Exact neutrino oscillation probabilities with the time evolution operator

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Motivation

- Neutrino oscillations are well-established and extensively studied, and it has proven beyond any doubt that neutrinos are mixed massive particles.
- Right now, the three-flavor neutrino model provides the best possible theoretical paradigm for describing the underlying physics of most present and future experiments.
- Matter effects need to be incorporated into the theory since, in most scenarios, neutrinos propagate in big chunks of terrestrial matter (NOVA, DUNE, T2K).
- Analytical solutions are needed to understand the size of matter effects. The exact solution to this problem can be achieved using the time evolution operator.
- Deriving approximate solutions from the exact expressions is more accessible than the perturbative approach.
- This method only requires the eigenvalues of the Hamiltonian. No eigenvectors are required. Approximate eigenvalues can work nicely, too.

Theory

Let $\hbar = c = 1$. The starting point is the Schrödinger like equation for the evolution operator $\hat{\mathscr{U}}(r, r_0)$.

$$i\frac{\mathrm{d}}{\mathrm{d}r}\hat{\mathcal{U}}(r,r_0) = \hat{H}(r)\hat{\mathcal{U}}(r,r_0) \tag{1}$$

Then, the flavor transition probabilities can be computed via the relation $P_{\alpha\beta} = |\hat{\mathscr{U}}_{\beta\alpha}|^2$. In the above equation, $\hat{H}(r)$ is the hamiltonian in matter in the flavor basis:

$$\hat{H}(r) = UH_0U^\dagger + V_{\mathrm{CC}}(r)Y = \mathcal{O}_{23}\Gamma\left[\mathcal{O}_{13}\mathcal{O}_{12}H_0\mathcal{O}_{12}^{\mathsf{t}}\mathcal{O}_{13}^{\mathsf{t}} + V_{\mathrm{CC}}(r)Y\right]\Gamma^\dagger\mathcal{O}_{23}^{\mathsf{t}} \tag{2}$$

Where U is the leptonic mixing matriz (PMNS), $\mathscr{O}_{ij}$ are orthogonal rotations matrices around the plane ij, $Y = \mathrm{diag}(1,0,0)$, and $V_{\mathrm{CC}}(r) = \sqrt{2}G_{\mathrm{F}}Y_e\frac{\rho(r)}{m_u}$ is the neutrino-charged potential.

$$H_0 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & \Delta_{21} & 0 \\ 0 & 0 & \Delta_{31} \end{pmatrix} \quad \mathcal{O}_{12} = \begin{pmatrix} \mathrm{c}_{12} & \mathrm{s}_{12} & 0 \\ -\mathrm{s}_{12} & \mathrm{c}_{12} & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad \mathcal{O}_{13} = \begin{pmatrix} \mathrm{c}_{13} & 0 & \mathrm{s}_{13} \\ 0 & 1 & 0 \\ -\mathrm{s}_{13} & 0 & \mathrm{c}_{13} \end{pmatrix} \quad \Gamma = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & e^{i\delta} \end{pmatrix} \quad \mathcal{O}_{23} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \mathrm{c}_{23} & \mathrm{s}_{23} \\ 0 & -\mathrm{s}_{23} & \mathrm{c}_{23} \end{pmatrix}$$

$$\Delta_{kj} \equiv \frac{\Delta m_{kj}^2}{2E} = \frac{m_k^2 - m_j^2}{2E}$$

Neutrino mass difference

Since matrices $\mathcal{O}_{23}$ and Γ can be factorized from both sides in the expression for effective Hamiltonian in the flavor basis, we can define an auxiliary operator $\tilde{\mathcal{U}}(r, r_0)$:

$$\tilde{\mathcal{U}}(r, r_0) \equiv \begin{pmatrix} \tilde{\mathcal{U}}_{ee} & \tilde{\mathcal{U}}_{ea} & \tilde{\mathcal{U}}_{eb} \\ \tilde{\mathcal{U}}_{ae} & \tilde{\mathcal{U}}_{aa} & \tilde{\mathcal{U}}_{ab} \\ \tilde{\mathcal{U}}_{be} & \tilde{\mathcal{U}}_{ba} & \tilde{\mathcal{U}}_{bb} \end{pmatrix} = \Gamma^{\dagger} \mathcal{O}_{23}^{\mathsf{t}} \begin{pmatrix} \hat{\mathcal{U}}_{\mathrm{ee}} & \hat{\mathcal{U}}_{\mathrm{e}\mu} & \hat{\mathcal{U}}_{\mathrm{e}\tau} \\ \hat{\mathcal{U}}_{\mu\mathrm{e}} & \hat{\mathcal{U}}_{\mu\mu} & \hat{\mathcal{U}}_{\mu\tau} \\ \hat{\mathcal{U}}_{\tau\mathrm{e}} & \hat{\mathcal{U}}_{\tau\mu} & \hat{\mathcal{U}}_{\tau\tau} \end{pmatrix} \mathcal{O}_{23} \Gamma \tag{3}$$

which obeys the evolution equation

$$i \frac{\mathrm{d}}{\mathrm{d}r} \tilde{\mathcal{U}}(r, r_0) = \tilde{H}(r) \tilde{\mathcal{U}}(r, r_0) \tag{4}$$

with the new Hamiltonian $\tilde{H}$ given by

$$\tilde{H}(r) \equiv \mathcal{O}_{13} \mathcal{O}_{12} H_0 \mathcal{O}_{12}^{\mathsf{t}} \mathcal{O}_{13}^{\mathsf{t}} + V_{\mathrm{CC}}(r) Y \tag{5}$$

Therefore, by solving equation (4) for $\tilde{\mathcal{U}}(r, r_0)$, we can compute all the neutrino probabilities.

For a symmetric potential, the matrix elements of the operator $\hat{\mathcal{U}}(r, r_0)$ can be expressed in terms of the components of $\tilde{\mathcal{U}}(r, r_0)$ as follow:

$$\hat{\mathcal{U}}_{ee} = \tilde{\mathcal{U}}_{ee},$$ No dependence on θ_{23} and δ

$$\hat{\mathcal{U}}_{\mu e} = c_{23}\tilde{\mathcal{U}}_{ea} + s_{23}e^{i\delta}\tilde{\mathcal{U}}_{eb},$$
$$\hat{\mathcal{U}}_{\tau e} = -s_{23}\tilde{\mathcal{U}}_{ea} + c_{23}e^{i\delta}\tilde{\mathcal{U}}_{eb},$$
$$\hat{\mathcal{U}}_{e\mu} = c_{23}\tilde{\mathcal{U}}_{ea} + s_{23}e^{-i\delta}\tilde{\mathcal{U}}_{eb},$$
$$\hat{\mathcal{U}}_{\mu\mu} = c_{23}^2\tilde{\mathcal{U}}_{aa} + s_{23}^2\tilde{\mathcal{U}}_{bb} + s_{2\theta_{23}}\tilde{\mathcal{U}}_{ab}\cos\delta,$$
$$\hat{\mathcal{U}}_{\tau\mu} = s_{23}c_{23}(\tilde{\mathcal{U}}_{bb} - \tilde{\mathcal{U}}_{aa}) + (c_{23}^2 e^{i\delta} - s_{23}^2 e^{-i\delta})\tilde{\mathcal{U}}_{ab},$$
$$\hat{\mathcal{U}}_{e\tau} = -s_{23}\tilde{\mathcal{U}}_{ea} + c_{23}e^{-i\delta}\tilde{\mathcal{U}}_{eb},$$
$$\hat{\mathcal{U}}_{\mu\tau} = s_{23}c_{23}(\tilde{\mathcal{U}}_{bb} - \tilde{\mathcal{U}}_{aa}) + (c_{23}^2 e^{-i\delta} - s_{23}^2 e^{i\delta})\tilde{\mathcal{U}}_{ab},$$
$$\hat{\mathcal{U}}_{\tau\tau} = s_{23}^2\tilde{\mathcal{U}}_{aa} + c_{23}^2\tilde{\mathcal{U}}_{bb} - s_{2\theta_{23}}\tilde{\mathcal{U}}_{ab}\cos\delta.$$

$$\hat{\mathcal{U}}_{\alpha\beta}(\delta) = \hat{\mathcal{U}}_{\beta\alpha}(-\delta) \qquad (6)$$

$$\sum_{\alpha} P_{\alpha\beta} = \sum_{\beta} P_{\alpha\beta} = 1 \qquad (7)$$

We only need to focus on three probabilities. The other ones can be derived using (6) or (7). We have chosen the following three: P_{ee}, $P_{\mu e}$, and $P_{\mu\tau}$.

Propagation Basis $\tilde{\mathcal{U}}(r, r_0)$

Blennow, M. & Smirnov, A. Y. Neutrino propagation in matter. *Advances in High Energy Physics* **2013**, 1–33 (2013).

Since $\tilde{\mathcal{U}}\tilde{\mathcal{U}}^\dagger = \mathbb{I}$, the electron survival probability can be written as:

$$P_{ee} = 1 - |\tilde{\mathcal{U}}_{ea}|^2 - |\tilde{\mathcal{U}}_{eb}|^2 \tag{8}$$

The muon to electron and to tau neutrino conversion probabilities can be written as:

$$P_{\mu e} = c_{23}^2 |\tilde{\mathcal{U}}_{ea}|^2 + s_{23}^2 |\tilde{\mathcal{U}}_{eb}|^2 + s_{2\theta_{23}} \,\mathrm{Re}(\tilde{\mathcal{U}}_{ea}\tilde{\mathcal{U}}_{eb}^*) \cos\delta - s_{2\theta_{23}} \,\mathrm{Im}(\tilde{\mathcal{U}}_{ea}\tilde{\mathcal{U}}_{eb}^*) \sin\delta \tag{9}$$

And

$$\begin{aligned} P_{\mu\tau} = \frac{1}{4} s_{2\theta_{23}}^2 |\tilde{\mathcal{U}}_{bb} - \tilde{\mathcal{U}}_{aa}|^2 + (1 - s_{2\theta_{23}}^2 \cos^2\delta)|\tilde{\mathcal{U}}_{ab}|^2 + \frac{1}{2} s_{4\theta_{23}} \,\mathrm{Re}[(\tilde{\mathcal{U}}_{bb} - \tilde{\mathcal{U}}_{aa})\tilde{\mathcal{U}}_{ab}^*] \cos\delta \\ + s_{2\theta_{23}} \,\mathrm{Im}[(\tilde{\mathcal{U}}_{bb} - \tilde{\mathcal{U}}_{aa})\tilde{\mathcal{U}}_{ab}^*] \sin\delta. \end{aligned} \tag{10}$$

For the antineutrinos, the substitutions $\delta \longrightarrow -\delta$ and $V_{\mathrm{CC}} \longrightarrow -V_{\mathrm{CC}}$ must be performed. Therefore, we need to compute $\tilde{\mathcal{U}}(r, r_0)$ and perform the operations indicated in expressions (8), (9) and (10) to completely solve the problem.

For a constant (i.e., symmetric) potential:

$$\tilde{\mathcal{U}}(L) = \exp\left(-iL\tilde{H}\right) \tag{11}$$

where $L = r - r_0$ (average path traveled in the media). This exponential function can be calculated explicitly using the *Caley-Hamilton* theorem. This method only needs the eigenvalues of $\tilde{H}$, a real symmetric matrix. Explicitly, the Hamiltonian $\tilde{H}$ is of the form:

$$\tilde{H} = \begin{pmatrix} s_{12}^2\Delta_{21} + s_{13}^2\Delta_{ee} + V_{CC} & \frac{1}{2}s_{2\theta_{12}}c_{13}\Delta_{21} & \frac{1}{2}s_{2\theta_{13}}\Delta_{ee} \\ \frac{1}{2}s_{2\theta_{12}}c_{13}\Delta_{21} & c_{12}^2\Delta_{21} & -\frac{1}{2}s_{2\theta_{12}}s_{13}\Delta_{21} \\ \frac{1}{2}s_{2\theta_{13}}\Delta_{ee} & -\frac{1}{2}s_{2\theta_{12}}s_{13}\Delta_{21} & s_{12}^2\Delta_{21} + c_{13}^2\Delta_{ee} \end{pmatrix} \tag{12}$$

with $\Delta_{ee} \equiv \Delta_{31} - s_{12}^2\Delta_{21}$. The eigenvalues of $\tilde{H}$ can be obtained by solving the characteristic equation

$$\xi^3 - \kappa_1\xi^2 + \kappa_2\xi - \kappa_3 = 0 \tag{13}$$

where κ_1, κ_2, and κ_3 are quantities that can be expressed in terms of the trace and the determinant of $\tilde{H}$:

$$\kappa_1 = \mathrm{Tr}\Big(\tilde{H}\Big), \qquad \kappa_2 = \tfrac{1}{2}\left[\mathrm{Tr}^2(\tilde{H}) - \mathrm{Tr}\Big(\tilde{H}^2\Big)\right], \quad \text{and} \quad \kappa_3 = \det\Big(\tilde{H}\Big). \tag{14}$$

Or explicitly given by

$$\kappa_1 = \Delta_{21} + \Delta_{31} + V_{\mathrm{CC}}, \qquad \kappa_2 = \Delta_{21}\Delta_{31} + V_{\mathrm{CC}}(\Delta_{21} + \mathrm{c}_{13}^2\Delta_{ee}), \quad \text{and} \quad \kappa_3 = \mathrm{c}_{12}^2\mathrm{c}_{13}^2\Delta_{21}\Delta_{31}V_{\mathrm{CC}}. \tag{15}$$

The eigenvalues are given in closed form from the three real solution to cubic equation:

$$\xi_k = \frac{\kappa_1}{3} + \frac{2}{3}\sqrt{\kappa_1^2 - 3\kappa_2}\cos\left[\frac{1}{3}\arccos\left(\frac{2\kappa_1^3 + 27\kappa_3 - 9\kappa_1\kappa_2}{2(\kappa_1^2 - 3\kappa_2)^{3/2}}\right) - \frac{2\pi}{3}(3-k)\right] \tag{16}$$

In the previous expression, we have the order $\xi_1 < \xi_2 < \xi_3$. In the case of inverted mass ordering, one must reassign the subscripts labels to achieve the vacuum condition $\Delta_{32} < 0 < \Delta_{21}$. This can be done by defining $\xi_1^{\mathrm{IO}} = \xi_2(-|\Delta_{32}|)$, $\xi_2^{\mathrm{IO}} = \xi_3(-|\Delta_{32}|)$, and $\xi_3^{\mathrm{IO}} = \xi_1(-|\Delta_{32}|)$.

Now that we have the exact eigenvalues at our disposal, we can compute $\tilde{\mathscr{U}}(L)$ as an application of the Caley-Hamilton theorem. This theorem allows us to write the time evolution operator as

$$\tilde{\mathcal{U}}(L) = e^{-iL\tilde{H}} = \mathsf{F}_0\mathbb{I} + \mathsf{F}_1\tilde{H} + \mathsf{F}_2\tilde{H}^2 \tag{17}$$

Where the coefficients F_0, F_1, and F_3 are given by

$$\begin{pmatrix} \mathsf{F}_0 \\ \mathsf{F}_1 \\ \mathsf{F}_2 \end{pmatrix} = \frac{e^{-iL\xi_1}}{\omega_{21}\omega_{31}\omega_{32}} \begin{pmatrix} \xi_2\xi_3\omega_{32} - \xi_1\xi_3\omega_{31}e^{-i\phi_{21}} + \xi_1\xi_2\omega_{21}e^{-i\phi_{31}} \\ -(\xi_2+\xi_3)\omega_{32} + (\xi_1+\xi_3)\omega_{31}e^{-i\phi_{21}} - (\xi_1+\xi_2)\omega_{21}e^{-i\phi_{31}} \\ \omega_{32} - \omega_{31}e^{-i\phi_{21}} + \omega_{21}e^{-i\phi_{31}} \end{pmatrix} \tag{18}$$

Here, $\omega_{kj} \equiv \xi_k - \xi_j$ and $\phi_{kj} = \omega_{kj}L$ is the eigenvalue difference in matter and the product of this three quantities $(\omega_{21}\omega_{31}\omega_{32})$ gives an exact quantity

$$\frac{1}{\omega_{21}\omega_{31}\omega_{32}} \equiv \frac{1}{\Omega} = \frac{3\sqrt{3}}{\sqrt{4\left(\kappa_1^2 - 3\kappa_2\right)^3 - \left(2\kappa_1^3 - 9\kappa_1\kappa_2 + 27\kappa_3\right)^2}} \tag{19}$$

By using the relations given in expression (14), one can simplify the structure of the matrix elements of F_1 and $\tilde{H}^2$ to achieve compact expressions for the non-diagonal elements. For example, $\tilde{\mathcal{U}}_{ea}$, $\tilde{\mathcal{U}}_{eb}$, and $\tilde{\mathcal{U}}_{ab}$ are given by

$$\begin{aligned}
\tilde{\mathcal{U}}_{ea} &= \mathrm{c}_{13}\mathrm{s}_{2\theta_{12}}\frac{\Delta_{21}}{2\Omega}\left[\omega_{32}\left(\xi_1-\Delta_{31}\right)-\omega_{31}\left(\xi_2-\Delta_{31}\right)e^{-i\omega_{21}L}+\omega_{21}\left(\xi_3-\Delta_{31}\right)e^{-i\omega_{31}L}\right]e^{-i\xi_1 L}\\
\tilde{\mathcal{U}}_{eb} &= \mathrm{s}_{2\theta_{13}}\frac{\Delta_{ee}}{2\Omega}\left[\omega_{32}\left(\xi_1-\mathrm{c}_{12}^2\tfrac{\Delta_{31}}{\Delta_{ee}}\Delta_{21}\right)-\omega_{31}\left(\xi_2-\mathrm{c}_{12}^2\tfrac{\Delta_{31}}{\Delta_{ee}}\Delta_{21}\right)e^{-i\omega_{21}L}+\omega_{21}\left(\xi_3-\mathrm{c}_{12}^2\tfrac{\Delta_{31}}{\Delta_{ee}}\Delta_{21}\right)e^{-i\omega_{31}L}\right]e^{-i\xi_1 L}\\
\tilde{\mathcal{U}}_{ab} &= -\mathrm{s}_{13}\mathrm{s}_{2\theta_{12}}\frac{\Delta_{21}}{2\Omega}\left[\omega_{32}\left(\xi_1-\Delta_{31}-V_{\mathrm{CC}}\right)-\omega_{31}\left(\xi_2-\Delta_{31}-V_{\mathrm{CC}}\right)e^{-i\omega_{21}L}+\omega_{21}\left(\xi_3-\Delta_{31}-V_{\mathrm{CC}}\right)e^{-i\omega_{31}L}\right]e^{-i\xi_1 L}
\end{aligned} \tag{20}$$

From the first two amplitudes, we can write neutrino electron survival probability as

$$P_{ee} = 1-\Lambda_{21}^{ee}\sin^2\left(\frac{\omega_{21}L}{2}\right)-\Lambda_{31}^{ee}\sin^2\left(\frac{\omega_{31}L}{2}\right)-\Lambda_{32}^{ee}\sin^2\left(\frac{\omega_{32}L}{2}\right) \tag{21}$$

While muon to electron to electron can be written as

$$P_{\mu e} = \Lambda_{21}^{\mu e}\sin^2\left(\frac{\omega_{21}L}{2}\right)+\Lambda_{31}^{\mu e}\sin^2\left(\frac{\omega_{31}L}{2}\right)+\Lambda_{32}^{\mu e}\sin^2\left(\frac{\omega_{32}L}{2}\right)-8\frac{\Delta_{21}\Delta_{31}\Delta_{32}}{\omega_{21}\omega_{31}\omega_{32}}\mathcal{J}\sin\left(\frac{\omega_{21}L}{2}\right)\sin\left(\frac{\omega_{31}L}{2}\right)\sin\left(\frac{\omega_{32}L}{2}\right) \tag{22}$$

In the previous equations

P_{ee}

$$\Lambda^{ee}_{21} = \frac{1}{\Omega\omega_{21}}\left[c^2_{13}s^2_{2\theta_{12}}\Delta^2_{21}(\xi_1-\Delta_{31})(\xi_2-\Delta_{31}) + s^2_{2\theta_{13}}\Delta^2_{ee}\left(\xi_1 - c^2_{12}\frac{\Delta_{31}}{\Delta_{ee}}\Delta_{21}\right)\left(\xi_2 - c^2_{12}\frac{\Delta_{31}}{\Delta_{ee}}\Delta_{21}\right)\right]$$
$$\Lambda^{ee}_{31} = \frac{-1}{\Omega\omega_{31}}\left[c^2_{13}s^2_{2\theta_{12}}\Delta^2_{21}(\xi_1-\Delta_{31})(\xi_3-\Delta_{31}) + s^2_{2\theta_{13}}\Delta^2_{ee}\left(\xi_1 - c^2_{12}\frac{\Delta_{31}}{\Delta_{ee}}\Delta_{21}\right)\left(\xi_3 - c^2_{12}\frac{\Delta_{31}}{\Delta_{ee}}\Delta_{21}\right)\right] \quad (23)$$
$$\Lambda^{ee}_{32} = \frac{1}{\Omega\omega_{32}}\left[c^2_{13}s^2_{2\theta_{12}}\Delta^2_{21}(\xi_2-\Delta_{31})(\xi_3-\Delta_{31}) + s^2_{2\theta_{13}}\Delta^2_{ee}\left(\xi_2 - c^2_{12}\frac{\Delta_{31}}{\Delta_{ee}}\Delta_{21}\right)\left(\xi_3 - c^2_{12}\frac{\Delta_{31}}{\Delta_{ee}}\Delta_{21}\right)\right]$$

$P_{\mu e}$

$$\Lambda^{\mu e}_{21} = \frac{1}{\Omega\omega_{21}}\left\{c^2_{23}c^2_{13}s^2_{2\theta_{12}}\Delta^2_{21}(\xi_1-\Delta_{31})(\xi_2-\Delta_{31}) + s^2_{23}s^2_{2\theta_{13}}\Delta^2_{ee}\left(\xi_1 - c^2_{12}\frac{\Delta_{31}}{\Delta_{ee}}\Delta_{21}\right)\left(\xi_2 - c^2_{12}\frac{\Delta_{31}}{\Delta_{ee}}\Delta_{21}\right)\right.$$
$$\left. + 4\Delta_{21}\Delta_{ee}\left[\left(\xi_1 - c^2_{12}\frac{\Delta_{31}}{\Delta_{ee}}\Delta_{21}\right)(\xi_2-\Delta_{31}) + (\xi_1-\Delta_{31})\left(\xi_2 - c^2_{12}\frac{\Delta_{31}}{\Delta_{ee}}\Delta_{21}\right)\right]\mathcal{J}_r c_\delta\right\}$$
$$\Lambda^{\mu e}_{31} = \frac{-1}{\Omega\omega_{31}}\left\{c^2_{23}c^2_{13}s^2_{2\theta_{12}}\Delta^2_{21}(\xi_1-\Delta_{31})(\xi_3-\Delta_{31}) + s^2_{23}s^2_{2\theta_{13}}\Delta^2_{ee}\left(\xi_1 - c^2_{12}\frac{\Delta_{31}}{\Delta_{ee}}\Delta_{21}\right)\left(\xi_3 - c^2_{12}\frac{\Delta_{31}}{\Delta_{ee}}\Delta_{21}\right)\right.$$
$$\left. + 4\Delta_{21}\Delta_{ee}\left[\left(\xi_1 - c^2_{12}\frac{\Delta_{31}}{\Delta_{ee}}\Delta_{21}\right)(\xi_3-\Delta_{31}) + (\xi_1-\Delta_{31})\left(\xi_3 - c^2_{12}\frac{\Delta_{31}}{\Delta_{ee}}\Delta_{21}\right)\right]\mathcal{J}_r c_\delta\right\} \quad (24)$$
$$\Lambda^{\mu e}_{32} = \frac{1}{\Omega\omega_{32}}\left\{c^2_{23}c^2_{13}s^2_{2\theta_{12}}\Delta^2_{21}(\xi_2-\Delta_{31})(\xi_3-\Delta_{31}) + s^2_{23}s^2_{2\theta_{13}}\Delta^2_{ee}\left(\xi_2 - c^2_{12}\frac{\Delta_{31}}{\Delta_{ee}}\Delta_{21}\right)\left(\xi_3 - c^2_{12}\frac{\Delta_{31}}{\Delta_{ee}}\Delta_{21}\right)\right.$$
$$\left. + 4\Delta_{21}\Delta_{ee}\left[\left(\xi_2 - c^2_{12}\frac{\Delta_{31}}{\Delta_{ee}}\Delta_{21}\right)(\xi_3-\Delta_{31}) + (\xi_2-\Delta_{31})\left(\xi_3 - c^2_{12}\frac{\Delta_{31}}{\Delta_{ee}}\Delta_{21}\right)\right]\mathcal{J}_r c_\delta\right\}$$

$$\mathcal{J} = \mathrm{Im}(U^*_{e3}U_{\mu 3}U_{e2}U^*_{\mu 2}) = \mathcal{J}_r \sin\delta$$

Jarlskog invariant

$$\mathcal{J}_r = \frac{1}{8}\sin 2\theta_{23}\sin 2\theta_{13}\cos\theta_{13}\sin 2\theta_{12}$$

Reduced Jarlskog invariant

Applications

For the configuration of the DUNE experiment, the eigenvalues of the problem can be approximated as

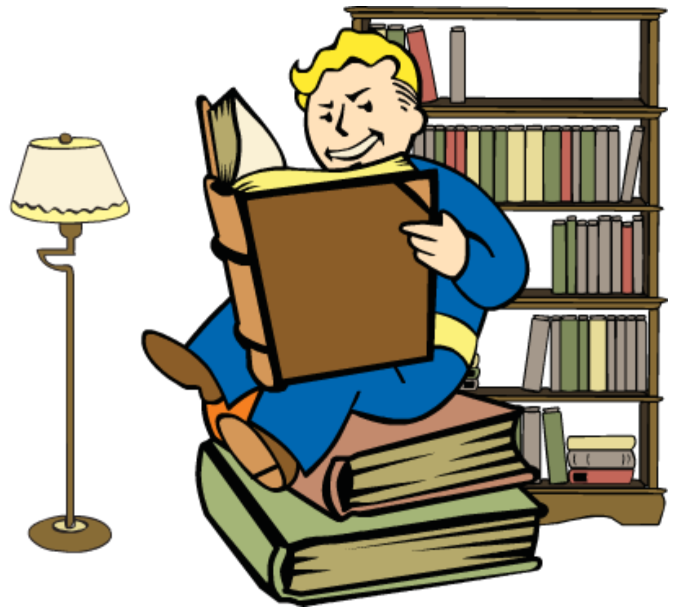
$$\xi_1 \simeq \frac{1}{2}\left(\Delta_{21} + V_{CC} - \sqrt{(c_{2\theta_{12}}\Delta_{21} - V_{CC})^2 + (s_{2\theta_{12}}\Delta_{21})^2}\right)$$
$$\xi_2 \simeq \frac{1}{2}\left(\Delta_{21} + V_{CC} + \sqrt{(c_{2\theta_{12}}\Delta_{21} - V_{CC})^2 + (s_{2\theta_{12}}\Delta_{21})^2}\right)$$
$$\xi_3 \simeq \Delta_{31}$$

$$\Delta_1 = \sqrt{(c_{2\theta_{12}}\Delta_{21} - V_{CC})^2 + (s_{2\theta_{12}}\Delta_{21})^2}$$

$$\Lambda_{21}^{\mu e} \simeq c_{23}^2 c_{13}^2 s_{2\theta_{12}}^2 \frac{\Delta_{21}^2}{\Delta_1^2} + s_{23}^2 s_{2\theta_{13}}^2 \frac{\Delta_{ee}^2}{\Delta_1^2 \omega_{31}\omega_{32}} \left(\xi_1 - c_{12}^2 \tfrac{\Delta_{31}}{\Delta_{ee}}\Delta_{21}\right)\left(\xi_2 - c_{12}^2 \tfrac{\Delta_{31}}{\Delta_{ee}}\Delta_{21}\right)$$
$$+ 4\frac{\Delta_{21}\Delta_{ee}}{\Delta_1^2}\left(\frac{c_{12}^2 \frac{\Delta_{31}}{\Delta_{ee}}\Delta_{21} - \xi_1}{\omega_{31}} + \frac{c_{12}^2 \frac{\Delta_{31}}{\Delta_{ee}}\Delta_{21} - \xi_2}{\omega_{32}}\right)\mathcal{J}_r c_\delta$$
$$\Lambda_{31}^{\mu e} \simeq s_{23}^2 s_{2\theta_{13}}^2 \frac{\Delta_{ee}\Delta_{31}\Delta_{32}}{\Delta_1 \omega_{31}^2 \omega_{32}}\left(c_{12}^2 \tfrac{\Delta_{31}}{\Delta_{ee}}\Delta_{21} - \xi_1\right) + 4\frac{\Delta_{21}\Delta_{31}\Delta_{32}}{\Delta_1\omega_{31}\omega_{32}}\mathcal{J}_r c_\delta$$
$$\Lambda_{32}^{\mu e} \simeq s_{23}^2 s_{2\theta_{13}}^2 \frac{\Delta_{ee}\Delta_{31}\Delta_{32}}{\Delta_1 \omega_{31}\omega_{32}^2}\left(\xi_2 - c_{12}^2 \tfrac{\Delta_{31}}{\Delta_{ee}}\Delta_{21}\right) - 4\frac{\Delta_{21}\Delta_{31}\Delta_{32}}{\Delta_1\omega_{31}\omega_{32}}\mathcal{J}_r c_\delta$$

$$P_{\mu e} = \Lambda_{21}^{\mu e}\sin^2\left(\frac{\omega_{21}L}{2}\right) + \Lambda_{31}^{\mu e}\sin^2\left(\frac{\omega_{31}L}{2}\right) + \Lambda_{32}^{\mu e}\sin^2\left(\frac{\omega_{32}L}{2}\right) - 8\frac{\Delta_{21}\Delta_{31}\Delta_{32}}{\omega_{21}\omega_{31}\omega_{32}}\mathcal{J}\sin\left(\frac{\omega_{21}L}{2}\right)\sin\left(\frac{\omega_{31}L}{2}\right)\sin\left(\frac{\omega_{32}L}{2}\right)$$

$$P_{\mu e} \approx c_{23}^2 s_{2\theta_{12}}^2 \frac{\Delta_{21}^2}{V_{CC}^2}\sin^2\left(\frac{V_{CC}L}{2}\right) + s_{23}^2 s_{2\theta_{13}}^2 \frac{\Delta_{31}^2}{(\Delta_{31} - V_{CC})^2}\sin^2\left(\frac{(\Delta_{31} - V_{CC})L}{2}\right)$$
$$+ s_{2\theta_{23}} s_{2\theta_{13}} s_{2\theta_{12}} \frac{\Delta_{21}}{V_{CC}}\frac{\Delta_{31}}{\Delta_{31} - V_{CC}}\sin\left(\frac{V_{CC}L}{2}\right)\sin\left(\frac{(\Delta_{31} - V_{CC})L}{2}\right)\cos\left(\frac{\Delta_{31}L}{2} + \delta\right)$$



H. Nunokawa, S. J. Parke, and J. W. Valle, Prog. Part. Nucl. Phys. **60,** 338 (2008).


$P_{\mu e}$
E [GeV]
$P_{\mu e}^{\rm NO}$
$\bar{P}_{\mu e}^{\rm NO}$
$P_{\mu e}^{\rm IO}$
$\bar{P}_{\mu e}^{\rm IO}$
$\left|\frac{\Delta P_{\mu e}}{P_{\mu e}}\right|$ [%]
$\left|\frac{\Delta P_{\mu e}^{\rm NO}}{P_{\mu e}^{\rm NO}}\right|$
$\left|\frac{\Delta \bar{P}_{\mu e}^{\rm NO}}{\bar{P}_{\mu e}^{\rm NO}}\right|$
$\left|\frac{\Delta P_{\mu e}^{\rm IO}}{P_{\mu e}^{\rm IO}}\right|$
$\left|\frac{\Delta \bar{P}_{\mu e}^{\rm IO}}{\bar{P}_{\mu e}^{\rm IO}}\right|$

Thank you!