Multiple Realizations of Models of Finite Modular Symmetries

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Outline

The Standard Model

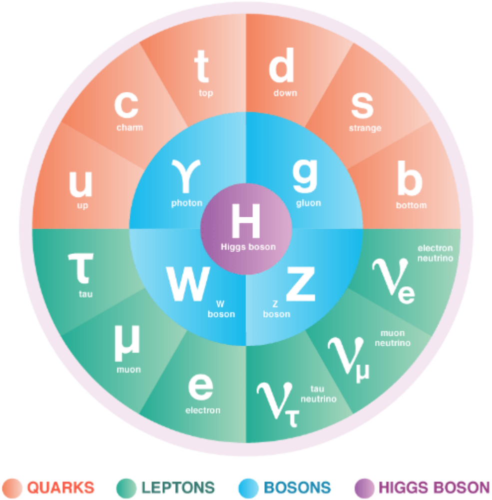


Figure: Standard Model in a Pie

The Theory is not Perfect

Achievements:

- Passed high precision tests.
- Predicted the W and Z bosons, as well as the Higgs, way before their detection.
- Explains a wide variety of phenomena.

Problems:

- Doesn't contain gravity.
- Too many free parameters.
- Why three flavors?.
- Is the neutrino Dirac or Majoranna?.
- What is the explanation of Yukawa couplings?.
- What is the explanation of lepton and quark mixing and hierarchies.

Ingredients

Field	$\mathrm{SU}(3)_{\text{color}}$	$\mathrm{SU}(2)_L$	$\mathrm{U}(1)_Y$
$L = (\nu_L, e_L)^T$	**1**	**2**	-1
e_R	**1**	**1**	-2
$Q = (u_L, d_L)^T$	**3**	**2**	$\frac{1}{3}$
u_R	**3**	**1**	$\frac{4}{3}$
d_R	**3**	**1**	$-\frac{2}{3}$

Table: Charges with electric charge given by $Q = I_3 + \frac{Y}{2}$.

A flavor or family is a copy of this table with different labels and measured masses.

The Lagrangian

$$\begin{aligned}\mathcal{L} = &\sum_{f\in\{u,c,t\}} \mathrm{i}\overline{q}^{U}_{fR} \not{\mathcal{D}} q^{U}_{fR} + \sum_{f\in\{d,s,b\}} \mathrm{i}\overline{q}^{D}_{fR} \not{\mathcal{D}} q^{D}_{fR} + \sum_{f\in\{e,\mu,\tau\}} \mathrm{i}\overline{E}_{fR} \not{\mathcal{D}} E_{fR} \\ &+ \sum_{f\in\{e,\mu,\tau\}} \mathrm{i}\overline{L}_{fL} \not{\mathcal{D}} L_{fL} + \sum_{f\in\{1,2,3\}} \mathrm{i}\overline{Q}_{fL} \not{\mathcal{D}} Q_{fL} \\ &- \frac{1}{2} \mathrm{Tr}\left(G_{\alpha\beta} G^{\alpha\beta}\right) - \frac{1}{2} \mathrm{Tr}\left(A_{\alpha\beta} A^{\alpha\beta}\right) - \frac{1}{4} B_{\alpha\beta} B^{\alpha\beta} \\ &+ \frac{1}{2} \left(\mathcal{D}^{\alpha} H\right)^{\dagger} \left(\mathcal{D}_{\alpha} H\right) - \frac{m^2}{2} H^{\dagger} H - \frac{\lambda}{4!} \left(H^{\dagger} H\right)^2 + \mathcal{L}_{\mathrm{Yukawa}}. \end{aligned} \tag{1}$$

Yukawa terms, the problem

$$\mathcal{L}_{\text{Yukawa}} = \sum_{i,j\in\{e,\mu,\tau\}} Y^L_{ij}\overline{L}_{iL} H E_{jR} + \sum_{i\in\{1,2,3\}} \sum_{d,s,b} Y^D_{ij}\overline{Q}_{iL} H q^D_{jR} \\ + \sum_{i\in\{1,2,3\}} \sum_{u,c,t} Y^U_{ij}\overline{Q}_{iL}\tilde{H} q^U_{jR} + \text{h.c.} \tag{2}$$

HIstory of the Modular Framework

In the 60's the Electroweak Model was conceived.

In the 70's the CKM matrix was used to describe quark masses.

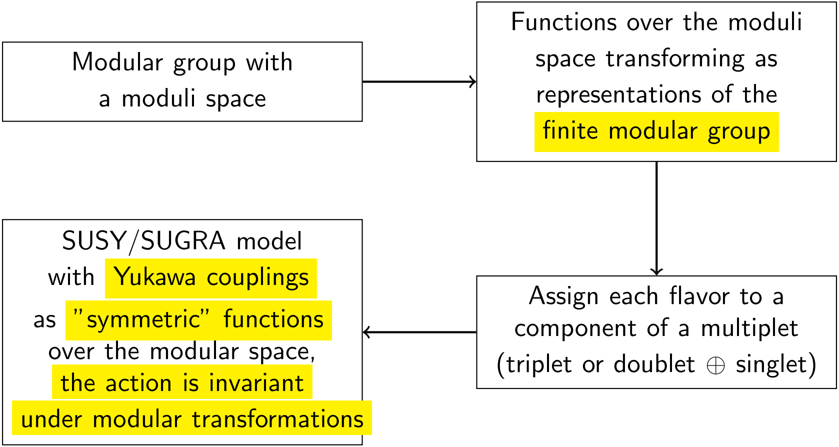
Late 90's and early 2000's neutrinos oscillate.

2017 Feruglio proposed Yukawa couplings to be modular forms.

Functions with "a special type of symmetry".

Onwards. Exploration of the modular framework.

Summary of the idea



The only symmetry capable of expanding spacetime symmetries is one relating fields of integer spin with fields of half-integer spin, called supersymmetry, in general the Lagrangian for $\mathcal{N} = 1$ local SUSY is given by

$$\mathcal{L} \; = \; \int d^2\theta d^2\overline{\theta}\mathcal{K}(\Phi, \overline{\Phi}) + \int d^2\theta \left(\mathcal{W}(\Phi) + \text{h.c.}\right). \tag{3}$$

SUGRA

If we make SUSY local we obtain supergravity, lagrangians now depend only on the Kähler function

$$G(\Phi, \overline{\Phi}) \;=\; \frac{\mathcal{K}(\Phi, \overline{\Phi})}{M_P^2} + \log|\mathcal{W}(\Phi)|^2. \tag{4}$$

Constructing Lagrangians in SUGRA theories requires more elaborate constructions. Hereafter we use Planck units $M_P^2 \;=\; 1$.

We define

$$\mathrm{SL}(2,\mathbb{Z}) \;=\; \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \,\middle|\, a,b,c,d \in \mathbb{Z} \text{ y } ad - bc = 1 \right\}. \tag{5}$$

The level N congruence subgroups are defined as

$$\Gamma(N) \;=\; \left\{ \gamma \in \mathrm{SL}(2,\mathbb{Z}) \middle|\; \gamma \bmod N \;=\; \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \right\}, \tag{6}$$

are normal and $\mathrm{SL}(2,\mathbb{Z})/\Gamma(N) \;:=\; \Gamma'_N$ is finite.

Complex Upper-Half Plane

The domain of our functions will be only the complex upper-half plane $\mathcal{H}$ consisting on complex numbers τ with $\operatorname{Im}\tau > 0$. The modular group acts on numbers $\tau \in \mathcal{H}$ as

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \tau = \frac{a\tau + b}{c\tau + d}, \tag{7}$$

the fundamental domain is the set of all numbers τ inequivalent under modular transformations.

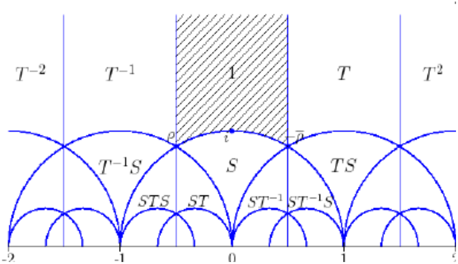


Figure: Fundamental Domain of $\mathrm{SL}(2, \mathbb{Z})$.

Modular Forms

Functions on $\mathcal{H}$ satisfying

Transforms as

$$f\left(\frac{a\tau+b}{c\tau+d}\right) \;=\; (c\tau+d)^k f(\tau).$$

f holomorphic in $\mathcal{H}$ with $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathcal{N} \subset \mathrm{SL}(2,\mathbb{Z})$

f is of weight k, if $\Gamma(N) \subset \mathcal{N}$ we say f is of level N.

Modular forms of weight k form a vector space.

Finite Group Transformation

Weight *k* modular forms of a normal subgroup $\mathcal{N}$ of $\mathrm{SL}(2,\mathbb{Z})$ (of finite index) transform as representations of

$$\mathrm{SL}(2,\mathbb{Z})/\mathcal{N}.$$

When we arrange modular forms like this we call them Vector Valued Modular Forms (VVMF's).

Congruence subgroups are not all normal subgroups of $\mathrm{SL}(2,\mathbb{Z})$.

Other Finite Modular Groups

$\mathrm{SL}(2,\mathbb{Z})$ is not the only source of finite modular groups. From the chinese remainder theorem we know that the quotient

$$\Gamma(N)/\Gamma(M)$$

such that N and M come from the prime factorization of some number

$$\mathrm{SL}(2,\mathbb{Z})/\Gamma(3) \cong \Gamma(2)/\Gamma(6) \cong \Gamma(4)/\Gamma(12) \cong \cdots . \tag{8}$$

These also possess VVMF's.

Do these different realizations of groups lead to different models?

Example: Group with GAP ID [24, 3]

There are infinite instances of [24, 3], two of them arise from $\mathrm{SL}(2,\mathbb{Z})$

$$\Gamma_3' \equiv \langle\ S, T \mid S^4 = (ST)^3 = T^3 = \mathbb{1},\ S^2 T = TS^2\ \rangle \cong T'\,, \qquad (9)$$

y

$$2T \equiv \langle\ S, T \mid S^4 = (ST)^3 = S^2 T^3 = \mathbb{1},\ S^2 T = TS^2\ \rangle \cong T'\,. \qquad (10)$$

One comes from $\Gamma(2)/\Gamma(6)$, its presentation being

$$\Gamma(2)/\Gamma(6) \equiv \langle\ a, b \mid a^4 = (ab)^3 = b^3 = \mathbb{1},\ a^2 b = ba^2\ \rangle \cong T'\,. \qquad (11)$$

Irreps for Γ_3' and $\Gamma(2)/\Gamma(6)$

$\boldsymbol{r}$	$\rho_{\boldsymbol{r}}^{(\Gamma_3')}(S), \rho_{\boldsymbol{r}}(a)$	$\rho_{\boldsymbol{r}}^{(\Gamma_3')}(T), \rho_{\boldsymbol{r}}(b)$
$\mathbf{1}$	1	1
$\mathbf{1'}$	1	ω
$\mathbf{1''}$	1	ω^2
$\mathbf{2}$	$\frac{\mathrm{i}}{\sqrt{3}}\begin{pmatrix} 1 & \sqrt{2} \\ \sqrt{2} & -1 \end{pmatrix}$	$\omega\begin{pmatrix} 1 & 0 \\ 0 & \omega \end{pmatrix}$
$\mathbf{2'}$	$\frac{\mathrm{i}}{\sqrt{3}}\begin{pmatrix} 1 & \sqrt{2} \\ \sqrt{2} & -1 \end{pmatrix}$	$\omega^2\begin{pmatrix} 1 & 0 \\ 0 & \omega \end{pmatrix}$
$\mathbf{2''}$	$\frac{\mathrm{i}}{\sqrt{3}}\begin{pmatrix} 1 & \sqrt{2} \\ \sqrt{2} & -1 \end{pmatrix}$	$\begin{pmatrix} 1 & 0 \\ 0 & \omega \end{pmatrix}$
$\mathbf{3}$	$\frac{1}{3}\begin{pmatrix} -1 & 2 & 2 \\ 2 & -1 & 2 \\ 2 & 2 & -1 \end{pmatrix}$	$\begin{pmatrix} 1 & 0 & 0 \\ 0 & \omega & 0 \\ 0 & 0 & \omega^2 \end{pmatrix}$

Irreps for $2T$

$\boldsymbol{r}$	$\rho_{\boldsymbol{r}}^{(2T)}(S)$	$\rho_{\boldsymbol{r}}^{(2T)}(T)$
$\mathbf{1}$	1	1
$\mathbf{1}'$	1	ω
$\mathbf{1}''$	1	ω^2
$\mathbf{2}$	$-\frac{\mathrm{i}}{\sqrt{3}}\begin{pmatrix} 1 & \sqrt{2} \\ \sqrt{2} & -1 \end{pmatrix}$	$-\omega\begin{pmatrix} 1 & 0 \\ 0 & \omega \end{pmatrix}$
$\mathbf{2}'$	$-\frac{\mathrm{i}}{\sqrt{3}}\begin{pmatrix} 1 & \sqrt{2} \\ \sqrt{2} & -1 \end{pmatrix}$	$-\omega^2\begin{pmatrix} 1 & 0 \\ 0 & \omega \end{pmatrix}$
$\mathbf{2}''$	$-\frac{\mathrm{i}}{\sqrt{3}}\begin{pmatrix} 1 & \sqrt{2} \\ \sqrt{2} & -1 \end{pmatrix}$	$-\begin{pmatrix} 1 & 0 \\ 0 & \omega \end{pmatrix}$
$\mathbf{3}$	$\frac{1}{3}\begin{pmatrix} -1 & 2 & 2 \\ 2 & -1 & 2 \\ 2 & 2 & -1 \end{pmatrix}$	$\begin{pmatrix} 1 & 0 & 0 \\ 0 & \omega & 0 \\ 0 & 0 & \omega^2 \end{pmatrix}$

Shared Modular Forms

The groups Γ_3' and $2T$ share the following modular forms of minimal weight

r	k_0	Analytic Form	Fourier Series $q = e^{2\pi i\tau}$
1	0	1	1
1′	4	η^8	$q^{1/3}(1 - 8q + 20q^2 - 70q^4 + \cdots)$
1″	8	η^{16}	$q^{2/3}(1 - 16q + 104q^2 - 320q^3 + 260q^4 + \cdots)$
3	2	$\begin{pmatrix} \eta^4 \left(\frac{K}{1728}\right)^{-\frac{1}{6}} {}_2F_1\left(-\frac{1}{6}, \frac{1}{6}, \frac{1}{2}; \frac{2}{3}, \frac{1}{3}\right) \\ -6\eta^4 \left(\frac{K}{1728}\right)^{\frac{1}{6}} {}_2F_1\left(\frac{1}{6}, \frac{1}{2}, \frac{5}{6}; \frac{2}{3}, \frac{4}{3}\right) \\ -18\eta^4 \left(\frac{K}{1728}\right)^{\frac{1}{2}} {}_2F_1\left(\frac{1}{2}, \frac{5}{6}, \frac{7}{6}; \frac{5}{3}, \frac{4}{3}\right) \end{pmatrix}$	$\begin{pmatrix} 1 + 12q + 36q^2 + 12q^3 + 84q^4 + 72q^5 + \cdots \\ -6q^{1/3}(1 + 7q + 8q^2 + 18q^3 + 14q^4 + \cdots) \\ -18q^{2/3}(1 + 2q + 5q^2 + 4q^3 + 8q^4 + \cdots) \end{pmatrix}$

VVMF's for Γ_3'

r	k_0	Analytic Form	Fourier Series $q = e^{2\pi i\tau}$
2	5	$\begin{pmatrix} \eta^{10}\left(\frac{K}{1728}\right)^{-\frac{1}{12}} {}_2F_1\left(-\frac{1}{12}, \frac{1}{4}; \frac{2}{3}\right) \\ 3\sqrt{2}\eta^{10}\left(\frac{K}{1728}\right)^{\frac{1}{4}} {}_2F_1\left(\frac{1}{4}, \frac{7}{12}; \frac{4}{3}\right) \end{pmatrix}$	$\begin{pmatrix} q^{1/3}(1 - 2q - 28q^2 + 126q^3 - 112q^4 + \cdots) \\ 3\sqrt{2}q^{2/3}(1 - 7q + 14q^2 + 4q^3 - 28q^4 + \cdots) \end{pmatrix}$
2′	3	$\begin{pmatrix} \eta^{6}\left(\frac{K}{1728}\right)^{\frac{5}{12}} {}_2F_1\left(\frac{5}{12}, \frac{3}{4}; \frac{5}{3}\right) \\ -\frac{1}{27\sqrt{2}}\eta^{6}\left(\frac{K}{1728}\right)^{-\frac{1}{4}} {}_2F_1\left(-\frac{1}{4}, \frac{1}{12}; \frac{1}{3}\right) \end{pmatrix}$	$\begin{pmatrix} q^{2/3}(1 + 8q + 17q^2 + 40q^3 + 50q^4 + \cdots) \\ -\frac{1}{27\sqrt{2}}(1 + 72q + 270q^2 + 720q^3 + 936q^4 + 2160q^5 + \cdots) \end{pmatrix}$
2″	1	$\begin{pmatrix} \eta^{2}\left(\frac{K}{1728}\right)^{-\frac{1}{12}} {}_2F_1\left(-\frac{1}{12}, \frac{1}{4}; \frac{2}{3}\right) \\ 3\sqrt{2}\eta^{2}\left(\frac{K}{1728}\right)^{\frac{1}{4}} {}_2F_1\left(\frac{1}{4}, \frac{7}{12}; \frac{4}{3}\right) \end{pmatrix}$	$\begin{pmatrix} 1 + 6q + 6q^3 + 6q^4 + \cdots \\ 3\sqrt{2}q^{1/3}(1 + q + 2q^2 + 2q^4 + \cdots) \end{pmatrix}$

VVMF's for $2T$

r	k_0	Analytic Form	Fourier Series $q = e^{2\pi i\tau}$
2	5	$\begin{pmatrix} \eta^{10}\left(\frac{K}{1728}\right)^{\frac{5}{12}} {}_2F_1\left(\frac{5}{12},\frac{3}{4};\frac{5}{3}\right) \\ -\frac{1}{27\sqrt{2}}\eta^{10}\left(\frac{K}{1728}\right)^{-\frac{1}{4}} {}_2F_1\left(-\frac{1}{4},\frac{1}{12};\frac{1}{3}\right) \end{pmatrix}$	$\begin{pmatrix} q^{5/6}(1+4q-13q^2-4q^3-17q^4+\cdots) \\ -\frac{1}{27\sqrt{2}}q^{1/6}(1+68q-16q^2-208q^3-833q^4+\cdots) \end{pmatrix}$
2′	3	$\begin{pmatrix} \eta^{6}\left(\frac{K}{1728}\right)^{-\frac{1}{12}} {}_2F_1\left(-\frac{1}{12},\frac{1}{4};\frac{2}{3}\right) \\ 3\sqrt{2}\eta^{6}\left(\frac{K}{1728}\right)^{\frac{1}{4}} {}_2F_1\left(\frac{1}{4},\frac{7}{12};\frac{4}{3}\right) \end{pmatrix}$	$\begin{pmatrix} q^{1/6}(1+2q-22q^2+26q^3+25q^4+\cdots) \\ 3\sqrt{2}q^{1/2}(1-3q+2q^3+9q^4\cdots) \end{pmatrix}$
2″	7	$\begin{pmatrix} \eta^{14}\left(\frac{K}{1728}\right)^{-\frac{1}{12}} {}_2F_1\left(-\frac{1}{12},\frac{1}{4};\frac{2}{3}\right) \\ 3\sqrt{2}\eta^{14}\left(\frac{K}{1728}\right)^{\frac{1}{4}} {}_2F_1\left(\frac{1}{4},\frac{7}{12};\frac{4}{3}\right) \end{pmatrix}$	$\begin{pmatrix} q^{1/2}(1-6q-18q^2+242q^3-693q^4+\cdots) \\ 3\sqrt{2}q^{5/6}(1-11q+44q^2-58q^3-77q^4+\cdots) \end{pmatrix}$

$\Gamma(6)$ Modular Forms

We know the (scalar) modular forms of $\Gamma(6)$

$$\begin{aligned} e_1(\tau) &:= \frac{\eta^3(3\tau)}{\eta(\tau)}, \quad e_2(\tau) := \frac{\eta^3(\tau/3)}{\eta(\tau)}, \quad e_3(\tau) := \frac{\eta^3(6\tau)}{\eta(2\tau)}, \\ e_4(\tau) &:= \frac{\eta^3(\tau/6)}{\eta(\tau/2)}, \quad e_5(\tau) := \frac{\eta^3(2\tau/3)}{\eta(2\tau)}, \quad e_6(\tau) := \frac{\eta^3(3\tau/2)}{\eta(\tau/2)}. \end{aligned} \tag{12}$$

Different moduli space for these functions.

VVMF's for $\Gamma(2)/\Gamma(6)$

Scalar Modular Forms of $\Gamma(6)$ arrange as

$$
\begin{aligned}
Y^{(1)}_{\mathbf{2}''}(\tau) &:= \begin{pmatrix} Y_1 \\ Y_2 \end{pmatrix} = \begin{pmatrix} 3e_1(\tau) + e_2(\tau) \\ 3\sqrt{2}e_1(\tau) \end{pmatrix}, \\
Y^{(1)}_{\mathbf{2}'I}(\tau) &:= \begin{pmatrix} Y_3 \\ Y_4 \end{pmatrix} = \begin{pmatrix} 3\sqrt{2}e_3(\tau) \\ -3e_3(\tau) - e_5(\tau) \end{pmatrix}, \\
Y^{(1)}_{\mathbf{2}'II}(\tau) &:= \begin{pmatrix} Y_5 \\ Y_6 \end{pmatrix} = \begin{pmatrix} \sqrt{6}\,(e_3(\tau) - e_6(\tau)) \\ -\sqrt{3}e_3(\tau) + \frac{1}{\sqrt{3}}e_4(\tau) - \frac{1}{\sqrt{3}}e_5(\tau) + \sqrt{3}e_6(\tau) \end{pmatrix}.
\end{aligned}
\tag{13}
$$

Different weight distribution.

Why would models be different?

Duality of spaces of VVMF's. VVMF's of Γ_3' y $2T$ are related in the following way

$$\left(Y^{(3)}_{\mathbf{2}'(2T)}\, Y^{(3)}_{\mathbf{2}'(2T)}\right)^{(6)}_{\mathbf{3}(2T)} \;=\; -\frac{\sqrt{2}}{108}\left(18 D_2^2\, Y^{(2)}_{\mathbf{3}} - E_4\, Y^{(2)}_{\mathbf{3}}\right). \tag{14}$$

$$\left(Y^{(1)}_{\mathbf{2}''(\Gamma_3')}\, Y^{(1)}_{\mathbf{2}''(\Gamma_3')}\right)^{(2)}_{\mathbf{3}(\Gamma_3')} \;=\; -\sqrt{2}\, Y^{(2)}_{\mathbf{3}}. \tag{15}$$

Modular forms or $2T$ are related to those of Γ_3'.

Example models

We choose representations and weights

$$\begin{aligned} &\rho_{u^c} = \mathbf{2}'' \oplus \mathbf{1}, \quad && k_{u^c_{1,2,3}} = 3, 3, 2, \quad && \rho_{d^c} = \mathbf{3}, \quad && k_{d^c} = 2\,, \\ &\rho_Q = \mathbf{3}, && k_Q = 1, && \rho_{H_{u/d}} = \mathbf{1}, && k_{H_{u/d}} = 0\,. \end{aligned} \tag{16}$$

We builld example models with each of the finite groups mentioned here.

Γ'_3 Superpotential

$$
\begin{aligned}
\mathcal{W} \supset \; & \left[\alpha_1 Y^{(3)}_{\mathbf{2'}} (u^c_D Q)_{\mathbf{2''}} + \alpha_2 Y^{(3)}_{\mathbf{2''}} (u^c_D Q)_{\mathbf{2'}} + \alpha_3 Y^{(2)}_{\mathbf{3}} (u^c_3 Q)_{\mathbf{3}}\right]_{\mathbf{1}} H_u \\
& + \left[\beta_1 Y^{(2)}_{\mathbf{3}} \, (d^c Q)_{\mathbf{3}_\mathrm{S}} + \beta_2 Y^{(2)}_{\mathbf{3}} \, (d^c Q)_{\mathbf{3}_\mathrm{A}}\right]_{\mathbf{1}} H_d \, .
\end{aligned}
\tag{17}
$$

This model does not agree with experiments.

$2T$ Superpotential

$$\mathcal{W} \supset \left[\alpha_1 Y^{(3)}_{\mathbf{2}'}(u^c_D Q)_{\mathbf{2}''} + \alpha_2 Y^{(2)}_{\mathbf{3}}(u^c_3 Q)_{\mathbf{3}}\right]_{\mathbf{1}} H_u + \left[\beta_1 Y^{(2)}_{\mathbf{3}}\,(d^c Q)_{\mathbf{3}_\mathrm{S}} + \beta_2 Y^{(2)}_{\mathbf{3}}\,(d^c Q)_{\mathbf{3}_\mathrm{A}}\right]_{\mathbf{1}} H_d\,. \quad (18)$$

This model does not agree with experiment.

Mass Matrices

$$M_{u(\Gamma_3')} = \begin{pmatrix} -\alpha_1 Y^{(3)}_{\mathbf{2'},2} - \alpha_2\sqrt{2}Y^{(3)}_{\mathbf{2''},1} & \alpha_1\sqrt{2}Y^{(3)}_{\mathbf{2'},1} & \alpha_2 Y^{(3)}_{\mathbf{2''},2} \\ -\alpha_1 Y^{(3)}_{\mathbf{2'},1} & \alpha_2\sqrt{2}Y^{(3)}_{\mathbf{2''},2} & \alpha_2 Y^{(3)}_{\mathbf{2''},1} - \alpha_1\sqrt{2}Y^{(3)}_{\mathbf{2'},2} \\ \alpha_3 Y^{(2)}_{\mathbf{3},1} & \alpha_3 Y^{(2)}_{\mathbf{3},3} & \alpha_3 Y^{(2)}_{\mathbf{3},2} \end{pmatrix} v_u, \tag{19}$$

$$M_{u(2T)} = \begin{pmatrix} -\alpha_1 Y^{(3)}_{\mathbf{2'},2} & \alpha_1\sqrt{2}Y^{(3)}_{\mathbf{2'},1} & 0 \\ -\alpha_1 Y^{(3)}_{\mathbf{2'},1} & 0 & -\alpha_1\sqrt{2}Y^{(3)}_{\mathbf{2'},2} \\ \alpha_2 Y^{(2)}_{\mathbf{3},1} & \alpha_2 Y^{(2)}_{\mathbf{3},3} & \alpha_2 Y^{(2)}_{\mathbf{3},2} \end{pmatrix} v_u, \tag{20}$$

$$M_{d(2T,\Gamma_3')} = \begin{pmatrix} 2\beta_1 Y^{(2)}_{\mathbf{3},1} & \beta_2 Y^{(2)}_{\mathbf{3},3} - \beta_1 Y^{(2)}_{\mathbf{3},3} & -\beta_1 Y^{(2)}_{\mathbf{3},2} - \beta_2 Y^{(2)}_{\mathbf{3},2} \\ -\beta_1 Y^{(2)}_{\mathbf{3},3} - \beta_2 Y^{(2)}_{\mathbf{3},3} & 2\beta_1 Y^{(2)}_{\mathbf{3},2} & \beta_2 Y^{(2)}_{\mathbf{3},1} - \beta_1 Y^{(2)}_{\mathbf{3},1} \\ \beta_2 Y^{(2)}_{\mathbf{3},2} - \beta_1 Y^{(2)}_{\mathbf{3},2} & -\beta_1 Y^{(2)}_{\mathbf{3},1} - \beta_2 Y^{(2)}_{\mathbf{3},1} & 2\beta_1 Y^{(2)}_{\mathbf{3},3} \end{pmatrix} \tag{21}$$

Determinants distinguish the models

Here η is the Dedekind eta function

$$\det(M_{u(\Gamma_3')}) = \frac{4}{81}\sqrt{2}(\alpha_1 - 108\alpha_2)(\alpha_1 + 54\alpha_2)\alpha_3\eta^{16}, \tag{22}$$

$$\det(M_{u(2T)}) = -12\sqrt{2}\alpha_1^2\alpha_2\eta^{16}, \tag{23}$$

$$\det(M_{d(2T,\Gamma_3')}) = -2\beta_1\left(\beta_1^2 - \beta_2^2\right)E_6. \tag{24}$$

Γ(2)/Γ(6) Superpotential

$$\mathcal{W} \supset \left[\mathcal{Y}^{(3)}_{\mathbf{2'}}(u^c_D Q)_{\mathbf{2''}} + \mathcal{Y}^{(3)}_{\mathbf{2''}}(u^c_D Q)_{\mathbf{2'}} + \mathcal{Y}^{(2)}_{\mathbf{3}}(u^c_3 Q)_{\mathbf{3}}\right]_{\mathbf{1}} H_u$$
$$+ \left[\mathcal{Y}^{(2)}_{\mathbf{3}}\,(d^c Q)_{\mathbf{3}_\mathrm{S}} + \mathcal{X}^{(2)}_{\mathbf{3}}\,(d^c Q)_{\mathbf{3}_\mathrm{A}} + \mathcal{Y}^{(2)}_{\mathbf{1'}}(d^c Q)_{\mathbf{1''}} + \mathcal{Y}^{(2)}_{\mathbf{1}}(d^c Q)_{\mathbf{1}}\right]_{\mathbf{1}} H_d\,. \tag{25}$$

This model agrees with experiment but has more free parameters than the standard model.

Mass Matrices

$$M_u = \begin{pmatrix} -\sqrt{2}\mathcal{Y}^{(3)}_{\mathbf{2''},1} - \mathcal{Y}^{(3)}_{\mathbf{2'},2} & \sqrt{2}\mathcal{Y}^{(3)}_{\mathbf{2'},1} + \mathcal{Y}^{(3)}_{\mathbf{2},2} & -\sqrt{2}\mathcal{Y}^{(3)}_{\mathbf{2},1} + \mathcal{Y}^{(3)}_{\mathbf{2''},2} \\ -\mathcal{Y}^{(3)}_{\mathbf{2'},1} + \sqrt{2}\mathcal{Y}^{(3)}_{\mathbf{2},2} & \mathcal{Y}^{(3)}_{\mathbf{2''},2} + \mathcal{Y}^{(3)}_{\mathbf{2},1} & -\sqrt{2}\mathcal{Y}^{(3)}_{\mathbf{2'},2} + \mathcal{Y}^{(3)}_{\mathbf{2''},1} \\ \sqrt{2}\mathcal{Y}^{(2)}_{\mathbf{3},1} & \sqrt{2}\mathcal{Y}^{(2)}_{\mathbf{3},3} & \sqrt{2}\mathcal{Y}^{(2)}_{\mathbf{3},2} \end{pmatrix} v_u,$$

$$M_d = \begin{pmatrix} \sqrt{2}\mathcal{Y}^{(2)}_{\mathbf{1},1} + 2\mathcal{Y}^{(2)}_{\mathbf{3},1} & \sqrt{2}\mathcal{Y}^{(2)}_{\mathbf{1''},1} - \mathcal{Y}^{(2)}_{\mathbf{3},3} + \mathcal{X}^{(2)}_{\mathbf{3},3} & -\mathcal{Y}^{(2)}_{\mathbf{3},2} - \mathcal{X}^{(2)}_{\mathbf{3},2} \\ \sqrt{2}\mathcal{Y}^{(2)}_{\mathbf{1''},1} - \mathcal{Y}^{(2)}_{\mathbf{3},3} - \mathcal{X}^{(2)}_{\mathbf{3},3} & 2\mathcal{Y}^{(2)}_{\mathbf{3},2} & \sqrt{2}\mathcal{Y}^{(2)}_{\mathbf{1},1} - \mathcal{Y}^{(2)}_{\mathbf{3},1} + \mathcal{X}^{(2)}_{\mathbf{3},1} \\ -\mathcal{Y}^{(2)}_{\mathbf{3},2} + \mathcal{X}^{(2)}_{\mathbf{3},2} & \sqrt{2}\mathcal{Y}^{(2)}_{\mathbf{1},1} - \mathcal{X}^{(2)}_{\mathbf{3},1} - \mathcal{Y}^{(2)}_{\mathbf{3},1} & \sqrt{2}\mathcal{Y}^{(2)}_{\mathbf{1''},1} + 2\mathcal{Y}^{(2)}_{\mathbf{3},3} \end{pmatrix} \quad (26)$$

Conclusion

We demonstrated the inequivalence of the infinite manifestations of finite modular groups.

Many theoretical problems yet to be solved.

Still a way to go for realistic phenomenology.