# Dispersion relations in conjunction with the renormalization group

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work in collaboration with
Sergey Mikhailov and Alexander Pimikov (BLTP, JINR)
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UNIVERSIDAD DE TARAPACÁ
Universidad del Estado

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- Conclusions

# Infrared safe effective couplings

- **Different effective couplings in Euclidean, $\mathcal{A}_n$, and Minkowskian, $\mathfrak{A}_n$, regions**

$$\overline{\alpha_s}^n \to \{\mathcal{A}_n, \mathfrak{A}_n\} \tag{1}$$

- **Based on**

$$\textit{RG} \quad + \quad \textit{Causality} \tag{2}$$

RG → UV asymptotics

Causality → Dispersive represent.

- Euclidean: $-q^2 = Q^2,\ L = \ln(Q^2/\Lambda^2),\ a_s^n[L] \to \{\mathcal{A}_n[L]\}_{n\in\mathbb{N}}$
- Minkowskian: $q^2 = s,\ L_s = \ln(s/\Lambda^2),\ a_s^n[L] \to \{\mathfrak{A}_n[L_s]\}_{n\in\mathbb{N}}$
- $\mathcal{A}_n^{(l)} = \hat{D}[\mathfrak{A}_n^{(l)}] \equiv Q^2 \int_0^\infty \frac{\mathfrak{A}_n^{(l)}}{(\sigma+Q^2)^2} d\sigma$

  $\mathfrak{A}_n^{(l)} = \hat{R}[\mathcal{A}_n^{(l)}] \equiv \int_{-s-i\varepsilon}^{-s+i\varepsilon} \frac{\mathcal{A}_n^{(l)}}{\sigma} d\sigma$
- On the set of the pars $\{\mathcal{A}_n, \mathfrak{A}_n\} : \hat{D}\hat{R} = \hat{R}\hat{D} = 1$

# Infrared safe effective couplings

New non-power perturbation theory [ MS-scheme ] – Analytic PT

- **PT** $\sum_m d_m a_s^m(Q^2) \Rightarrow \sum_m d_m \mathcal{A}_m(Q^2)$ **APT**
- **APT**: $\mathcal{A}_n \cdot \mathcal{A}_m \neq \mathcal{A}_{n+m}$ : No algebra
- **FAPT: concept generalization for $\forall\nu$ - real**

$$\overline{\alpha_s}^{\nu} \to \{\mathcal{A}_\nu, \mathfrak{A}_\nu\} \quad (3)$$

New functions $f(a_s)$ : $(a_s)^\nu, (a_s)^\nu \ln(a_s), (a_s)^\nu L^m, e^{-a_s}, \ldots$

FAPT: Karanikas A.& Stefanis N., PLB504 (2001), 225; 636 (2006) 330; Bakulev A. & M.S. & Stefanis N.,PRD72 (2005) 074014; PRD75 (2007) 056005; JHEP06 (2010) 085
G. Cvetic & A. Kotikov, J.Phys.G39 (2012) 065005

# Dispersive "Källen–Lehmann" representation

By **analytization** we mean "Källen–Lehmann" representation

$$\left[f\left(Q^2\right)\right]_{\rm an} = \int_0^\infty \frac{\rho_f(\sigma)}{\sigma + Q^2 - i\epsilon} d\sigma, \quad \rho_1(\sigma) = \frac{\textrm{Im}}{\pi}\left[a_s^1(-\sigma)\right]\beta_0 \qquad (4)$$

For 1 loop run (here pole remover ):

$$\begin{aligned}
\rho_1(\sigma) &\overset{1l}{=} \frac{1}{L_\sigma^2 + \pi^2} \\
\mathcal{A}_1[L] &= \int_0^\infty \frac{\rho_1(\sigma)}{\sigma + Q^2} d\sigma \overset{1l}{=} \frac{1}{L} - \frac{1}{e^L - 1} \\
\mathfrak{A}_1\left[L_s\right] &= \int_s^\infty \frac{\rho_1(\sigma)}{\sigma} d\sigma \quad \overset{1l}{=} \frac{1}{\pi} \arccos \frac{L_s}{\sqrt{\pi^2 + L_s^2}}
\end{aligned} \qquad (5)$$

**Inequality:** $a_s^n[L] \geqslant (\mathcal{A}_n[L], \mathfrak{A}_n[L]) \xrightarrow{L\to\infty} a_s^n[L]$

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**Inequality:** $a_s^n[L] \geqslant (\mathcal{A}_n[L], \mathfrak{A}_n[L]) \xrightarrow{L\to\infty} a_s^n[L]$

# APT graphics: Distorting mirror

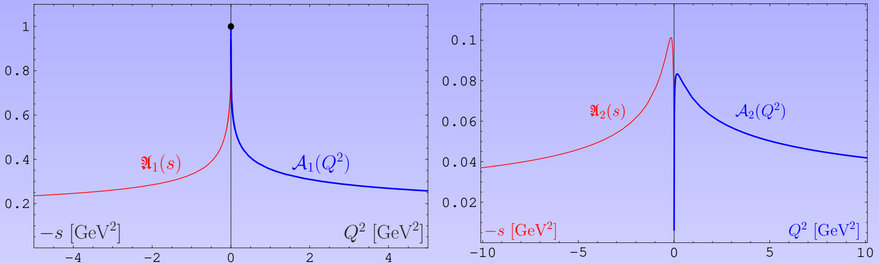


Figure: APT distorting mirror from [Shirkov&Solovtsov1997-2007]. Coupling images of $\mathfrak{A}_1(s) and \mathcal{A}_1(Q^2)$ in left pannel; $\mathfrak{A}_2(s) and \mathcal{A}_2(Q^2)$ in the right pannel

# FAPT(Eucl): Properties of $\mathcal{A}_\nu[L]$

First, Euclidean coupling ( pole remover ):

$$\mathcal{A}_\nu[L] = \frac{1}{L^\nu} - \frac{F\left(e^{-L}, 1-\nu\right)}{\Gamma(\nu)} \tag{6}$$

Here, $F(z,\nu)$ is reduced Lerch transcendental function. It is analytic function in $\nu$. Properties: The charge $\mathcal{A}_\nu$ is bounded for $\nu \geq 1$,

- $\mathcal{A}_0[L] = 1$;
- $\mathcal{A}_{-m}[L] = L^m$ for $m \in \mathbb{N}$;
- $\mathcal{A}_m[L] = (-1)^m \mathcal{A}_m[-L]$ for $m \geq 2\,,\ m \in \mathbb{N}$;
- $\mathcal{A}_\nu[\pm\infty] = 0$ for $\nu > 1\,$;

$\mathcal{A}_\nu[-\infty] = (\infty)^{1-\nu}$ for $\nu < 1\,$, *i.e.*, $\mathcal{A}_\nu(Q^2 \to 0)$ becomes Unbounded

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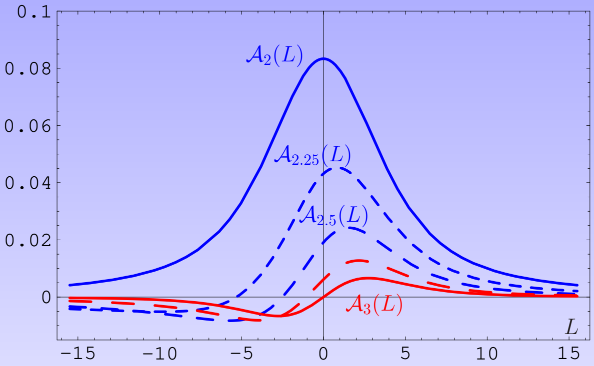
# FAPT(Eucl): $\mathcal{A}_\nu[L]$ versus $L$

Euclidean Coupling: $\mathcal{A}_\nu[L] = \frac{1}{L^\nu} - \frac{F\left(e^{-L}, 1-\nu\right)}{\Gamma(\nu)}$




$\bar{a}_s^{1+\nu}[L] \gg \mathcal{A}_{1+\nu}[L] \gg \mathcal{A}_{2+\nu}[L]$ at $L \sim 1$

To hold the correspondence with PT asymptotics we put "calibrated FAPT" condition: $\mathcal{A}_\nu(0) = \mathfrak{A}_\nu(0) = 0$ for $0 < \nu \leq 1$

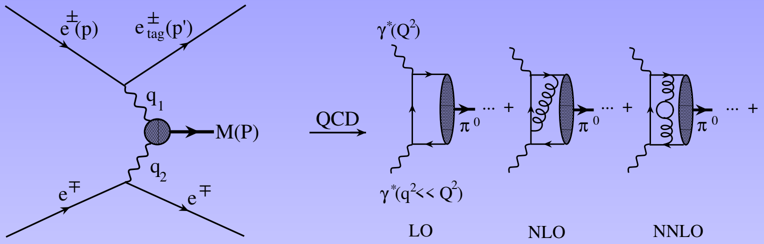
# Exclusive hard process

$$\gamma(q^2 \simeq 0)\gamma^*(Q^2) \to \pi^0$$

Light Cone Sum Rules within FAPT,
New prediction for the pion TFF

[Mikhailov S. Pimikov A. Stefanis N., PRD93 (2016) 114018; A.C. Mikhailov S. Stefanis N., 1806.07790]

# Pion Transition Form Factor (TFF)




Why it is interesting for QCD?

- The measurements of TFF experiments have the best accuracy (BESIII) among others exclusive hard reactions
- Significant theoretical advances in QCD here: high order NNLO$_\beta$ contribution $O(\alpha_s^2\beta_0)$ to the hard part; Distribution Amplitude (DA) of twist-2 for hadron part; contributions from twist-4 and corrections a'la twist-6

3 steps to TFF: Perturbative & twist expansion;
Dispersive representation; LCSR with duality interval $s_0$ for $q^2 \to 0$.

# Pion TFF in pQCD

Hard process at $-Q^2, -q^2 \gg m_\rho^2 \Rightarrow$ collinear factorization

$$F_{\text{FOPT}}^{(\text{tw}=2)}(Q^2, q^2) = N_{\text{T}} \left( T_{\text{LO}} + a_s T_{\text{NLO}} + a_s^2 T_{\text{NNLO}} + \dots \right) \otimes \varphi_\pi^{(2)}$$

where $N_T = \sqrt{2} f_\pi / 3$, and hard the hard coefficient functions given by

$$\begin{aligned}
T_{\text{LO}} &= a_s^0 \; T_0(x) \equiv 1/\left(q^2 \bar{y} + Q^2 y\right) \\
a_s T_{\text{NLO}} &= a_s^1 \; T_0(y) \otimes \left[\mathcal{T}^{(1)} + \underline{L \; V_0}\right](y, x)\,, \\
a_s^2 T_{\text{NNLO}} &= a_s^2 \; T_0(y) \otimes \left[\mathcal{T}^{(2)} - \underline{L \, \mathcal{T}^{(1)} \beta_0} + \underline{L \, \mathcal{T}^{(1)} \otimes V_0} - \underline{\frac{L^2}{2} \, \beta_0 V_0} \right. \\
&\qquad \left. + \underline{\frac{L^2}{2} \; V_0 \otimes V_0} + \underline{\underline{L \; V_1}}\right](y, x)\,,
\end{aligned}$$

$L = L(y) = \ln\left[(q^2 \bar{y} + Q^2 y)/\mu_F^2\right]$ Plain terms $\mathcal{T}^{(1)}, \mathcal{T}^{(2)}$ - corrections to parton subprocess; Underlined terms due to $\bar{a}_s(y)$ and ERBL, $V_0$, $V_1$ - kernels; underlined term - two loops ERBL.

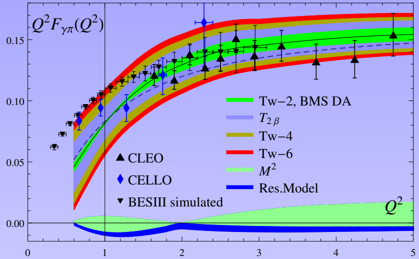
# Pion-photon TFF in LCSR vs exp. data

The evolution of the Distribution Amplitude (DA) is taken into account in terms of the conformal expansion

$$\varphi_{\pi}^{(2)}(z,\mu^2) = \psi_0(x) + \sum_{n=2,4,\ldots}^{\infty} b_n(\mu^2)\psi_n(x)\,, \qquad (7)$$

where $\varphi_{\pi}^{\text{asy}} = \psi_0(x) = 6x(1-x) \equiv 6x\bar{x}$ is the asymptotic pion DA.

# Pion-photon TFF in LCSR vs exp. data




| Total rad. correctios | -18 % at 3 GeV$^2$ |
|---|---|
| Source | Uncertainty (%) |
| Unknown NNLO term $\mathcal{T}_c^{(2)}$ | $\mp 5$ |
| Range of Tw-2 BMS DAs | $-3.4 \div 4.1$ |
| Tw-4 coupling $\delta^2 = [0.152 - 0.228]$ GeV$^2$ | $\pm 3.0$ |
| Tw-6 parameter variation | $-2.4 \div 3.0$ |
| Total | $-13.6 \div 14.9$ |

# Pion TFF in pQCD with RG improvement

Collecting all the "underlined" term of RG-evolution into
$a_s(\mu^2) \to \bar{a}_s(y) \equiv \bar{a}_s\left(q^2\bar{y} + Q^2 y\right)$ and ERBL–factor.

$$
\begin{aligned}
F^{(\mathrm{tw}=2)}(Q^2,q^2) \;&=\; N_{\mathrm{T}}\, T_0(y) \underset{y}{\otimes} \Big\{ \Big[1 + \bar{a}_s(y)\mathcal{T}^{(1)}(y,x) + \bar{a}_s^2(y)\mathcal{T}^{(2)}(y,x) + \ldots\Big] \\
&\quad \underset{x}{\otimes} \exp\left[-\int_{a_s}^{\bar{a}_s(y)} d\alpha\, \frac{V(\alpha;x,z)}{\beta(\alpha)}\right]\Big\} \underset{z}{\otimes} \varphi_\pi^{(2)}(z,\mu^2)\,, \\
&=\; F^{\mathrm{RG}}_{(l)0} + \sum_{n=2,4,\ldots}^{\infty} b_n(\mu^2) F^{\mathrm{RG}}_{(l)n}(Q^2,q^2)
\end{aligned}
$$

$$
F^{\mathrm{RG}}_{(1l)n}(Q^2,q^2) = N_{\mathrm{T}}\, T_0(y) \underset{y}{\otimes} \left\{ \left[1 + \bar{a}_s(y)\mathcal{T}^{(1)}(y,x)\right] \left(\frac{\bar{a}_s(y)}{a_s(\mu^2)}\right)^{\nu_n} \right\} \underset{x}{\otimes} \psi_n(x)
$$

where $\nu_n = \gamma_n/2\beta_0$

# Dispersive form of TFF leads to fractional APT

$$\left[F_{(1l)n}(Q^2,q^2)\right]_{\mathbf{an}} = \int_{m^2}^{\infty} \frac{\rho_F(Q^2,\sigma)}{\sigma+q^2-i\epsilon}\, d\sigma,\ \ \rho_F(\sigma) = \frac{\mathbf{Im}}{\pi}\left[F_{(1l)n}(Q^2,-\sigma)\right]$$

$$\nu(0)=0;\, F^{\text{FAPT}}_{(1l),0}(Q^2,q^2;m^2) = \ N_T\, T_0(Q^2,q^2;y) \underset{y}{\otimes} \left\{1+\mathbb{A}_1(m^2,y)\mathcal{T}^{(1)}(y,x)\right\} \underset{x}{\otimes} \psi_0(x)$$

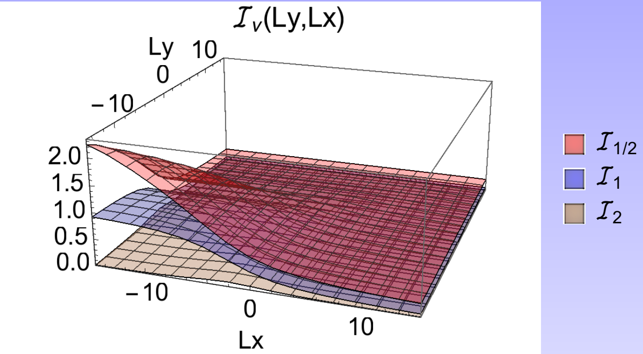
$$\nu(n)\neq 0;\, F^{\text{FAPT}}_{(1l),n}(Q^2,q^2;m^2) = \frac{N_T}{a_s^{\nu_n}(\mu^2)}\, T_0(Q^2,q^2;y) \underset{y}{\otimes} \left\{\mathbb{A}_{\nu_n}(m^2,y)1+\mathbb{A}_{1+\nu_n}(m^2,y)\mathcal{T}^{(1)}(y,x)\right\} \underset{x}{\otimes} \psi_n(x)$$

The same expression as for RG-case, $\mathbb{A}_\nu(m^2,y) \Leftrightarrow \bar{a}_s^{\nu}(y)$

$$\mathbb{A}_\nu(m^2,y) = \mathcal{I}_\nu(m^2,Q(y)) - \mathfrak{A}_\nu(m^2);\ \ \mathbb{A}_\nu(0,y) = \mathcal{A}_\nu(Q(y)) - \mathfrak{A}_\nu(0)$$

the certain kinematics enters by means of $Q(y) = q^2\bar{y} + Q^2 y$

# Generalization of FAPT coupling




$$\mathcal{I}_\nu(y,x) \stackrel{def}{=} \int_y^\infty \frac{d\sigma}{\sigma+x}\rho_\nu^{(l)}(\sigma)$$

$$\mathcal{A}_\nu(x) = \mathcal{I}_\nu(y \to 0, x),\ \mathfrak{A}_\nu(Y) = \mathcal{I}_\nu(y, x \to 0),\ \mathcal{A}_1(0) = \mathfrak{A}_1(0)$$

# Partial $\text{TFF}_{\text{LCSR}}$

Restricting for simplicity our attention to the $\text{NNLO}_\beta$ approximation of the partial form factors $F_n$ within FAPT, we derive in the limits $q^2 \to 0$, $Q(y) \to yQ^2$ the following expression

$$Q^2 F^{(\text{tw-2})}_{\text{FAPT};n}(Q^2) \approx \frac{N_{\text{T}}}{\underline{[a_s(\mu^2)]^{\nu_n}}\,[1+c_1 a_s(\mu^2)]^{\omega_n}} \Big\{ \underline{\frac{\mathbb{A}_{\nu_n}(m^2,x)}{x} + \left(\frac{\mathbb{A}_{1+\nu_n}(m^2,y)}{y}\right) \underset{y}{\otimes} \mathcal{T}^{(1)}(y,x)}$$
$$+\omega_n c_1 \left[\frac{\mathbb{A}_{1+\nu_n}(m^2,x)}{x} + \frac{\mathbb{A}_{2+\nu_n}(m^2,x)}{x}\frac{c_1(\omega_n-1)}{2} + \left(\frac{\mathbb{A}_{2+\nu_n}(m^2,y)}{y}\right) \underset{y}{\otimes} \mathcal{T}^{(1)}(y,x)\right]$$
$$+\underline{\underline{\left(\frac{\mathbb{A}_{2+\nu_n}(m^2,y)}{y}\right) \underset{y}{\otimes} \mathcal{T}^{(2)}(y,x)}}\Big\} \underset{x}{\otimes} \psi_n(x)\,,$$

$$\begin{aligned}\mathbb{A}_\nu(m^2,y) &= \theta(y \geqslant y_m)\left[\mathcal{A}_\nu(Q(y)) - \mathfrak{A}_\nu(0)\right] + \\ &\quad \theta(y < y_m)\left[\mathcal{I}_\nu(m(y),Q(y)) - \mathfrak{A}_\nu(m(y))\right]\end{aligned}$$

$c_1 = \beta_1/\beta_0$; $\omega_n = [\gamma_1(n)\beta_0 - \gamma_0(n)\beta_1]/[2\beta_0\beta_1]$.

# $\mathrm{TFF}_{\mathrm{LCSR}}$ in comparison with the experimental data

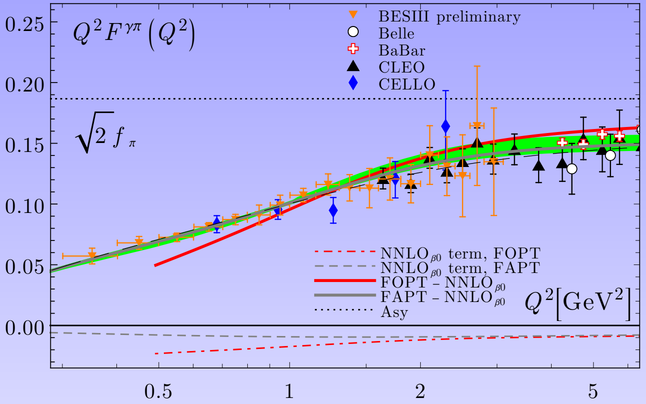



Figure: Theoretical predictions for the scaled $\gamma^*\gamma\pi^0$ transition form factor $Q^2 F^{\gamma\pi}_{\mathrm{FAPT}}(Q^2)$ [GeV] using different DAs. The displayed FAPT/FOPT TFF results employ the best-fit nonperturbative higher-twist parameters $\delta^2_{\text{tw-4}} = 0.19\,\mathrm{GeV}^2$ and $\delta^2_{\text{tw-6}} = 1.61 \times 10^{-4}\,\mathrm{GeV}^6$.

# Exclusive hard process

$$\gamma^*(q^2)\pi^+(p_1^2) \to \pi^+(p_2^2)$$

Light Cone Sum Rules within FAPT,
New prediction for the pion electromagnetic FF

[A.C. Mikhailov S Pimikov A., Work in progress]

# Status of the Pion electromagnetic FF

- We invoke the knowledge of transition FF because of their analogy, i.e., replacing $j_{\mu 5}$ by $j_{\mu}^{em}$ where the light cone dominance leaves to an expansion on nonlocal operators and then to pion distribution amplitudes (DAs).

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- For this reason and due to the same structural form as TFF, we collected together they constitute the self-consistent and completed result $F_\pi^{\text{RG,LO}}$,

$$F_\pi^{\text{RG,LO}}(Q^2) = F_\pi^{(2)\text{RG,LO}}(Q^2) + F_\pi^{(4)\text{RG}}(Q^2) + F_\pi^{(6)}(Q^2)\,. \qquad (8)$$

# Pion electromagnetic FF: FAPT

This $F_{\pi}^{\text{RG,LO}}$ is valuable and can be compared with the experimental data independently. Here

$$
\begin{aligned}
F_{\pi(0)}^{(2)\text{RG,LO}}(Q^2, M^2) &= \int_{u_0}^{1} \exp\left(-\frac{Q^2}{M^2}\frac{\bar{u}}{u}\right)\psi_0(u)du,\ (n=0); \\
F_{\pi(n)}^{(2)\text{RG,LO}}(Q^2, M^2) &= \frac{1}{a_s^{\nu_n}(\mu_0^2)}\left[\mathfrak{A}_{\nu_n}(s(u)) + \int_0^{s(u)} \rho_{\nu_n}(s)\omega_1(s)\,ds\right]\exp\left(-\frac{Q^2}{M^2}\frac{\bar{u}}{u}\right) \\
&\quad \otimes\left(\theta(u \geqslant u_0)\psi_n(u)\right),\ (n>0),
\end{aligned}
$$

The NLO will be incorporate within the FOPT formalism following [Bijnens J. Khodjamirian A. EPJC26,67(2002)] and we call this as “hybrid”.

# Pion electromagnetic FF: Higher Twists

In order to keep the same form of the OPE, we arrive at the following expression to the twist-4 term

$$F_\pi^{(4)\text{RG}}(Q^2, M^2) = \frac{\delta^2_{\text{tw-4}}(\mu_0^2)}{a_s^{\nu_{t4}}(\mu_0^2)} \cdot \left[\mathcal{I}_{\nu_{t4}}(s(u),0) + \frac{1}{M^2 u}\mathcal{I}'_{\nu_{t4}}(s(u),0) + \int_0^{s(u)} \rho_{\nu_{t4}}(s)\omega_2(s)ds\right] \exp\left(-\frac{Q^2}{M^2}\frac{\bar{u}}{u}\right) \otimes \left(\theta(u \geqslant u_0)\varphi_\pi^{(4)}(u)\right),$$

where the weight $\omega_2(s) = \frac{1}{s^2}\left[\exp\left(-\frac{s}{M^2 u}\right) - \left(1 - \frac{s}{M^2 u}\right)\right]$ contains two subtractions of the expansion of exponential.

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$$F_\pi^{(4)\mathrm{RG}}(Q^2, M^2) \;=\; \frac{\delta^2_{\text{tw-4}}(\mu_0^2)}{a_s^{\nu_{t4}}(\mu_0^2)} \cdot \Bigg[\mathcal{I}_{\nu_{t4}}(s(u), 0) + \frac{1}{M^2 u}\mathcal{I}'_{\nu_{t4}}(s(u), 0) + \int_0^{s(u)} \rho_{\nu_{t4}}(s)\omega_2(s)ds\Bigg] \exp\left(-\frac{Q^2}{M^2}\frac{\bar{u}}{u}\right) \otimes \left(\theta(u \geqslant u_0)\varphi_\pi^{(4)}(u)\right),$$

where the weight $\omega_2(s) = \frac{1}{s^2}\left[\exp\left(-\frac{s}{M^2 u}\right) - \left(1 - \frac{s}{M^2 u}\right)\right]$ contains two subtractions of the expansion of exponential. The factorizable twist-6 contribution is given by [PRD61,073004(2000)]

$$F_\pi^{(6)}(Q^2) = \frac{4\pi C_{\mathrm{F}}}{3 f_\pi^2\, Q^4}\; \delta^2_{\text{tw-6}}(\mu_0^2),$$

where $\delta^2_{\text{tw-6}}(\mu_0^2 = 1\,\mathrm{GeV}^2) = \langle\sqrt{\alpha_s}\bar{q}q\rangle^2 = \mathbf{(1.61 \pm 2 \cdot 0.26)} \times 10^{-4}\,\mathrm{GeV}^6$ – "the best fit" taken from [PRD103,096003(2021)] and
$\langle\sqrt{\alpha_s}\bar{q}q\rangle^2 = \left(\;1.84^{+0.84}_{-0.24}\;\right) \times 10^{-4}\,\mathrm{GeV}^6$ [PRD102,074022(2020)].

# Pion electromagnetic FF: Higher Twists

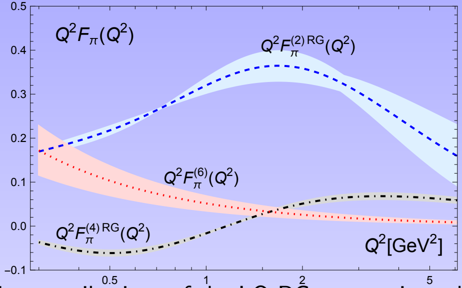


Figure: Different twist contributions of the LO RG summation: the tw-2 $F_\pi^{(2)RG,LO}$ – the upper widening, blue strip; tw-4 $F_\pi^{(4)RG}$ – thin, dashed-dotted grey line; numerically corrected tw-6 $F_\pi^{(6)}$ – red narrowing strip with dots.

# emFF$_{LCSR}$incomparisonwithexperimentaldata

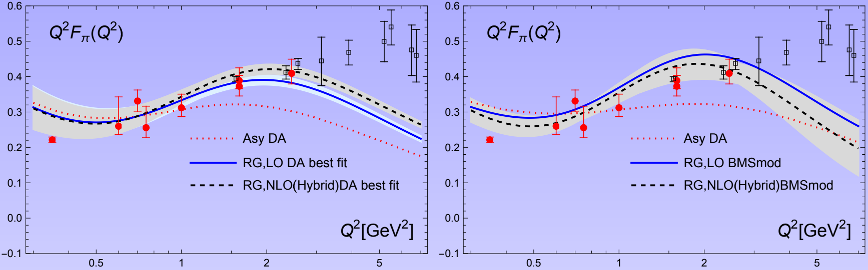


Figure: Our prediction for LO RG, $F_\pi^{\mathrm{RG,LO}}$ for DA$^{bf}$, BMSmod, and Asy DA – dotted red line in both panels, recent lattice results [PRL133(2024)18,181902] are shown as open boxes. **Left.** $F_\pi^{\mathrm{RG,LO}}$ at DA$^{bf}$ – solid black line and in addition to them the standard NLO LCSR [EPJC26,67(2002)] corrections – short dashed upper blue line. **Right.** The same legends for the curves for $F_\pi^{\mathrm{RG,LO}}$ and NLO LCSR corrections at BMSmod DA.

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THANKS!