The role of Goldstone modes to determine the sound velocity in isospin imbalance strongly interacting matter

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Motivation: Understanding the Sound Velocity Peak in Isospin-Imbalanced Matter

- **LQCD** has become a powerful tool, providing high-precision results for strongly interacting matter with isospin imbalance, especially at zero baryonic density where the sign problem is absent.
- LQCD reports a peak in the **speed of sound** in isospin-imbalanced matter, hinting at critical physical behaviors and potential phase transitions, making it an intriguing phenomenon in QCD.
- A correct understanding of this peak could impact multiple fields in physics, from nuclear matter in **neutron stars** to early-universe conditions.
- To fully interpret LQCD results, especially for phenomena like the sound velocity peak, we need a complementary **analytical theory**.

Sound Velocity Peaks in Isospin-Imbalanced Matter

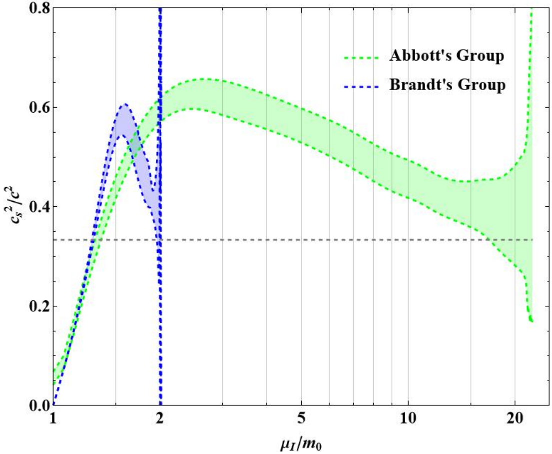

Figura: The squared speed of sound, c_s^2/c^2, as a function of the isospin chemical potential μ_I/m_0. Abbott's Group (green) shows a peak around 2.5 times the pion mass followed by a smooth decline, while Brandt's Group (blue) exhibits a sharper peak at a lower value of 1.6 times the pion mass.

Approach and the Role of the Sigma Model

- **Numerical Foundation:** LQCD provides reliable data for QCD at specific regimes, but is computationally **limited**, especially in high-density environments.

- **Analytical Insight with the Sigma Model:** The linear sigma model with quarks (LSMq) approximates QCD physics within about **20 % accuracy** for two flavors ($N_f = 2$), while offering a simplicity that allows for deeper exploration of the **underlying physics**.

- **Qualitative Utility:** Despite its approximations, the LSMq captures **essential features** of QCD, providing valuable qualitative insights, especially for studying phase transitions and sound velocity in isospin-imbalanced matter.

Goal: To develop a theoretical framework that aligns with LQCD data and reveals the physics behind the sound velocity peak, offering a clearer understanding of its implications across various areas in physics.

Theoretical Framework

The model is based on the **linear sigma model with two-flavor quarks**, including mesonic fields σ and $\vec{\pi}$, as well as fermionic fields ψ. The Lagrangian density is given by:

$$\mathcal{L} = \frac{1}{2}\left[(\partial_\mu\sigma)^2 + (\partial_\mu\vec{\pi})^2\right] - \frac{\mu^2}{2}(\sigma^2 + \vec{\pi}) - \frac{\lambda}{4}(\sigma^2 + \vec{\pi}^2)^2 + i\bar{\psi}\gamma^\mu\partial_\mu\psi - ig\bar{\psi}\gamma^5\vec{\tau}\cdot\vec{\pi}\psi - g\bar{\psi}\psi\sigma \quad (1)$$

- The kinetic terms of σ and $\vec{\pi}$ describe the mesonic degrees of freedom.
- The potential $-\frac{\mu^2}{2}(\sigma^2 + \vec{\pi}^2) - \frac{\lambda}{4}(\sigma^2 + \vec{\pi}^2)^2$ leads to spontaneous symmetry breaking $\sigma \to \sigma + v$.
- The interaction terms $-ig\bar{\psi}\gamma^5\vec{\tau}\cdot\vec{\pi}\psi$ and $-g\bar{\psi}\psi\sigma$ represent the interactions between mesons and fermions.

In this framework, the **isospin chemical potential** will be introduced to study the system's behavior under isospin imbalance, leading to **pion condensation**.

Isospin Chemical Potential

To introduce the isospin chemical potential, we use the covariant derivative for the charged pions as:

$$\partial_\mu \pi_\pm \to \partial_\mu \pi_\pm \pm i\mu_I \delta_\mu^0 \pi_\pm \tag{2}$$

This modification affects the energies of the charged pion states. For fermions, a similar procedure is applied:

$$\partial_\mu \psi \to \left(\partial_\mu - i\tau_3 \frac{\mu_I}{2} \gamma^0 \delta_\mu^0\right) \psi \tag{3}$$

When the chemical potential exceeds the pion mass in vacuum, π^+ becomes a Goldstone boson, allowing the charged pion condensation $\langle \pi_\pm \rangle \neq 0$:

$$\pi_\pm \to \pi_\pm + \frac{\Delta}{\sqrt{2}} e^{\pm i\theta} \tag{4}$$

where θ is a free parameter associated with the remaining symmetry of the broken $U(1)_{I_3}$ group.

Key Elements from the Lagrangian

From the resulting Lagrangian, we can extract three essential components that are crucial for analyzing the system:

- **Tree-level Potential**: Determines the ground state configuration and drives spontaneous symmetry breaking.

$$V_{tree} = -\frac{a^2}{2}\left(v^2 + \Delta^2\right) + \frac{\lambda}{4}\left(v^2 + \Delta^2\right)^2 - \frac{1}{2}\mu_I^2\Delta^2 - hv. \quad (5)$$

- **Fermionic Propagator** S^{-1}: Encodes the dynamics of fermions under the influence of the pion condensate and isospin chemical potential.

- **Mesonic Propagator** D^{-1}: Describes the behavior of mesons, particularly the modified dispersion relations due to the isospin chemical potential.

These components provide the groundwork for calculating interactions, scattering amplitudes, and effective masses within the system.

Inverse Propagator of Mesons

Taking for simplicity $\theta = 0$, the inverse propagator of the mesons is given by:

$$\begin{pmatrix} K^2 - m_\sigma^2 & -\sqrt{2}\lambda v\Delta & -\sqrt{2}\lambda v\Delta & 0 \\ -\sqrt{2}\lambda v\Delta & K^2 - m_{ch}^2 + \mu_I^2 + 2\mu_I k_0 & -\lambda\Delta^2 & 0 \\ -\sqrt{2}\lambda v\Delta & -\lambda\Delta^2 & K^2 - m_{ch}^2 + \mu_I^2 - 2\mu_I k_0 & 0 \\ 0 & 0 & 0 & K^2 - m_{\pi_0}^2 \end{pmatrix} \tag{6}$$

where we have used the shorthand notation

$$m_\sigma^2 = \lambda(3v^2 + \Delta^2) - a^2 \tag{7}$$

$$m_{\pi_0}^2 = \lambda(v^2 + \Delta^2) - a^2 \tag{8}$$

$$m_{ch}^2 = \lambda(v^2 + 2\Delta^2) - a^2. \tag{9}$$

This expression captures the mixing of the mesons, which is a key factor in our analysis. The off-diagonal elements illustrate how the charged mesons couple with the sigma meson, influencing the dynamics of the system.

One-Loop Corrections, Fermions

The contribution to the one-loop effective potential from fermions is given by:

$$V_f^1 = i \int \frac{d^4 k}{(2\pi)^4} \ln\left(\det\left\{S_f^{-1}\right\}\right), \tag{10}$$

where S_f^{-1} is also a non-diagonal 2×2 matrix in flavor space. It can be shown that at $T = 0$:

$$V_f^1 = -2N_c \int \frac{d^3 k}{(2\pi)^3} \left[E_\Delta^u + E_\Delta^d\right], \tag{11}$$

where $N_c = 3$ is the number of colors, and:

$$E_\Delta^{\substack{d\\u}} = \left\{\left(\sqrt{k^2 + m_f^2} \mp \mu_I/2\right)^2 + g^2\Delta^2\right\}^{1/2}. \tag{12}$$

Here, we have chosen $\mu_d = -\mu_I/2$ and $\mu_u = \mu_I/2$.

The contribution to the one-loop effective potential from bosons is expressed as:

$$V_b^1 = -\frac{i}{2}\int \frac{d^4 k}{(2\pi)^4} \ln\left(\det\left\{D_b^{-1}\right\}\right), \tag{13}$$

where D_b^{-1} is the non-diagonal boson inverse propagator.
Note that the structure of the boson inverse propagator makes it evident that the only boson that does not mix with the rest is the neutral pion. The charged pions and the sigma do mix. In this sense, only $m_{\pi_0}^2$, corresponds to the square of the neutral pion mass, while m_σ^2 and m_{ch}^2 are not the squares of the sigma and charged pion masses, but rather useful combinations of the parameters that appear in the analysis.

Goldstone Modes and Symmetry Breaking

The breaking of the $U(1)_I$ global symmetry is accompanied by the emergence of a Goldstone boson. To determine the restrictions imposed on v and Δ by the development of a massless mode, we examine the limit where the components of $K^\mu \to 0$ and set the determinant of the inverse boson propagator to zero:

$$\det_{K^\mu \to 0} D_b^{-1} = m_{\pi^0}^2 m_\sigma^2 \left(m_{\pi^0}^2 - \mu_I^2\right) \left(m_{\pi^0}^2 + 2\Delta^2 \lambda \frac{m_{\pi^0}^2}{m_\sigma^2} - \mu_I^2\right) = 0, \tag{14}$$

which leads to the conditions:

$$\left(m_{\pi^0} - \mu_I\right) \left(\sqrt{m_{\pi^0}^2 + 2\Delta^2 \lambda \frac{m_{\pi^0}^2}{m_\sigma^2}} - \mu_I\right) = 0. \tag{15}$$

From these equations, we find three possible solutions, of which only two are real across the $\mu_I \geq m_\pi$ domain, given by:

$$\begin{aligned}
\Delta_1 &= \sqrt{\frac{\mu_I^2 - 2(3\lambda v^2 - 2a^2) + \sqrt{4a^4 + 4\mu_I^2(6\lambda v^2 - a^2) + \mu_I^4}}{6\lambda}}, && (16)\\
\Delta_2 &= \sqrt{\frac{\mu_I^2 - (\lambda v^2 - a^2)}{\lambda}}. && (17)
\end{aligned}$$

Notice that Δ_2 corresponds to the condition for the existence of a massless mode at tree-level, while Δ_1 arises from loop corrections, indicating a non-trivial relation between v, μ_I, and Δ.

We claim that any physical state must be described by one of these two solutions; any other solution for Δ would violate the existence of a Goldstone boson.

Relationship Between Couplings

From the renormalization procedure we found a crucial relationship between the couplings in the model. Due the Ward-Takahashi identity ensures that:

$$4g^4 N_c = 3\lambda^2, \tag{18}$$

which connects the boson-fermion coupling g and the boson self-coupling λ.

Additionally, at tree level, these couplings can be expressed in terms of physical masses:

$$\lambda = \frac{m_\sigma^2 - m_\pi^2}{2f_\pi^2}, \qquad g = \frac{m_f}{f_\pi}. \tag{19}$$

For $N_c = 3$, this leads to the mass relation:

$$m_\sigma^2 = 4m_f^2 + m_\pi^2. \tag{20}$$

This relation is vital for ensuring the consistency of the model and connects low-energy effective theory with underlying microscopic parameters.

Squared Speed of Sound vs. Isospin Chemical Potential

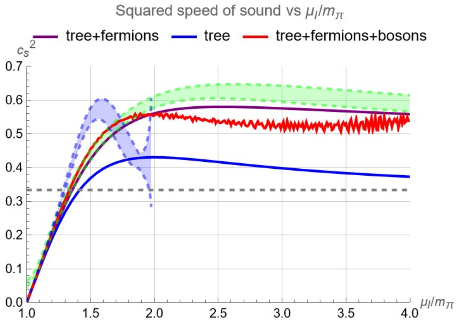

Figura: At the tree level, a peak is already evident, indicating that the theory predicts a non-trivial behavior even without further corrections. When only fermions are included, the peak becomes very smooth and barely noticeable. However, upon incorporating mesons into the model, the peak re-emerges prominently. This observation suggests that the origin of the peak is fundamentally linked to the intrinsic dynamics of the mesons. It is only through their proper and self-consistent treatment that we can reveal their nature and the underlying physics contributing to this behavior.

Robustness of the Model

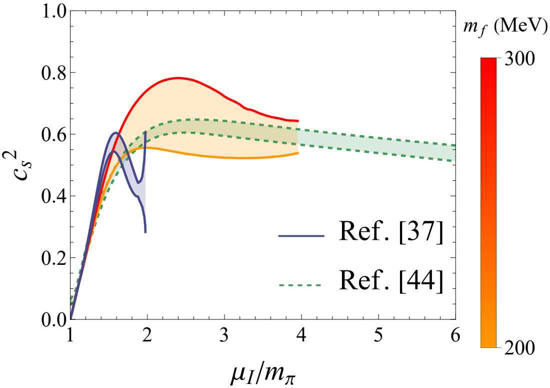

Figura: The system after have used the physical restrictions previuosly shown becomes a only two free parameters model the pion, and the quark mass. Fixing the pion mass with Variations in m_f do not lead to the disappearance of the peak. The peak persists across a wide range of physcical quark masses, suggesting stability in the behavior of the system. This robustness further reinforces the importance of mesonic dynamics in determining the properties of the system.

Conclusions

- The analysis within the linear sigma model demonstrates that the sound velocity peak in isospin-imbalanced matter is closely linked to the emergence of Goldstone modes, which arise from spontaneous symmetry breaking.
- Our theoretical framework aligns well with LQCD results, providing a comprehensive understanding of the peak's behavior and its implications for strongly interacting matter.
- The interplay between the isospin chemical potential and the pion condensation mechanism plays a crucial role in shaping the dynamics of the system, impacting various physical observables.
- Future work will focus on refining our calculations, increasing the number of free parameters including the ρ mesons, and extending it to finite temperature in order to explore more interesting phenomena such as BEC-BCS transition, chiral restoration, the triple point, and the behavior of the peak as a function of T and μ_I.

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Questions

Thank you for your attention!

Any questions?