Flowing behind the horizon

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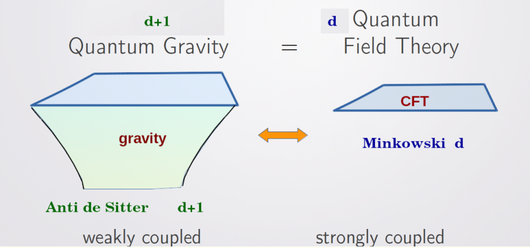
Holography ∼ AdS/CFT

- Since 1997 [Maldacena][Gubser,Klebanov, Polyakov, Witten :] useful tool to study some strongly coupled quantum field theories

- Equivalence of two very different theories



- Connection with other areas of physics
- More recently: Connection between quantum gravity and quantum information

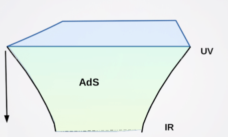
  Key ingredient: entagnlement entropy $S_A = \mathrm{min} A$

- But we **do not know** yet
  - How is gravity encoded in the boundary quantum degrees of freedom ?
  - How do we “reconstruct” spacetime -including behind a black hole horizon – just from boundary data?

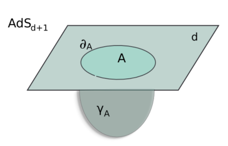
- We know is that this duality geometrizes field theoretical concepts

  Examples:

  Energy $\leftrightarrow$ radial, emergent direction



  Entanglement entropy = Minimal area in the bulk



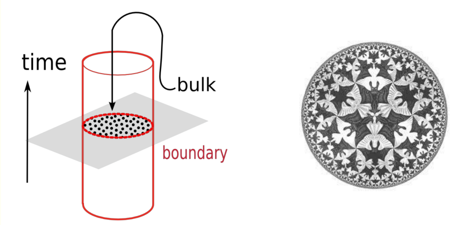
- Many holographic constructs explore behind the horizon
- But often, not all the way to the singularity
- Here: black hole interior as trans-IR RG flow?

1. AdS/CFT, main ideas
2. Holography behind the horizon
   - Entanglement and Complexity-Volume are not enough
3. Trans-IR flows
   - Monotonicity and the null energy condition
   - Toward the singularity, connection to BKL
4. Open questions

AdS/CFT

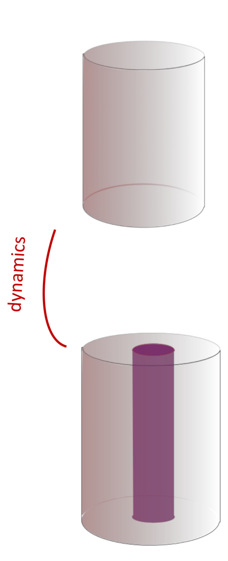
AdS: Constant negative curvature space

CFT "lives at the boundary"



Gravitational theory maps to non-gravitational one

- Strong/weak coupling duality
- Gauge theory lives in fewer dimensions, *holographic*



Asymptotically AdS geometry $\Leftrightarrow$ CFT state

AdS $\Leftrightarrow$ vacuum state

Black hole formation $\Leftrightarrow$ thermalization

Black hole in AdS $\Leftrightarrow$ thermal state

Radial direction $\Leftrightarrow$ energy in the boundary (quantum) theory

Solving bulk equations of motion $\Rightarrow$ enough to reconstruct the bulk outside the event horizon

But this method is not able to describe regions behind causal horizons

What holographic constructs explore **inside the black hole**? All the way to the **singularity**?

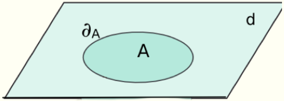
Holography behind the horizon

Holographic entanglement entropy

Entanglement in QM

Density matrix ρ describes the state of the whole system

Partition the system, construct the reduced density matrix ρ_A by tracing over degrees of freedom in B, $\rho_A = \mathrm{Tr}_B \rho$



Entanglement entropy

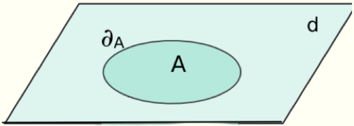
$$S_A = -\mathrm{Tr}\left[\rho_A \log \rho_A\right]$$

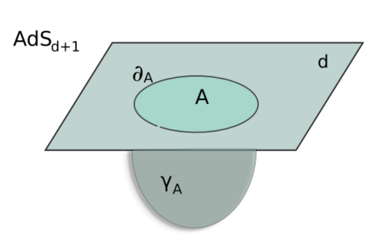
Entanglement in Conformal Field Theory

- Non-local object
- Useful in quantum many-body systems
- Plays crucial role in quantum information (cryptography, teleportation .....)
- But
  - difficult to measure
  - difficult to calculate, especially in strongly coupled field theories

AdS/CFT $\rightarrow$ entanglement entropy can be described geometrically

Holographic Entanglement Entropy





Entanglement in the boundary quantum, theory encodes the geometry of spacetime

$$\boxed{\mathsf{S}_A = \mathrm{Area}_{min}(\gamma_A)}$$

[Ryu,Takayanagi 2007]

Evidence

- Agrees with analytical 2D CFT results [Holzhey-Larsen-Wilczek][Calabrese, Cardy]
- Holographic proof of strong subadditivity,

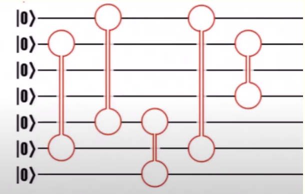
$$S(AB) + S(BC) \geq S(ABC) + S(B)$$

  [Headrick, Takayanagi] and many other properties known from QI
- Proof –assuming holography [Maldacena, Lewkowycz]
- Time evolution

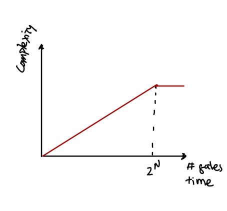
Holographic Complexity

- Other important boundary quantity: **circuit or gate complexity**

  Starting in a reference state $|\Psi_0\rangle$, what is the minimmum number of fundamental gates (unitaries that act on 2 qubits) needed to make a target state $|\Psi_f\rangle$ within accuracy ϵ.

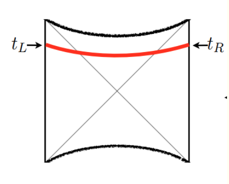


Defining property of complexity: linear growth



Complexity = Volume (CV): maximal codimension-1 bulk slice [Susskind 2014] [Susskind, Stanford 2014]

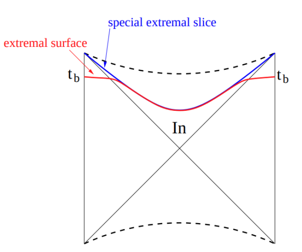
$$C \sim Vol_{\max}(B)$$



- Other complexities: complexity-action (CA), CV 2.0, complexity=anything [Brown, Roberts, Susskind, Swingle, Zhao 2016][Couch, Fischler, Nguyen 2016][Belin, Myers, Ruan, Sarosi, Speranza 2021]

However,

- There are extremal surface **barriers** [Engelhardt, Wall 2013]
- Extremal surfaces do not probe past these barriers
- $\Rightarrow$ it is **not possible** to reconstruct the entire bulk, to reach the singularity, using extremal surfaces (entanglement, CV)

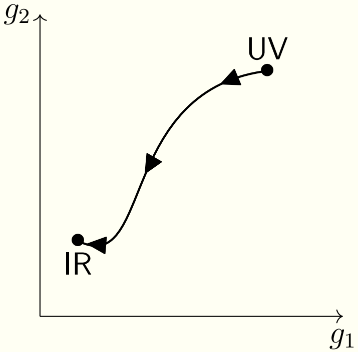


Trans-IR Flows

Motivation

- **RG flow:** Interpolation between theories at different energy scales
  - Goes from UV fixed point to IR fixed point
- View as a curve in the space of couplings parameterized by energy

scale Λ



Holography:

- Radial direction $\leftrightarrow$ energy
- Holographic RG flow UV-IR: from boundary to horizon
- Extensive literature on holographic RG flows of vacuum states, seminal papers:
  - [Freedman, S. S. Gubser, K. Pilch, Warner 1999]
  - [Myers, Sinha 2010–11]

- RG flows in black hole backgrounds ? [Frenkel, Hartnoll, Kruthoff, Shi 2021]
- Black holes are dual to a thermal state
- "RG" flow into the black hole? Trans-IR

Holographic RG Flow

- In AdS/CFT, to trigger an RG flow we need to deform the boundary CFT *i.e.* add matter
  - Example: Scalar field ϕ is dual to operator $\mathcal{O}$

$$\int d^{d+1}X\sqrt{|g|}\left[\nabla_\mu\phi\nabla^\mu\phi+V(\phi)\right]\longleftrightarrow\int d^dx\,\phi_0\mathcal{O} \tag{3.1}$$

- Relevant deformations trigger RG flows
  - Flow is encoded by classical bulk dynamics

[Balasubramanian, Kraus 1999] [de Boer, Verlinde, Verlinde 2000]

RG Flows of the Vacuum

- Start with domain wall ansatz with flat slicing

$$ds^2 = e^{2A(\rho)}\left(-dt^2 + d\vec{x}^2\right) + d\rho^2, \quad ((t,\vec{x}) \in \mathbb{R}^d, \quad \rho \geq 0)$$

  Get AdS_{d+1} with curvature ℓ when $A(\rho) = \rho/\ell$

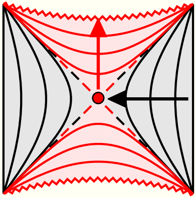
- Trace anomaly coefficient is [Freedman et. al][Myers et. al.]

$$a_{\mathsf{UV}} = \frac{\pi^{d/2}}{\Gamma\left(\frac{d}{2}\right)}\left(\frac{\ell}{\ell_P}\right)^{d-1} = \frac{\pi^{d/2}}{\Gamma\left(\frac{d}{2}\right)\ell_P^{d-1}}\left[\frac{1}{A'(\rho)}\right]^{d-1}$$

  - Identify RHS as holographic a-function $a(\rho)$
  - For general warp factor $e^{A(\rho)}$ and Einstein gravity, can prove monotonicity if NEC is obeyed

Trans-IR Flows as Black Hole Interiors

- AdS/CFT: energy scale is the bulk radial extra dimension
- black holes
  - RG flow of some UV thermal state (bdry.) to IR (horizon)
  - In the interior the radial coordinate becomes timelike $\Longrightarrow$ trans-IR [Frenkel, Hartnoll, Kruthoff, Shi 2020]



- Trans-IR flow: analytic continuation to imaginary Λ
- Analytically continuation to study the black hole interior is not a new idea: Maldacena, Hubeny, etc.

- Can we define an a-function that is monotonic in the interior? [EC, Kundu,Patra, Shashi 2022][EC, Kundu,Patra, Shashi 2022] [EC, Shashi 2022] [EC, Sashi, Sun 2023] [EC, Castillo, Landsteiner, Salazar-Landea 2023] [EC, Patra, Pedraza 2023]

More questions

1. What happens to the degrees of freedom at a trans-IR endpoint?
   - Symmetric backgrounds: we lose all degrees of freedom at the singularity. Anisotropic backgrounds: unclear
2. Connection to Belinski–Khalatnikov–Lifshitz (BKL) behavior close to the singularity?

Trans-IR flows

- Holographic RG flows of thermal states
- The trans-IR explores the interior of a black hole geometry
- Start with "blackened" ansatz

$$ds^2 = e^{2A(\rho)} \left(-f(\rho)^2 dt^2 + d\vec{x}^2\right) + d\rho^2,$$

  - Exterior is $(t, \vec{x}) \in \mathbb{R}^d$, $\rho \geq 0$, with horizon at $\rho = 0$

- The Null Energy condition allows us to define an a function even for **less symmetric backgrounds** [EC, Shashi][EC, Castillo, Landsteiner, Salazar-Landea]

- And we can prove that this a function is also monotonic in the interior

Null Energy Condition (NEC)

$$T^{\mu\nu} k_\mu k_\nu > 0$$

Key idea: if NEC along $k^\mu = \frac{e^{-A(\rho)}}{f(\rho)} \partial_t^\mu + \partial_\rho^\mu$ can be written as

$$\mathcal{C}(\rho) \frac{d}{d\rho} \left[\tilde{a}(\rho)\right] - \mathcal{K}(\rho)^2 \geq 0,$$

where $\mathcal{C}(\rho)$ is positive outside the horizon

$$\implies \qquad \frac{d}{d\rho} \left[\tilde{a}(\rho)\right] \geq \frac{\mathcal{K}(\rho)^2}{\mathcal{C}(\rho)} \geq 0.$$

$\implies \tilde{a}(\rho)$ is a candidate a-function.

The a_T-Function

- In the the case of a metric

$$ds^2 = e^{2A(\rho)}\left(-f(\rho)^2 dt^2 + d\vec{x}^2\right) + d\rho^2,$$

we obtain

$$\boxed{a_T(\rho) = \frac{\pi^{d/2}}{\Gamma\left(\frac{d}{2}\right)\ell_P^{d-1}}\left[\frac{f(\rho)}{A'(\rho)}\right]^{d-1}}$$

The a_T-Function (cont.)

- We can prove that
  - Stationary at horizon:

$$\left.\frac{da_T}{d\rho}\right|_{\text{hor}} = 0$$

  - Monotonicity condition:

$$\text{UV} \to \text{IR} : \frac{da_T}{d\rho} \geq 0,$$
$$\text{Trans-IR} : \frac{da_T}{d\kappa} \leq 0$$

Example: Free Kasner Flows

Einstein + Free Scalar Theory

- Einstein gravity + scalar

$$I = \int d^{d+1}x\sqrt{-g}\left(R + d(d-1) - \frac{1}{2}(\nabla^\alpha\phi\nabla_\alpha\phi + m^2\phi^2)\right)$$

$\phi = \phi(r)$ dual to relevant operator $\mathcal{O}$ with dimension Δ,

$$I_{\mathcal{O}} = \int d^dx\,\phi_0\,\mathcal{O}$$

Schwarzschild-like metric ansatz with horizon $r = r_h$

$$ds^2 = \frac{1}{r^2}\left[-F(r)e^{-\chi(r)}dt^2 + \frac{dr^2}{F(r)} + d\vec{x}^2\right], \quad (r \in \mathbb{R})$$

Exterior: $r \leq r_h$ $(F \geq 0)$; Bdry at $r = 0$
Interior: $r \geq r_h$ $(F \leq 0)$

Solving for Black Hole Solutions

- Equations of motion with this ansatz:

$$\phi'' + \left(\frac{F'}{F} - \frac{d-1}{r} - \frac{\chi'}{2}\right)\phi' + \frac{\Delta(d-\Delta)}{r^2 F}\phi = 0 \quad (4.1)$$

$$\chi' - \frac{2F'}{F} - \frac{\Delta(d-\Delta)\phi^2}{(d-1)rF} - \frac{2d}{rF} + \frac{2d}{r} = 0 \quad (4.2)$$

$$\chi' - \frac{r}{d-1}(\phi')^2 = 0, \quad (4.3)$$

  - Solve numerically for $\{F, \chi, \phi\}$
    - Solutions labeled by "strength" of deformation measured by the dimensionless parameter $\phi_0 / T^{d-\Delta}$

a_T

- Can use coordinate transformation from blackened domain wall ansatz to Schwarzschild-like ansatz to write

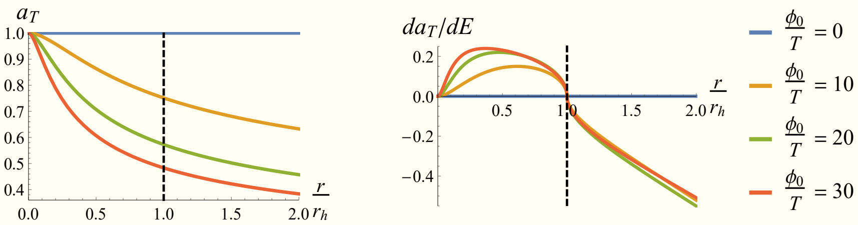
$$a_T(r) = \frac{\pi^{d/2}}{\Gamma\left(\frac{d}{2}\right)\ell_P^{d-1}} e^{-(d-1)\chi(r)/2} \tag{4.4}$$

  - Can go further and obtain the evolution of a_T along the full flow:

$$\frac{da_T}{dE} = \begin{cases} \dfrac{da_T}{d\rho} = -r\sqrt{F(r)}\dfrac{da_T}{dr}, & \text{if } r \leq r_h \\ \dfrac{da_T}{d\kappa} = r\sqrt{|F(r)|}\dfrac{da_T}{dr}, & \text{if } r \geq r_h \end{cases} \tag{4.5}$$

a_T (cont.)

- Set $d = 3$, $\Delta = 2$ and plot a_T, da_T/dE for various deformations



Close to the singularity; Kasner Flows

- Near-singularity ($r \to \infty$) geometry is a **Kasner(-like) universe**

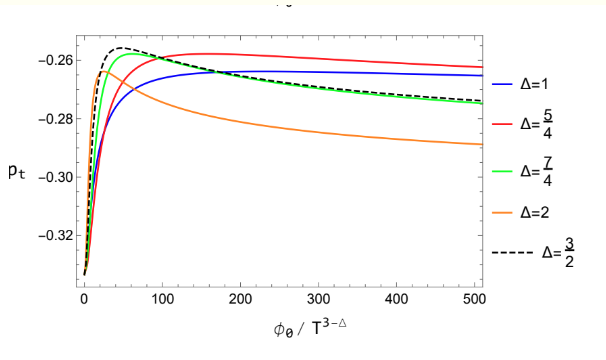
$$ds^2 \sim -d\tau^2 + \tau^{2p_t} dt^2 + \tau^{2p_x} d\vec{x}^2, \quad \phi \sim -\sqrt{2} p_\phi \log \tau$$

$$p_t + (d-1)p_x = 1, \quad p_\phi^2 + p_t^2 + (d-1)p_x^2 = 1$$

  - Thus, call the geometries **Kasner flows** [Frenkel, Hartnoll, Kruthoff, Shi 2020] [**Caceres**, Kundu, Patra, Shashi 2021]

Connecting singularity and boundary

Recall that the boundary deformation deformation is
$I_{\mathcal{O}} = \int d^d x \, \phi_0 \, \mathcal{O}$



Example: Free Kasner Flows

a_T close to the singularity

- As we approach singularity ($r \to \infty$) we have $a_T \to 0$

$$a_T(r) \sim r^{-(d-1)^2 q^2/2}$$

  where q is a function of p_t and d
- $\Longrightarrow$ Lose all d.o.f. at trans-IR endpoint. Complexity? [Caputa, Das, Das 2021]

- Study near-singularity backreacting geometry in greater generality
  - Can use **Belinskii-Khalatnikov-Lifshitz (BKL) approximation**
  - Can also examine numerical solutions to theories with more matter [Hartnoll, Horowitz, Kruthoff, Santos 2021]

Anisotropic Flows

Is the monotonic trans IR a-function an artifact of having too much symmetry?

- Non-isotropic RG flows [D.Giataganas, U. Gürsoy, J.F. Pedraza 2018][C.S. Chu, D. Giataganas 2020]
- NEC
- **Trans-IR non-isotropic** flows

Consider a background that breaks the rotational symmetry of the constant-ρ slices,

$$ds^2 = e^{2A(\rho)}\left[-f(\rho)^2 dt^2 + e^{2\mathcal{X}(\rho)} d\vec{x}_1^2 + d\vec{x}_2^2\right] + d\rho^2$$

Asymptotically AdS

$$A(\rho) \sim \frac{\rho}{\ell} \;\; (\rho \to \infty), \quad \lim_{\rho\to\infty} \mathcal{X}(\rho) = 0, \quad \lim_{\rho\to\infty} f(\rho) = 1.$$

Key observation: NEC along $k^\mu = \dfrac{e^{-A(\rho)}}{f(\rho)}\partial_t^\mu + \partial_\rho^\mu$ can be written as

$$\mathcal{C}(\rho)\frac{d}{d\rho}\left[\tilde{a}(\rho)\right] - \mathcal{K}(\rho)^2 \geq 0,$$

where $\mathcal{C}(\rho)$ is manifestly positive outside the horizon.

$\Longrightarrow$

$$\frac{d}{d\rho}\left[\tilde{a}(\rho)\right] \geq \frac{\mathcal{K}(\rho)^2}{\mathcal{C}(\rho)} \geq 0.$$

$\Longrightarrow \tilde{a}(\rho)$ is a candidate a-function.

$d+1$ dimensional space, $d_1+d_2=d-1$

$$ds^2 = e^{2A(\rho)}\left[-f(\rho)^2 dt^2 + e^{2\mathcal{X}(\rho)} d\vec{x}_1^2 + d\vec{x}_2^2\right] + d\rho^2$$

we have,

$$\mathcal{C}(\rho) = \frac{1}{(d-1)f(\rho)}\left[d_1\big(A'(\rho)+\mathcal{X}'(\rho)\big) + d_2 A'(\rho)\right]^2 e^{d_1\mathcal{X}(\rho)/(d-1)},$$
$$\mathcal{K}(\rho) = \sqrt{\frac{d_1 d_2}{d-1}}\mathcal{X}'(\rho).$$

and,

$$\boxed{a(\rho) \sim e^{-d_1 \mathcal{X}(\rho)} \left[\frac{(d-1) f(\rho)}{d_1 \Big(A'(\rho) + \mathcal{X}'(\rho) \Big) + d_2 A'(\rho)} \right]^{d-1}}$$

- Asymptotes to the appropriate UV trace anomaly ℓ^{d-1}
- Monotonicity outside the horizon $\checkmark$
  NEC
- Monotonicity inside the horizon is more involved $\checkmark$

Example: $I_{\mathsf{EYM}} = \int d^5x\sqrt{-g}\left[\frac{1}{2\ell_{\mathsf{P}}^3}\left(R+12\right) - \frac{1}{4\hat{g}^2}F^a_{\mu\nu}F^{a\mu\nu}\right].$

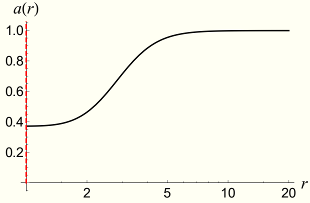
$$ds^2 = e^{2A(\rho)}\left[-f(\rho)^2dt^2 + e^{2\mathcal{X}(\rho)}dx^2 + dy^2 + dz^2\right] + d\rho^2.$$

$d_1 = 1$ and $d_2 = 2$

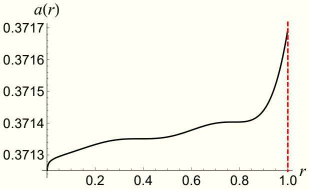
The a-function is

$$a_T^{(1)}(\rho) \sim e^{-\mathcal{X}(\rho)}\left[\frac{f(\rho)}{A'(\rho) + \frac{1}{3}\mathcal{X}'(\rho)}\right]^3.$$

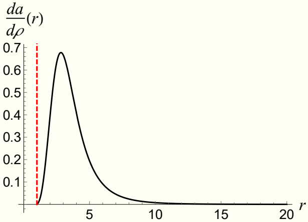
Solve EOMs numerically,



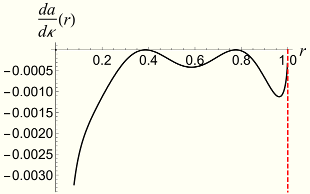
(a) Exterior a-function



(b) Interior a-function



(a) Derivative, exterior . $\frac{da_T}{d\rho}$



(b) Derivative, interior. $\frac{da_T}{d\kappa}$

Conclusions

Takeaways

- Trans-IR flows seem rather abstract in typical QFT.
  - Naturally emerge in holographic RG framework as black hole interiors
- We can define an a-function that is monotonic along the flows, even in anisotropic backgrounds
- We can correlate behavior at the boundary with deformation at the singularity

Remaining Mysteries

- Broad question: trans-IR flow in the quantum side

- Better understanding more black hole interiors/near-singularity geometries with more complicated matter profiles (BKL analysis using cosmological billiards [Damour, Henneaux, Nicolai 2003]).

- a_T close to the singularity? Losing degrees of freedom?

Many more questions...

Gracias!

Example: p-wave superfluid

$(4+1)$-dimensional Einstein-Yang-Mills theory with $SU(2)$ gauge symmetry whose bulk action is (setting $\ell = 1$)

$$I_{\mathsf{EYM}} = \int d^5x\sqrt{-g}\left[\frac{1}{2\ell_{\mathsf{P}}^3}\left(R+12\right) - \frac{1}{4\hat{g}^2}F^a_{\mu\nu}F^{a\mu\nu}\right].$$

$a = 1, 2, 3$ runs over the generators of the (three-dimensional representation of the) $SU(2)$ gauge group

$$ds^2 = -N(r)\sigma(r)^2 dt^2 + \frac{1}{N(r)} dr^2 + r^2 h(r)^{-4} dx^2 + r^2 h(r)^2 (dy^2 + dz^2)$$

$$\mathcal{A} = \phi(r)\tau^3 dt + w(r)\tau^1 dx,$$

Solve EOMs numerically

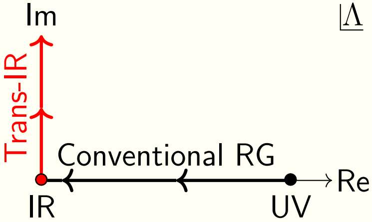
Note: To prove monotonicity in the interior it is easier to change to Schawrzchild -like coordinates,

$$ds^2 = \frac{1}{r^2}\left[-F(r)e^{-\chi(r)}dt^2 + \frac{dr^2}{F(r)} + d\vec{x}^2\right],$$

$$e^{2A(\rho)} = \frac{1}{r^2}, \quad f(\rho)^2 = F(r)e^{-\chi(r)}, \quad \frac{dr}{d\rho} = -r\sqrt{F(r)}.$$

What Is a Trans-IR Flow?

- Assume gapless system for now ($\Lambda_{IR} = 0$)
- Trans-IR flow accessed by analytically continuing to imaginary Λ

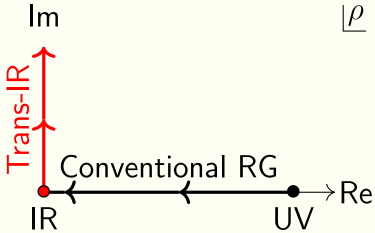


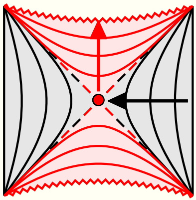
Trans-IR Flows as Black Hole Interiors

- Energy scale is the bulk radial extra dimension
- RG flows: UV at the boundary, IR at the horizon

Q: Black holes?

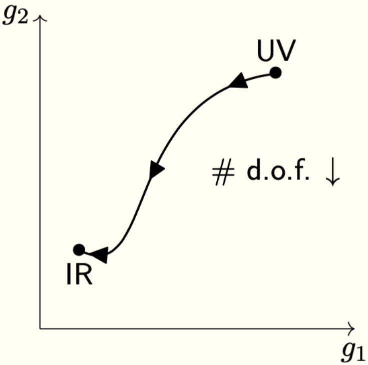
- RG flow of some UV thermal state
- Radial coordinate becomes timelike $\implies$ interior is trans-IR! [Frenkel, Hartnoll, Kruthoff, Shi 2020]





Counting Degrees of Freedom

- Count degrees of freedom along flow with a monotonically decreasing function of energy
- Zamolodchikov c-theorem (2d)
  - Evaluates to central charge at fixed points
- Cardy a-theorem ($2n$-d)
  - Evaluates to trace anomaly coefficient at fixed points



RG Flows of Thermal States

- To probe trans-IR, need to use a black hole geometry
- Start with "blackened" domain wall ansatz with warped flat slicing

$$ds^2 = e^{2A(\rho)}\left(-f(\rho)^2 dt^2 + d\vec{x}^2\right) + d\rho^2, \;\; (f(\rho) = f_1\rho + O(\rho^3))$$

  - Exterior is $(t, \vec{x}) \in \mathbb{R}^d$, $\rho \geq 0$, with horizon at $\rho = 0$

- Interior accessed via analytic continuation

$$\boxed{\rho = i\kappa}, \;\; t = t_I - \mathsf{sgn}(t_I)\frac{i\gamma}{2T}, \;\; \left(\kappa \geq 0, \quad \gamma \in \mathbb{Z} + \frac{1}{2}\right)$$

- Get AdS$_{d+1}$-Schwarzschild with curvature ℓ when

$$e^{A(\rho)} = \frac{2}{d}\cosh\left(\frac{d\rho}{2\ell}\right)^{2/d}, \quad f(\rho) = \tanh\left(\frac{d\rho}{2\ell}\right)$$