Quantization of a Klein-Gordon spin $j = 1$ field

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Homogeneous Lorentz Group

The **Homogeneous Lorentz Group (HLG)** is the group $O(1,3)$, whose elements, $L^{\mu}{}_{\rho}$, are defined by

$$L^{\mu}{}_{\rho} g_{\mu\nu} L^{\nu}{}_{\sigma} = g_{\rho\sigma}. \tag{1}$$

This group can be separated into 4 disconnected components:

- Proper ortochronous: $\det L = 1$ & $L^{0}{}_{0} > 0$. It's the subgroup $SO^{+}(1,3)$, aka **Restricted Lorentz Group (RLG)**. Its elements are $\Lambda^{\mu}{}_{\nu}$.
- Improper ortochronous: $\det L = -1$ & $L^{0}{}_{0} > 0$. Its elements are

$$[\mathcal{P}\Lambda]^{\mu}{}_{\nu}, \qquad \mathcal{P} = \text{Diag}(1,-1,-1,-1). \tag{2}$$

- Improper heterochronous: $\det L = -1$ & $L^{0}{}_{0} < 0$. Its elements are

$$[\mathcal{T}\Lambda]^{\mu}{}_{\nu}, \qquad \mathcal{T} = \text{Diag}(-1,1,1,1). \tag{3}$$

- Proper heterochronous: $\det L = 1$ & $L^{0}{}_{0} < 0$. Its elements are

$$[\mathcal{P}\mathcal{T}\Lambda]^{\mu}{}_{\nu}, \qquad \mathcal{P}\mathcal{T} = \text{Diag}(-1,-1,-1,-1). \tag{4}$$

Homogeneous Lorentz Group

In general, the RLG is a six-parameter Lie group whose elements can be written as

$$\Lambda(\boldsymbol{\theta}, \boldsymbol{\phi}) = \exp\left[-\frac{i}{2}\Omega_{\mu\nu}J^{\mu\nu}\right] \tag{5}$$

where the generators $J^{\mu\nu}$ satisfy the algebra

$$[J^{\mu\nu}, J^{\rho\sigma}] = i(\eta^{\nu\rho}J^{\mu\sigma} - \eta^{\mu\rho}J^{\nu\sigma} + \eta^{\nu\sigma}J^{\mu\rho} - \eta^{\mu\sigma}J^{\nu\rho}). \tag{6}$$

It can be rewritten as

$$[J^i, J^j] = i\varepsilon^{ijk}J^k, \qquad [J^i, K^j] = i\varepsilon^{ijk}K^k, \qquad [K^i, K^j] = -i\varepsilon^{ijk}J^k, \tag{7}$$

where $J^i = \frac{1}{2}\varepsilon^{ijk}J^{jk}$, $K^i = J^{i0}$. Defining the operators $\boldsymbol{A}, \boldsymbol{B}$ as

$$\boldsymbol{A} = \frac{1}{2}(\boldsymbol{J} - i\boldsymbol{K}), \qquad \boldsymbol{B} = \frac{1}{2}(\boldsymbol{J} + i\boldsymbol{K}), \tag{8}$$

it simplifies to two copies of the $SU(2)$ algebra

$$[A^i, A^j] = i\varepsilon^{ijk}A^k, \qquad [A^i, B^j] = 0, \qquad [B^i, B^j] = i\varepsilon^{ijk}B^k. \tag{9}$$

In this sense

$$SO^+(1,3) \simeq SU(2)_A \otimes SU(2)_B \tag{10}$$

The irreps of the RLG are therefore

- Defined by (a, b), where $a, b = 0, 1/2, 1, 3/2, \cdots$.
- Have dimension $(2a+1)(2b+1)$.
- Have eigenstates $\{|a, b, m_a, m_b\rangle = |a, m_a\rangle\, |b, m_b\rangle\}$, with

$$\boldsymbol{A}^2 |a, m_a\rangle = a(a+1) |a, m_a\rangle \qquad \boldsymbol{B}^2 |b, m_b\rangle = b(b+1) |b, m_b\rangle \tag{11}$$

$$A_3 |a, m_a\rangle = m_a |a, m_a\rangle \qquad B_3 |b, m_b\rangle = m_b |b, m_b\rangle \tag{12}$$

$$(0,0)$$

$$\left(\tfrac{1}{2},0\right) \qquad \left(0,\tfrac{1}{2}\right)$$

$$(1,0) \qquad \left(\tfrac{1}{2},\tfrac{1}{2}\right) \qquad (0,1)$$

$$\left(\tfrac{3}{2},0\right) \qquad \left(1,\tfrac{1}{2}\right) \qquad \left(\tfrac{1}{2},1\right) \qquad \left(0,\tfrac{3}{2}\right)$$

$$\vdots \qquad \vdots \qquad \vdots \qquad \vdots \qquad \vdots$$

Homogeneous Lorentz Group

There is an infinite number of irreps for the RLG, however, the Standard Model only uses a few of them:

- $(0,0)$: Higgs.
- $\left(\frac{1}{2},0\right)$, $\left(0,\frac{1}{2}\right)$: Leptons and quarks.
- $\left(\frac{1}{2},\frac{1}{2}\right)$: Gauge bosons.

$$(0,0)$$

$$\left(\frac{1}{2},0\right) \qquad \left(0,\frac{1}{2}\right)$$

$$(1,0) \qquad \left(\frac{1}{2},\frac{1}{2}\right) \qquad (0,1)$$

$$\left(\frac{3}{2},0\right) \qquad \left(1,\frac{1}{2}\right) \qquad \left(\frac{1}{2},1\right) \qquad \left(0,\frac{3}{2}\right)$$

$$\vdots \qquad \vdots \qquad \vdots \qquad \vdots \qquad \vdots$$

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The dynamics of a spin $j = 1/2$ field $\psi \in \left(\frac{1}{2}, 0\right) \oplus \left(0, \frac{1}{2}\right)$ are conventionally described by the Dirac equation

$$(i\gamma^\mu \partial_\mu - m)\psi(x) = 0. \tag{13}$$

- **Dirac solution $\Rightarrow$ KG solution.**
  Every solution of the Dirac equation solves the Klein-Gordon equation

$$(-i\gamma^\mu \partial_\mu - m)(i\gamma^\mu \partial_\mu - m)\psi(x) = (\partial^2 + m^2)\psi(x) = 0. \tag{14}$$

  However the converse is not true, i.e. not every solution of the Klein-Gordon equation is a solution to the Dirac equation.

- **KG solution $\not\Rightarrow$ Dirac solution.**
  The most general solution of the Klein-Gordon equation is not a solution of the Dirac equation, but can be written in terms of two Dirac solutions[1]

$$\psi = \frac{1}{2}\left(1 + \frac{\not\partial}{m}\right)\psi + \frac{1}{2}\left(1 - \frac{\not\partial}{m}\right)\psi = \frac{1}{2}\left(1 + \frac{\not\partial}{m}\right)\psi + \gamma^5 \frac{1}{2}\left(1 + \frac{\not\partial}{m}\right)\gamma^5\psi = \psi_1 + \gamma^5\psi_2 \tag{15}$$

Is it possible to describe the dynamics of a spin $j = 1/2$ particle $\psi \in \left(\frac{1}{2}, 0\right) \oplus \left(0, \frac{1}{2}\right)$ by using just the Klein-Gordon equation?

[1] N. Cufaro Petroni et al. "Second order wave equation for spin 1/2 fields". In: *Phys. Rev. D* 31 (1985), pp. 3157–3161. DOI: 10.1103/PhysRevD.31.3157.

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Klein-Gordon $j = 1/2$ field. KG hermitian theory

The Klein-Gordon Lagrangian for the spin $j = 1/2$ field $\psi \in \left(\frac{1}{2}, 0\right) \oplus \left(0, \frac{1}{2}\right)$ is

$$\mathcal{L} = \partial^\mu \bar{\psi} \partial_\mu \psi - m^2 \bar{\psi}\psi, \tag{16}$$

where the Dirac dual is $\bar{\psi} = \psi^\dagger \gamma^0$. The conjugated momenta $\pi_\psi, \pi_{\bar{\psi}}$ are

$$\pi_\psi = \frac{\partial \mathcal{L}}{\partial(\partial_0 \psi)} = \dot{\bar{\psi}}, \qquad \pi_{\bar{\psi}} = \frac{\partial \mathcal{L}}{\partial(\partial_0 \bar{\psi})} = \dot{\psi}. \tag{17}$$

Observations

- The field has 8 degrees of freedom, twice those of the Dirac field.
- The mass dimension of this field is 1, not the 3/2 dimension of the Dirac field.

Klein-Gordon $j = 1/2$ field. KG hermitian theory

The Dirac decomposition of the Klein-Gordon solution is

$$\psi = \frac{1}{\sqrt{2m}} \left(\psi_1 + \gamma^5 \psi_2 \right), \qquad \bar{\psi} = \frac{1}{\sqrt{2m}} \left(\bar{\psi}_1 - \bar{\psi}_2 \gamma^5 \right), \tag{18}$$

$$\psi(x) = \frac{1}{\sqrt{2m}} \int \frac{d^3p}{(2\pi)^3 \sqrt{2E_{\boldsymbol{p}}}} \sum_{s=1}^{2} \left\{ \left[a^1_{\boldsymbol{p},s} + a^2_{\boldsymbol{p},s} \gamma^5 \right] u_{\boldsymbol{p},s} e^{-ip\cdot x} + \left[b^{1\dagger}_{\boldsymbol{p},s} + b^{2\dagger}_{\boldsymbol{p},s} \gamma^5 \right] v_{\boldsymbol{p},s} e^{ip\cdot x} \right\} \Big|_{p^0 = E_{\boldsymbol{p}}},$$

$$\bar{\psi}(x) = \frac{1}{\sqrt{2m}} \int \frac{d^3p}{(2\pi)^3 \sqrt{2E_{\boldsymbol{p}}}} \sum_{s=1}^{2} \left\{ \bar{u}_{\boldsymbol{p},s} \left[a^{1\dagger}_{\boldsymbol{p},s} - a^{2\dagger}_{\boldsymbol{p},s} \gamma^5 \right] e^{ip\cdot x} + \bar{v}_{\boldsymbol{p},s} \left[b^1_{\boldsymbol{p},s} - b^2_{\boldsymbol{p},s} \gamma^5 \right] e^{-ip\cdot x} \right\} \Big|_{p^0 = E_{\boldsymbol{p}}}.$$

The canonical quantization starts by imposing the anticommutation relations at equal times

$$\left\{ \psi_a(\boldsymbol{x}, t), \pi_{\psi,b}(\boldsymbol{y}, t) \right\} = -\left\{ \bar{\psi}_a(\boldsymbol{x}, t), \pi_{\bar{\psi},b}(\boldsymbol{y}, t) \right\} = i\delta_{ab} \delta^{(3)}(\boldsymbol{x} - \boldsymbol{y}), \tag{19}$$

which imply anticommutation relations with the wrong sign $\Rightarrow$ negative-norm states

$$\{a^1_{\boldsymbol{p},s}, a^{1\dagger}_{\boldsymbol{q},r}\} = (2\pi)^3 \delta_{sr} \delta^{(3)}(\boldsymbol{p} - \boldsymbol{q}), \qquad \{b^1_{\boldsymbol{p},s}, b^{1\dagger}_{\boldsymbol{q},r}\} = (2\pi)^3 \delta_{sr} \delta^{(3)}(\boldsymbol{p} - \boldsymbol{q}), \tag{20}$$

$$\{a^2_{\boldsymbol{p},s}, a^{2\dagger}_{\boldsymbol{q},r}\} = -(2\pi)^3 \delta_{sr} \delta^{(3)}(\boldsymbol{p} - \boldsymbol{q}), \qquad \{b^2_{\boldsymbol{p},s}, b^{2\dagger}_{\boldsymbol{q},r}\} = -(2\pi)^3 \delta_{sr} \delta^{(3)}(\boldsymbol{p} - \boldsymbol{q}). \tag{21}$$

It has been shown recently[2] that this problem can be fixed, and the Klein-Gordon $j = 1/2$ field can be consistently quantized as a pseudohermitian QFT.

[2] Rodolfo Ferro-Hernández et al. "Quantization of second-order fermions". In: *Phys. Rev. D* 109 (8 Apr. 2024), p. 085003. DOI: 10.1103/PhysRevD.109.085003. URL: https://link.aps.org/doi/10.1103/PhysRevD.109.085003.

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Klein-Gordon $j = 1/2$ field. KG pseudohermitian theory

Pseudohermiticity in quantum theories was proposed and studied in[3]. An operator H is pseudo-hermitian if it satisfies

$$H^{\#} = \eta^{-1} H^{\dagger} \eta = H, \tag{22}$$

where η is a linear and invertible operator. By redefining the inner product between two states as

$$\langle a(t)|b(t)\rangle_{\eta} = \langle a(t)|\eta|b(t)\rangle\,, \tag{23}$$

two features emerge:

- The probability amplitudes are preserved in time

$$\langle a(t)|b(t)\rangle_{\eta} = \langle a|e^{-iH^{\dagger}t}\eta e^{iHt}|b\rangle = \langle a|\eta e^{-iHt}e^{iHt}|b\rangle = \langle a|b\rangle_{\eta}\,. \tag{24}$$

- The energy spectrum is real

$$(E - E^{*})\,\langle a_E|a_E\rangle_{\eta} = \langle a_E|\,(\eta H - H^{\dagger}\eta)\,|a_E\rangle = 0. \tag{25}$$

where $|a_E\rangle$ are energy eigenestates: $H\,|a_E\rangle = E\,|a_E\rangle$.

[3] Ali Mostafazadeh. "Pseudo-Hermiticity versus PT symmetry: The necessary condition for the reality of the spectrum of a non-Hermitian Hamiltonian". In: *Journal of Mathematical Physics* 43.1 (Jan. 2002), pp. 205–214. ISSN: 0022-2488. DOI: 10.1063/1.1418246. eprint: https://pubs.aip.org/aip/jmp/article-pdf/43/1/205/19019524/205_1_online.pdf. URL: https://doi.org/10.1063/1.1418246.

Klein-Gordon $j = 1/2$ field. KG pseudohermitian theory

The Klein-Gordon theory can be turned pseudohermitian by redefining the dual of the field ψ as $\hat{\psi}$

$$\hat{\psi} = \eta^{-1}\bar{\psi}\eta, \qquad \mathcal{L} = \partial^{\mu}\hat{\psi}\partial_{\mu}\psi - m^2\hat{\psi}\psi, \qquad \mathcal{L}^{\#} = \eta^{-1}\mathcal{L}^{\dagger}\eta = \mathcal{L}. \tag{26}$$

The fields are now

$$\psi(x) = \frac{1}{\sqrt{2m}}\int \frac{d^3p}{(2\pi)^3\sqrt{2E_{\boldsymbol{p}}}}\sum_{s=1}^{2}\left\{\left[a^1_{\boldsymbol{p},s} + a^2_{\boldsymbol{p},s}\gamma^5\right]u_{\boldsymbol{p},s}e^{-ip\cdot x} + \left[b^{1\dagger}_{\boldsymbol{p},s} + b^{2\dagger}_{\boldsymbol{p},s}\gamma^5\right]v_{\boldsymbol{p},s}e^{ip\cdot x}\right\},$$

$$\hat{\psi}(x) = \frac{1}{\sqrt{2m}}\int \frac{d^3p}{(2\pi)^3\sqrt{2E_{\boldsymbol{p}}}}\sum_{s=1}^{2}\left\{\bar{u}_{\boldsymbol{p},s}\left[\eta^{-1}a^{1\dagger}_{\boldsymbol{p},s}\eta - \eta^{-1}a^{2\dagger}_{\boldsymbol{p},s}\eta\gamma^5\right]e^{ip\cdot x}\right.$$

$$\left. + \bar{v}_{\boldsymbol{p},s}\left[\eta^{-1}b^{1}_{\boldsymbol{p},s}\eta - \eta^{-1}b^{2}_{\boldsymbol{p},s}\eta\gamma^5\right]e^{-ip\cdot x}\right\}.$$

Defining the action of η on the creation/anihilation operators as

$$\eta^{-1}a^1_{\boldsymbol{p},s}\eta = a^1_{\boldsymbol{p},s}, \qquad \eta^{-1}b^{1\dagger}_{\boldsymbol{p},s}\eta = b^{1\dagger}_{\boldsymbol{p},s}, \qquad \eta^{-1}a^2_{\boldsymbol{p},s}\eta = -a^2_{\boldsymbol{p},s}, \qquad \eta^{-1}b^{2\dagger}_{\boldsymbol{p},s}\eta = -b^{2\dagger}_{\boldsymbol{p},s}, \tag{27}$$

the dual becomes

$$\hat{\psi}(x) = \frac{1}{\sqrt{2m}}\int \frac{d^3p}{(2\pi)^3\sqrt{2E_{\boldsymbol{p}}}}\sum_{s=1}^{2}\left\{\bar{u}_{\boldsymbol{p},s}\left[a^{1\dagger}_{\boldsymbol{p},s} + a^{2\dagger}_{\boldsymbol{p},s}\gamma^5\right]e^{ip\cdot x} + \bar{v}_{\boldsymbol{p},s}\left[b^1_{\boldsymbol{p},s} + b^2_{\boldsymbol{p},s}\gamma^5\right]e^{-ip\cdot x}\right\}. \tag{28}$$

An explicit expression for this operator η is

$$\eta = \exp\left[i\pi\int\frac{d^3\boldsymbol{p}}{(2\pi)^3}\sum_s\left(a^{2\dagger}_{\boldsymbol{p},s}a^2_{\boldsymbol{p},s} + b^{2\dagger}_{\boldsymbol{p},s}b^2_{\boldsymbol{p},s}\right)\right], \qquad \eta^{\dagger} = \eta \qquad \eta^{\dagger}\eta = 1. \tag{29}$$

The Klein-Gordon Lagrangian is now

$$\mathcal{L} = \partial^\mu \hat{\psi} \partial_\mu \psi - m^2 \bar{\psi} \psi, \tag{30}$$

where $\hat{\psi} = \eta^{-1} \bar{\psi} \eta$. The conjugated momenta $\pi_\psi, \pi_{\hat{\psi}}$ are

$$\pi_\psi = \frac{\partial \mathcal{L}}{\partial(\partial_0 \psi)} = \dot{\hat{\psi}}, \qquad \pi_{\hat{\psi}} = \frac{\partial \mathcal{L}}{\partial(\partial_0 \hat{\psi})} = \dot{\psi}. \tag{31}$$

The canonical anticommutation relations at equal times

$$\{\psi_a(\boldsymbol{x}, t), \pi_{\psi,b}(\boldsymbol{y}, t)\} = -\left\{\hat{\psi}_a(\boldsymbol{x}, t), \pi_{\hat{\psi},b}(\boldsymbol{y}, t)\right\} = i\delta_{ab}\delta^{(3)}(\boldsymbol{x} - \boldsymbol{y}), \tag{32}$$

imply now the anticommutation relations with the right sign $\Rightarrow$ no negative-norm states

$$\{a^1_{\boldsymbol{p},s}, a^{1\dagger}_{\boldsymbol{q},r}\} = (2\pi)^3 \delta_{sr} \delta^{(3)}(\boldsymbol{p} - \boldsymbol{q}), \qquad \{b^1_{\boldsymbol{p},s}, b^{1\dagger}_{\boldsymbol{q},r}\} = (2\pi)^3 \delta_{sr} \delta^{(3)}(\boldsymbol{p} - \boldsymbol{q}), \tag{33}$$

$$\{a^2_{\boldsymbol{p},s}, a^{2\dagger}_{\boldsymbol{q},r}\} = (2\pi)^3 \delta_{sr} \delta^{(3)}(\boldsymbol{p} - \boldsymbol{q}), \qquad \{b^2_{\boldsymbol{p},s}, b^{2\dagger}_{\boldsymbol{q},r}\} = (2\pi)^3 \delta_{sr} \delta^{(3)}(\boldsymbol{p} - \boldsymbol{q}). \tag{34}$$

In addition, this pseudohermitian QFT has the following features

- Microcausality

$$\{\psi_a(x), \hat{\psi}_b(y)\} = \Delta(x-y)\delta_{ab}, \qquad \{\psi_a(x), \psi_b(y)\} = 0, \qquad \{\hat{\psi}_a(x), \hat{\psi}_b(y)\} = 0, \tag{35}$$

  where $\Delta(x-y)$ is the Lorentz invariant and causal Schwinger's Green function

$$\Delta(x-y) = \int \frac{d^3p}{(2\pi)^3 2E_{\boldsymbol{p}}} \left[e^{-ip\cdot(x-y)} - e^{ip\cdot(x-y)} \right] \tag{36}$$

- Hamiltonian

$$\begin{aligned} H &= : \int d^3x \left(\partial_0\hat{\psi}\partial_0\psi + \partial_i\hat{\psi}\partial_i\psi + m^2\hat{\psi}\psi \right) : \\ &= \int \frac{d^3q}{(2\pi)^3} E_{\boldsymbol{q}} \sum_{r=1}^{2} \left\{ a^{1\dagger}_{\boldsymbol{q},r} a^{1}_{\boldsymbol{q},r} + a^{2\dagger}_{\boldsymbol{q},r} a^{2}_{\boldsymbol{q},r} + b^{1\dagger}_{\boldsymbol{q},r} b^{1}_{\boldsymbol{q},r} + b^{2\dagger}_{\boldsymbol{q},r} b^{2}_{\boldsymbol{q},r} \right\} \end{aligned} \tag{37}$$

- Momentum

$$\begin{aligned} P_i &= : -\int d^3x \left(\partial_0\hat{\psi}\partial_i\psi + \partial_i\hat{\psi}\partial_0\psi \right) : \\ &= \int \frac{d^3q}{(2\pi)^3} \boldsymbol{q} \sum_{r=1}^{2} \left\{ a^{1\dagger}_{\boldsymbol{q},r} a^{1}_{\boldsymbol{q},r} + a^{2\dagger}_{\boldsymbol{q},r} a^{2}_{\boldsymbol{q},r} + b^{1\dagger}_{\boldsymbol{q},r} b^{1}_{\boldsymbol{q},r} + b^{2\dagger}_{\boldsymbol{q},r} b^{2}_{\boldsymbol{q},r} \right\} \end{aligned} \tag{38}$$

- $U(1)$-charge

$$Q_{U(1)} = : i \int d^3x \left(\hat{\psi}\partial_0\psi - \partial_0\hat{\psi}\psi \right) : \\ = \int \frac{d^3q}{(2\pi)^3} \sum_{r=1}^{2} \left\{ a^{1\dagger}_{\boldsymbol{q},r} a^{1}_{\boldsymbol{q},r} + a^{2\dagger}_{\boldsymbol{q},r} a^{2}_{\boldsymbol{q},r} - b^{1\dagger}_{\boldsymbol{q},r} b^{1}_{\boldsymbol{q},r} - b^{2\dagger}_{\boldsymbol{q},r} b^{2}_{\boldsymbol{q},r} \right\} \tag{39}$$

- Discrete symmetries

$$P\psi(x)P^{-1} = i\gamma^0\psi(\mathcal{P}x), \qquad C\psi(x)C^{-1} = \mathcal{C}\hat{\psi}^T(x), \qquad T\psi(x)T^{-1} = \mathcal{C}\gamma^5\psi(\mathcal{T}x),$$

  where $\mathcal{C} = -i\gamma^2\gamma^0$. The theory is invariant under C, P, and T.
- Can have dimension-4 fermion self-interactions

$$\mathcal{L}_{\text{int}} = \frac{\lambda_1}{2}\left(\hat{\psi}\psi\right)^2 + \frac{\lambda_2}{2}\left(\hat{\psi}\gamma^5\psi\right)\left(\hat{\psi}\gamma^5\psi\right) + \frac{\lambda_3}{2}\left(\hat{\psi}M^{\mu\nu}\psi\right)\left(\hat{\psi}M_{\mu\nu}\psi\right) \tag{40}$$

- Renormalizable. It has been shown in[4] that its electrodynamics and self-interactions are renormalizable at one-loop.

[4] Carlos A. Vaquera-Araujo, Mauro Napsuciale, and René Ángeles-Martınez. "Renormalization of the QED of self-interacting second order spin 1/2 fermions.". In: *Journal of High Energy Physics* 2013.1 (Jan. 2013), p. 11. ISSN: 1029-8479. DOI: 10.1007/JHEP01(2013)011. URL: https://doi.org/10.1007/JHEP01(2013)011.

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4 Conclusions

Is it possible to describe the dynamics of a spin $j = 1$ matter particle $\Psi \in (1, 0) \oplus (0, 1)$ by using the Klein-Gordon equation?

$$(0, 0)$$

$$\left(\tfrac{1}{2}, 0\right) \qquad \left(0, \tfrac{1}{2}\right)$$

$$(1, 0) \qquad \left(\tfrac{1}{2}, \tfrac{1}{2}\right) \qquad (0, 1)$$

$$\vdots \qquad \vdots \qquad \vdots \qquad \vdots$$

Observations:

- It is analogous to a Dirac field, but for spin $j = 1$

$$\text{Dirac: } \psi(x) \in \left(\frac{1}{2}, 0\right) \oplus \left(0, \frac{1}{2}\right) \qquad \sim \qquad \Psi(x) \in (1, 0) \oplus (0, 1)$$

$\psi(x)$: 4-dim spinor object $\qquad$ $\Psi(x)$: 6-dim spinor-like object

- Free EOM:

$$\text{Dirac: } (i\gamma^\mu \partial_\mu - m)\psi = 0 \qquad \sim \qquad \mathcal{O}(\partial)\Psi(x) = 0?$$

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There exists a covariant basis for operators in the $(1,0) \oplus (0,1)$ rep.

- For the $(\frac{1}{2},0) \oplus (0,\frac{1}{2})$ rep: Dirac basis $\{\mathbb{1}, \gamma^5, \gamma^\mu, \gamma^5\gamma^\mu, \sigma^{\mu\nu}\}$.
- For the $(1,0) \oplus (0,1)$ rep: basis $\{\mathbb{1}, \chi, S^{\mu\nu}, \chi S^{\mu\nu}, M^{\mu\nu}, C^{\mu\nu\alpha\beta}\}$[5].
- A comparison of these two bases:

Operator	Rep $(\frac{1}{2},0) \oplus (0,\frac{1}{2})$	Rep $(1,0) \oplus (0,1)$
Dimension	4×4	6×6
Unit	$\mathbb{1}$	$\mathbb{1}$
Chirality	γ^5	χ
Vector/tensor	γ^μ	$S^{\mu\nu}$
Pseudo vector/tensor	$\gamma^5\gamma^\mu$	$\chi S^{\mu\nu}$
Lorentz-generators	$M^{\mu\nu}$	$M^{\mu\nu}$
Four-rank tensor	—	$C^{\mu\nu\alpha\beta}$

Table: Comparison of covariant basis for the $(\frac{1}{2},0) \oplus (0,\frac{1}{2})$ and $(1,0) \oplus (0,1)$ reps.

[5] Selim Gómez-Ávila and M. Napsuciale. "Covariant basis induced by parity for the $(j,0)\bigoplus(0,j)$ representation". In: *Phys. Rev. D* 88 (9 Nov. 2013), p. 096012. DOI: 10.1103/PhysRevD.88.096012. URL: https://link.aps.org/doi/10.1103/PhysRevD.88.096012.

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4. Conclusions

Klein-Gordon $j = 1$ field. Mauro equation

It has been shown in[6,7] that the free EOM for this spin $j = 1$ field $\Psi \in (1,0) \oplus (0,1)$ is

$$(\Sigma^{\mu\nu}\partial_\mu\partial_\nu + m^2)\Psi(x) = 0, \tag{41}$$

which we will call Mauro equation for simplicity, where

$$\Sigma^{\mu\nu} \equiv \frac{1}{2}\left(g^{\mu\nu} + S^{\mu\nu}\right), \tag{42}$$

and $S^{\mu\nu}$ is the tensor operator of the basis. Its components are given by the $SU(2)$ generators J^i for $j = 1$

$$S^{00} = \Pi = \begin{pmatrix} 0 & \mathbb{1} \\ \mathbb{1} & 0 \end{pmatrix}, \quad S^{0i} = \begin{pmatrix} 0 & J^i \\ -J^i & 0 \end{pmatrix}, \quad S^{ij} = \begin{pmatrix} 0 & g^{ij} + \{J^i, J^j\} \\ g^{ij} + \{J^i, J^j\} & 0 \end{pmatrix}. \tag{43}$$

The classical aspects and canonical quantization have been already studied. It has been found that it yields a well defined QFT.

[6] M. Napsuciale et al. "Spin one matter fields". In: *Phys. Rev. D* 93.7 (2016), p. 076003. DOI: 10.1103/PhysRevD.93.076003. arXiv: 1509.07938 [hep-ph].

[7] Mauro Napsuciale. "Space-time origin of gauge symmetry". In: *Physica Scripta* 98.9 (Aug. 2023), p. 095305. DOI: 10.1088/1402-4896/acecb5. URL: https://dx.doi.org/10.1088/1402-4896/acecb5.

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4. Conclusions

Klein-Gordon $j = 1$ field. Klein-Gordon vs Mauro

The dynamics of a spin $j = 1$ field $\Psi \in (1,0) \oplus (0,1)$ are described by the Mauro equation

$$(\Sigma^{\mu\nu}\partial_\mu\partial_\nu + m^2)\Psi(x) = 0. \tag{44}$$

- **Mauro solution $\Rightarrow$ KG solution**.
  Every solution of the Mauro equation solves the Klein-Gordon equation

$$(R^{\mu\nu}\partial_\mu\partial_\nu + m^2)(\Sigma^{\mu\nu}\partial_\mu\partial_\nu + m^2)\Psi(x) = m^2(\partial^2 + m^2)\Psi(x) = 0, \tag{45}$$

  where $R_{\mu\nu} = \frac{1}{2}(g_{\mu\nu} + S_{\mu\nu})$ and $(S^{\mu\nu}\partial_\mu\partial_\nu)^2 = \partial^4$. However the converse is not true, i.e. not every solution of the Klein-Gordon equation is a solution to the Mauro equation.
- **KG solution $\not\Rightarrow$ Mauro solution**.
  The most general solution of the Klein-Gordon equation is not a solution of the Mauro equation, but can be written in terms of two Mauro solutions ($S(\partial) = S^{\mu\nu}\partial_\mu\partial_\nu$)

$$\begin{aligned}\Psi &= \left[1 + \frac{1}{2m^2}\left(\partial^2 - S(\partial)\right)\right]\Psi + \left[1 + \frac{1}{2m^2}\left(\partial^2 + S(\partial)\right)\right]\Psi \\ &= \left[1 + \frac{1}{2m^2}\left(\partial^2 - S(\partial)\right)\right]\Psi + \chi\left[1 + \frac{1}{2m^2}\left(\partial^2 - S(\partial)\right)\right]\chi\Psi = \Psi_1 + \chi\Psi_2\end{aligned} \tag{46}$$

Is it possible to describe the dynamics of a spin $j = 1$ particle $\Psi \in (1,0) \oplus (0,1)$ by using just the Klein-Gordon equation?

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The Klein-Gordon Lagrangian for the spin $j = 1$ field $\Psi \in (1,0) \oplus (0,1)$ is

$$\mathcal{L} = \partial^\mu \bar{\Psi} \partial_\mu \Psi - m^2 \bar{\Psi}\Psi, \tag{47}$$

where the dual is $\bar{\Psi} = \Psi^\dagger S^{00}$. The conjugated momenta $\Pi_\Psi, \Pi_{\bar{\Psi}}$ are

$$\Pi_\Psi = \frac{\partial \mathcal{L}}{\partial(\partial_0 \Psi)} = \dot{\bar{\Psi}}, \qquad \Pi_{\bar{\Psi}} = \frac{\partial \mathcal{L}}{\partial(\partial_0 \bar{\Psi})} = \dot{\Psi}. \tag{48}$$

Observation

- The field has 12 degrees of freedom, twice those of the Mauro field.
- The mass dimension of this field is 1, the same as the Mauro field.

The Mauro decomposition of the Klein-Gordon solution is

$$\Psi = \Psi_1 + \chi\Psi_2, \qquad \bar{\Psi} = \bar{\Psi}_1 - \bar{\Psi}_2\chi, \tag{49}$$

$$\Psi(x) = \int \frac{d^3p}{(2\pi)^3\sqrt{2E_{\boldsymbol{p}}}} \sum_{s=1}^{3} \left\{ \left[a^1_{\boldsymbol{p},s} + a^2_{\boldsymbol{p},s}\chi \right] U_{\boldsymbol{p},s} e^{-ip\cdot x} + \left[b^{1\dagger}_{\boldsymbol{p},s} + b^{2\dagger}_{\boldsymbol{p},s}\chi \right] U^c_{\boldsymbol{p},s} e^{ip\cdot x} \right\} \Big|_{p^0=E_{\boldsymbol{p}}}, \tag{50}$$

$$\bar{\Psi}(x) = \int \frac{d^3p}{(2\pi)^3\sqrt{2E_{\boldsymbol{p}}}} \sum_{s=1}^{3} \left\{ \bar{U}_{\boldsymbol{p},s} \left[a^{1\dagger}_{\boldsymbol{p},s} - a^{2\dagger}_{\boldsymbol{p},s}\chi \right] e^{ip\cdot x} + \bar{U}^c_{\boldsymbol{p},s} \left[b^1_{\boldsymbol{p},s} - b^2_{\boldsymbol{p},s}\chi \right] e^{-ip\cdot x} \right\} \Big|_{p^0=E_{\boldsymbol{p}}}. \tag{51}$$

The canonical quantization starts by imposing the commutation relations at equal times

$$\left[\Psi_a(\boldsymbol{x},t), \Pi_{\Psi,b}(\boldsymbol{y},t)\right] = i\delta_{ab}\delta^{(3)}(\boldsymbol{x}-\boldsymbol{y}), \qquad \left[\bar{\Psi}_a(\boldsymbol{x},t), \Pi_{\bar{\Psi},b}(\boldsymbol{y},t)\right] = i\delta_{ab}\delta^{(3)}(\boldsymbol{x}-\boldsymbol{y}); \tag{52}$$

which imply commutation relations with the wrong sign ⇒ negative-norm states

$$[a^1_{\boldsymbol{p},s}, a^{1\dagger}_{\boldsymbol{q},r}] = (2\pi)^3\delta_{sr}\delta^{(3)}(\boldsymbol{p}-\boldsymbol{q}), \qquad [b^1_{\boldsymbol{p},s}, b^{1\dagger}_{\boldsymbol{q},r}] = (2\pi)^3\delta_{sr}\delta^{(3)}(\boldsymbol{p}-\boldsymbol{q}), \tag{53}$$

$$[a^2_{\boldsymbol{p},s}, a^{2\dagger}_{\boldsymbol{q},r}] = -(2\pi)^3\delta_{sr}\delta^{(3)}(\boldsymbol{p}-\boldsymbol{q}), \qquad [b^2_{\boldsymbol{p},s}, b^{2\dagger}_{\boldsymbol{q},r}] = -(2\pi)^3\delta_{sr}\delta^{(3)}(\boldsymbol{p}-\boldsymbol{q}). \tag{54}$$

⇒ same problem as second-order fermions

Is it possible to consistently quantize this theory as a pseudohermitian QFT?

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Klein-Gordon $j = 1$ field. KG pseudohermitian theory

The Klein-Gordon theory can be turned pseudohermitian by redefining the dual of the field Ψ as $\hat{\Psi}$

$$\hat{\Psi} = \eta^{-1}\bar{\Psi}\eta, \qquad \mathcal{L} = \partial^\mu \hat{\Psi}\partial_\mu \Psi - m^2 \hat{\Psi}\Psi, \qquad \mathcal{L}^{\#} = \eta^{-1}\mathcal{L}^\dagger \eta = \mathcal{L}. \tag{55}$$

The fields are now

$$\Psi(x) = \int \frac{d^3p}{(2\pi)^3\sqrt{2E_{\boldsymbol{p}}}} \sum_{s=1}^{3} \left\{ \left[a^1_{\boldsymbol{p},s} + a^2_{\boldsymbol{p},s}\chi \right] U_{\boldsymbol{p},s} e^{-ip\cdot x} + \left[b^{1\dagger}_{\boldsymbol{p},s} + b^{2\dagger}_{\boldsymbol{p},s}\chi \right] U^c_{\boldsymbol{p},s} e^{ip\cdot x} \right\},$$

$$\hat{\Psi}(x) = \int \frac{d^3p}{(2\pi)^3\sqrt{2E_{\boldsymbol{p}}}} \sum_{s=1}^{3} \left\{ \bar{U}_{\boldsymbol{p},s} \left[\eta^{-1} a^{1\dagger}_{\boldsymbol{p},s}\eta - \eta^{-1} a^{2\dagger}_{\boldsymbol{p},s}\eta\chi \right] e^{ip\cdot x} \right.$$

$$\left. + \bar{U}^c_{\boldsymbol{p},s} \left[\eta^{-1} b^1_{\boldsymbol{p},s}\eta - \eta^{-1} b^2_{\boldsymbol{p},s}\eta\chi \right] e^{-ip\cdot x} \right\}.$$

Defining the action of η on the creation/anihilation operators as

$$\eta^{-1} a^1_{\boldsymbol{p},s}\eta = a^1_{\boldsymbol{p},s}, \qquad \eta^{-1} b^{1\dagger}_{\boldsymbol{p},s}\eta = b^{1\dagger}_{\boldsymbol{p},s}, \qquad \eta^{-1} a^2_{\boldsymbol{p},s}\eta = -a^2_{\boldsymbol{p},s}, \qquad \eta^{-1} b^{2\dagger}_{\boldsymbol{p},s}\eta = -b^{2\dagger}_{\boldsymbol{p},s}, \tag{56}$$

the dual becomes

$$\hat{\Psi}(x) = \int \frac{d^3p}{(2\pi)^3\sqrt{2E_{\boldsymbol{p}}}} \sum_{s=1}^{3} \left\{ \bar{U}_{\boldsymbol{p},s} \left[a^{1\dagger}_{\boldsymbol{p},s} + a^{2\dagger}_{\boldsymbol{p},s}\chi \right] e^{ip\cdot x} + \bar{U}^c_{\boldsymbol{p},s} \left[b^1_{\boldsymbol{p},s} + b^2_{\boldsymbol{p},s}\chi \right] e^{-ip\cdot x} \right\} \tag{57}$$

An explicit expression for this operator η is

$$\eta = \exp\left[i\pi \int \frac{d^3\boldsymbol{p}}{(2\pi)^3} \sum_s \left(a^{2\dagger}_{\boldsymbol{p},s} a^2_{\boldsymbol{p},s} + b^{2\dagger}_{\boldsymbol{p},s} b^2_{\boldsymbol{p},s} \right) \right], \qquad \eta^\dagger = \eta \qquad \eta^\dagger\eta = 1. \tag{58}$$

Klein-Gordon $j = 1$ field. KG pseudohermitian theory

The Klein-Gordon Lagrangian is now

$$\mathcal{L} = \partial^\mu \hat{\Psi} \partial_\mu \Psi - m^2 \bar{\psi} \Psi, \tag{59}$$

where $\hat{\Psi} = \eta^{-1} \bar{\Psi} \eta$. The conjugated momenta $\Pi_\Psi, \Pi_{\hat{\Psi}}$ are

$$\Pi_\psi = \frac{\partial \mathcal{L}}{\partial(\partial_0 \Psi)} = \dot{\hat{\Psi}}, \qquad \Pi_{\hat{\Psi}} = \frac{\partial \mathcal{L}}{\partial(\partial_0 \hat{\Psi})} = \dot{\Psi}. \tag{60}$$

The canonical commutation relations at equal times

$$\left[\Psi_a(\boldsymbol{x}, t), \Pi_{\Psi,b}(\boldsymbol{y}, t)\right] = i\delta_{ab}\delta^{(3)}(\boldsymbol{x} - \boldsymbol{y}), \qquad \left[\bar{\Psi}_a(\boldsymbol{x}, t), \Pi_{\bar{\Psi},b}(\boldsymbol{y}, t)\right] = i\delta_{ab}\delta^{(3)}(\boldsymbol{x} - \boldsymbol{y}); \tag{61}$$

imply now the commutation relations with the right sign $\Rightarrow$ no negative-norm states

$$[a^1_{\boldsymbol{p},s}, a^{1\dagger}_{\boldsymbol{q},r}] = (2\pi)^3 \delta_{sr} \delta^{(3)}(\boldsymbol{p} - \boldsymbol{q}), \qquad [b^1_{\boldsymbol{p},s}, b^{1\dagger}_{\boldsymbol{q},r}] = (2\pi)^3 \delta_{sr} \delta^{(3)}(\boldsymbol{p} - \boldsymbol{q}), \tag{62}$$

$$[a^2_{\boldsymbol{p},s}, a^{2\dagger}_{\boldsymbol{q},r}] = (2\pi)^3 \delta_{sr} \delta^{(3)}(\boldsymbol{p} - \boldsymbol{q}), \qquad [b^2_{\boldsymbol{p},s}, b^{2\dagger}_{\boldsymbol{q},r}] = (2\pi)^3 \delta_{sr} \delta^{(3)}(\boldsymbol{p} - \boldsymbol{q}). \tag{63}$$

In addition, this pseudohermitian QFT has the following features

- Microcausality

$$[\Psi_a(x), \hat{\Psi}_b(y)] = \Delta(x-y)\delta_{ab}, \qquad [\Psi_a(x), \Psi_b(y)] = 0, \qquad [\hat{\Psi}_a(x), \hat{\Psi}_b(y)] = 0, \tag{64}$$

where $\Delta(x-y)$ is the Lorentz invariant and causal Schwinger's Green function

$$\Delta(x-y) = \int \frac{d^3p}{(2\pi)^3 2E_{\boldsymbol{p}}} \left[e^{-ip\cdot(x-y)} - e^{ip\cdot(x-y)} \right] \tag{65}$$

- Hamiltonian

$$\begin{aligned} H &= : \int d^3x \left(\partial_0 \hat{\Psi} \partial_0 \Psi + \partial_i \hat{\Psi} \partial_i \Psi + m^2 \hat{\Psi} \Psi \right) : \\ &= \int \frac{d^3q}{(2\pi)^3} E_{\boldsymbol{q}} \sum_{r=1}^{3} \left\{ a^{1\dagger}_{\boldsymbol{q},r} a^{1}_{\boldsymbol{q},r} + a^{2\dagger}_{\boldsymbol{q},r} a^{2}_{\boldsymbol{q},r} + b^{1\dagger}_{\boldsymbol{q},r} b^{1}_{\boldsymbol{q},r} + b^{2\dagger}_{\boldsymbol{q},r} b^{2}_{\boldsymbol{q},r} \right\} \end{aligned} \tag{66}$$

- Momentum

$$\begin{aligned} P_i &= : - \int d^3x \left(\partial_0 \hat{\Psi} \partial_i \Psi + \partial_i \hat{\Psi} \partial_0 \Psi \right) : \\ &= \int \frac{d^3q}{(2\pi)^3} \boldsymbol{q} \sum_{r=1}^{3} \left\{ a^{1\dagger}_{\boldsymbol{q},r} a^{1}_{\boldsymbol{q},r} + a^{2\dagger}_{\boldsymbol{q},r} a^{2}_{\boldsymbol{q},r} + b^{1\dagger}_{\boldsymbol{q},r} b^{1}_{\boldsymbol{q},r} + b^{2\dagger}_{\boldsymbol{q},r} b^{2}_{\boldsymbol{q},r} \right\} \end{aligned} \tag{67}$$

- $U(1)$-charge

$$\begin{aligned} Q_{U(1)} &= : i \int d^3x \left(\hat{\Psi}\partial_0\Psi - \partial_0\hat{\Psi}\Psi \right) : \\ &= \int \frac{d^3q}{(2\pi)^3} \sum_{r=1}^{3} \left\{ a^{1\dagger}_{\boldsymbol{q},r} a^{1}_{\boldsymbol{q},r} + a^{2\dagger}_{\boldsymbol{q},r} a^{2}_{\boldsymbol{q},r} - b^{1\dagger}_{\boldsymbol{q},r} b^{1}_{\boldsymbol{q},r} - b^{2\dagger}_{\boldsymbol{q},r} b^{2}_{\boldsymbol{q},r} \right\} \end{aligned} \tag{68}$$

- Discrete symmetries (up to a phase)

$$P\Psi(x)P^{-1} = S^{00}\Psi(\mathcal{P}x), \quad C\Psi(x)C^{-1} = -S^{22}S^{00}\hat{\Psi}^{T}(x), \quad T\Psi(x)T^{-1} = S^{11}S^{33}\Psi(\mathcal{T}x),$$

  The theory is invariant under C, P, and T.

- Can have dimension-4 self-interactions

$$\mathcal{L}_{\text{int}} = \frac{\lambda_1}{2}\left(\hat{\Psi}\Psi\right)^2 + \frac{\lambda_2}{2}\left(\hat{\Psi}\chi\Psi\right)\left(\hat{\Psi}\chi\Psi\right) + \frac{\lambda_3}{2}\left(\hat{\Psi}M^{\mu\nu}\Psi\right)\left(\hat{\Psi}M_{\mu\nu}\Psi\right) + \frac{\lambda_3}{2}\left(\hat{\Psi}S^{\mu\nu}\Psi\right)\left(\hat{\Psi}S_{\mu\nu}\Psi\right)$$

- Renormalizable. It has been shown in[8] that the electrodynamics and self-interactions are renormalizable at one-loop.

[8] Ailier Rivero-Acosta and Carlos A. Vaquera-Araujo. "Renormalization of a model for spin-1 matter fields". In: *The European Physical Journal C* 80.7 (July 2020), p. 618. ISSN: 1434-6052. DOI: 10.1140/epjc/s10052-020-8190-5. URL: https://doi.org/10.1140/epjc/s10052-020-8190-5.

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Conclusions

For the KG hermitian theory:

- The theory has negative-norm states. Inconsistent QFT.

Alternative approach: pseudohermiticity

- An operator H is pseudohermitian if it satisfies $H^{\#} = \eta^{-1} H^{\dagger} \eta = H$.
- Redefinition of the inner product of states $\langle a(t)|b(t)\rangle_{\eta} = \langle a(t)|\eta|b(t)\rangle$.
- Real energy spectrum.
- Unitary time evolution.

For the KG pseudohermitian theory:

- The theory doesn't have negative-norm states.
- Causal theory.
- Real energy spectrum.
- Unitary time evolution.
- Hamlitonian bounded from below.
- C,P,T invariant.
- Renormalizable.

Thanks

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The commutation relations for bosons are not the same as the anticommutation relations for fermions. However, the relations needed to calculate the commutators/anticommutators of creation and anihilation operators (shown in blue) are exactly the same for bosons and fermions, this is why both theories are quite similar.

$(1/2, 0) \oplus (1/2, 0)$ Fermions

$$\left\{\psi_a(\boldsymbol{x},t), \pi_{\psi,b}(\boldsymbol{y},t)\right\} = i\delta_{ab}\delta^{(3)}(\boldsymbol{x}-\boldsymbol{y}) \qquad \left\{\bar{\psi}_a(\boldsymbol{x},t), \pi_{\bar{\psi},b}(\boldsymbol{y},t)\right\} = -i\delta_{ab}\delta^{(3)}(\boldsymbol{x}-\boldsymbol{y}) \tag{69}$$

$$\left\{\pi_{\psi,b}(\boldsymbol{y},t), \psi_a(\boldsymbol{x},t)\right\} = i\delta_{ab}\delta^{(3)}(\boldsymbol{x}-\boldsymbol{y}) \qquad \left\{\pi_{\bar{\psi},b}(\boldsymbol{y},t), \bar{\psi}_a(\boldsymbol{x},t)\right\} = -i\delta_{ab}\delta^{(3)}(\boldsymbol{x}-\boldsymbol{y}) \tag{70}$$

$(1, 0) \oplus (1, 0)$ Bosons

$$\left[\Psi_a(\boldsymbol{x},t), \Pi_{\Psi,b}(\boldsymbol{y},t)\right] = i\delta_{ab}\delta^{(3)}(\boldsymbol{x}-\boldsymbol{y}) \qquad \left[\bar{\Psi}_a(\boldsymbol{x},t), \Pi_{\bar{\Psi},b}(\boldsymbol{y},t)\right] = i\delta_{ab}\delta^{(3)}(\boldsymbol{x}-\boldsymbol{y}) \tag{71}$$

$$\left[\Pi_{\Psi,b}(\boldsymbol{y},t), \Psi_a(\boldsymbol{x},t)\right] = -i\delta_{ab}\delta^{(3)}(\boldsymbol{x}-\boldsymbol{y}) \qquad \left[\Pi_{\bar{\Psi},b}(\boldsymbol{y},t), \bar{\Psi}_a(\boldsymbol{x},t)\right] = -i\delta_{ab}\delta^{(3)}(\boldsymbol{x}-\boldsymbol{y}) \tag{72}$$