Super Statistics and the QCD Critical End Point

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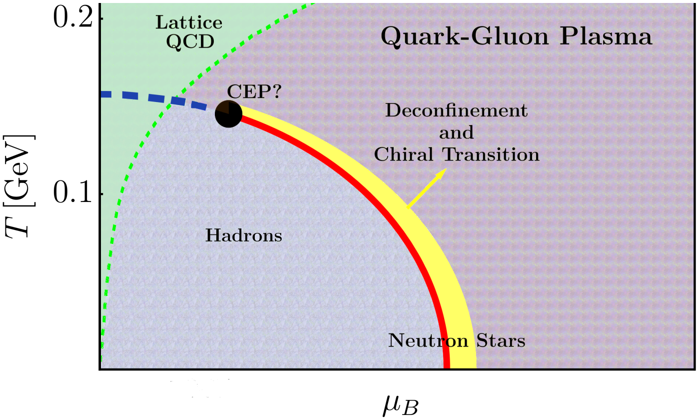
- Impact of finite volume and thermal fluctuations on the Critical End Point (CEP) of the QCD phase diagram.
- Implementation of the Super Statistic framework to a Linear σ Model coupled to quarks.
- We compute an effective thermodynamic potential Ω_{T} that depends explicitly on the volume V.
- The potential is also a function of a parameter q that models the temperature fluctuations.

Hypothetical QCD phase diagram



Super Statistics

Some intensive parameter $\widetilde{\beta}$ may fluctuate by following a certain probability distribution $f(\widetilde{\beta})$, [1]. It is possible to construct a modified Boltzmann factor

$$\hat{B} \equiv \int_0^\infty d\widetilde{\beta} f(\widetilde{\beta}) e^{-\widetilde{\beta}\hat{H}}.$$

- $f(\widetilde{\beta})$ must be normalized.
- The new statistic must be normalizable.
- Reduction to Boltzmann-Gibbs statistics if there are no fluctuations of intensive quantities.

[1] C. Beck and E. G. D. Cohen, Physica 322A, 267 (2003).

The Gamma distribution

Using the Gamma distribution function

$$f(\tilde{\beta}) = \frac{1}{b\Gamma(c)} \left(\frac{\tilde{\beta}}{b}\right)^{c-1} e^{-\tilde{\beta}/b}.$$

The average of $\widetilde{\beta}$ is

$$\beta = \int_0^\infty \widetilde{\beta} f(\widetilde{\beta}) d\widetilde{\beta} = bc,$$

which we take as the inverse temperature. $\hat{B}$ can be written as the q-exponential

$$\hat{B} = \left(1 - (1-q)\beta\hat{H}\right)^{1/(1-q)} \equiv e_q^{-\beta\hat{H}}, \quad \text{with } c = -\frac{1}{1-q}.$$

Boltzmann factor of Tsallis statistics [2]. Ordinary statistics at $q \to 1$.

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[2] C. Tsallis, R. S. Mendes, and A. R. Plastino, Physica 261A, 534 (1998).

The Gamma distribution

The partition function is given by

$$\mathcal{Z} = \mathrm{Tr}\,\hat{\rho},$$

where $\hat{\rho}$ is a density operator defined as

$$\hat{\rho} = \frac{1}{\mathcal{Z}} e_q^{-\beta \hat{H}}.$$

Tsallis prescription for the effective potential density

$$\Omega_T = -\frac{1}{V\beta} \ln_q \mathcal{Z} = -\frac{1}{V\beta} \frac{\mathcal{Z}^{1-q} - 1}{1-q},$$

preserves the Legendre structure.

In order to find an expression for the Super Statistics partition function, we expand around $q = 1$,

$$\mathcal{Z} = \mathcal{Z}_0 + \frac{q-1}{2}\beta^2\frac{\partial^2\mathcal{Z}_0}{\partial\beta^2} + \frac{(q-1)^2}{24}\left(8\beta^3\frac{\partial^3\mathcal{Z}_0}{\partial\beta^3} + 3\beta^4\frac{\partial^4\mathcal{Z}_0}{\partial\beta^4}\right) + \mathcal{O}\left((q-1)^3\right).$$

Boltzmann partition function

$$\mathcal{Z}_0 = e^{-\beta V \Omega_0},$$

where Ω_0 is the equilibrium effective potential density. With this,

$$\mathcal{Z} = \mathcal{Z}(q, V, \beta, \Omega_0).$$

The Linear σ Model

We use the LσM coupled to quarks

$$\mathcal{L} = \frac{1}{2}(\partial_\mu \sigma)^2 + \frac{1}{2}(\partial_\mu \boldsymbol{\pi})^2 + \frac{a^2}{2}(\sigma^2 + \boldsymbol{\pi}^2) - \frac{\lambda}{4}(\sigma^2 + \boldsymbol{\pi}^2)^2 + \mathrm{i}\bar{\psi}\gamma^\mu \partial_\mu \psi - g\bar{\psi}(\sigma + \mathrm{i}\gamma_5 \boldsymbol{\tau} \cdot \boldsymbol{\pi})\psi,$$

where ψ is an $SU(2)$ isospin doublet, and $\boldsymbol{\pi}$, σ are isospin triplet and singlet, respectively. Classical groundstate determined by

$$V = \frac{a^2}{2}(\sigma^2 + \boldsymbol{\pi}^2) - \frac{\lambda}{4}(\sigma^2 + \boldsymbol{\pi}^2)^2,$$

at $\sigma^2 + \boldsymbol{\pi}^2 = a^2/\lambda \equiv v_0$. The model admits SSB. Shifting $\sigma \to \sigma' + v$, where v is taken as a variable. The fields acquire

$$m_{\sigma'}^2 = 3\lambda v^2 - a^2,$$
$$m_\pi^2 = \lambda v^2 - a^2,$$
$$m_f = gv.$$

The Effective Potential

This effective potential density in a high-temperature limit [3,4], is

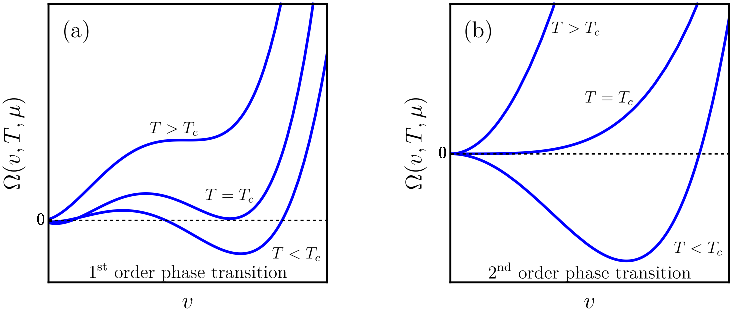
$$\Omega_0 = -\frac{a^2}{2}v^2 + \frac{\lambda}{4}v^4$$
$$+ \sum_{i=\sigma,\boldsymbol{\pi}} \left\{ \frac{m_i^4}{64\pi^2}\left[\ln\left(\frac{16\pi^2 T^2}{2a^2}\right) - 2\gamma_{\mathsf{e}} + 1\right] - \frac{\pi^2 T^4}{90} + \frac{m_i^2 T^2}{24} - \frac{T}{12\pi}\left[m_i^2 + \Pi(T,\mu)\right]^{3/2} \right\}$$
$$- \frac{N_c}{16\pi^2}\sum_{f=u,d}\left\{ m_f^4\left[\ln\left(\frac{8\pi^2 T^2}{a^2}\right) + \psi_0\left(\frac{1}{2} + \frac{\mathrm{i}\mu}{2\pi T}\right) + \psi_0\left(\frac{1}{2} - \frac{\mathrm{i}\mu}{2\pi T}\right) + 1\right]\right.$$
$$\left. + 8m_f^2 T^2\left[\mathrm{Li}_2\left(-e^{\mu/T}\right) + \mathrm{Li}_2\left(-e^{-\mu/T}\right)\right] - 32T^4\left[\mathrm{Li}_4\left(-e^{\mu/T}\right) + \mathrm{Li}_4\left(-e^{-\mu/T}\right)\right]\right\},$$

where,

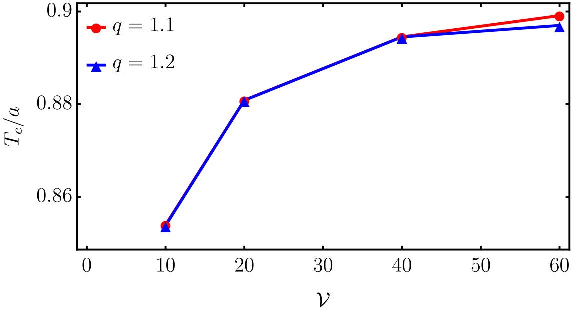
$$\Pi(T,\mu) = \frac{\lambda T^2}{2} - \frac{N_f N_c g^2 T^2}{\pi^2}\left[\mathrm{Li}_2\left(-e^{\mu/T}\right) + \mathrm{Li}_2\left(-e^{-\mu/T}\right)\right],$$

[3] A. Ayala et. al., Nucl. Phys. B897, 77 (2015).
[4] A. Ayala et. al., Int. J. Mod. Phys. A 31, 1650199 (2016).

Our goal is to describe the chiral symmetry restoration, where the quark mass vanishes so that the σ-field vacuum expectation value v is promoted as the order parameter for the transition.

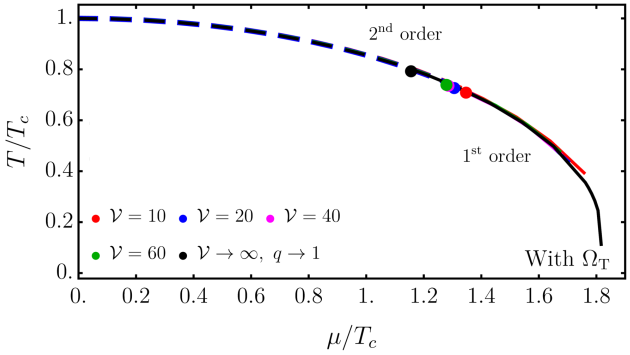


Shapes of the effective potential.



Pseudo-critical temperature T_c/a at $\mu = 0$, as a function of the dimensionless volume $\mathcal{V} = a^3 V$ for $q = 1.1$ and $q = 1.2$. The reference temperature $T_c^0/a \approx 0.9$ is the transition temperature at $\mu = 0$ in the equilibrium case $q \to 1$. A similar behavior is found with $q = 0.8$, and $q = 0.9$.

CEP location for $q = 1.2$



Effective QCD phase diagram obtained from Ω_{T} and several values of $\mathcal{V}$. The points are the CEP location for each volume. The values of T and μ in the critical line are normalized to their own T_c which is volume-dependent.

What about using other distributions?

Considering a distribution function $f(\widetilde{\beta})$, in general we expand

$$\hat{B} = e^{-\beta \hat{H}} \left[1 + \frac{q-1}{2} \beta^2 \hat{H}^2 + \eta(q) \beta^3 \hat{H}^3 + \dots \right],$$

where

$$\beta = \langle \widetilde{\beta} \rangle, \quad q = \frac{\langle \widetilde{\beta}^2 \rangle}{\langle \widetilde{\beta} \rangle^2}.$$

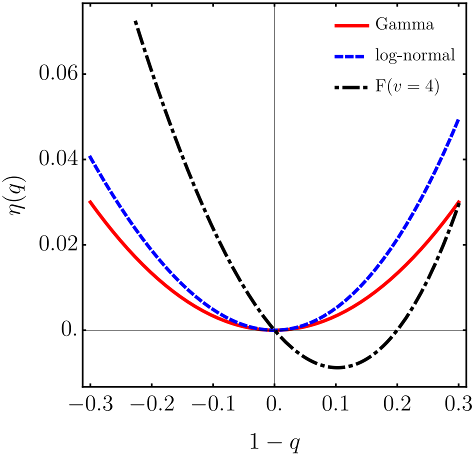
- Gamma or χ^2 distribution
- Log-normal distribution

$$f(\tilde{\beta}) = \frac{1}{\sqrt{2\pi} \tilde{\beta} u} \exp \left[- \frac{\log^2 \left(\tilde{\beta}/v \right)}{2u^2} \right].$$

- F distribution

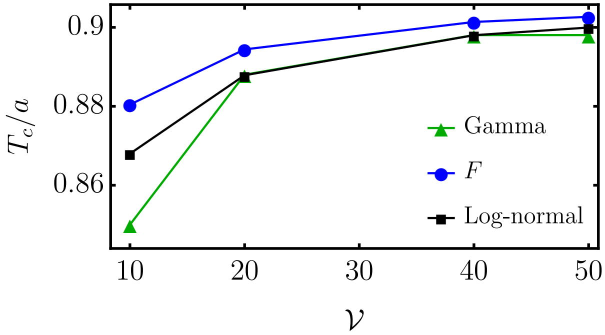
$$f(\tilde{\beta}) = \frac{\Gamma[(v+w)/2]}{\Gamma(v/2)\Gamma(w/2)} \left(\frac{bv}{w} \right)^{v/2} \frac{\tilde{\beta}^{v/2-1}}{\left(1 + \frac{bv}{w} \tilde{\beta} \right)^{(v+w)/2}}.$$

$$\eta(q) = \begin{cases} -\frac{1}{3}(q-1)^2, & \text{Gamma distribution} \\ -\frac{1}{6}\left(q^3 - 3q + 2\right), & \text{Log-normal distribution} \\ -\frac{1}{3}\frac{(q-1)(5q-6)}{3-q}, & F \text{ distribution with } v = 4. \end{cases},$$



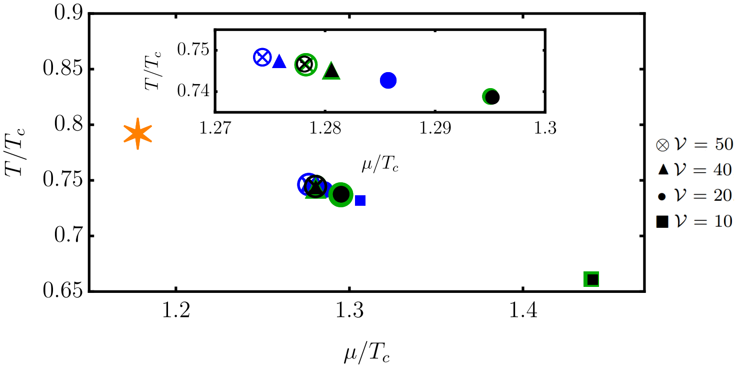
Similarly, we get an expression for the partition function in terms of the effective potential density Ω_0 and use the LσM coupled to quarks,

$$\mathcal{Z} \approx e^{-\beta V \Omega_0} \left\{ 1 + \frac{q-1}{2} \beta^2 \left[V^2 \left(\Omega_0 + \beta \frac{\partial \Omega_0}{\partial \beta} \right)^2 - 2V \frac{\partial \Omega_0}{\partial \beta} - \beta V \frac{\partial^2 \Omega_0}{\partial \beta^2} \right] \right.$$
$$- \eta(q) \beta^3 V \left[-V^2 \left(\Omega_0 + \beta \frac{\partial \Omega_0}{\partial \beta} \right)^3 - 3 \frac{\partial^2 \Omega_0}{\partial \beta^2} - \beta \frac{\partial^3 \Omega_0}{\partial \beta^3} \right.$$
$$\left.\left. + 3V \left(\Omega_0 + \beta \frac{\partial \Omega_0}{\partial \beta} \right) \left(2 \frac{\partial \Omega_0}{\partial \beta} + \beta \frac{\partial^2 \Omega_0}{\partial \beta^2} \right) \right] \right\}.$$



Pseudo-critical temperature T_c/a at $\mu = 0$, as a function of the dimensionless volume $\mathcal{V} = a^3V$ for $q = 1.2$ and different distribution functions.

CEP location for $q = 1.2$



CEP locations for different distribution functions: Gamma (green), F (blue), and Log-normal (black). The yellow star is the pure Boltzmann result. The results are independent of q.

Conclusions

- Pseudocritical temperature T_c at $\mu = 0$ changes with the volume, around 7% for the smaller volume.
- The CEP location moves to low temperatures and high chemical potential as the volume decreases.
- For volumes $\mathcal{V}$ around 20-50 we find CEP location at T/T_c around 1.27-1.3, μ/T_c around 0.73-0.75.
- We do not find significant changes in the explored region for $q \in [0.8, 1.2]$, thus we conclude that chiral symmetry restoration is robust against thermal fluctuations.