quad \text{NNLO}$$

- We provide differential equations in ϵ-factorized form
- We find a new analytic structure: Nested square roots!

Outlook:

- Computation of one-fold integral solution of DEs

following [Chicherin, Sotnikov '20]

- Finding canonical form of remaining two elliptic PentaBox families