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THESIS: Polarization effects in $\tau^- \to \pi^- \ell^+ \ell^- \nu_\tau$ decays.

A Thesis submitted by Jesús Javier Rendón Castañeda as partial fullfilment to obtain the degree of master in sciences.

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Resumen

En esta Tesis estudiamos el decaimiento $\tau^- \to \pi^- \ell^+ \ell^- \nu_\tau$ cuando la polarización del vector tau es tomada en cuenta, ℓ denota un electrón o un muón. Nuestro principal objetivo en este trabajo fue obtener la amplitud para este proceso, esto se llevó a cabo satisfactoriamente en el capítulo 4 y en el apéndice A. Nuestros resultados para la amplitud serán incluídos en el generador Monte Carlo TAUOLA usado por la colaboración Belle(-II).

Este proceso requiere un conocimiento del vértice $W^* \gamma^* \pi^-$ en el cual tanto el bosón W como el fotón son virtuales. Estudiamos dicho vértice usando una teoría efectiva denominada Teoría Quiral de Resonancias.

Para las integraciones numéricas se usó la rutina vegas en Fortran. Encontramos algunos resultados como los BR para el caso no polarizado (como una prueba de consistencia) y los BR correspondientes a una dirección particular del vector de polarización del leptón tau $\vec{s}$. Usando estos BR encontramos asimetrías de polarización para las diferentes contribuciones a la amplitud polarizada.

Abstract

In this Thesis we study the $\tau^- \to \pi^- \ell^+ \ell^- \nu_\tau$ decays in the case where the polarization of the decaying tau lepton can be measured, ℓ denotes an electron or a muon. Our main goal in this work was to obtain the amplitude for this process, this was successfully done in chapter 4 and appendix A. Our results for the amplitude will be included in the Monte Carlo generator TAUOLA used by the Belle(-II) collaboration.

This process requires knowledge of the $W^*\gamma^*\pi^-$ vertex in which the W boson and the photon are virtual. We study such vertex using an effective field theory called Resonance Chiral Theory ($R\chi T$).

For the numerical integrations we used a vegas routine in Fortran. We found several results like the unpolarized BR for the process (as a consistency check) , and the branching ratios corresponding to some particular and simplified cases for the direction of the tau polarization vector $\vec{s}$. Using these branching fractions we found polarization asymmetries for the different contributions to the polarized amplitude.

Contents

Chapter 1

Introduction

The τ lepton is the only lepton that can decay into hadrons, this property makes the study of the τ lepton crucial for the understanding of the fundamental interactions of nature. In fact, due to its importance, there is a complete area of specialization in particle physics called τ physics, whose purpose is to study in detail τ lepton decays and interactions, this makes the tau lepton privileged, in the sense that one can study the hadronization of QCD currents in a clean environment where the electroweak part of the process is under good theoretical control, which makes semileptonic tau decays an ideal arena to learn about low-energy strong interactions.

This thesis studies the $\tau^-(p_\tau, s) \to \pi^-(p)\ell^+(p_+, s_+)\ell^-(p_-, s_-)\nu_\tau(q, s_\nu)$ decays in the case where the polarization of the decaying τ lepton is taken into account. The $\ell^+\ell^-$ denotes an electron-positron pair (e^-e^+) or a muon-antimuon pair ($\mu^-\mu^+$).

Our goal in this work is to calculate the total amplitude for the process mentioned above, we found this in chapter 4, where we wrote the amplitude in a compact form (the full expression for this amplitude is shown in appendix A), then we use our result for the amplitude to obtain the branching ratios for some particular cases in the direction of the polarization vector $\vec{s}$ and to study polarization asymmetries in these particular cases.

It is important to mention that the analytical expressions that we obtained in this work will be included in the Monte Carlo generator TAUOLA used by the Belle II colaboration, this will help the experimentalists in the simulation of backgrounds for the analysis of tau physics in the Belle II experiment.

The organization of the Thesis is as follows: in chapter 2 we briefly describe the standard model of particle physics; in chapter 3 we point out some of the motivations for studying this decay and its importance in the search for new physics; chapter 4 is the heart of this Thesis, and as I wrote at the beginning of this section, it deals with the calculation of the amplitude, branching ratios, and polarization asymmetries for the process we are concerned with; chapter 5 is about some simplifications in the Levi Civita terms in our amplitude; and finally we state our conclusions in chapter 6.

Chapter 2

The Standard Model

The standard model of particle physics (SM) is a theory that explains with astonishing precision the electromagnetic, weak, and strong interactions. That is, it describes three of the four fundamental forces of nature. It only fails at explaining gravity, which has not been incorporated in the theory yet.

The success of the standard model is notorious due to its predictive power, in fact the part of the SM that explains the electromagnetic interaction, the so called Quantum Electrodynamics (QED) is known as the jewel of physics because it is the most precise physical theory that we have. Fortunately for the young (and not so young) physicist the SM is not the end of the story, there are unanswered questions and big challenges that are waiting for new ideas.

Rigorously speaking the standard model is a quantum field theory that is invariant under transformations of the $SU(3)_c \times SU(2)_L \times U(1)_Y$ group. The part invariant under the $SU(2)_L \times U(1)_Y$ group refers to the electroweak theory and the part invariant under the $SU(3)_c$ group refers to Quantum Chromodynamics. I will describe each of these in what follows.

The following two sections will be written based on reference [1].

2.1 Electroweak theory

As I wrote previously the electroweak theory is invariant under the $SU(2)_L \times U(1)_Y$ group. This theory is also a chiral one, which means that left-handed and right-handed spinors are not treated in the same way, this particular feature of electroweak interactions makes this theory so special.

The Lagrangian density for the Electroweak Theory is written as:

$$\mathcal{L} = \mathcal{L}_f + \mathcal{L}_{gauge} + \mathcal{L}_{\phi} + \mathcal{L}_{Yukawa} \, , \tag{2.1}$$

where the terms in the right hand side refer to the contributions due to fermions, the gauge bosons, the Higgs boson, and the Yukawa term respectively. The gauge terms are,

$$\mathcal{L}_{gauge} = -\frac{1}{4}W^i_{\mu\nu}W^{\mu\nu i} - \frac{1}{4}B_{\mu\nu}B^{\mu\nu}\,. \tag{2.2}$$

The terms $W^i_{\mu\nu}$, and $B_{\mu\nu}$ are respectively the field strength tensors for $SU(2)$, and $U(1)$ and are given by the following equations:

$$W^i_{\mu\nu} = \partial_\mu W^i_\nu - \partial_\nu W^i_\mu - g\epsilon^{ijk}W_{\mu j}W_{\nu k}, \quad i,j,k = 1,2,3 \tag{2.3a}$$

$$B_{\mu\nu} = \partial_\mu B_\nu - \partial_\nu B_\mu\,. \tag{2.3b}$$

The Lagrangian for the scalar Higgs field is,

$$\mathcal{L} = (D^\mu\phi)^\dagger D_\mu\phi - V(\phi)\,, \tag{2.4}$$

where, $\phi = \begin{pmatrix}\phi^+\\ \phi^0\end{pmatrix}$ is a complex scalar field called the Higgs field.

The full expression for the gauge covariant derivative that appears in (2.4) is shown in the following equation,

$$D_\mu\phi = (\partial_\mu + i\frac{g}{2}\vec{\tau}\cdot\vec{W}_\mu + i\frac{g'}{2}B_\mu)\phi\,. \tag{2.5}$$

The potential $V(\phi)$ in (2.4) is called the Higgs potential and is given by the following equation,

$$V(\phi) = \mu^2\phi^\dagger\phi + \lambda(\phi^\dagger\phi)^2\,. \tag{2.6}$$

For $\mu^2 < 0$ there will be spontaneous symmetry breaking and the vacuum expectation value of $\langle 0|\phi^0|0\rangle$ will generate the masses of the W and Z bosons. The Lagrangian corresponding to the fermions denoted as $\mathcal{L}_f$ is

$$\begin{aligned}\mathcal{L}_f = \sum_{m=1}^{3} \Big[& \bar{q}_{mL} i\not{D} q_{mL} + \bar{l}_{mL} i\not{D} l_{mL} + \bar{u}_{mR} i\not{D} u_{mR} + \bar{d}_{mR} i\not{D} d_{mR} \\ & + \bar{\nu}_{mR} i\not{D} \nu_{mR} + \bar{e}_{mR} i\not{D} e_{mR} \Big]\,,\end{aligned} \tag{2.7}$$

where, $q_{mL} = \begin{pmatrix}u_m\\ d_m\end{pmatrix}$, and $l_{mL} = \begin{pmatrix}\nu_m\\ e^-_m\end{pmatrix}$ are $SU(2)$ doblets and u_{mR},d_{mR},ν_{mR} and e^-_{mR} are singlets. The m subscript labels the family, as far as we know there are just three families so $m = 3$.

I have to say that right-handed neutrinos are not part of the standard model, so ν_{mR} should be absent above, however they are included in some extensions of the SM. Probably the main application of right-handed neutrinos is as candidates for explaining neutrino masses.

The full expressions for the gauge covariant derivatives in (2.7) are given in the following equations,

$$D_\mu q_{mL} = (\partial_\mu + i\frac{g}{2}\vec{\tau}\cdot\vec{W}_\mu + i\frac{g'}{6}B_\mu)q_{mL}\,, \tag{2.8a}$$

$$D_\mu l_{mL} = (\partial_\mu + i\frac{g}{2}\vec{\tau}\cdot\vec{W}_\mu - i\frac{g'}{2}B_\mu)l_{mL}\,, \tag{2.8b}$$

$$D_\mu u_{mR} = (\partial_\mu + 2i\frac{g'}{3}B_\mu)u_{mR}\,, \tag{2.8c}$$

$$D_\mu d_{mR} = (\partial_\mu - i\frac{g'}{3}B_\mu)d_{mR}\,, \tag{2.8d}$$

$$D_\mu e_{mR} = (\partial_\mu - ig'B_\mu)e_{mR}\,, \tag{2.8e}$$

$$D_\mu \nu_{mR} = \partial_\mu \nu_{mR}\,. \tag{2.8f}$$

From (2.7) we can see that right-handed and left-handed spinors are treated in different ways.

2.2 Quantum Chromodynamics

Quantum chromodynamics (QCD) is the theory that describes the strong interactions. The basic ingredients of this theory are quarks and gluons, quarks come in six different flavors: u, d, c, s, t, and b, each of which can have three different colors: Red (R), Green (G), and Blue (B), there are eight different gluons ($G^i, i = 1, ..., 8$) which are the force mediators, and there is also a gauge coupling (g_s) that determines the strength of the force.

The QCD Lagrangian is given by the following equation:

$$\mathcal{L}_{QCD} = -\frac{1}{4}G^i_{\mu\nu}G^{\mu\nu i} + \sum_{r=1}^{6}\overline{q}_r[i\not{D} - m_r]q_r\,, \tag{2.4}$$

where:

$$G^i_{\mu\nu} = \partial_\mu G^i_\nu - \partial_\nu G^i_\mu - g_s f_{ijk}G^j_\mu G^k_\nu, \quad i,j,k = 1, ..., 8 \tag{2.5a}$$

and

$$D_\mu = \partial_\mu + i\frac{g_s}{2}\lambda^i G^i\,. \tag{2.5b}$$

From the QCD Lagrangian we can see that QCD is a non-chiral theory, that is, right-handed and left-handed spinors are treated on the same footing, and from (2.5a) we can also see that gluons have self interactions due to the fact that the structure constants are not zero. This is a consequence of the non-abelian theory of the $SU(3)$ group.

2.3 Non perturbative QCD

Perturbation theory is the favorite tool of high energy physicists for making calculations in particle physics. It works very well most of the time, however, it is well known that one cannot treat perturbatively low-energy QCD phenomena due to the fact that the strong coupling constant α_s is greater than one (in fact it blows up) at low energies making perturbative calculations not treatable in this regime.

This feature of strong interactions is related to an effect called asymptotic freedom which states that at low distances (compared to the size of a proton) the quarks inside of it behave as free particles while at distances of the order of the size of a proton or greater the quarks are strongly coupled to each other making energetically favorable to create $q\overline{q}$ pairs instead of separating the corresponding free quarks, this is called confinement.

In figure 2.1 we can see different measurements for the α_s coupling compared with the QCD prediction, there we can appreciate easily the phenomenon of asymptotic freedom.

We can also see from figure 2.1 that at the scale of mass of the τ lepton $\alpha_s(M_\tau) = (0.327^{+0.019}_{-0.016})$ [2] and below ($Q \approx 1.77$ Gev), α_s grows rapidly, so that the perturbative calculations are not an option in this regime. In section 4.4 we will explain how to handle this problem.

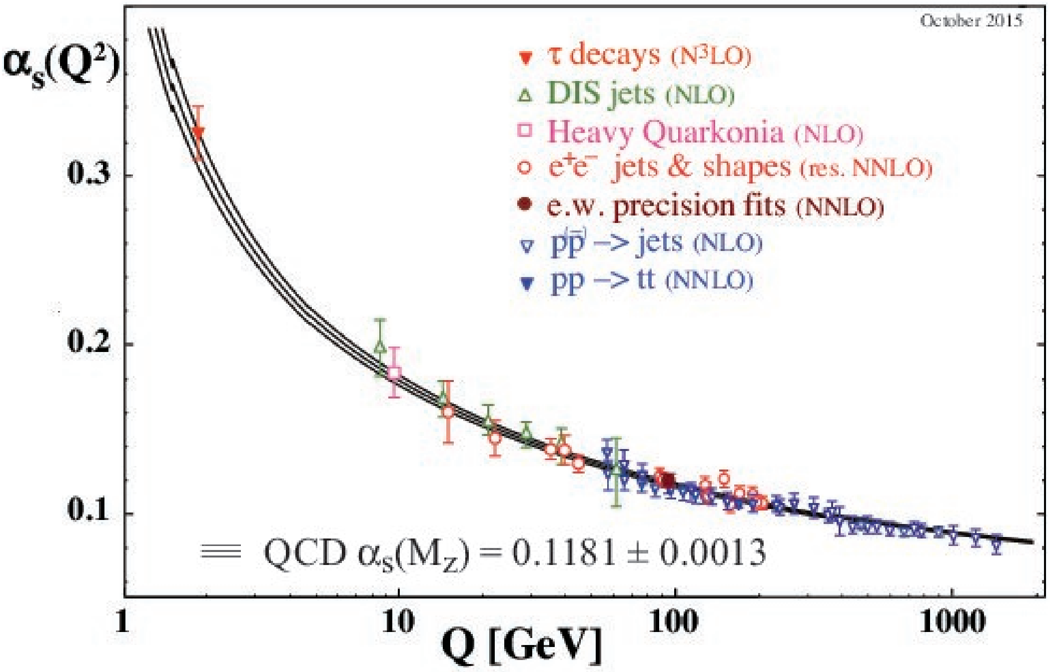

Figure 2.1: Measurements of the α_s coupling as a function of energy Q (experimental dots) compared with QCD prediction (solid lines).

Chapter 3

Motivation for the study of the $\tau^- \to \pi^- \ell^+ \ell^- \nu_\tau$ decay

In this chapter we will discuss the motivation for the studying of the $\tau^- \to \pi^- \ell^+ \ell^- \nu_\tau$ decay, but first we will explain the importance of τ physics in the next paragraph.

The τ lepton has the peculiarity of having an enormous mass ($M_\tau = 1.777\text{GeV}$) compared to the other leptons (e^-, and μ^-), this peculiarity allows the τ lepton to decay into hadrons, these decays are called hadronic or semileptonic decays and are fundamental to study QCD at low energies, which is not well understood in the Standard Model. This makes the study of the τ lepton the perfect tool to explore some connections between leptons and quarks that cannot be explored with the muon (μ) or the electron (e), the τ lepton is also a good candidate in the search of new physics (for example some possible decays of the τ lepton can be studied to look for lepton flavor violation as I will mention below). For this and many other reasons the study of τ decays is crucial in particle physics.

The calculation of the branching ratio for the $\tau^- \to \pi^- \ell^+ \ell^- \nu_\tau$ decay and its measurement are important for the following reasons [3]:

- Clean enviroment for the study of low energy hadronic interactions
- Important for the study of $\tau \to \pi\gamma\nu$, which is notable background for $\tau \to \mu\gamma$ (due to mis-ID of $\pi \to \mu$). $\tau \to \pi\gamma\nu$ is also important in the search of $\tau \to \pi\eta\nu$ because the η can be detected in its desintegration to two photons, so that one can confuse the photon in $\tau \to \pi\gamma\nu$ together with another photon with the signal of a η particle. I must say that there are mexican physicist working in the search of $\tau \to \pi\eta\nu$, so the work done in this Thesis is an important background for them

- $\tau \to \ell^+\ell^-\pi\nu_\tau$ is also background for $\tau \to \ell^-\ell^-\ell^+$ (again due to mis-ID of $\pi \to \mu$)
- Chiral perturbation theory predicts the branching ratio of the $\tau^- \to \pi^-\ell^+\ell^-\nu_\tau$ precisely for high energies of ν_τ. Its confirmation is important
- Search for long lived Sterile neutrino [4, 5]

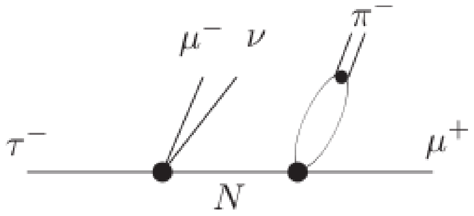

Figure 3.1: Possible sterile neutrino interaction.

The conclusion of the previous comments is that it is important to study this decay because it is background for several processes in the search of new physics (lepton flavor violation, sterile neutrinos and genuine second class currents, for example).

Charged lepton flavor violation, sterile neutrinos, and genuine second class currents are three hot topics in high energy physics nowadays. The observation of any of these would represent evidence of new physics, that is why there is too much effort and excitement in the high energy physics community in the search of these phenomena. Now we will describe briefly these three topics.

It is well known that neutrinos oscillate, so that there is no doubt that lepton flavor violation is real, but there remains the question if the same occurs in the charged lepton sector. The decays $\tau^- \to \mu^-\gamma$, and $\tau^- \to \ell^-\ell^-\ell^+$ that we wrote previously do not conserve lepton flavor. These and any other charged lepton flavor violating decay have never been observed in nature. The standard model does not incorporate such interactions, so observations are consistent with the standard model so far, but in principle there is nothing that prevents this kind of decays.

As far as we know all neutrinos are left-handed, and all antineutrinos are right-handed, this property makes neutrinos special, they are not like the other particles which can have both right-handed or left-handed helicities. On the other hand we know that neutrinos have a non-zero mass because this is requiered to explain the phenomenon of neutrino oscillations. The absence of right-handed neutrinos would require that its mass is generated in a way which differs to the one that governs the rest of massive particles. The questions of how neutrinos aquire mass and why it is so tiny remain open. The search for rigt-handed neutrinos is an active area, they have never been observed so far, which again is in accordance with the standard model. If right-handed neutrinos existed, they would interact only via gravity, that is why they are called sterile neutrinos. This property makes these hypothetical particles good candidates for dark matter.

At this point we have not mentioned the importance of polarization effects in the analysis of this decay. In what follows we will comment about additional observables that can be studied taking into account the polarization of the τ lepton.

The main reason for doing this Thesis work is to calculate the amplitude for the process in the case where the polarization of the τ lepton is taken into account. This is important because the results that we obtained will be included in the Monte Carlo generator TAUOLA used by the Belle II colaboration, and they will be helpfull in the simulation of backgrounds for the analysis of τ physics in the Belle II experiment.

A measurement of this process will also help us in the understanging of low-energy QCD, in fact this decay is a perfect scenario for testing different models for the calculation of the form factors involved in the process. The model we used in this work for that purpose is called chiral perturbation theory (including resonances). Here, polarization effects will be of fundamental importance because in this way we have more chances to test theoretical results (in addition to the unpolarized case, we have more freedom to study many possible polarization cases) by comparing with experiments. In section 4.5 we calculated several branching ratios corresponding to different directions of the polarization vector ($\vec{s}$) in this process, and with those BR we obtained some polarization asymmetries. These results could be a possible route for testing chiral perturbation theory in the near future. We have plans for studying this polarization asymmetries more deeply in a continuation of this work, so that we can have more theoretical results for the experimentalists to compare with.

Chapter 4

Polarization effects in $\tau^- \to \pi^-\ell^+\ell^-\nu_\tau$ decays

4.1 Calculation of the amplitude for the process

There are five Feynmann diagrams contributing to the amplitude in this process as can be seen in figure 4.1: the first three are model independent, and together they contribute to the so called inner bremsstralhung (IB) amplitude, the other two diagrams depend on the model we use for describing the vertex $\gamma^* W\pi^-$, and for this reason the amplitude corresponding to those is called model dependent (also called structure dependent, because it corresponds to the non point-like part of the interaction) amplitude.

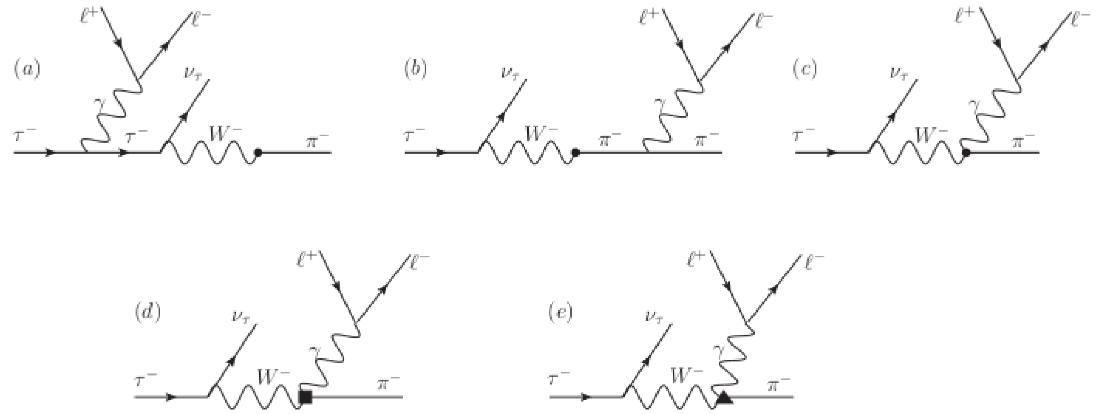

Figure 4.1: Feynman diagrams for the $\tau^- \to \pi^-\ell^+\ell^-\nu_\tau$ decay

The amplitude for diagram (a) in figure 4.1 is,

$$\mathcal{M}_{IB\tau} = -iG_F V_{ud}^* e^2 F_\pi p_\mu \frac{\ell_\nu}{k^2} \overline{u}(q)\gamma^\mu(1-\gamma^5)\Big(\frac{\not{p}_t - \not{k} + M_\tau}{(p_\tau - k)^2 - M_\tau^2}\Big)\gamma^\nu u_{(}p_\tau)\,. \quad (4.1)$$

The amplitude for diagrams (b) and (c) in figure 4.1 is,

$$\mathcal{M}_{IB\pi} = -iG_F V_{ud}^* e^2 F_\pi \frac{\ell^\nu}{k^2}\Big(\frac{2p_\nu(\not{p} + \not{k})}{(p+k)^2 - m_\pi^2} - g_{\mu\nu}\Big)\overline{u}(q)(1+\gamma^5)\gamma^\mu u(p_\tau)\,. \quad (4.2)$$

We can check gauge invariance in the case of the real photon by substituting $\frac{\ell^\nu}{k^2} \to \epsilon^\nu$ in the total inner bremsstrahlung amplitude, this must vanish when $\epsilon \to k$,

$$\lim_{\epsilon \to k}(\mathcal{M}_{IB\tau} + \mathcal{M}_{IB\pi}) = 0\,. \quad (4.3)$$

The model dependent amplitudes are given by the following equations,

$$\mathcal{M}_V = -G_F V_{ud}^* \frac{e^2}{k^2} F_V(p\cdot k, k^2)\epsilon_{\mu\nu\rho\sigma}k^\rho p^\sigma \ell^\nu \tau^\mu\,, \quad (4.4)$$

$$\begin{aligned}\mathcal{M}_A =& iG_F V_{ud}^* \frac{2e^2}{k^2}\ell_\nu \Bigg(F_A(p\cdot k, k^2)\left[(k^2 + p\cdot k)g^{\mu\nu} - k^\mu p^\nu\right] \\ &+ B(k^2)k^2\left[g^{\mu\nu} - \frac{(p+k)^\mu p^\nu}{k^2 + 2p\cdot k}\right]\Bigg)\tau_\mu\,,\end{aligned} \quad (4.5)$$

where we have used $\ell_\mu := \bar{u}(p_-)\gamma_\mu v(p_+)$ and $\tau_\mu := \bar{u}(q)\gamma_\mu(1-\gamma_5)u(p_\tau)$ as short for the electromagnetic and weak spinor currents.

4.2 Modulus squared of the amplitude

As I have argued before, the aim of this work is to calculate the branching ratio of the $\tau^-(p_\tau, s) \to \pi^-(p)\ell^+(p_+, s_+)\ell^-(p_-, s_-)\nu_\tau(q, s_\nu)$ decays in the case where the polarization of the decaying tau can be measured. Conventions for momenta and spins will be used as just indicated in the following. The total amplitude for the process is the sum of the amplitudes corresponding to the inner bremmstrahlung and the structure dependent contributions, including the vector and axial vector form factors, which is shown in the next equation:

$$\mathcal{M} = \mathcal{M}_{IB} + \mathcal{M}_V + \mathcal{M}_A\,, \quad (4.6)$$

where each of the contributions is given by the following equations:

$$\mathcal{M}_{IB} = -iG_F V_{ud}^* \frac{e^2}{k^2} F_\pi M_\tau \ell_\mu \bar{u}(q)(1+\gamma_5)\left[\frac{2p^\mu}{2p\cdot k + k^2} + \frac{2p_\tau^\mu - \not{k}\gamma^\mu}{k^2 - 2p_\tau \cdot k}\right] u(p_\tau)\,, \tag{4.7}$$

$$\mathcal{M}_V = -G_F V_{ud}^* \frac{e^2}{k^2} F_V(p\cdot k, k^2)\epsilon_{\mu\nu\rho\sigma} k^\rho p^\sigma \ell^\nu \tau^\mu\,, \tag{4.8}$$

$$\begin{aligned}\mathcal{M}_A =& iG_F V_{ud}^* \frac{2e^2}{k^2}\ell_\nu \Bigg(F_A(p\cdot k, k^2)\left[(k^2 + p\cdot k)g^{\mu\nu} - k^\mu p^\nu\right] \\ &+ B(k^2)k^2\left[g^{\mu\nu} - \frac{(p+k)^\mu p^\nu}{k^2 + 2p\cdot k}\right]\Bigg)\tau_\mu\,,\end{aligned} \tag{4.9}$$

where again, we have used $\ell_\mu := \bar{u}(p_-)\gamma_\mu v(p_+)$ and $\tau_\mu := \bar{u}(q)\gamma_\mu(1-\gamma_5)u(p_\tau)$ as short for the electromagnetic and weak spinor currents. For the calculation of the branching ratio we need the square of the amplitude in equation (4.6), which is given by the following expression.

$$|\mathcal{M}|^2 = |\mathcal{M}_{IB}|^2 + |\mathcal{M}_V|^2 + |\mathcal{M}_A|^2 + 2\Re e(\mathcal{M}_{IB}\mathcal{M}_V^\dagger) + 2\Re e(\mathcal{M}_{IB}\mathcal{M}_A^\dagger) + 2\Re e(\mathcal{M}_V\mathcal{M}_A^\dagger)\,. \tag{4.10}$$

I will work the different pieces of this amplitude in what follows. I will start with $\mathcal{M}_{IB}$. From equation (4.7) we have:

$$\begin{aligned}|\mathcal{M}_{IB}|^2 = &{G_F}^2 |V_{ud}|^2 \frac{e^4}{k^4}{F_\pi}^2 {M_\tau}^2 [\bar{u}(p_-)\gamma_\mu v(p_+)][\bar{u}(p_-)\gamma_\nu v(p_+)]^\dagger \\ &\left[\bar{u}(q)(1+\gamma_5)\left(\frac{2p^\mu}{2p\cdot k + k^2} + \frac{2p_\tau^\mu - \not{k}\gamma^\mu}{k^2 - 2p_\tau\cdot k}\right)u(p_\tau)\right] \\ &\left[\bar{u}(q)(1+\gamma_5)\left(\frac{2p^\nu}{2p\cdot k + k^2} + \frac{2p_\tau^\nu - \not{k}\gamma^\nu}{k^2 - 2p_\tau\cdot k}\right)u(p_\tau)\right]^\dagger\,.\end{aligned} \tag{4.11}$$

I will simplify equation (4.11) using the following identities:

$$\sum_{spins}[\bar{u}(a)\Gamma_1 u(b)][\bar{u}(a)\Gamma_2 u(b)]^\dagger = Tr[\Gamma_1(\not{p}_b + m_b)\bar{\Gamma}_2(\not{p}_a + m_a)]\,, \tag{4.12a}$$

$$\sum_{spins}[\bar{u}(a)\Gamma_1 v(b)][\bar{u}(a)\Gamma_2 v(b)]^\dagger = Tr[\Gamma_1(\not{p}_b - m_b)\bar{\Gamma}_2(\not{p}_a + m_a)]\,, \tag{4.12b}$$

$$\sum_{spins}[\bar{v}(a)\Gamma_1 u(b)][\bar{v}(a)\Gamma_2 u(b)]^\dagger = Tr[\Gamma_1(\not{p}_b + m_b)\bar{\Gamma}_2(\not{p}_a - m_a)]\,, \tag{4.12c}$$

where: $\bar{\Gamma}_2 = \gamma^0 {\Gamma_2}^\dagger \gamma^0$.

So using the identities stated before we can easily simplify the following term in (4.11):

$$\begin{aligned}\sum_{spins}\ell_\mu \ell_\nu^\dagger &= \sum_{spins}[\bar{u}(p_-)\gamma_\mu v(p_+)][\bar{u}(p_-)\gamma_\nu v(p_+)]^\dagger \\ &= Tr[\gamma_\mu(\not{p}_+ - m_\ell)\bar{\gamma}_\nu(\not{p}_- + m_\ell)] \\ &= 4[p_{+\mu}p_{-\nu} + p_{+\nu}p_{-\mu} - g_{\mu\nu}({m_\ell}^2 + p_-\cdot p_+)]\,. \\ &= 4\ell_{\mu\nu}\end{aligned} \tag{4.13}$$

In the following we will be summing over the ν_τ, ℓ^- and ℓ^+ spins, but not over that of the decaying τ^-. The results for the unpolarized case, where the sum over spins affects all four fermions can be found in [6, 7]. We have verified this result as a check of our calculation. Equation (4.13) is one of the terms we need as a consecuence of working equation (4.11). We can easily see from (4.11) that there will be another 9 terms as a result of each of the products, I will denote each of these terms with T_1, T_2, ..., T_9: $\sum_{s_\nu,s_{\ell+},s_{\ell-}}|M_{IB}|^2 = 4{G_F}^2|V_{ud}|^2\frac{e^4}{k^4}{F_\pi}^2{M_\tau}^2\sum_{i=1,...,9}T_i^{\mu\nu}\ell_{\mu\nu}$.

When we have a definite polarization in one of the particles involved in the process (as is our case with the τ lepton) instead of summing over spins we have to make the following substitution,

$$u(p,s)\overline{u}(p,s) = (\not{p} + m)\left(\frac{1+\gamma^5\not{s}}{2}\right)\,. \tag{4.14}$$

$$\begin{aligned}T_1^{\mu\nu} &= \frac{4p^\mu p^\nu}{(2p\cdot k + k^2)^2}Tr[(1+\gamma_5)(\not{p}_t + M_\tau)\Big(\frac{1+\gamma_5\not{s}}{2}\Big)(1-\gamma_5)\not{q}] \\ &= \frac{16p^\mu p^\nu}{(2p\cdot k + k^2)^2}\left[p_\tau\cdot q + M_\tau s\cdot q\right]\,.\end{aligned} \tag{4.15}$$

$$\Rightarrow {T_1}^{\mu\nu}\ell_{\mu\nu} = \frac{16}{(2p\cdot k + k^2)^2}\left[2(p\cdot p_+)(p\cdot p_-) - p^2(p_+\cdot p_- + m_\ell^2)\right]\left[p_\tau\cdot q + M_\tau s\cdot q\right]\,. \tag{4.16}$$

$$\begin{aligned} T_2^{\mu\nu} &= \frac{4p^\mu p_\tau^\nu}{(2p\cdot k+k^2)(k^2-2p_\tau\cdot k)} Tr[(1+\gamma_5)(\not{p}_t+M_\tau)\Big(\frac{1+\gamma_5\not{s}}{2}\Big)(1-\gamma_5)\not{q}] \\ &= \frac{16p^\mu p_\tau^\nu}{(2p\cdot k+k^2)(k^2-2p_\tau\cdot k)}\big[p_\tau\cdot q+M_\tau s\cdot q\big] \,. \end{aligned} \tag{4.17}$$

$$\begin{aligned} \Rightarrow T_2{}^{\mu\nu}\ell_{\mu\nu} = \frac{16}{(2p\cdot k+k^2)(k^2-2p_\tau\cdot k)}\big[p_\tau\cdot q+M_\tau s\cdot q\big] \\ \big[(p\cdot p_-)(p_\tau\cdot p_+) - p\cdot p_\tau(p_+\cdot p_- + m_\ell^2) + (p\cdot p_+)(p_\tau\cdot p_-)\big] \,. \end{aligned} \tag{4.18}$$

$$\begin{aligned} T_3^{\mu\nu} &= -\frac{2p^\mu k_\lambda}{(2p\cdot k+k^2)(k^2-2p_\tau\cdot k)} Tr[(1+\gamma_5)(\not{p}_t+M_\tau)\Big(\frac{1+\gamma_5\not{s}}{2}\Big)\gamma^\nu\gamma^\lambda(1-\gamma_5)\not{q}] \\ &= \frac{-8}{(2p\cdot k+k^2)(k^2-2p_\tau\cdot k)}\big[p^\mu p_\tau^\nu(k\cdot q) - p^\mu q^\nu(p_\tau\cdot k) + p^\mu k^\nu(p_\tau\cdot q) + ip_{\tau a}q_b p^\mu k_\lambda \epsilon^{\lambda\nu ab} \\ &+ M_\tau\big(-ip^\mu k_\lambda s_a q_b\epsilon^{\lambda\nu ba} + p^\mu s^\nu(k\cdot q) - p^\mu q^\nu(k\cdot s) + p^\mu k^\nu(s\cdot q)\big)\big] \,. \end{aligned} \tag{4.19}$$

$$\begin{aligned} \Rightarrow T_3{}^{\mu\nu}\ell_{\mu\nu} &= \frac{8}{(2p\cdot k+k^2)(k^2-2p_\tau\cdot k)}\Big[(p_+\cdot p_- + m_\ell^2)[(k\cdot q)(p\cdot p_\tau) - (k\cdot p_\tau)(p\cdot q) \\ &+ (k\cdot p)(p_\tau\cdot q) + ik_\mu p_\nu p_{\tau\lambda}q_\sigma\epsilon^{\mu\nu\lambda\sigma}] + (k\cdot p_\tau)(p\cdot p_+)(p_-\cdot q) - (k\cdot q)(p\cdot p_+)(p_-\cdot p_\tau) \\ &- (k\cdot p_-)(p\cdot p_+)(p_\tau\cdot q) - (k\cdot q)(p\cdot p_-)(p_+\cdot p_\tau) + (k\cdot p_\tau)(p\cdot p_-)(p_+\cdot q) \\ &- (k\cdot p_+)(p\cdot p_-)(p_\tau\cdot q) - i(p\cdot p_+)k_\mu p_{-\nu}p_{\tau\lambda}q_\sigma\epsilon^{\mu\nu\lambda\sigma} - i(p\cdot p_-)k_\mu p_{+\nu}p_{\tau\lambda}q_\sigma\epsilon^{\mu\nu\lambda\sigma}\Big] \\ &+ \frac{8M_\tau}{(2p\cdot k+k^2)(k^2-2p_\tau\cdot k)}\Big[(p_+\cdot p_- + m_\ell^2)[(k\cdot q)(p\cdot s) - (k\cdot s)(p\cdot q) \\ &+ (k\cdot p)(s\cdot q) - ik_\mu p_\nu q_\lambda s_\sigma\epsilon^{\mu\nu\lambda\sigma}] + (k\cdot s)(p\cdot p_+)(p_-\cdot q) - (k\cdot q)(p\cdot p_+)(p_-\cdot s) \\ &- (k\cdot p_-)(p\cdot p_+)(s\cdot q) - (k\cdot q)(p\cdot p_-)(p_+\cdot s) + (k\cdot s)(p\cdot p_-)(p_+\cdot q) \\ &- (k\cdot p_+)(p\cdot p_-)(s\cdot q) + i(p\cdot p_+)k_\mu p_{-\nu}q_\lambda s_\sigma\epsilon^{\mu\nu\lambda\sigma} + i(p\cdot p_-)k_\mu p_{+\nu}q_\lambda s_\sigma\epsilon^{\mu\nu\lambda\sigma}\Big] \,. \end{aligned} \tag{4.20}$$

$$\begin{aligned} T_4^{\mu\nu} &= \frac{4p_\tau^\mu p_\tau^\nu}{(k^2-2p_\tau\cdot k)^2} Tr[(1+\gamma_5)(\not{p}_t+M_\tau)\Big(\frac{1+\gamma_5\not{s}}{2}\Big)(1-\gamma_5)\not{q}] \\ &= \frac{16p_\tau^\mu p_\tau^\nu}{(k^2-2p_\tau\cdot k)^2}\big[p_\tau\cdot q+M_\tau s\cdot q\big] \,. \end{aligned} \tag{4.21}$$

$$\Rightarrow T_4{}^{\mu\nu}\ell_{\mu\nu} = \frac{16}{(2p\cdot k+k^2)^2}\big[2(p\cdot p_+)(p\cdot p_-)-p^2(p_+\cdot p_-+m_\ell^2)\big]\big[p_\tau\cdot q+M_\tau s\cdot q\big]\,. \tag{4.22}$$

$$\begin{aligned}T_5^{\mu\nu} &= \frac{4p_\tau^\mu p^\nu}{(2p\cdot k+k^2)(k^2-2p_\tau\cdot k)}Tr[(1+\gamma_5)(\not{p}_t+M_\tau)\Big(\frac{1+\gamma_5\not{s}}{2}\Big)(1-\gamma_5)\not{q}] \\ &= \frac{16p_\tau^\mu p^\nu}{(2p\cdot k+k^2)(k^2-2p_\tau\cdot k)}\big[p_\tau\cdot q+M_\tau s\cdot q\big]\,.\end{aligned} \tag{4.23}$$

$$\begin{aligned}\Rightarrow T_5{}^{\mu\nu}\ell_{\mu\nu} = \frac{16}{(2p\cdot k+k^2)(k^2-2p_\tau\cdot k)}&\big[p_\tau\cdot q+M_\tau s\cdot q\big] \\ \big[(p\cdot p_-)(p_\tau\cdot p_+)-p\cdot p_\tau(p_+\cdot p_-+m_\ell^2)&+(p\cdot p_+)(p_\tau\cdot p_-)\big]\,.\end{aligned} \tag{4.24}$$

$$\begin{aligned}T_6^{\mu\nu} &= -\frac{2p_\tau^\mu k_\lambda}{(k^2-2p_\tau\cdot k)^2}Tr[(1+\gamma_5)(\not{p}_t+M_\tau)\Big(\frac{1+\gamma_5\not{s}}{2}\Big)\gamma^\nu\gamma^\lambda(1-\gamma_5)\not{q}] \\ &= \frac{-8}{(k^2-2p_\tau\cdot k)^2}\big[p_\tau^\mu p_\tau^\nu(k\cdot q)-p_\tau^\mu q^\nu(p_\tau\cdot k)+p_\tau^\mu k^\nu(p_\tau\cdot q)+ip_{\tau a}q_b p_\tau^\mu k_\lambda\epsilon^{\lambda\nu ab} \\ &+M_\tau\big(p_\tau^\mu s^\nu(k\cdot q)-p_\tau^\mu q^\nu(k\cdot s)+p_\tau^\mu k^\nu(s\cdot q)-ip_\tau^\mu k_\lambda s_a q_b\epsilon^{\lambda\nu ba}\big)\big]\,.\end{aligned} \tag{4.25}$$

$$\begin{aligned}&\Rightarrow T_6{}^{\mu\nu}\ell_{\mu\nu} = \frac{8}{(k^2-2p_\tau\cdot k)^2}\Big[(p_+\cdot p_-+m_\ell^2)(k\cdot q)p_\tau^2-2(k\cdot q)(p_+\cdot p_\tau)(p_-\cdot p_\tau) \\ &+(k\cdot p_\tau)(p_+\cdot q)(p_-\cdot p_\tau)+(k\cdot p_\tau)(p_+\cdot p_\tau)(p_-\cdot q)-(k\cdot p_-)(p_+\cdot p_\tau)(p_\tau\cdot q) \\ &-(k\cdot p_+)(p_\tau\cdot p_-)(p_\tau\cdot q)-i(p_\tau\cdot p_+)k_\mu p_{-\nu}p_{\tau\lambda}q_\sigma\epsilon^{\mu\nu\lambda\sigma}-i(p_\tau\cdot p_-)k_\mu p_{+\nu}p_{\tau\lambda}q_\sigma\epsilon^{\mu\nu\lambda\sigma}\Big] \\ &+\frac{8M_\tau}{(k^2-2p_\tau\cdot k)^2}\Big[(p_+\cdot p_-+m_\ell^2)[(k\cdot q)(p_\tau\cdot s)-(k\cdot s)(p_\tau\cdot q) \\ &+(k\cdot p_\tau)(s\cdot q)-ik_\mu p_{\tau\nu}q_\lambda s_\sigma\epsilon^{\mu\nu\lambda\sigma}]+(k\cdot s)(p_\tau\cdot p_+)(p_-\cdot q)-(k\cdot q)(p_\tau\cdot p_+)(p_-\cdot s) \\ &-(k\cdot p_-)(p_\tau\cdot p_+)(s\cdot q)-(k\cdot q)(p_\tau\cdot p_-)(p_+\cdot s)+(k\cdot s)(p_\tau\cdot p_-)(p_+\cdot q) \\ &-(k\cdot p_+)(p_\tau\cdot p_-)(s\cdot q)+i(p_\tau\cdot p_+)k_\mu p_{-\nu}q_\lambda s_\sigma\epsilon^{\mu\nu\lambda\sigma}+i(p_\tau\cdot p_-)k_\mu p_{+\nu}q_\lambda s_\sigma\epsilon^{\mu\nu\lambda\sigma}\Big]\,.\end{aligned} \tag{4.26}$$

$$\begin{aligned} T_7^{\mu\nu} &= -\frac{2p^\nu k_\lambda}{(2p\cdot k+k^2)(k^2-2p_\tau\cdot k)} Tr[(1+\gamma_5)\gamma^\lambda\gamma^\mu(\not{p}_t+M_\tau)\Big(\frac{1+\gamma_5\not{s}}{2}\Big)(1-\gamma_5)\not{q}] \\ &= \frac{-8}{(2p\cdot k+k^2)(k^2-2p_\tau\cdot k)}\big[p^\nu p_\tau^\mu(k\cdot q)-p^\nu q^\mu(p_\tau\cdot k)+p^\nu k^\mu(p_\tau\cdot q)-ip_{\tau a}q_b p^\nu k_\lambda\epsilon^{\lambda\mu ab} \\ &+ M_\tau\big(ip^\nu k_\lambda s_a q_b\epsilon^{\lambda\mu ba}+p^\nu s^\mu(k\cdot q)-p^\nu q^\mu(k\cdot s)+p^\nu k^\mu(s\cdot q)\big)\big]\,. \end{aligned} \tag{4.27}$$

$$\begin{aligned} &\Rightarrow T_7{}^{\mu\nu}\ell_{\mu\nu} = \frac{8}{(2p\cdot k+k^2)(k^2-2p_\tau\cdot k)}\Big[(p_+\cdot p_-+m_\ell^2)[(k\cdot q)(p\cdot p_\tau)-(k\cdot p_\tau)(p\cdot q) \\ &+(k\cdot p)(p_\tau\cdot q)-ik_\mu p_\nu p_{\tau\lambda}q_\sigma\epsilon^{\mu\nu\lambda\sigma}]+(k\cdot p_\tau)(p\cdot p_+)(p_-\cdot q)-(k\cdot q)(p\cdot p_+)(p_-\cdot p_\tau) \\ &-(k\cdot p_-)(p\cdot p_+)(p_\tau\cdot q)-(k\cdot q)(p\cdot p_-)(p_+\cdot p_\tau)+(k\cdot p_\tau)(p\cdot p_-)(p_+\cdot q) \\ &-(k\cdot p_+)(p\cdot p_-)(p_\tau\cdot q)+i(p\cdot p_+)k_\mu p_{-\nu}p_{\tau\lambda}q_\sigma\epsilon^{\mu\nu\lambda\sigma}+i(p\cdot p_-)k_\mu p_{+\nu}p_{\tau\lambda}q_\sigma\epsilon^{\mu\nu\lambda\sigma}\Big] \\ &+\frac{8M_\tau}{(2p\cdot k+k^2)(k^2-2p_\tau\cdot k)}\Big[(p_+\cdot p_-+m_\ell^2)[(k\cdot q)(p\cdot s)-(k\cdot s)(p\cdot q) \\ &+(k\cdot p)(s\cdot q)+ik_\mu p_\nu q_\lambda s_\sigma\epsilon^{\mu\nu\lambda\sigma}]+(k\cdot s)(p\cdot p_+)(p_-\cdot q)-(k\cdot q)(p\cdot p_+)(p_-\cdot s) \\ &-(k\cdot p_-)(p\cdot p_+)(s\cdot q)-(k\cdot q)(p\cdot p_-)(p_+\cdot s)+(k\cdot s)(p\cdot p_-)(p_+\cdot q) \\ &-(k\cdot p_+)(p\cdot p_-)(s\cdot q)-i(p\cdot p_+)k_\mu p_{-\nu}q_\lambda s_\sigma\epsilon^{\mu\nu\lambda\sigma}-i(p\cdot p_-)k_\mu p_{+\nu}q_\lambda s_\sigma\epsilon^{\mu\nu\lambda\sigma}\Big]\,. \end{aligned} \tag{4.28}$$

$$\begin{aligned} T_8^{\mu\nu} &= -\frac{2p_\tau^\nu k_\lambda}{(k^2-2p_\tau\cdot k)^2} Tr[(1+\gamma_5)\gamma^\lambda\gamma^\mu(\not{p}_t+M_\tau)\Big(\frac{1+\gamma_5\not{s}}{2}\Big)(1-\gamma_5)\not{q}] \\ &= \frac{-8}{(k^2-2p_\tau\cdot k)^2}\big[p_\tau^\nu p_\tau^\mu(k\cdot q)-p_\tau^\nu q^\mu(p_\tau\cdot k)+p_\tau^\nu k^\mu(p_\tau\cdot q)-ip_{\tau a}q_b p_\tau^\nu k_\lambda\epsilon^{\lambda\mu ab} \\ &+ M_\tau\big(p_\tau^\nu s^\mu(k\cdot q)-p_\tau^\nu q^\mu(k\cdot s)+p_\tau^\nu k^\mu(s\cdot q)+ip_\tau^\nu k_\lambda s_a q_\sigma\epsilon^{\lambda\mu ba}\big)\big]\,. \end{aligned} \tag{4.29}$$

$$
\begin{aligned}
&\Rightarrow {T_8}^{\mu\nu}\ell_{\mu\nu} = \frac{8}{(k^2-2p_\tau\cdot k)^2}\Big[(p_+\cdot p_- + m_\ell^2)(k\cdot q)p_\tau^2 - 2(k\cdot q)(p_+\cdot p_\tau)(p_-\cdot p_\tau) \\
&+ (k\cdot p_\tau)(p_+\cdot q)(p_-\cdot p_\tau) + (k\cdot p_\tau)(p_+\cdot p_\tau)(p_-\cdot q) - (k\cdot p_-)(p_+\cdot p_\tau)(p_\tau\cdot q) \\
&- (k\cdot p_+)(p_\tau\cdot p_-)(p_\tau\cdot q) + i(p_\tau\cdot p_+)k_\mu p_{-\nu} p_{\tau\lambda} q_\sigma \epsilon^{\mu\nu\lambda\sigma} + i(p_\tau\cdot p_-)k_\mu p_{+\nu} p_{\tau\lambda} q_\sigma \epsilon^{\mu\nu\lambda\sigma}\Big] \\
&+ \frac{8M_\tau}{(k^2-2p_\tau\cdot k)^2}\Big[(p_+\cdot p_- + m_\ell^2)[(k\cdot q)(p_\tau\cdot s) - (k\cdot s)(p_\tau\cdot q) \\
&+ (k\cdot p_\tau)(s\cdot q) + ik_\mu p_{\tau\nu} q_\lambda s_\sigma \epsilon^{\mu\nu\lambda\sigma}] + (k\cdot s)(p_\tau\cdot p_+)(p_-\cdot q) - (k\cdot q)(p_\tau\cdot p_+)(p_-\cdot s) \\
&- (k\cdot p_-)(p_\tau\cdot p_+)(s\cdot q) - (k\cdot q)(p_\tau\cdot p_-)(p_+\cdot s) + (k\cdot s)(p_\tau\cdot p_-)(p_+\cdot q) \\
&- (k\cdot p_+)(p_\tau\cdot p_-)(s\cdot q) - i(p_\tau\cdot p_+)k_\mu p_{-\nu} q_\lambda s_\sigma \epsilon^{\mu\nu\lambda\sigma} - i(p_\tau\cdot p_-)k_\mu p_{+\nu} q_\lambda s_\sigma \epsilon^{\mu\nu\lambda\sigma}\Big]\,.
\end{aligned}
\tag{4.30}
$$

$$
\begin{aligned}
T_9^{\mu\nu} &= \frac{1}{(k^2-2p_\tau\cdot k)^2} Tr[(1+\gamma_5)\not{k}\gamma^\mu(\not{p}_t + M_\tau)\Big(\frac{1+\gamma_5\not{s}}{2}\Big)\gamma^\nu\not{k}(1-\gamma_5)\not{q}] \\
&= \frac{4}{(k^2-2p_\tau\cdot k)^2}\big[-\tau^{\mu\nu}k^2 - 2g^{\mu\nu}(p_\tau\cdot k)(k\cdot q) + 4k^\mu p_\tau^\nu(k\cdot q) \\
&+ i\big(-2k^\mu p_{\tau b}k_a q_c \epsilon^{\nu abc} + 2k^\nu p_{\tau b}k_a q_c\epsilon^{\mu abc} - k^2 p_{\tau b} q_c \epsilon^{\mu\nu bc} + 2(p_\tau\cdot k)k_a q_b \epsilon^{\mu\nu ab}\big)\big] \\
&+ \frac{4M_\tau}{(k^2-2p_\tau\cdot k)^2}\big[-\Omega^{\mu\nu}k^2 - 2g^{\mu\nu}(s\cdot k)(k\cdot q) + 4k^\mu s^\nu(k\cdot q) \\
&+ i\big(2k^\mu k_a q_b s_c \epsilon^{\nu abc} - 2k^\nu k_a q_b s_c \epsilon^{\mu abc} + k^2 q_b s_c \epsilon^{\mu\nu bc} + 2(s\cdot k)k_a q_b \epsilon^{\mu\nu ab}\big)\big]\,.
\end{aligned}
\tag{4.31}
$$

In the previous equation, we introduced following the short-hand notation

$$\tau^{\mu\nu} = p_\tau^\mu q^\nu + p_\tau^\nu q^\mu - g^{\mu\nu} p_\tau\cdot q\,. \tag{4.32a}$$

$$\Omega^{\mu\nu} = s^\mu q^\nu + s^\nu q^\mu - g^{\mu\nu} s\cdot q\,. \tag{4.32b}$$

$$
\begin{aligned}
\Rightarrow {T_9}^{\mu\nu}\ell_{\mu\nu} = {} & \frac{8}{(k^2-2p_\tau\cdot k)^2}\Big[(p_+\cdot p_- + m_\ell^2)[-k^2(p_\tau\cdot q)+2(k\cdot p_\tau)(k\cdot q)] \\
& + k^2(p_+\cdot p_-)(p_\tau\cdot q) - 2(k\cdot p_\tau)(k\cdot q)(p_+\cdot p_-) + 2(k\cdot p_-)(k\cdot q)(p_+\cdot p_\tau) \\
& + 2(k\cdot p_+)(k\cdot q)(p_-\cdot p_\tau) - k^2(p_+\cdot q)(p_-\cdot p_\tau) - k^2(p_+\cdot p_\tau)(p_-\cdot q)\Big] \\
& + \frac{8M_\tau}{(k^2-2p_\tau\cdot k)^2}\Big[(p_+\cdot p_- + m_\ell^2)[-k^2(q\cdot s)+2(k\cdot s)(k\cdot q)] \\
& + k^2(p_+\cdot p_-)(s\cdot q) - 2(k\cdot s)(k\cdot q)(p_+\cdot p_-) + 2(k\cdot p_-)(k\cdot q)(p_+\cdot s) \\
& + 2(k\cdot p_+)(k\cdot q)(p_-\cdot s) - k^2(p_+\cdot q)(p_-\cdot s) - k^2(p_+\cdot s)(p_-\cdot q)\Big]\,.
\end{aligned}
\tag{4.33}
$$

Substituting the terms T_1 to T_9 in equation (4.11) and taking into account the symmetric character of $\ell_{\mu\nu}$ we have the following result for $|\mathcal{M}_{IB}|^2$:

$$
\begin{aligned}
\overline{|\mathcal{M}_{IB}|^2} = 16{G_F}^2{V_{ud}}^2\frac{e^4}{k^4}{F_\pi}^2{M_\tau}^2\ell_{\mu\nu}\Big[& -\frac{\tau^{\mu\nu}k^2}{(k^2-2p_\tau\cdot k)^2} + \frac{4p^\mu q^\nu(p_\tau\cdot k)}{(k^2-2p_\tau\cdot k)(k^2+2p\cdot k)} \\
& + \frac{4p_\tau^\mu q^\nu(p_\tau\cdot k)}{(k^2-2p_\tau\cdot k)^2} - \frac{2g^{\mu\nu}(p_\tau\cdot k)(k\cdot q)}{(k^2-2p_\tau\cdot k)^2} - \frac{4p^\mu p_\tau^\nu(k\cdot q)}{(k^2+2p\cdot k)(k^2-2p_\tau\cdot k)} \\
& - \frac{4p_\tau^\mu p_\tau^\nu(k\cdot q)}{(k^2-2p_\tau\cdot k)^2} + \frac{8p^\mu p_\tau^\nu(p_\tau\cdot q)}{(k^2+2p\cdot k)(k^2-2p_\tau\cdot k)} \\
& + \frac{4p^\mu p^\nu(p_\tau\cdot q)}{(k^2+2p\cdot k)^2} + \frac{4p_\tau^\mu p_\tau^\nu(p_\tau\cdot q)}{(k^2-2p_\tau\cdot k)^2} \\
& + M_\tau\Big(-\frac{\Omega^{\mu\nu}k^2}{(k^2-2p_\tau\cdot k)^2} + \frac{4p^\mu q^\nu(k\cdot s)}{(k^2-2p_\tau\cdot k)(k^2+2p\cdot k)} \\
& + \frac{4p_\tau^\mu q^\nu(k\cdot s)}{(k^2-2p_\tau\cdot k)^2} - \frac{2g^{\mu\nu}(k\cdot s)(k\cdot q)}{(k^2-2p_\tau\cdot k)^2} - \frac{4p^\mu s^\nu(k\cdot q)}{(k^2+2p\cdot k)(k^2-2p_\tau\cdot k)} \\
& - \frac{4p_\tau^\mu s^\nu(k\cdot q)}{(k^2-2p_\tau\cdot k)^2} + \frac{8p^\mu p_\tau^\nu(s\cdot q)}{(k^2+2p\cdot k)(k^2-2p_\tau\cdot k)} \\
& + \frac{4p^\mu p^\nu(s\cdot q)}{(k^2+2p\cdot k)^2} + \frac{4p_\tau^\mu p_\tau^\nu(s\cdot q)}{(k^2-2p_\tau\cdot k)^2}\Big)\Big]\,.
\end{aligned}
\tag{4.34}
$$

We can simplify some of the contractions worked out previously, for example we can easily see that the terms in ${T_3}^{\mu\nu}\ell_{\mu\nu}$ and ${T_7}^{\mu\nu}\ell_{\mu\nu}$ corresponding to the Levi-Civita symbol will cancel each other in the sum, exactly the same occurs when we sum the terms $T_6{}^{\mu\nu}\ell_{\mu\nu}$ and $T_8{}^{\mu\nu}\ell_{\mu\nu}$ so that we obtain the following simplifications:

$$\begin{aligned}
&T_3{}^{\mu\nu}\ell_{\mu\nu} + T_7{}^{\mu\nu}\ell_{\mu\nu} = \frac{8}{(2p\cdot k + k^2)(k^2 - 2p_\tau \cdot k)}\Big[(p_+ \cdot p_- + m_\ell^2)[(k\cdot q)(p\cdot p_\tau) \\
&- (k\cdot p_\tau)(p\cdot q) + (k\cdot p)(p_\tau \cdot q)] + (k\cdot p_\tau)(p\cdot p_+)(p_-\cdot q) \\
&- (k\cdot q)(p\cdot p_+)(p_-\cdot p_\tau) - (k\cdot p_-)(p\cdot p_+)(p_\tau\cdot q) \\
&- (k\cdot q)(p\cdot p_-)(p_+\cdot p_\tau) + (k\cdot p_\tau)(p\cdot p_-)(p_+\cdot q) - (k\cdot p_+)(p\cdot p_-)(p_\tau\cdot q)\Big] \\
&+ \frac{8M_\tau}{(2p\cdot k + k^2)(k^2 - 2p_\tau\cdot k)}\Big[(p_+\cdot p_- + m_\ell^2)[(k\cdot q)(p\cdot s) - (k\cdot s)(p\cdot q) \\
&+ (k\cdot p)(s\cdot q)] + (k\cdot s)(p\cdot p_+)(p_-\cdot q) - (k\cdot q)(p\cdot p_+)(p_-\cdot s) \\
&- (k\cdot p_-)(p\cdot p_+)(s\cdot q) - (k\cdot q)(p\cdot p_-)(p_+\cdot s) + (k\cdot s)(p\cdot p_-)(p_+\cdot q) \\
&- (k\cdot p_+)(p\cdot p_-)(s\cdot q)\Big]\,.
\end{aligned} \tag{4.35}$$

$$\begin{aligned}
&T_6{}^{\mu\nu}\ell_{\mu\nu} + T_8{}^{\mu\nu}\ell_{\mu\nu} = \frac{8}{(k^2 - 2p_\tau\cdot k)^2}\Big[(p_+\cdot p_- + m_\ell^2)(k\cdot q)p_\tau^2 \\
&- 2(k\cdot q)(p_+\cdot p_\tau)(p_-\cdot p_\tau) + (k\cdot p_\tau)(p_+\cdot q)(p_-\cdot p_\tau) + (k\cdot p_\tau)(p_+\cdot p_\tau)(p_-\cdot q) \\
&- (k\cdot p_-)(p_+\cdot p_\tau)(p_\tau\cdot q) - (k\cdot p_+)(p_\tau\cdot p_-)(p_\tau\cdot q)\Big] \\
&+ \frac{8M_\tau}{(k^2 - 2p_\tau\cdot k)^2}\Big[(p_+\cdot p_- + m_\ell^2)[(k\cdot q)(p_\tau\cdot s) - (k\cdot s)(p_\tau\cdot q) \\
&+ (k\cdot p_\tau)(s\cdot q)] + (k\cdot s)(p_\tau\cdot p_+)(p_-\cdot q) - (k\cdot q)(p_\tau\cdot p_+)(p_-\cdot s) \\
&- (k\cdot p_-)(p_\tau\cdot p_+)(s\cdot q) - (k\cdot q)(p_\tau\cdot p_-)(p_+\cdot s) + (k\cdot s)(p_\tau\cdot p_-)(p_+\cdot q) \\
&- (k\cdot p_+)(p_\tau\cdot p_-)(s\cdot q)\Big]\,.
\end{aligned} \tag{4.36}$$

These cancellations are due to the fact that the imaginary part of $(T_3 + T_6)^{\mu\nu}$ is antisymmetric in (μ, ν), while $\ell_{\mu\nu}$ is symmetric in these indices. The same reasoning applies to the imaginary part of $(T_7 + T_8)^{\mu\nu}$.

We can also simplify the contractions corresponding to the terms T_2 and T_5:

$$
\begin{aligned}
T_2{}^{\mu\nu}\ell_{\mu\nu} + T_5{}^{\mu\nu}\ell_{\mu\nu} = \frac{32}{(2p\cdot k + k^2)(k^2 - 2p_\tau\cdot k)}\left[p_\tau\cdot q + M_\tau s\cdot q\right] \\
\left[(p\cdot p_-)(p_\tau\cdot p_+) - p\cdot p_\tau(p_+\cdot p_- + m_\ell^2) + (p\cdot p_+)(p_\tau\cdot p_-)\right].
\end{aligned}
\tag{4.37}
$$

We cannot simplify the other terms, so the amplitude corresponding to the inner bremsstrahlung is just the sum:

$$
\sum_{s_\nu, s_{\ell+}, s_{\ell-}} |\mathcal{M}_{IB}|^2 = 4G_F{}^2|V_{ud}|^2\frac{e^4}{k^4}F_\pi{}^2M_\tau{}^2\sum_{i=1,\ldots,9}T_i^{\mu\nu}\ell_{\mu\nu}\,. \tag{4.38}
$$

Its explicit expression is presented in appendix A.

Now I will work out the element $\sum_{s_\nu, s_{\ell+}, s_{\ell-}}|\mathcal{M}_V|^2$. From equation (4.8) we see that:

$$
\begin{aligned}
&|\mathcal{M}_V|^2 = G_F{}^2|V_{ud}|^2\frac{e^4}{k^4}|F_V(p\cdot k, k^2)|^2\epsilon_{\mu'\nu'\rho'\sigma'}\epsilon_{\mu\nu\rho\sigma}k^\rho p^\sigma k^{\rho'}p^{\sigma'}\times \\
&[\bar{u}(p_-)\gamma^\nu v(p_+)][\bar{u}(p_-)\gamma^{\nu'}v(p_+)]^\dagger[\bar{u}(q)\gamma^\mu(1-\gamma_5)u(p_\tau)][\bar{u}(q)\gamma^{\mu'}(1-\gamma_5)u(p_\tau)]^\dagger\,.
\end{aligned}
\tag{4.39}
$$

The part corresponding to the electromagnetic current has been worked out previously in eq. (4.13). We just need to calculate the part corresponding to the weak current:

$$
\begin{aligned}
\sum_{s_\nu}\tau^\mu\tau^{\mu'\dagger} &= \sum_{s_\nu}[\bar{u}(q)\gamma^\mu(1-\gamma_5)u(p_\tau)][\bar{u}(q)\gamma^{\mu'}(1-\gamma_5)u(p_\tau)]^\dagger \\
&= Tr[\gamma^\mu(1-\gamma_5)(\not{p}_t + M_\tau)\left(\frac{1+\gamma_5\not{s}}{2}\right)(1+\gamma_5)\gamma^{\mu'}\not{q}] \\
&= 4[\tau^{\mu\mu'} + ip_{\tau a}q_b\epsilon^{\mu\mu' ab} - M_\tau(\Omega^{\mu\mu'} + is_a q_b\epsilon^{\mu\mu' ab})]\,.
\end{aligned}
\tag{4.40}
$$

$$
\begin{aligned}
\Rightarrow \sum_{s_\nu, s_{\ell+}, s_{\ell-}}|\mathcal{M}_V|^2 &= 16G_F{}^2|V_{ud}|^2\frac{e^4}{k^4}|F_V(p\cdot k, k^2)|^2\epsilon_{\mu'\nu'\rho'\sigma'}\epsilon_{\mu\nu\rho\sigma}k^\rho p^\sigma k^{\rho'}p^{\sigma'}\ell^{\nu\nu'}\times \\
&[\tau^{\mu\mu'} - M_\tau\Omega^{\mu\mu'}]\,.
\end{aligned}
\tag{4.41}
$$

We note that $k^\rho p^\sigma k^{\rho'}p^{\sigma'}\ell^{\nu\nu'}$ is symmetric under exchange of the pairs (ρ, ρ'), (σ, σ'), and (ν, ν'). This would require symmetry under the exchange (μ, μ') as well. However, $\epsilon^{\mu\mu' ab}$ is obviously antisymmetric under this exchange, which implies that the terms with a Levi-Civita symbol in eq. (4.40) do not contribute to the eq. (4.41).

For the element $\sum_{s_\nu, s_{\ell+}, s_{\ell-}} |\mathcal{M}_A|^2$ we have from eq. (4.9):

$$\sum_{s_\nu, s_{\ell+}, s_{\ell-}} |\mathcal{M}_\mathcal{A}|^2 = 4{G_F}^2 |V_{ud}|^2 \frac{e^4}{k^4} \sum_{spins} \ell_\nu \ell^\dagger_{\nu'} \sum_{s_\nu} \tau_\mu \tau^\dagger_{\mu'} \times$$
$$\left\{ F_A(p\cdot k, k^2) \left[(k^2 + p\cdot k) g^{\mu\nu} - k^\mu p^\nu \right] + B(k^2) k^2 \left[g^{\mu\nu} - \frac{(p+k)^\mu p^\nu}{k^2 + 2p\cdot k} \right] \right\}$$
$$\left\{ F_A^*(p\cdot k, k^2) \left[(k^2 + p\cdot k) g^{\mu'\nu'} - k^{\mu'} p^{\nu'} \right] + B^*(k^2) k^2 \left[g^{\mu'\nu'} - \frac{(p+k)^{\mu'} p^{\nu'}}{k^2 + 2p\cdot k} \right] \right\} . \tag{4.42}$$

Substituting equations (4.13) and (4.40) in (4.42) we have:

$$\sum_{s_\nu, s_{\ell+}, s_{\ell-}} \left|\mathcal{M}_A\right|^2 = 64{G_F}^2 |V_{ud}|^2 \frac{e^4}{k^4} \ell_{\nu\nu'} \times$$
$$(\tau_{\mu\mu'} + i p_\tau^a q^b \epsilon_{\mu\mu' ab} - M_\tau \Omega_{\mu\mu'} - i M_\tau s^a q^b \epsilon_{\mu\mu' ab}) \mathcal{A}^{\mu\nu} \mathcal{A}^{\mu'\nu' *} , \tag{4.43}$$

where:

$$\mathcal{A}^{\mu\nu} = F_A(p\cdot k, k^2) \left[(k^2 + p\cdot k) g^{\mu\nu} - k^\mu p^\nu \right] + B(k^2) k^2 \left[g^{\mu\nu} - \frac{(p+k)^\mu p^\nu}{k^2 + 2p\cdot k} \right] . \tag{4.44}$$

Now I will work out the interference terms in the square of the amplitude given by eq. (4.6):

$$
\begin{aligned}
&\sum_{s_\nu, s_{\ell+}, s_{\ell-}} 2\Re e(\mathcal{M}_{IB}{\mathcal{M}_V}^\dagger) = \\
&= 2\Re e\Bigg(- iG_F V_{ud}^* \frac{e^2}{k^2} F_\pi M_\tau \ell^\mu \bar{u}(q)(1+\gamma_5)\left[\frac{2p_\mu}{2p\cdot k + k^2} + \frac{2p_{\tau\mu} - \not{k}\gamma_\mu}{k^2 - 2p_\tau \cdot k}\right] u(p_\tau) \\
&\left\{ -G_F V_{ud} \frac{e^2}{k^2} {F_V}^*(p\cdot k, k^2)\epsilon^{\mu'\nu'\rho'\sigma'} k_{\rho'} p_{\sigma'} \ell^\dagger_{\nu'} \tau^\dagger_{\mu'} \right\} \Bigg) \\
&= 2\Re e\Bigg[i{G_F}^2 |{V_{ud}}^2| \frac{e^4}{k^4} F_\pi M_\tau \Bigg(\sum_{spins} \ell^\mu \ell^\dagger_{\nu'} \Bigg) {F_V}^*(p\cdot k, k^2)\epsilon^{\mu'\nu'\rho'\sigma'} k_{\rho'} p_{\sigma'} \\
&\Bigg(\sum_{spins} [\bar{u}(q)(1+\gamma_5)u(p_\tau)][\bar{u}(q)\gamma_{\mu'}(1-\gamma_5)u(p_\tau)]^\dagger \left[\frac{2p_\mu}{2p\cdot k + k^2} + \frac{2p_{\tau\mu}}{k^2 - 2p_\tau \cdot k}\right] \\
&- \sum_{spins} [\bar{u}(q)(1+\gamma_5)\gamma_\lambda\gamma_\mu u(p_\tau)][\bar{u}(q)\gamma_{\mu'}(1-\gamma_5)u(p_\tau)]^\dagger \left[\frac{k^\lambda}{k^2 - 2p_{\tau\cdot k}}\right] \Bigg) \Bigg] .
\end{aligned}
\tag{4.45}
$$

Now using trace techniques we can simplify the spin sums as follows:

$$
\begin{aligned}
&\sum_{spins} [\bar{u}(q)(1+\gamma_5)u(p_\tau)][\bar{u}(q)\gamma_{\mu'}(1-\gamma_5)u(p_\tau)]^\dagger \\
&= Tr[(1+\gamma_5)(\not{p}_t + M_\tau)\Big(\frac{1+\gamma_5\not{s}}{2}\Big)(1+\gamma_5)\gamma_{\mu'}\not{q}] ,
\end{aligned}
\tag{4.46}
$$

$$
\begin{aligned}
&\sum_{spins} [\bar{u}(q)(1+\gamma_5)\gamma_\lambda\gamma_\mu u(p_\tau)][\bar{u}(q)\gamma_{\mu'}(1-\gamma_5)u(p_\tau)]^\dagger \\
&= Tr[(1+\gamma_5)\gamma_\lambda\gamma_\mu(\not{p}_t + M_\tau)\Big(\frac{1+\gamma_5\not{s}}{2}\Big)(1+\gamma_5)\gamma_{\mu'}\not{q}] .
\end{aligned}
\tag{4.47}
$$

Substituting equations (4.46) and (4.47) in equation (4.45) we obtain the following expression for $\sum_{spins} 2\Re e(\mathcal{M}_{IB}{\mathcal{M}_V}^\dagger)$:

$$
\begin{aligned}
\sum_{s_\nu, s_{\ell+}, s_{\ell-}} 2\Re e(\mathcal{M}_{IB}{\mathcal{M}_V}^\dagger) = &- 32{G_F}^2|{V_{ud}}^2| \frac{e^4}{k^4} F_\pi M_\tau \times \\
&\Im m\left[{F_V}^*(p\cdot k, k^2) {\ell^\mu}_{\nu'} \epsilon^{\mu'\nu'\rho'\sigma'} k_{\rho'} p_{\sigma'} \mathcal{V}_{\mu\mu'} \right] ,
\end{aligned}
\tag{4.48}
$$

where:

$$
\begin{aligned}
\mathcal{V}^{\mu\mu'} = \frac{1}{ab}\Bigg[& (q \cdot s)\left({p_\tau}^{\mu'}\left(2bp^\mu - ak^\mu\right) + ag^{\mu\mu'}\left(k \cdot p_\tau\right) - a{p_\tau}^\mu\left(k^{\mu'} - 2{p_\tau}^{\mu'}\right) - ia\epsilon^{\mu\mu' k p_\tau}\right) \\
& + i\epsilon^{\mu' p_\tau q s}\left(ak^\mu - 2\left(ap_\tau^\mu + bp^\mu\right)\right) - aMg^{\mu\mu'}\left(k \cdot q\right) - ag^{\mu\mu'}\left(k \cdot s\right)\left(p_\tau \cdot q\right) \\
& + ag^{\mu\mu'}\left(k \cdot q\right)\left(p_\tau \cdot s\right) + aMk^{\mu'}q^\mu - aMk^\mu q^{\mu'} + iaM\epsilon^{\mu\mu' k q} - aq^{\mu'}s^\mu\left(k \cdot p_\tau\right) \\
& - aq^\mu s^{\mu'}\left(k \cdot p_\tau\right) - ap_\tau^{\mu'}s^\mu\left(k \cdot q\right) + ap_\tau^\mu s^{\mu'}\left(k \cdot q\right) + ap_\tau^{\mu'}q^\mu\left(k \cdot s\right) + ap_\tau^\mu q^{\mu'}\left(k \cdot s\right) \\
& + ak^{\mu'}s^\mu\left(p_\tau \cdot q\right) + ak^\mu s^{\mu'}\left(p_\tau \cdot q\right) - ak^{\mu'}q^\mu\left(p_\tau \cdot s\right) + ak^\mu q^{\mu'}\left(p_\tau \cdot s\right) + iap_\tau^\mu\epsilon^{\mu' k q s} \\
& + iaq^{\mu'}\epsilon^{\mu k p_\tau s} + ias^{\mu'}\epsilon^{\mu k p_\tau q} + ia\left(k \cdot p_\tau\right)\epsilon^{\mu\mu' q s} + 2aMp_\tau^\mu q^{\mu'} - 2ap_\tau^\mu s^{\mu'}\left(p_\tau \cdot q\right) \\
& - 2a{p_\tau}^\mu q^{\mu'}\left(p_\tau \cdot s\right) + 2bMp^\mu q^{\mu'} - 2bp^\mu s^{\mu'}\left(p_\tau \cdot q\right) - 2bp^\mu q^{\mu'}\left(p_\tau \cdot s\right)\Bigg] .
\end{aligned}
\tag{4.49}
$$

In the previous equation, we introduced the following short-hand notation:

$$
a = k^2 + 2p \cdot k \,, \tag{4.50a}
$$

$$
b = k^2 - 2p_\tau \cdot k \,. \tag{4.50b}
$$

For the term $\sum_{spins} 2\Re e(\mathcal{M}_{IB}{\mathcal{M}_A}^\dagger)$ we have:

$$
\begin{aligned}
&\sum_{s_\nu, s_{\ell+}, s_{\ell-}} 2\Re e(\mathcal{M}_{IB}{\mathcal{M}_A}^\dagger) = \\
&= 2\Re e\Bigg(- iG_F V_{ud}^* \frac{e^2}{k^2} F_\pi M_\tau \ell_\mu \bar{u}(q)(1+\gamma_5)\left[\frac{2p^\mu}{2p\cdot k + k^2} + \frac{2p_\tau^\mu - \not{k}\gamma^\mu}{k^2 - 2p_\tau \cdot k}\right] u(p_\tau) \\
&\Bigg[- iG_F V_{ud} \frac{2e^2}{k^2} \ell^{\nu'} \Bigg\{ F_A^*(p\cdot k, k^2)\left[(k^2 + p\cdot k) g_{\mu'\nu'} - k_{\mu'} p_{\nu'}\right] \\
&+ B^*(k^2) k^2 \left[g_{\mu'\nu'} - \frac{(p+k)_{\mu'} p_{\nu'}}{k^2 + 2p\cdot k}\right] \Bigg\} {\tau_{\mu'}}^\dagger \Bigg] \Bigg) \\
&= 2\Re e\Bigg[- {G_F}^2 |{V_{ud}}^2| \frac{2e^4}{k^4} F_\pi M_\tau \Bigg(\sum_{spins} \ell_\mu {\ell^{\nu'}}^\dagger \Bigg) \\
&\Bigg\{ F_A^*(p\cdot k, k^2)\left[(k^2 + p\cdot k) g_{\mu'\nu'} - k_{\mu'} p_{\nu'}\right] + B^*(k^2) k^2 \left[g_{\mu'\nu'} - \frac{(p+k)_{\mu'} p_{\nu'}}{k^2 + 2p\cdot k}\right] \Bigg\} \\
&\Bigg(\sum_{spins} [\bar{u}(q)(1+\gamma_5) u(p_\tau)][\bar{u}(q)\gamma^{\mu'}(1-\gamma_5)u(p_\tau)]^\dagger \left[\frac{2p^\mu}{2p\cdot k + k^2} + \frac{2p_\tau^\mu}{k^2 - 2p_\tau \cdot k}\right] \\
&- \sum_{spins} [\bar{u}(q)(1+\gamma_5)\gamma_\lambda \gamma^\mu u(p_\tau)][\bar{u}(q)\gamma^{\mu'}(1-\gamma_5)u(p_\tau)]^\dagger \left[\frac{k^\lambda}{k^2 - 2p_\tau \cdot k}\right] \Bigg) \Bigg] .
\end{aligned}
\tag{4.51}
$$

We do not need to do too much to compute this interference term because we have worked out all we need previously, we just use equations (4.13), (4.46) and (4.47) to obtain the following expression:

$$
\sum_{s_\nu, s_{\ell+}, s_{\ell-}} 2\Re e(\mathcal{M}_{IB}{\mathcal{M}_A}^\dagger) = -64{G_F}^2 |{V_{ud}}^2| \frac{e^4}{k^4} F_\pi M_\tau {\ell_\mu}^{\nu'} \Re e[{\mathcal{A}^*}_{\mu'\nu'} \mathcal{V}^{\mu\mu'}] ,
\tag{4.52}
$$

where $\mathcal{A}_{\mu'\nu'}$ and $\mathcal{V}^{\mu\mu'}$ are given by equations (4.44) and (4.49) as we have defined them before.

Finally I will work out the term $\sum_{spins} 2\Re e(\mathcal{M}_V {\mathcal{M}_A}^\dagger)$:

$$
\begin{aligned}
&\sum_{s_\nu, s_{\ell+}, s_{\ell-}} 2\Re e(\mathcal{M}_V \mathcal{M}_A{}^\dagger) = \\
&= \sum_{s_\nu, s_{\ell+}, s_{\ell-}} 2\Re e\Bigg(-G_F V_{ud}^* \frac{e^2}{k^2} F_V(p\cdot k, k^2)\epsilon_{\mu\nu\rho\sigma} k^\rho p^\sigma \ell^\nu \tau^\mu \Bigg[-iG_F V_{ud} \frac{2e^2}{k^2} \ell_{\nu'} \\
&\Bigg\{ F_A^*(p\cdot k, k^2)\Big[(k^2 + p\cdot k) g_{\mu'}^{\nu'} - k_{\mu'} p^{\nu'}\Big] + B^*(k^2) k^2 \left[g_{\mu'}^{\nu'} - \frac{(p+k)_{\mu'} p^{\nu'}}{k^2 + 2p\cdot k} \right] \Bigg\} \tau^{\mu'\dagger} \Bigg] \Bigg) \\
&= 2\Re e\Bigg[iG_F{}^2 |V_{ud}{}^2| \frac{2e^4}{k^4} \Big(\sum_{spins} \ell^\nu \ell_{\nu'}{}^\dagger \Big) \Big(\sum_{s_\nu} \tau^\mu \tau^{\mu'\dagger} \Big) \epsilon_{\mu\nu\rho\sigma} k^\rho p^\sigma F_V(p\cdot k, k^2) \\
&\Bigg\{ F_A^*(p\cdot k, k^2)\Big[(k^2 + p\cdot k) g_{\mu'}^{\nu'} - k_{\mu'} p^{\nu'}\Big] + B^*(k^2) k^2 \left[g_{\mu'}^{\nu'} - \frac{(p+k)_{\mu'} p^{\nu'}}{k^2 + 2p\cdot k} \right] \Bigg\} \Bigg] .
\end{aligned}
\tag{4.53}
$$

We do not need to do too much in this term either. Using (4.13), (4.40) and (4.44) we obtain the following result:

$$
\begin{aligned}
\sum_{s_\nu, s_{\ell+}, s_{\ell-}} 2\Re e(\mathcal{M}_V \mathcal{M}_A{}^\dagger) = &- 2G_F^2 |V_{ud}|^2 \frac{2e^4}{k^4} (4\ell_{\nu'}^{\nu}) \Im m\Big[F_V(p\cdot k, k^2) \epsilon_{\mu\nu\rho\sigma} k^\rho p^\sigma \\
&4[\tau^{\mu\mu'} + ip_{\tau a} q_b \epsilon^{\mu\mu' ab} - M_\tau(\Omega^{\mu\mu'} + is_a q_b \epsilon^{\mu\mu' ab})] \mathcal{A}_{\mu'}^{\nu' *} \Big] .
\end{aligned}
\tag{4.54}
$$

$$
\begin{aligned}
\Rightarrow \sum_{s_\nu, s_{\ell+}, s_{\ell-}} 2\Re e(\mathcal{M}_V \mathcal{M}_A{}^\dagger) = &- 64 G_F^2 |V_{ud}|^2 \frac{e^4}{k^4} \ell_{\nu'}^{\nu} \Im m\Big[F_V(p\cdot k, k^2) \epsilon_{\mu\nu\rho\sigma} k^\rho p^\sigma \\
&[\tau^{\mu\mu'} + ip_{\tau a} q_b \epsilon^{\mu\mu' ab} - M_\tau(\Omega^{\mu\mu'} + is_a q_b \epsilon^{\mu\mu' ab})] \mathcal{A}_{\mu'}^{\nu' *} \Big] .
\end{aligned}
\tag{4.55}
$$

4.3 Kinematics

The decay we are analyzing is an example of a one to four body decay. In these kind of decays it can be shown [8] that the square of the amplitude $|\mathcal{M}|^2$ can be written as a function of five independent variables. The choice of these variables is not unique, so we are free to choose a convenient way we find. We follow the convention used in [8, 6], which states the following:

- $s_{12} = p_{12}^2 = (p+q)^2$, The invariant mass of the pion-neutrino system
- $s_{34} = p_{34}^2 = (p_- + p_+)^2 = k^2$, The invariant mass of the lepton pair $\ell^+\ell^-$
- θ_1, the angle between the neutrino trajectory and the 3-momentum vector $\vec{k'} = \vec{p} + \vec{q}$
- θ_3, the angle between the trajectory of the ℓ^+ lepton and the 3-momentum vector $\vec{k}$ in the rest frame of the center of mass of the lepton pair.
- ϕ, The angle between the planes of the pion-neutrino and the lepton pair systems

The kinematic limits for these variables were calculated in [8] and are given by the following expressions:

$$(2m_\ell)^2 \leq s_{34} \leq (M_\tau - m_\pi)^2 \tag{4.56a}$$

$$(m_\pi)^2 \leq s_{12} \leq (M_\tau - \sqrt{s_{34}})^2 \tag{4.56b}$$

$$-1 \leq \cos\theta_{1,3} \leq 1 \tag{4.56c}$$

$$0 \leq \phi \leq 2\pi \tag{4.56d}$$

Finally the branching ratio can be written in terms of the five variables defined before as is shown in the following equation [8, 6]:

$$d\Gamma = \frac{X\beta_{12}\beta_{34}}{4(4\pi)^6 M_\tau^3}\overline{|\mathcal{M}|^2} ds_{12} ds_{34} \sin(\theta_1) d\theta_1 \sin(\theta_3) d\theta_3 d\phi\,, \tag{4.57}$$

where:

$$\beta_{12} = \frac{\sqrt{s_{12}^2 + m_\pi^4 - 2s_{12}m_\pi^2}}{s_{12}}\,, \tag{4.58a}$$

$$\beta_{34} = \frac{\sqrt{s_{34}^2 - 4s_{34}m_\ell^2}}{s_{34}}\,, \tag{4.58b}$$

$$X = \frac{\sqrt{s_{12}^2 + s_{34}^2 + M_\tau^4 - 2s_{12}s_{34} - 2s_{12}M_\tau^2 - 2s_{34}M_\tau^2}}{2}\,. \tag{4.58c}$$

From conservation of energy and momentum we have that

$$p_\tau = p + p_+ + p_- + q\,. \tag{4.59}$$

When the decaying particle has a definite polarization (as is our case with the tau lepton) $|\mathcal{M}|^2$ depends on the polarization four vector (s) of the decaying particle, in addition to be a function of the invariants formed with all the four independent momenta available (among p_τ, p, p_+, p_- and q in our case). Then $|\mathcal{M}|^2$ will also be a function of products like: $p \cdot s$, $p_+ \cdot s$, $p_- \cdot s$, and $q \cdot s$, this can be seen in all the expressions that we found for the different contributions to $|\mathcal{M}|^2$ in the previous chapter and in appendix A.
With the purpose of calculating $p{\cdot}s$, $p_+{\cdot}s$, $p_-{\cdot}s$, and $q{\cdot}s$, it will be convenient to construct a specific reference frame in which I will find definite expressions for the four momenta p_τ, p, p_+, p_-, and q, I will follow the technique used in [9].
To simplify things I will do the following substitutions:
$p_\tau \to P$, $p_- \to p_1$, $p_+ \to p_2$, $p \to p_3$, and $q \to p_4$, $M_\tau \to M$.
First, I will choose the reference frame as the rest frame of the decaying particle, then I will choose the z-axis along the direction of $\vec{p_1}$ (the three momentum of the particle with p_1), and finally I will choose the x-axis in such a way that $\vec{p_2}$ lies on the x-z plane. The four momentum p_3, and p_4 will be found using the previous ones.
With the conventions stated before we get the following expressions for the four momentum,

$$P = (M, 0, 0, 0)\,, \tag{4.60a}$$

$$p_1 = (E_1, \vec{p_1}) = (E_1, 0, 0, |\vec{p_1}|)\,, \tag{4.60b}$$

$$p_2 = (E_2, \vec{p_2}) = (E_2, |\vec{p_2}|\text{sin}\theta, 0, |\vec{p_2}|\text{cos}\theta)\,, \tag{4.60c}$$

$$p_3 = (E_3, \vec{p_3}) = (E_3, a_3, b_3, c_3)\,, \tag{4.60d}$$

$$p_4 = (E_4, \vec{p_4}) = (E_4, a_4, b_4, c_4)\,, \tag{4.60e}$$

$$s = (0, a_5, b_5, c_5)\,, \tag{4.60f}$$

where M is the mass of the decaying particle, θ is the angle between the z-axis and the $\vec{p_2}$ direction. The components a_i, b_i, and c_i, $i = 3, 4$ will be calculated using the energy and momentum conservation.

s is a polarization vector, so we have the following constraint ($s^2 = -1 \Rightarrow |\vec{s}| = 1$),

$$(a_5)^2 + (b_5)^2 + (c_5)^2 = 1\,. \tag{4.61}$$

The energies E_k, $k = 1, 2, 3, 4$ can be calculated as follows,

$$E_k = \frac{P \cdot p_k}{M}, \quad k = 1, 2, 3\,. \tag{4.62}$$

And $E_4 = M - E_1 - E_2 - E_3$. For $|\vec{p_1}|$, $|\vec{p_1}|$, $\cos\theta$, and $\sin\theta$ we have,

$$|\vec{p_1}| = \left[\frac{(P \cdot p_1)^2}{M^2} - m_1^2\right]^{\frac{1}{2}}, \tag{4.63a}$$

$$|\vec{p_2}| = \left[\frac{(P \cdot p_2)^2}{M^2} - m_2^2\right]^{\frac{1}{2}}, \tag{4.63b}$$

$$\cos\theta = \frac{E_1 E_2 - p_1 \cdot p_2}{|\vec{p_1}||\vec{p_2}|} = \frac{(P \cdot p_1)(P \cdot p_2) - M^2 p_1 \cdot p_2}{\Big[\big((P \cdot p_1)^2 - M^2 m_1^2\big)\big((P \cdot p_2)^2 - M^2 m_2^2\big)\Big]^{\frac{1}{2}}}, \tag{4.63c}$$

$$\sin\theta = [1 - \cos^2\theta]^{\frac{1}{2}}\,. \tag{4.63d}$$

For c_3 and c_4 we have,

$$c_3 = \frac{E_1 E_3 - p_1 \cdot p_3}{|\vec{p_1}|}, \tag{4.64a}$$

$$c_4 = \frac{E_1 E_4 - p_1 \cdot p_4}{|\vec{p_1}|} = \frac{E_1(M - E_1 - E_2 - E_3) - P \cdot p_1 + m_1^2 + p_1 \cdot p_2 + p_1 \cdot p_3}{|\vec{p_1}|}\,. \tag{4.64b}$$

For a_3 and a_4 we have,

$$a_3 = \frac{E_2 E_3 - p_2 \cdot p_3 - c_3 |\vec{p_2}| \cos\theta}{|\vec{p_2}| \sin\theta}, \tag{4.65a}$$

$$\begin{aligned} a_4 =& \frac{E_2 E_4 - p_2 \cdot p_4 - c_4 |\vec{p_2}| \cos\theta}{|\vec{p_2}| \sin\theta} \\ =& \frac{E_2(M - E_1 - E_2 - E_3) - p_2 \cdot P + p_1 \cdot p_2 + m_2^2 + p_2 \cdot p_3 - c_4 |\vec{p_2}| \cos\theta}{|\vec{p_2}| \sin\theta}\,. \end{aligned} \tag{4.65b}$$

Finally for b_3, and b_4 we have the following expressions,

$$b_3 = [E_3^2 - a_3^2 - c_3^2 - m_3^2]^{\frac{1}{2}}, \tag{4.66a}$$

$$b_4 = [E_4^2 - a_4^2 - c_4^2 - m_4^2]^{\frac{1}{2}} = [(M - E_1 - E_2 - E_3)^2 - a_4^2 - c_4^2 - m_4^2]^{\frac{1}{2}} \,. \tag{4.66b}$$

Now we have everything we need, the four momenta P, p_1, p_2, p_3, and p_4 can be written in terms of the Lorentz invariant quantities that we found previosly, as it is shown in the following equations,

$$P = (M, 0, 0, 0)\,, \tag{4.67}$$

$$p_1 = \left(\frac{P \cdot p_1}{M}, 0, 0, \left[-\frac{\begin{bmatrix} M^2 & P \cdot p_1 \\ P \cdot p_1 & m_1^2 \end{bmatrix}}{M^2} \right]^{\frac{1}{2}} \right)\,, \tag{4.68}$$

$$p_2 = \left(\frac{P \cdot p_2}{M}, \left(-\frac{\begin{bmatrix} M^2 & P \cdot p_1 & P \cdot p_2 \\ P \cdot p_1 & m_1^2 & p_1 \cdot p_2 \\ P \cdot p_2 & p_1 \cdot p_2 & m_2^2 \end{bmatrix}}{\begin{bmatrix} M^2 & P \cdot p_1 \\ P \cdot p_1 & m_1^2 \end{bmatrix}} \right)^{\frac{1}{2}}, 0, -\frac{\begin{bmatrix} M^2 & P \cdot p_2 \\ P \cdot p_1 & p_1 \cdot p_2 \end{bmatrix}}{\left(-m_1^2 \begin{bmatrix} M^2 & P \cdot p_1 \\ P \cdot p_1 & m_1^2 \end{bmatrix} \right)^{\frac{1}{2}}} \right)\,, \tag{4.69}$$

$$p_3 = \left(\frac{P \cdot p_3}{M}, a_3, b_3, c_3 \right)\,, \tag{4.70a}$$

where a_3, b_3, and c_3 can be written in the compact form,

$$a_3 = \frac{\begin{vmatrix} M^2 & P \cdot p_1 & P \cdot p_3 \\ P \cdot p_1 & m_1^2 & p_1 \cdot p_3 \\ P \cdot p_2 & p_1 \cdot p_2 & p_2 \cdot p_3 \end{vmatrix}}{\left(-\begin{vmatrix} M^2 & P \cdot p_1 \\ P \cdot p_1 & m_1^2 \end{vmatrix} \begin{vmatrix} M^2 & P \cdot p_1 & P \cdot p_2 \\ P \cdot p_1 & m_1^2 & p_1 \cdot p_2 \\ P \cdot p_2 & p_1 \cdot p_2 & m_2^2 \end{vmatrix} \right)^{\frac{1}{2}}}\,, \tag{4.70b}$$

$$b_3 = \left(-\frac{\begin{vmatrix} M^2 & P \cdot p_1 & P \cdot p_2 & P \cdot p_3 \\ P \cdot p_1 & m_1^2 & p_1 \cdot p_2 & p_1 \cdot p_3 \\ P \cdot p_2 & p_1 \cdot p_2 & m_2^2 & p_2 \cdot p_3 \\ P \cdot p_3 & p_1 \cdot p_3 & p_2 \cdot p_3 & m_3^2 \end{vmatrix}}{\begin{vmatrix} M^2 & P \cdot p_1 & P \cdot p_2 \\ P \cdot p_1 & m_1^2 & p_1 \cdot p_2 \\ P \cdot p_2 & p_1 \cdot p_2 & m_2^2 \end{vmatrix}} \right)^{\frac{1}{2}}\,, \tag{4.70c}$$

$$c_3 = -\frac{\begin{vmatrix} M^2 & P\cdot p_3 \\ P\cdot p_1 & p_1\cdot p_3 \end{vmatrix}}{\left(-m_1^2 \begin{vmatrix} M^2 & P\cdot p_1 \\ P\cdot p_1 & m_1^2 \end{vmatrix}\right)^{\frac{1}{2}}} \,. \tag{4.70d}$$

The term p_4 has the same form as p_3, we just need to replace 3 → 4.
The previous expressions can be further simplified noticing that $m_1 = m_2 = m_\ell$, and $m_4^2 = 0$.
I have to calculate polarization effects, these will be given in terms of $p\cdot s$, $p_-\cdot s$, $p_+\cdot s$, and $q\cdot s$, these terms can be easily calculated using the expressions obtained previously, once I define a direction for the polarization vector. I will simplify things by choosing this polarization vector orthogonal to $\vec{p_-}$, and $\vec{p_+}$, so that $p_-\cdot s = 0$, and $p_+\cdot s = 0$, then the explicit form of the vector s is, $s = (0,0,1,0)$, given this form, the products $p\cdot s$, and $q\cdot s$ are given by the following equations:

$$p\cdot s = b_3 \,. \tag{4.71}$$

Equation (4.59) fixes the value of $q\cdot s$ according to,

$$q\cdot s = -p\cdot s = -b_3 \,. \tag{4.72}$$

At this point we have succeeded in finding all the products between the polarization four vector and the available momenta in terms of invariants. Now it is time to write those invariants as fuctions of the variables we chose at the beginning of this section, that is s_{12}, s_{34}, θ_1, θ_3, and ϕ. Here I will follow the conventions used in [8]:

$$p_{12}^2 := s_{12} \,, \tag{4.73a}$$

$$p_{34}^2 := s_{34} \,, \tag{4.73b}$$

$$q_{12}^2 = 2(m_1^2 + m_2^2) - s_{12} \,, \tag{4.73c}$$

$$q_{34}^2 = 2(m_3^2 + m_4^2) - s_{34} \,, \tag{4.73d}$$

$$p_{12}\cdot q_{12} = m_2^2 - m_1^2 \,, \tag{4.73e}$$

$$p_{34}\cdot q_{34} = m_4^2 - m_3^2 \,, \tag{4.73f}$$

$$p_{12}\cdot p_{34} = \frac{1}{2}(M^2 - s_{12} - s_{34}) \,, \tag{4.73g}$$

$$p_{12}\cdot q_{34} = -X\beta_{34}\cos\theta_3 + \frac{m_4^2 - m_3^2}{s_{34}} p_{12}\cdot p_{34} \,, \tag{4.73h}$$

$$p_{34}\cdot q_{12} = X\beta_{12}\cos\theta_1 + \frac{m_2^2 - m_1^2}{s_{12}} p_{12}\cdot p_{34} \,, \tag{4.73i}$$

$$\begin{aligned} q_{12} \cdot q_{34} =& \frac{m_1^2 - m_2^2}{s_{12}} \frac{m_3^2 - m_4^2}{s_{34}} + \frac{m_1^2 - m_2^2}{s_{12}} X \beta_{34} \mathrm{cos}\theta_3 \\ &+ \beta_{12}\beta_{34}[p_{12}p_{34}\mathrm{cos}\theta_1\mathrm{cos}\theta_3 - \sqrt{s_{12}s_{34}}\mathrm{sin}\theta_1\mathrm{sin}\theta_3\mathrm{cos}\phi] \qquad (4.73\mathrm{j}) \\ &\frac{m_3^2 - m_4^2}{s_{34}} X \beta_{12} \mathrm{cos}\theta_1 \,, \end{aligned}$$

$$\epsilon_{\alpha\beta\gamma\delta} p_{12}^{\alpha} p_{34}^{\beta} q_{12}^{\gamma} q_{34}^{\delta} = -\sqrt{s_{12}s_{34}}\beta_{12}\beta_{34}\mathrm{sin}\theta_1\mathrm{sin}\theta_3\mathrm{sin}\phi \,, \qquad (4.73\mathrm{k})$$

where,

$$p_{12} = p_1 + p_2 \,, \qquad (4.74\mathrm{a})$$

$$p_{34} = p_3 + p_4 \,, \qquad (4.74\mathrm{b})$$

$$q_{12} = p_1 - p_2 \,, \qquad (4.74\mathrm{c})$$

$$q_{34} = p_3 - p_4 \,. \qquad (4.74\mathrm{d})$$

4.4 Form Factors

In section 2.3 we mentioned that we cannot use perturbative QCD to make calculations, this is because the α_s coupling grows rapidly at the energy scales involved in the process that we are studying. There are other alternatives to handle this problem. I will follow the tecniques used in references [10, 7, 6], the approach they use is an effective field Theory called Resonance Chiral Theory (RχT) that I introduce briefly in the following.
The basis of this approach comes from the seminal paper by Weinberg [11], where it is put forward that one can choose the most convenient degrees of freedom for the problem at hand and -provided all their symmetries are implemented in the corresponding Lagrangian- the predicted observables will be the most general ones consistent with axiomatic field theory properties (analyticity, perturbative unitarity, crossing symmetry and cluster decomposition) and the symmetries of the problem. From this point of view, the key is then to correctly identify the symmetries of low-energy QCD, to which we turn next.
In the limit of massless quarks (which is a reasonable approximation for the light u, d and s quarks) the QCD Lagrangian exhibits a global symmetry under the independent rotation of the left- and right-handed quark components, the so-called chiral symmetry. As a consequence of this symmetry meson states with opposite parity would have the same mass, a feature that is not seen in the hadron spectrum. In order to understand this, one must remember that there are two ways of realizing a symmetry: in the Wigner-Weyl

way the symmetry would be apparent in the spectrum; while in the less intuitive Nambu-Goldstone way, it will not. This seeming paradox comes from the fact that the symmetry group of the Lagrangian (G) is not shared by the vacuum state of the theory, which only has a subgroup of it ($H \in G$) as a symmetry. In this case the Goldstone theorem applies, stating that for each broken generator of G, there will be a massless spin-zero state, the so-called Nambu-Goldstone boson. If that has anything to do with the absence of parity doublets with similar masses in the lightest meson spectrum, the latter should include massless spinless particles. Although this is not exactly the case, the pions stand out as being much lighter than any other states, which supports the conjecture that they are the pseudo-Goldstone bosons associated to the spontaneous symmetry breakdown of chiral symmetry (in the two-flavor u and d case). Pions are light but not massless because the non-vanishing vacuum expectation value of the light-quark condensate $\langle \bar{q}q \rangle$ and because of the explicit breaking of chiral symmetry given the small light-quark masses.

Based on these ideas, Chiral Perturbation Theory was developed as the quantum effective field theory dual to QCD at low energies by Gasser and Leutwyler in the mid eighties [12, 13]. It is built using the lightest pseudo-Goldstone bosons as dynamical fields ($\pi^{\pm}$ and π^0 for two flavors and adding $K^0, \bar{K^0}, K^{\pm}, \eta$ when strangeness is included). The perturbative expansion is organized in powers of the squared momenta/masses of these fields over the chiral symmetry breaking scale, which is slightly above the GeV, warranting a good convergence of the perturbative series, specially in the two-flavor case. Chiral Perturbation Theory has been extended to next-to-next-to-leading order both in the even- [14, 15] and in the odd-intrinsic [16] parity sectors.

The tau mass is, however, larger than the chiral symmetry breaking scale, making Chiral Perturbation Theory insufficient to describe the hadron form factors entering their semileptonic decays. Following Weinberg's theorem, we should add to the Chiral Perturbation Theory Lagrangian the next heavier states (the lowest-lying light-flavor resonances) as explicit degrees of freedom. This brings in an associated problem: since the momenta and masses of these resonances are of the order of the chiral symmetry breaking scale, the expansion parameter of Chiral Perturbation Theory is no longer useful in its extended version. Solving this issue is still an open problem. Nevertheless, the large-N_C limit of QCD [17, 18, 19] applied to a theory of mesons [20] has yielded many interesting results resembling very much its phenomenology. Because of this, Resonance Chiral Theory [21, 22] adds to the pseudo-Goldstone bosons of chiral symmetry breakdown the ligthest resonances in such a way that Chiral Perturbation Theory results are recovered at low energies and the $1/N_C$ expansion allows to extend them to the GeV region (and also offers a deeper understanding of the η and η' meson system). A fundamental requirement within this program is the fulfilment of the short-distance QCD constraints on the relevant Green functions and form factors [23, 24]. For more details on Chiral Perturbation Theory including resonances one can consult e.g. refs. [25, 26].

The effective lagrangian is [10, 7, 6],

$$\begin{aligned}\mathcal{L}_{R\chi T} = \mathcal{L}_{WZW} + \mathcal{L}^{V}_{Kin} + \frac{F_\pi^2}{4}\langle u_\mu u^\mu + \chi_+\rangle + \frac{F_V}{2\sqrt{2}}\langle V_{\mu\nu} f_+^{\mu\nu}\rangle + \frac{F_A}{2\sqrt{2}}\langle A_{\mu\nu} f_-^{\mu\nu}\rangle \\ + i\frac{G_V}{\sqrt{2}}\langle V_{\mu\nu} u^\mu u^\nu\rangle + \sum_{i=1}^{7}\frac{c_i}{M_V}\mathcal{O}^i_{VJP} + \sum_{i=1}^{4} d_i \mathcal{O}^i_{VVP} + \sum_{i=1}^{5}\lambda_i \mathcal{O}^i_{VAP}\,,\end{aligned} \tag{4.75}$$

where $f_\pm^{\mu\nu} = uF_L^{\mu\nu}u^\dagger \pm u^\dagger F_R^{\mu\nu}u$ and $F_{R,L}^{\mu\nu}$ are the field strenght tensors associated with the external right and left handed auxiliary fields. All coupling constants are real, and M_V is the mass of the lightest vector meson resonance nonet.

The parameters introduced in the previous equation are defined as follows,

$$u_\mu = i\left[u^\dagger(\partial_\mu - ir_\mu)u - u(\partial_\mu - i\ell_\mu)u^\dagger\right]\,, \tag{4.76a}$$

$$\chi_\pm = u^\dagger \chi u^\dagger \pm u\chi^\dagger u\,, \tag{4.76b}$$

$$\chi = 2B_0(s + ip)\,, \tag{4.76c}$$

$$\Phi(x) = \begin{pmatrix} \frac{1}{\sqrt{2}}\pi^0 & \pi^+ & K^+ \\ \pi^- & -\frac{1}{\sqrt{2}}\pi^0 + \frac{1}{\sqrt{6}}\eta_8 & K^0 \\ K^- & \overline{K}^0 & -\frac{2}{\sqrt{6}}\eta_8 \end{pmatrix} \tag{4.76d}$$

$$u(\phi) = exp\left[\frac{i}{\sqrt{2}F_\pi}\Phi(x)\right] \tag{4.76e}$$

where r_μ, ℓ_μ, s, and p are external fields that promote the global $SU(3)_L \times SU(3)_R$ symmetry to a local one.

The explicit forms of the $\mathcal{O}$ operators is shown in the following equations,

$$\mathcal{O}^1_{VJP} = \epsilon_{\mu\nu\rho\sigma}\langle\{V^{\mu\nu}, f_+^{\rho\alpha}\}\nabla_\alpha u^\sigma\rangle\,, \tag{4.77a}$$

$$\mathcal{O}^2_{VJP} = \epsilon_{\mu\nu\rho\sigma}\langle\{V^{\mu\alpha}, f_+^{\rho\sigma}\}\nabla_\alpha u^\nu\rangle\,, \tag{4.77b}$$

$$\mathcal{O}^3_{VJP} = i\epsilon_{\mu\nu\rho\sigma}\langle\{V^{\mu\nu}, f_+^{\rho\sigma}\}\chi_-\rangle\,, \tag{4.77c}$$

$$\mathcal{O}^4_{VJP} = i\epsilon_{\mu\nu\rho\sigma}\langle V^{\mu\nu}\,[f_-^{\rho\sigma}, \chi_+]\rangle\,, \tag{4.77d}$$

$$\mathcal{O}^5_{VJP} = \epsilon_{\mu\nu\rho\sigma}\langle\{\nabla_\alpha V^{\mu\nu}, f_+^{\rho\alpha}\}u^\sigma\rangle\,, \tag{4.77e}$$

$$\mathcal{O}^6_{VJP} = \epsilon_{\mu\nu\rho\sigma}\langle\{\nabla_\alpha V^{\mu\alpha}, f_+^{\rho\sigma}\}u^\nu\rangle\,, \tag{4.77f}$$
$$\mathcal{O}^7_{VJP} = \epsilon_{\mu\nu\rho\sigma}\langle\{\nabla^\sigma V^{\mu\nu}, f_+^{\rho\alpha}\}u_\alpha\rangle\,. \tag{4.77g}$$

$$\mathcal{O}^1_{VAP} = \langle[V^{\mu\nu}, A_{\mu\nu}]\,\chi_-\rangle\,, \tag{4.78a}$$
$$\mathcal{O}^2_{VAP} = i\langle[V^{\mu\nu}, A_{\nu\alpha}]\,h^\alpha_\mu\rangle\,, \tag{4.78b}$$
$$\mathcal{O}^3_{VAP} = i\langle[\nabla^\mu V_{\mu\nu}, A^{\nu\alpha}]\,u_\alpha\rangle\,, \tag{4.78c}$$
$$\mathcal{O}^4_{VAP} = i\langle[\nabla^\alpha V_{\mu\nu}, A^\nu_\alpha]\,u^\mu\rangle\,, \tag{4.78d}$$
$$\mathcal{O}^5_{VAP} = i\langle[\nabla^\alpha V_{\mu\nu}, A^{\nu\nu}]\,u_\alpha\rangle\,, \tag{4.78e}$$

where $h_{\mu\nu} = \nabla_\mu u_\nu + \nabla_\nu u_\mu$.

$$\mathcal{O}^1_{VVP} = \epsilon_{\mu\nu\rho\sigma}\langle\{V^{\mu\nu}, V^{\rho\alpha}\}\nabla_\alpha u^\sigma\rangle\,, \tag{4.79a}$$
$$\mathcal{O}^2_{VVP} = i\epsilon_{\mu\nu\rho\sigma}\langle\{V^{\mu\nu}, V^{\rho\sigma}\}\chi_-\rangle\,, \tag{4.79b}$$
$$\mathcal{O}^3_{VVP} = \epsilon_{\mu\nu\rho\sigma}\langle\{\nabla_\alpha V^{\mu\nu}, V^{\rho\alpha}\}u^\sigma\rangle \tag{4.79c}$$
$$\mathcal{O}^4_{VVP} = \epsilon_{\mu\nu\rho\sigma}\langle\{\nabla^\sigma V^{\mu\nu}, V^{\rho\alpha}\}u_\alpha\rangle \tag{4.79d}$$

The structure-dependent form factors that appear in the amplitudes in equations (4.8) and (4.9) can be obtained from the feynman diagrams shown in figure 4.2 and figure 4.3.

Figure 4.2: Vector current contributions to the $W^{-*} \to \pi^-\gamma^*$ vertex.

Figure 4.3: Axial-Vector current contributions to the $W^{-*} \to \pi^-\gamma^*$

The vector form factor $F_V(t,k^2)$ is given by the following equation,

$$\begin{aligned}F_V(t,k^2) = &-\frac{N_c}{24\pi^2 F_\pi} + \frac{2\sqrt{2}F_V}{3F_\pi M_V}\Big[(c_2 - c_1 - c_5)t + (c_5 - c_1 - c_2 - 8c_3)m_\pi^2 + 2(c_6 - c_5)k^2\Big] \\ &\Big[\frac{\cos^2\theta}{M_\phi^2 - k^2 - iM_\phi\Gamma_\phi}(1 - \sqrt{2}\mathrm{tg}\theta) + \frac{\sin^2\theta}{M_\omega^2 - k^2 - iM_\omega\Gamma_\omega}(1 + \sqrt{2}\mathrm{cotg}\theta)\Big] \\ &+ \frac{2\sqrt{2}F_V}{3F_{\pi M_V}}D_\rho(t)\Big[(c_1 - c_2 - c_5 + 2c_6)t + (c_5 - c_1 - c_2 - 8c_3)m_\pi^2 + (c_2 - c_1 - c_5)k^2\Big] \\ &+ \frac{4F_V^2}{3F_\pi}D_\rho(t)\Big[d_3(t + 4k^2) + (d_1 + 8d_2 - d_3)m_\pi^2\Big] \\ &\Big[\frac{\cos^2\theta}{M_\phi^2 - k^2 - iM_\phi\Gamma_\phi}(1 - \sqrt{2}\mathrm{tg}\theta) + \frac{\sin^2\theta}{M_\omega^2 - k^2 - iM_\omega\Gamma_\omega}(1 - \sqrt{2}\mathrm{cotg}\theta)\Big]\,,\end{aligned} \tag{4.80}$$

where,

$$D_\rho(t) = \frac{1}{M_\rho^2 - t - iM_\rho\Gamma_\rho(t)}\,, \tag{4.81}$$

and,

$$\Gamma_\rho(s) = \frac{sM_\rho}{96\pi F_\pi^2}\left[\sigma_\pi^3(s)(s - 4m_\pi^2) + \frac{1}{2}\sigma_k^3(s)\theta(s - 4m_k^2)\right]\,, \tag{4.82}$$

is the decay width of the $\rho(770)$ resonance with $\sigma_p(s) = \sqrt{1 - \frac{4m_p^2}{s}}$ (ref. [27]). We will assume the ideal mixing case for the vector resonances ω and ϕ in any numerical application:

$$\omega_1 = \cos\theta\omega - \sin\theta\phi \sim \sqrt{\frac{2}{3}}\omega - \sqrt{\frac{1}{3}}\phi \tag{4.83a}$$

$$\omega_8 = \sin\theta\omega + \cos\theta\phi \sim \sqrt{\frac{2}{3}}\phi + \sqrt{\frac{1}{3}}\omega \tag{4.83b}$$

Similarly, the axial-vector form factor $F_A(t,k^2)$ is given by,

$$\begin{aligned}F_A(t,k^2) =& \frac{F_V^2}{F_\pi}\left(1 - 2\frac{G_V}{F_V}\right)D_\rho(k^2) - \frac{F_A^2}{F_\pi}D_{a_1}(t) \\ &+ \frac{F_AF_V}{\sqrt{2}F_\pi}D_\rho(k^2)D_{a_1}(t)(-\lambda'' t + \lambda_0 m_\pi^2)\,,\end{aligned} \tag{4.84}$$

where,

$$\sqrt{2}\lambda_0 = -4\lambda_1 - \lambda_2 - \frac{\lambda_4}{2} - \lambda_5\,, \tag{4.85a}$$

and,

$$\sqrt{2}\lambda'' = \lambda_2 - \frac{\lambda_4}{2} - \lambda_5\,. \tag{4.85b}$$

Finally for the $B(k^2)$ form factor we have [28],

$$B(k^2) = \frac{F_\pi F_V^{\pi^+\pi^-}|_\rho(k^2) - 1}{k^2}\,, \tag{4.86}$$

where $F_V^{\pi^+\pi^-}|_\rho$ is the $I = 1$ part of the $\pi^+\pi^-$ vector form factor, and has the following form [29],

$$\langle \pi^+(p_+)\pi^-(p_-)|\overline{u}\gamma^\mu u + \overline{d}\gamma^\mu d|0\rangle = (p_+ - p_-)^\mu F_V^{\pi^+\pi^-}(k^2)\,. \tag{4.87}$$

The arguments in the form factors are $t := (p+k)^2$ and $k^2 = (p_+ + p_-)^2$.

4.5 Results

At this point we have finished the analytical calculations in this work, now we need to do some numerical analysis in order to predict observables like the branching ratio of the process.

After the decay we have four particles: the lepton-antilepton pair, the pion, and the neutrino. The momenta of the lepton and the one for the antilepton define a plane, similarly the momenta of the pion and the one for the neutrino also define a plane, so we can build two (in general) different planes for the process. The direction of the polarization vector ($\vec{s}$) can point anywhere, there is not a preferred direction, but to simplify things we will choose the direction of this vector in convenient ways.

First we analize the ideal case where the momenta of each of the four particles (p, p_+, p_-, and q) lie on the same plane with the polarization vector $\vec{s}$ perpendicular to that plane, this simplify things considerably because in this case $p \cdot s = p_+ \cdot s = p_- \cdot s = q \cdot s = 0$, so that the part of the amplitude that depends on the polarization vector almost vanishes as can be seen in appendix A, the only terms that survive are the levi civita symbols. I have to say that this simplified situation is almost impossible to occur in reality, but it is important to mention for completeness and because it would be the cleanest enviroment to isolate polarization effects.

There is another situation even more restrictive than the previous one in which polarization effects disappear completely, in this case the momenta of each of the four particles not only lie on the same plane but in addition they are all collinear, two of them in one direction, and the other two in the opposite direction, again with the polarization vector perpendicular to all four momenta, this makes that the survivor levi civita terms in the previous paragraph vanish in this case, having as a consequence that the amplitude does not depend on the polarization vector $\vec{s}$ at all.

Taking into account the comments of the previous paragraph we see that to maximize polarization effects, any of the four momenta must be parallel, in fact we will study the case where $\vec{p_+}$, $\vec{p_-}$ and $\vec{s}$ are orthogonal, we will follow the convention of section 4.3, there we fixed the values of the vectors $\vec{p}_-$, and $\vec{s}$ as shown in the following equations,

$$\vec{p_-} = (0, 0, |\vec{p_-}|) = |\vec{p_-}|\hat{k}\,, \tag{4.88a}$$

$$\vec{s} = (0, 1, 0) = \hat{\jmath}\,. \tag{4.88b}$$

Now we can fix the value of $\vec{p}_+$ because we want it to be orthogonal to $\vec{p}_-$ and $\vec{s}$, a convenient way to choose it is,

$$\vec{p_+} = (|p_+|, 0, 0) = |p_+|\hat{\imath}\,. \tag{4.88c}$$

With this assumptions the scalar triple products that come from the Levi Civita terms (appendix A) take the following form,

$$\vec{p}_+ \cdot \vec{p}_- \times \vec{s} = -|\vec{p}_+||\vec{p}_-||\vec{s}| = -|\vec{p}_+||\vec{p}_-|\,, \tag{4.89a}$$

$$\vec{p} \cdot \vec{p}_- \times \vec{s} = \vec{p} \cdot (-|\vec{p}_-||\vec{s}|\hat{\imath}) = -|\vec{p}_-|\vec{p} \cdot \hat{\imath} = -|\vec{p}_-|a_3\,, \tag{4.89b}$$

$$\vec{p} \cdot \vec{p}_+ \times \vec{s} = \vec{p} \cdot (|\vec{p}_+||\vec{s}|\hat{k}) = |\vec{p}_+|\vec{p} \cdot \hat{k} = |\vec{p}_+|c_3\,, \tag{4.89c}$$

where a_3, c_3, p_-, and p_+ are given by equations (4.70b), (4.70d), (4.63a), and (4.63b) respectively.

With the purpose of studying polarization asymmetries we will define one observable, we denote it as A_{pol}, and it is defined in the following way,

$$A_{pol} = \frac{\Gamma^+ - \Gamma^-}{\Gamma^+ + \Gamma^-}\,, \tag{4.90}$$

where Γ^+ denotes the branching ratio when $\vec{s}$ is pointing in a direction called arbitrarily the up direction, and Γ^- denotes the branching ratio when $\vec{s}$ is pointing in a direction opposite to the one defined for Γ^+, called the down direction.

It is important to point out again that for simplicity we are studying the case where $\vec{p}_-$, $\vec{p}_+$, and $\vec{s}$ are orthogonal. For this case, the results of our observable are discussed in what follows.

In table 4.1 we show the branching ratios for the different parts of the process $(IBIB, VV, AA, IB-V, IB-A, V-A)$ for the polarized and unpolarized cases when we have a muon as a product. IB stands for the inner

Table 4.1: Different contributions for the branching ratio for the polarized and unpolarized cases ($\ell = \mu$).

	Non polarized τ	Polarized τ $\left(\frac{+1}{2}\right)$	Polarized τ $\left(\frac{-1}{2}\right)$	A_{pol}
IBIB	1.582×10^{-7}	1.224×10^{-7}	1.938×10^{-7}	-0.23
VV	6.219×10^{-7}	6.508×10^{-7}	5.920×10^{-7}	0.047
AA	1.033×10^{-6}	9.246×10^{-7}	1.142×10^{-6}	-0.105
IB-V	1.386×10^{-8}	-1.470×10^{-7}	-1.449×10^{-7}	0.007
IB-A	1.036×10^{-7}	1.337×10^{-7}	7.46×19^{-8}	0.284
V-A	-2.774×10^{-10}	-2.896×10^{-10}	-1.929×10^{-10}	0.200
TOTAL	1.939×10^{-6}	1.684×10^{-6}	1.857×10^{-6}	-0.049

Table 4.2: Different contributions for the branching ratio for the polarized and unpolarized cases ($\ell = e$).

	Non polarized τ	Polarized τ $\left(\frac{+1}{2}\right)$	Polarized τ $\left(\frac{-1}{2}\right)$	A_{pol}
IBIB	1.456×10^{-5}	8.553×10^{-6}	2.035×10^{-5}	-0.408
VV	1.167×10^{-6}	1.131×10^{-6}	1.138×10^{-6}	-0.05
AA	2.13×10^{-6}	1.294×10^{-6}	2.438×10^{-6}	-0.31
IB-V	-2.121×10^{-8}			
IB-A	-8.521×10^{-7}	-6.261×10^{-7}	-5.806×10^{-7}	0.042
V-A	2.374×10^{-10}	-1.1×10^{-9}	-4.602×10^{-9}	-0.614
TOTAL	1.7×10^{-5}	1.03×10^{-5}	2.334×10^{-5}	-0.38

bremsstrahlung contribution, V stands for the vector contribution, and A stands for the axial contribution. Here we also show the total branching ratios for the polarized and unpolarized cases (when $\ell = \mu$).

Similarly for $\ell^+\ell^- = e^+e^-$, we have table 4.2.

The errors coming from the integrations are small (at least 2 orders of magnitude less than the reported values in the tables) in almost all cases, except for the IBV part for the polarized cases when $\ell = e$, there we have some numerical problem, that is why I do dot report that quantity in the table above, fortunately the IBV contribution is supresed, so we can just ignore it and sum the other contributions.

To give an idea of the dependence of the branching ratios on the model used for the calculation of the form factors in table 4.3 we show the central values for the branching ratios in the unpolarized case (just as the second column in table 4.1 above) and the branching ratios for the unpolarized case when we vary the parameters 20 % for the case of the muon.

As a final comment, I must say that we want to extend this work in the

Table 4.3: Comparation of the central BR with the BR corresponding to a variation of 20 percent in the model parameters for the unpolarized case.

	Central ($\ell = \mu$)	20 %variation ($\ell = \mu$)
IBIB	1.582×10^{-7}	1.582×10^{-7}
VV	6.219×10^{-7}	2.541×10^{-7}
AA	1.033×10^{-6}	6.755×10^{-7}
IB-V	1.386×10^{-8}	-5.576×10^{-10}
IB-A	1.306×10^{-7}	-2.394×10^{-7}
V-A	-2.774×10^{-10}	-2.285×10^{-10}
TOTAL	1.939×10^{-6}	8.490×10^{-7}

near future. We are interested in studying polarization asymmetries with less restrictions in the directions of the momenta and the polarization vector, we will explore polarization asymmetries that come from triple products between $\vec{s}$, a vector in the plane of the charged leptons, and another vector in the plane formed by the pion and the neutrino, we will also explore the asymmetry that comes when we replace k^2 with the energy of the pion. Those are just some of the things in our future plans.

Chapter 5

Levi Civita simplifications

In chapter 4 we calculated $|\mathcal{M}|^2$, there we obtained simplified terms for each of the components of this amplitude. The full expressions for $|\mathcal{M}|^2$ can be found in appendix A, there we can see some terms with 4D Levi-civita symbols, in fact we find the following combinations:
$\epsilon^{p_\tau p p_+ s}$, $\epsilon^{p_\tau p p_- s}$, $\epsilon^{p_\tau p_+ p_- s}$, $\epsilon^{p_\tau p k s}$, $\epsilon^{p p_+ p_- s}$
Some of these terms are not independent because we have constraints between the different four momentum vectors in the process. From the kinematics we have that,

$$p_\tau = p + p_+ + p_- + q \tag{5.1}$$

From (5.1) we see that we really have four independent four momentum, we choose p_τ, p, p_+ and p_- as our four independent four vectors eliminating in this way q, which is fixed by the following equation:

$$q = p_\tau - p - p_+ - p_- \tag{5.2}$$

The four vector k is not independent either, because it it just the sum of the lepton-antilepton four momentum,

$$k = p_+ + p_- \tag{5.3}$$

With these simplifications, the independent levi-civita terms that we get are the following: $\epsilon^{p_\tau p p_+ s}$, $\epsilon^{p_\tau p p_- s}$, $\epsilon^{p_\tau p_+ p_- s}$, and $\epsilon^{p p_+ p_- s}$
We are going to study the process in the rest frame of reference of the decaying τ lepton, so the four momentum associated with the τ becomes $p_\tau^\mu = (M_\tau, \vec{0})$, this choice simplifies calculations, as we will see this allows us to write terms like the first three levi civitas written before as triple scalar products, the last levi civita term has a s^μ four vector instead of a p_τ^μ four vector, we do not want terms like that one because it complicates calculations, so we need to find a way to write this term in a more convenient form. Fortunately there

is an easy way to do this, it is called the Schouten identity and it states the following:

$$g_{\alpha\beta}\epsilon_{\gamma\delta\mu\nu} + g_{\alpha\gamma}\epsilon_{\delta\mu\nu\beta} + g_{\alpha\delta}\epsilon_{\mu\nu\beta\gamma} + g_{\alpha\mu}\epsilon_{\nu\beta\gamma\delta} + g_{\alpha\nu}\epsilon_{\beta\gamma\delta\mu} = 0\,. \tag{5.4}$$

Now we multiply the Schouten identity by all the four vectors that we have, that is we multiply it by $p_\tau^\beta p^\gamma p_+^\delta p_-^\mu s^\nu$, so we have the following result:

$$\begin{aligned} 0 =& p_{\tau\alpha} p^\gamma p_+^\delta p_-^\mu s^\nu \epsilon_{\gamma\delta\mu\nu} + p_\alpha p_\tau^\beta p_+^\delta p_-^\mu s^\nu \epsilon_{\delta\mu\nu\beta} + p_{+\alpha} p_\tau^\beta p^\gamma p_-^\mu s^\nu \epsilon_{\mu\nu\beta\gamma} \\ &+ p_{-\alpha} p_\tau^\beta p^\gamma p_+^\delta s^\nu \epsilon_{\nu\beta\gamma\delta} + s_\alpha p_\tau^\beta p^\gamma p_+^\delta p_-^\mu \epsilon_{\beta\gamma\delta\mu}\,. \end{aligned} \tag{5.2}$$

Multiplying the previous equation by p_τ^α we have the following result:

$$\begin{aligned} 0 =& M_\tau^2 p^\gamma p_+^\delta p_-^\mu s^\nu \epsilon_{\gamma\delta\mu\nu} + (p \cdot p_\tau) p_\tau^\beta p_+^\delta p_-^\mu s^\nu \epsilon_{\delta\mu\nu\beta} + (p_+ \cdot p_\tau) p_\tau^\beta p^\gamma p_-^\mu s^\nu \epsilon_{\mu\nu\beta\gamma} \\ &+ (p_- \cdot p_\tau) p_\tau^\beta p^\gamma p_+^\delta s^\nu \epsilon_{\nu\beta\gamma\delta} + 0\,. \end{aligned} \tag{5.3}$$

$$\begin{aligned} \Rightarrow p^\gamma p_+^\delta p_-^\mu s^\nu \epsilon_{\gamma\delta\mu\nu} =& -\frac{1}{M_\tau^2}\Big[(p \cdot p_\tau) p_\tau^\beta p_+^\delta p_-^\mu s^\nu \epsilon_{\delta\mu\nu\beta} + (p_+ \cdot p_\tau) p_\tau^\beta p^\gamma p_-^\mu s^\nu \epsilon_{\mu\nu\beta\gamma} \\ &+ (p_- \cdot p_\tau) p_\tau^\beta p^\gamma p_+^\delta s^\nu \epsilon_{\nu\beta\gamma\delta}\Big]\,. \end{aligned} \tag{5.4}$$

In (5.3) we have used that $p_\tau \cdot s = 0$.
Now using the fact that $p_\tau^\mu = (M_\tau, \vec{0})$ in (5.4) we arrive to the following equation:

$$\begin{aligned} p^\gamma p_+^\delta p_-^\mu s^\nu \epsilon_{\gamma\delta\mu\nu} =& -\frac{1}{M_\tau^2}\Big[M_\tau (p \cdot p_\tau) p_+^\delta p_-^\mu s^\nu \epsilon_{\delta\mu\nu 0} + M_\tau (p_+ \cdot p_\tau) p^\gamma p_-^\mu s^\nu \epsilon_{\mu\nu 0\gamma} \\ &+ M_\tau (p_- \cdot p_\tau) p^\gamma p_+^\delta s^\nu \epsilon_{\nu 0\gamma\delta}\Big]\,. \end{aligned} \tag{5.5}$$

We can simplify (5.5) using the explicit form of the polarization vector $s^\mu = (0, \vec{s})$ so that we can write it as triple scalar products, as we see in the following equation:

$$\begin{aligned} p^\gamma p_+^\delta p_-^\mu s^\nu \epsilon_{\gamma\delta\mu\nu} =& -\frac{1}{M_\tau}\Big[(p \cdot p_\tau)(\vec{p}_+ \cdot \vec{p}_- \times \vec{s}) + (p_+ \cdot p_\tau)(\vec{p} \cdot \vec{p}_- \times \vec{s}) \\ &+ (p_- \cdot p_\tau)(\vec{p} \cdot \vec{p}_+ \times \vec{s})\Big]\,. \end{aligned} \tag{5.6}$$

For triple scalar products we have that given three vectors $\vec{A}, \vec{B}$, and $\vec{C}$, the following property follows, $\vec{A} \cdot \vec{B} \times C = \vec{B} \cdot \vec{C} \times A = \vec{C} \cdot \vec{A} \times B$, so we can write (5.6) as follows,

$$\begin{aligned} p^{\gamma} p_{+}^{\delta} p_{-}^{\mu} s^{\nu} \epsilon_{\gamma\delta\mu\nu} = & -\frac{1}{M_{\tau}} \Big[(p \cdot p_{\tau})(\vec{s} \cdot \vec{p}_{+} \times \vec{p}_{-}) + (p_{+} \cdot p_{\tau})(\vec{s} \cdot \vec{p} \times \vec{p}_{-}) \\ & + (p_{-} \cdot p_{\tau})(\vec{s} \cdot \vec{p} \times \vec{p}_{+}) \Big] . \end{aligned} \tag{5.7}$$

Chapter 6

Conclusions

In this Thesis we studied the $\tau^- \to \pi^- \ell^+ \ell^- \nu_\tau$ decays in the case where the polarization of the τ lepton is taken into account, our main goal in this work was to obtain the amplitude for this process, this was succesfully done in chapter 4 and appendix A. Our results for the amplitude will be included in the Montecarlo generator TAUOLA used by the Belle II collaboration.

The amplitude naturally splits into two parts, one part is model independent and for this reason is under good theoretical control, this is the so called inner bremsstrahlung amplitude, the rest part of the amplitude must be treated with models because the complete theory of non-perturbative QCD is absent at present, for this purpose we used an effective field theory called Resonance Chiral Theory ($R\chi T$).

Once we obtained our theoretical calculation, we used a vegas routine in Fortran for doing the numerical integrations. We found several results like the unpolarized BR for the process (this is in agreement with ref [7, 6]), and the BR corresponding to some particular and simplified cases in the direction of the polarization vector $\vec{s}$, with these BR we found polarization assymetries for the different parts in the polarized amplitude.

The results for the asymetries could shed some light in future analysis in the search of new physics.

The work done here is just a first step in the study of this topic, there are many things to do for the near future. As part of the perspectives we have several things to do, for example, we will include an analysis of the invariant mass of the pair of leptons, and we will also study the asymetries in this process in a more general situation, that is with less restrictions in the vector $\vec{s}$.

Appendix A
Full expresions for the amplitude

In this appendix we collect the complete expressions for the different contributions to the squared matrix element, summed over the $\ell^+\ell^-\nu_\tau$ polarizations (but not over that of the decaying τ):

$$\begin{aligned}
&\sum_{s_\nu, s_{\ell+}, s_{\ell-}} |\mathcal{M}_{IB}|^2 = 32 G_F^2 |V_{ud}|^2 \frac{e^4}{k^4} F_\pi^2 M_\tau^2 \times \\
&\Bigg[\frac{2[(2p\cdot p_+)\,(p\cdot p_-) - p^2\,(m_\ell^2 + p_+\cdot p_-)]\,(p_\tau\cdot q + M_\tau q\cdot s)}{(k^2+2p\cdot k)^2} \\
&+ \frac{4\,(-\,(p\cdot p_\tau)\,(m_\ell^2 + p_+\cdot p_-) + (p\cdot p_-)\,(p_+\cdot p_\tau) + (p\cdot p_+)\,(p_-\cdot p_\tau))\,(p_\tau\cdot q + M_\tau\,(q\cdot s))}{(k^2+2p\cdot k)(k^2-2p_\tau\cdot k)} \\
&+ \frac{2\,(2\,(p_+\cdot p_\tau)\,(p_-\cdot p_\tau) - ({m_\ell}^2 + p_+\cdot p_-)\,{p_\tau}^2)\,(p_\tau\cdot q + M_\tau\,(q\cdot s))}{(k^2-2p_\tau\cdot k)^2} \\
&+ \frac{2}{(k^2+2p\cdot k)(k^2-2p_\tau\cdot k)}\Big(k\cdot q\big(M_\tau\left({m_\ell}^2 + p_+\cdot p_-\right)p\cdot s + \left({m_\ell}^2 + p_+\cdot p_-\right)p_\tau\cdot p \\
&- p\cdot p_-\,(M_\tau p_+\cdot s + p_\tau\cdot p_+) - M_\tau p\cdot p_+ p_-\cdot s - p\cdot p_+ p_\tau\cdot p_-\big) + {m_\ell}^2(-M_\tau)k\cdot s p\cdot q \\
&+ M_\tau q\cdot s\left(p\cdot k\left({m_\ell}^2 + p_+\cdot p_-\right) + p\cdot p_-\,(-\,(k\cdot p_+)) - p\cdot p_+ k\cdot p_-\right) + {m_\ell}^2 p\cdot k p_\tau\cdot q \\
&+ p_\tau\cdot k\left(-\left({m_\ell}^2 + p_+\cdot p_-\right)p\cdot q + p\cdot p_+ p_-\cdot q + p\cdot p_- p_+\cdot q\right) + M_\tau p\cdot p_+ k\cdot s p_-\cdot q \\
&+ M_\tau p\cdot p_- k\cdot s p_+\cdot q - M_\tau p_+\cdot p_- k\cdot s p\cdot q - p\cdot p_- k\cdot p_+ p_\tau\cdot q - p\cdot p_+ k\cdot p_- p_\tau\cdot q \\
&+ p_+\cdot p_- p\cdot k p_\tau\cdot q\Big) \\
&+ \frac{2}{\left(k^2-2p_\tau\cdot k\right)^2}\Big(k\cdot q\big(\left({m_\ell}^2 + p_+\cdot p_-\right)\left(M_\tau p_\tau\cdot s + {p_\tau}^2\right) - M_\tau p_+\cdot s p_\tau\cdot p_- \\
&- p_\tau\cdot p_+\,(M_\tau p_-\cdot s + 2p_\tau\cdot p_-)\big) + {m_\ell}^2(-M_\tau)k\cdot s p_\tau\cdot q + p_\tau\cdot k(M_\tau\left({m_\ell}^2 + p_+\cdot p_-\right)q\cdot s \\
&+ p_+\cdot q p_\tau\cdot p_- + p_-\cdot q p_\tau\cdot p_+) - M_\tau p_+\cdot p_- k\cdot s p_\tau\cdot q + M_\tau k\cdot s p_+\cdot q p_\tau\cdot p_- \\
&+ M_\tau k\cdot s p_-\cdot q p_\tau\cdot p_+ - M_\tau q\cdot s\,(k\cdot p_+ p_\tau\cdot p_- + k\cdot p_- p_\tau\cdot p_+) \\
&- k\cdot p_+ p_\tau\cdot p_- p_\tau\cdot q - k\cdot p_- p_\tau\cdot p_+ p_\tau\cdot q\Big) \\
&- \frac{1}{\left(k^2-2p_\tau\cdot k\right)^2}\Big(k^2({m_\ell}^2 M_\tau q\cdot s + {m_\ell}^2 p_\tau\cdot q + p_+\cdot q\,(M_\tau p_-\cdot s + p_\tau\cdot p_-) \\
&+ p_-\cdot q\,(M_\tau p_+\cdot s + p_\tau\cdot p_+)) - 2k\cdot q({m_\ell}^2 M_\tau k\cdot s + k\cdot p_+\,(M_\tau p_-\cdot s + p_\tau\cdot p_-) \\
&+ k\cdot p_-\,(M_\tau p_+\cdot s + p_\tau\cdot p_+)) - 2{m_\ell}^2 k\cdot q p_\tau\cdot k\Big)\Bigg]\,.
\end{aligned} \tag{A.1}$$

$$
\begin{aligned}
&\sum_{s_\nu, s_{\ell+}, s_{\ell-}} \left|\mathcal{M}_V\right|^2 = 32 G_F^2 |V_{ud}|^2 \frac{e^4}{k^4} \big| F_V(p\cdot k, k^2) \big| \times \\
&\Big[- M_\tau \left(k\cdot p\right)\left(k\cdot s\right)\left(p\cdot q\right) {m_\ell}^2 - k^2 \left(p\cdot p_\tau\right)\left(p\cdot q\right) {m_\ell}^2 + M_\tau k^2 \left(p\cdot q\right)\left(p\cdot s\right) {m_\ell}^2 \\
&- M_\tau \left(k\cdot p_-\right)\left(k\cdot s\right)\left(p\cdot p_+\right)\left(p\cdot q\right) - M_\tau \left(k\cdot p_+\right)\left(k\cdot s\right)\left(p\cdot p_-\right)\left(p\cdot q\right) \\
&- 2\left(k\cdot p_+\right)\left(k\cdot p_-\right)\left(p\cdot p_\tau\right)\left(p\cdot q\right) + 2 M_\tau \left(k\cdot p_+\right)\left(k\cdot p_-\right)\left(p\cdot q\right)\left(p\cdot s\right) \\
&+ M_\tau \left(k\cdot p\right)\left(k\cdot s\right)\left(p\cdot q\right)\left(p_+\cdot p_-\right) + k^2 \left(p\cdot p_\tau\right)\left(p\cdot q\right)\left(p_+\cdot p_-\right) - M_\tau k^2 \left(p\cdot q\right)\left(p\cdot s\right)\left(p_+\cdot p_-\right) \\
&+ \left(k\cdot p\right)\left(k\cdot p_-\right)\left(p\cdot q\right)\left(p_+\cdot p_\tau\right) - k^2 \left(p\cdot p_-\right)\left(p\cdot q\right)\left(p_+\cdot p_\tau\right) + M_\tau \left(k\cdot p_-\right)\left(k\cdot s\right) p^2 \left(p_+\cdot q\right) \\
&- M_\tau \left(k\cdot p\right)\left(k\cdot s\right)\left(p\cdot p_-\right)\left(p_+\cdot q\right) + \left(k\cdot p\right)\left(k\cdot p_-\right)\left(p\cdot p_\tau\right)\left(p_+\cdot q\right) - k^2 \left(p\cdot p_-\right)\left(p\cdot p_\tau\right)\left(p_+\cdot q\right) \\
&- M_\tau \left(k\cdot p\right)\left(k\cdot p_-\right)\left(p\cdot s\right)\left(p_+\cdot q\right) + M_\tau k^2 \left(p\cdot p_-\right)\left(p\cdot s\right)\left(p_+\cdot q\right) \\
&- M_\tau \left(k\cdot p\right)\left(k\cdot p_-\right)\left(p\cdot q\right)\left(p_+\cdot s\right) + M_\tau k^2 \left(p\cdot p_-\right)\left(p\cdot q\right)\left(p_+\cdot s\right) + \left(k\cdot p\right)\left(k\cdot p_+\right)\left(p\cdot q\right)\left(p_-\cdot p_\tau\right) \\
&- k^2 \left(p\cdot p_+\right)\left(p\cdot q\right)\left(p_-\cdot p_\tau\right) - (k\cdot p)^2 \left(p_+\cdot q\right)\left(p_-\cdot p_\tau\right) + k^2 p^2 \left(p_+\cdot q\right)\left(p_-\cdot p_\tau\right) \\
&+ M_\tau \left(k\cdot p_+\right)\left(k\cdot s\right) p^2 \left(p_-\cdot q\right) - M_\tau \left(k\cdot p\right)\left(k\cdot s\right)\left(p\cdot p_+\right)\left(p_-\cdot q\right) + \left(k\cdot p\right)\left(k\cdot p_+\right)\left(p\cdot p_\tau\right)\left(p_-\cdot q\right) \\
&- k^2 \left(p\cdot p_+\right)\left(p\cdot p_\tau\right)\left(p_-\cdot q\right) - M_\tau \left(k\cdot p\right)\left(k\cdot p_+\right)\left(p\cdot s\right)\left(p_-\cdot q\right) + M_\tau k^2 \left(p\cdot p_+\right)\left(p\cdot s\right)\left(p_-\cdot q\right) \\
&- (k\cdot p)^2 \left(p_+\cdot p_\tau\right)\left(p_-\cdot q\right) + k^2 p^2 \left(p_+\cdot p_\tau\right)\left(p_-\cdot q\right) + M_\tau (k\cdot p)^2 \left(p_+\cdot s\right)\left(p_-\cdot q\right) \\
&- M_\tau k^2 p^2 \left(p_+\cdot s\right)\left(p_-\cdot q\right) + \left(k\cdot p_\tau\right) \Big(\left(k\cdot p_-\right)\left(p\cdot p_+\right)\left(p\cdot q\right) + \left(k\cdot p_+\right)\left(p\cdot p_-\right)\left(p\cdot q\right) \\
&+ \left(k\cdot q\right) \big(p^2 \big(p_+\cdot p_- - {m_\ell}^2\big) - 2\left(p\cdot p_+\right)\left(p\cdot p_-\right)\big) - \left(k\cdot p_-\right) p^2 \left(p_+\cdot q\right) - \left(k\cdot p_+\right) p^2 \left(p_-\cdot q\right) \\
&+ \left(k\cdot p\right) \big(\left(p\cdot q\right) \big({m_\ell}^2 - p_+\cdot p_-\big) + \left(p\cdot p_-\right)\left(p_+\cdot q\right) + \left(p\cdot p_+\right)\left(p_-\cdot q\right)\big) \Big) \\
&- M_\tau \left(k\cdot p\right)\left(k\cdot p_+\right)\left(p\cdot q\right)\left(p_-\cdot s\right) + M_\tau k^2 \left(p\cdot p_+\right)\left(p\cdot q\right)\left(p_-\cdot s\right) \\
&+ M_\tau (k\cdot p)^2 \left(p_+\cdot q\right)\left(p_-\cdot s\right) - M_\tau k^2 p^2 \left(p_+\cdot q\right)\left(p_-\cdot s\right) + \left(k\cdot q\right) \Big(\left(k\cdot p_-\right)\left(p\cdot p_+\right)\left(p\cdot p_\tau\right) \\
&+ \left(k\cdot p_+\right)\left(p\cdot p_-\right)\left(p\cdot p_\tau\right) - M_\tau \left(k\cdot p_-\right)\left(p\cdot p_+\right)\left(p\cdot s\right) - M_\tau \left(k\cdot p_+\right)\left(p\cdot p_-\right)\left(p\cdot s\right) \\
&+ M_\tau \left(k\cdot s\right) \big(2\left(p\cdot p_+\right)\left(p\cdot p_-\right) + p^2 \big({m_\ell}^2 - p_+\cdot p_-\big)\big) - \left(k\cdot p_-\right) p^2 \left(p_+\cdot p_\tau\right) \\
&+ M_\tau \left(k\cdot p_-\right) p^2 \left(p_+\cdot s\right) - \left(k\cdot p_+\right) p^2 \left(p_-\cdot p_\tau\right) + M_\tau \left(k\cdot p_+\right) p^2 \left(p_-\cdot s\right) \\
&+ \left(k\cdot p\right) \Big(\left(p\cdot p_\tau\right) \big({m_\ell}^2 - p_+\cdot p_-\big) + M_\tau \left(p\cdot s\right) \big(p_+\cdot p_- - {m_\ell}^2\big) + \left(p\cdot p_-\right)\left(p_+\cdot p_\tau - M_\tau \left(p_+\cdot s\right)\right) \\
&+ \left(p\cdot p_+\right)\left(p_-\cdot p_\tau\right) - M_\tau \left(p\cdot p_+\right)\left(p_-\cdot s\right) \Big)\Big) + \left(k\cdot p_+\right)\left(k\cdot p_-\right) p^2 \left(p_\tau\cdot q\right) \\
&- \left(k\cdot p\right)\left(k\cdot p_-\right)\left(p\cdot p_+\right)\left(p_\tau\cdot q\right) - \left(k\cdot p\right)\left(k\cdot p_+\right)\left(p\cdot p_-\right)\left(p_\tau\cdot q\right) + k^2 \left(p\cdot p_+\right)\left(p\cdot p_-\right)\left(p_\tau\cdot q\right) \\
&+ (k\cdot p)^2 \left(p_+\cdot p_-\right)\left(p_\tau\cdot q\right) - k^2 p^2 \left(p_+\cdot p_-\right)\left(p_\tau\cdot q\right) - M_\tau \left(k\cdot p_+\right)\left(k\cdot p_-\right) p^2 \left(q\cdot s\right) \\
&+ M_\tau \left(k\cdot p\right)\left(k\cdot p_-\right)\left(p\cdot p_+\right)\left(q\cdot s\right) + M_\tau \left(k\cdot p\right)\left(k\cdot p_+\right)\left(p\cdot p_-\right)\left(q\cdot s\right) \\
&- M_\tau k^2 \left(p\cdot p_+\right)\left(p\cdot p_-\right)\left(q\cdot s\right) - M_\tau (k\cdot p)^2 \left(p_+\cdot p_-\right)\left(q\cdot s\right) + M_\tau k^2 p^2 \left(p_+\cdot p_-\right)\left(q\cdot s\right) \Big] .
\end{aligned}
\tag{A.2}
$$

In the remaining terms we introduce the following short-hand notation:

$$a = k^2 + 2p \cdot k \tag{6.1}$$

$$b = k^2 - 2p_\tau \cdot k \tag{6.2}$$

$$C_1 = k^2 + p \cdot k \tag{6.3}$$

$$C_2 = \frac{1}{k^2 + 2p \cdot k} = \frac{1}{a} \tag{6.4}$$

$$\sum_{s_\nu, s_{\ell+}, s_{\ell-}} \left|\mathcal{M}_A\right|^2 = 64{G_F}^2|V_{ud}|^2\frac{e^4}{k^4}\Big[2|F_A|^2\,(p_+\cdot q)\,(p_-\cdot p_\tau)\,{C_1}^2$$

$$+\,2|F_A|^2\,(p_+\cdot p_\tau)\,(p_-\cdot q)\,{C_1}^2$$

$$-\,2|F_A|^2M_\tau\,(p_+\cdot s)\,(p_-\cdot q)\,{C_1}^2 - 2|F_A|^2M_\tau\,(p_+\cdot q)\,(p_-\cdot s)\,{C_1}^2 - 2|F_A|^2\,(p_+\cdot p_-)\,(p_\tau\cdot q)\,{C_1}^2$$

$$+\,2|F_A|^2\left({m_\ell}^2 + p_+\cdot p_-\right)(p_\tau\cdot q)\,{C_1}^2 + iC_2(B^*F_A - B{F_A}^*)p_\mu p_{-\nu}p_{\tau\lambda}q_\sigma\epsilon^{\mu\nu\lambda\sigma}k^2\,(p\cdot p_+)\,C_1$$

$$+\,iC_2(B^*F_A - B{F_A}^*)M_\tau p_\mu p_{-\nu}s_\lambda q_\sigma\epsilon^{\mu\nu\sigma\lambda}k^2\,(p\cdot p_+)\,C_1$$

$$+\,iC_2(B^*F_A - B{F_A}^*)p_\mu p_{+\nu}p_{\tau\lambda}q_\sigma\epsilon^{\mu\nu\lambda\sigma}k^2\,(p\cdot p_-)\,C_1$$

$$+\,iC_2(B^*F_A - B{F_A}^*)M_\tau p_\mu p_{+\nu}s_\lambda q_\sigma\epsilon^{\mu\nu\sigma\lambda}k^2\,(p\cdot p_-)\,C_1 + 2|F_A|^2\,(k\cdot q)\,(p\cdot p_\tau)\left({m_\ell}^2 + p_+\cdot p_-\right)C_1$$

$$+\,2|F_A|^2\,(k\cdot p_\tau)\,(p\cdot q)\left({m_\ell}^2 + p_+\cdot p_-\right)C_1 - 2|F_A|^2M_\tau\,(k\cdot s)\,(p\cdot q)\left({m_\ell}^2 + p_+\cdot p_-\right)C_1$$

$$+\,B^*C_2F_Ak^2\,(k\cdot p_\tau + p\cdot p_\tau)\,(p\cdot q)\left({m_\ell}^2 + p_+\cdot p_-\right)C_1$$

$$+\,BC_2{F_A}^*k^2\,(k\cdot p_\tau + p\cdot p_\tau)\,(p\cdot q)\left({m_\ell}^2 + p_+\cdot p_-\right)C_1$$

$$+\,B^*C_2F_Ak^2\,(p\cdot p_\tau)\,(k\cdot q + p\cdot q)\left({m_\ell}^2 + p_+\cdot p_-\right)C_1$$

$$+\,BC_2{F_A}^*k^2\,(p\cdot p_\tau)\,(k\cdot q + p\cdot q)\left({m_\ell}^2 + p_+\cdot p_-\right)C_1 - 2|F_A|^2M_\tau\,(k\cdot q)\,(p\cdot s)\left({m_\ell}^2 + p_+\cdot p_-\right)C_1$$

$$-\,B^*C_2F_AM_\tau k^2\,(k\cdot q + p\cdot q)\,(p\cdot s)\left({m_\ell}^2 + p_+\cdot p_-\right)C_1$$

$$-\,BC_2{F_A}^*M_\tau k^2\,(k\cdot q + p\cdot q)\,(p\cdot s)\left({m_\ell}^2 + p_+\cdot p_-\right)C_1$$

$$-\,B^*C_2F_AM_\tau k^2\,(p\cdot q)\,(k\cdot s + p\cdot s)\left({m_\ell}^2 + p_+\cdot p_-\right)C_1$$

$$-\,BC_2{F_A}^*M_\tau k^2\,(p\cdot q)\,(k\cdot s + p\cdot s)\left({m_\ell}^2 + p_+\cdot p_-\right)C_1 - 2|F_A|^2\,(k\cdot q)\,(p\cdot p_-)\,(p_+\cdot p_\tau)\,C_1$$

$$-\,B^*C_2F_Ak^2\,(p\cdot p_-)\,(k\cdot q + p\cdot q)\,(p_+\cdot p_\tau)\,C_1$$

$$-\,BC_2{F_A}^*k^2\,(p\cdot p_-)\,(k\cdot q + p\cdot q)\,(p_+\cdot p_\tau)\,C_1 - 2|F_A|^2\,(k\cdot p_\tau)\,(p\cdot p_-)\,(p_+\cdot q)\,C_1$$

$$+\,2|F_A|^2M_\tau\,(k\cdot s)\,(p\cdot p_-)\,(p_+\cdot q)\,C_1 - B^*C_2F_Ak^2\,(p\cdot p_-)\,(k\cdot p_\tau + p\cdot p_\tau)\,(p_+\cdot q)\,C_1$$

$$-\,BC_2{F_A}^*k^2\,(p\cdot p_-)\,(k\cdot p_\tau + p\cdot p_\tau)\,(p_+\cdot q)\,C_1 + B^*C_2F_AM_\tau k^2\,(p\cdot p_-)\,(k\cdot s + p\cdot s)\,(p_+\cdot q)\,C_1$$

$$+\,BC_2{F_A}^*M_\tau k^2\,(p\cdot p_-)\,(k\cdot s + p\cdot s)\,(p_+\cdot q)\,C_1 + 2|F_A|^2M_\tau\,(k\cdot q)\,(p\cdot p_-)\,(p_+\cdot s)\,C_1$$

$$+\,B^*C_2F_AM_\tau k^2\,(p\cdot p_-)\,(k\cdot q + p\cdot q)\,(p_+\cdot s)\,C_1 + BC_2{F_A}^*M_\tau k^2\,(p\cdot p_-)\,(k\cdot q + p\cdot q)\,(p_+\cdot s)\,C_1$$

$$-\,2|F_A|^2\,(k\cdot q)\,(p\cdot p_+)\,(p_-\cdot p_\tau)\,C_1 - B^*C_2F_Ak^2\,(p\cdot p_+)\,(k\cdot q + p\cdot q)\,(p_-\cdot p_\tau)\,C_1$$

$$-\,BC_2{F_A}^*k^2\,(p\cdot p_+)\,(k\cdot q + p\cdot q)\,(p_-\cdot p_\tau)\,C_1 + 2B^*F_Ak^2\,(p_+\cdot q)\,(p_-\cdot p_\tau)\,C_1$$

$$+\,2B{F_A}^*k^2\,(p_+\cdot q)\,(p_-\cdot p_\tau)\,C_1 - 2|F_A|^2\,(k\cdot p_\tau)\,(p\cdot p_+)\,(p_-\cdot q)\,C_1$$

$$+\,2|F_A|^2M_\tau\,(k\cdot s)\,(p\cdot p_+)\,(p_-\cdot q)\,C_1 - B^*C_2F_Ak^2\,(p\cdot p_+)\,(k\cdot p_\tau + p\cdot p_\tau)\,(p_-\cdot q)\,C_1$$

$$-\,BC_2{F_A}^*k^2\,(p\cdot p_+)\,(k\cdot p_\tau + p\cdot p_\tau)\,(p_-\cdot q)\,C_1 + B^*C_2F_AM_\tau k^2\,(p\cdot p_+)\,(k\cdot s + p\cdot s)\,(p_-\cdot q)\,C_1$$

$$+\,BC_2{F_A}^*M_\tau k^2\,(p\cdot p_+)\,(k\cdot s + p\cdot s)\,(p_-\cdot q)\,C_1 + 2B^*F_Ak^2\,(p_+\cdot p_\tau)\,(p_-\cdot q)\,C_1$$

$$+\,2B{F_A}^*k^2\,(p_+\cdot p_\tau)\,(p_-\cdot q)\,C_1 - 2B^*F_AM_\tau k^2\,(p_+\cdot s)\,(p_-\cdot q)\,C_1 - 2B{F_A}^*M_\tau k^2\,(p_+\cdot s)\,(p_-\cdot q)\,C_1$$

$$+\,2|F_A|^2M_\tau\,(k\cdot q)\,(p\cdot p_+)\,(p_-\cdot s)\,C_1 + B^*C_2F_AM_\tau k^2\,(p\cdot p_+)\,(k\cdot q + p\cdot q)\,(p_-\cdot s)\,C_1$$

$$+\,BC_2{F_A}^*M_\tau k^2\,(p\cdot p_+)\,(k\cdot q + p\cdot q)\,(p_-\cdot s)\,C_1 - 2B^*F_AM_\tau k^2\,(p_+\cdot q)\,(p_-\cdot s)\,C_1$$

$$
\begin{aligned}
&+2|F_A|^2\left(k\cdot p_-\right)\left(p\cdot p_+\right)\left(p_\tau\cdot q\right)C_1+2|F_A|^2\left(k\cdot p_+\right)\left(p\cdot p_-\right)\left(p_\tau\cdot q\right)C_1\\
&+B^*C_2F_Ak^2\left(k\cdot p_++p\cdot p_+\right)\left(p\cdot p_-\right)\left(p_\tau\cdot q\right)C_1+BC_2{F_A}^*k^2\left(k\cdot p_++p\cdot p_+\right)\left(p\cdot p_-\right)\left(p_\tau\cdot q\right)C_1\\
&+B^*C_2F_Ak^2\left(p\cdot p_+\right)\left(k\cdot p_-+p\cdot p_-\right)\left(p_\tau\cdot q\right)C_1+BC_2{F_A}^*k^2\left(p\cdot p_+\right)\left(k\cdot p_-+p\cdot p_-\right)\left(p_\tau\cdot q\right)C_1\\
&-2B^*F_Ak^2\left(p_+\cdot p_-\right)\left(p_\tau\cdot q\right)C_1-2B{F_A}^*k^2\left(p_+\cdot p_-\right)\left(p_\tau\cdot q\right)C_1\\
&+2B^*F_Ak^2\left({m_\ell}^2+p_+\cdot p_-\right)\left(p_\tau\cdot q\right)C_1+2B{F_A}^*k^2\left({m_\ell}^2+p_+\cdot p_-\right)\left(p_\tau\cdot q\right)C_1\\
&-2|F_A|^2\left(k\cdot p\right)\left({m_\ell}^2+p_+\cdot p_-\right)\left(p_\tau\cdot q\right)C_1\\
&-B^*C_2F_Ak^2\left(k\cdot p+p^2\right)\left({m_\ell}^2+p_+\cdot p_-\right)\left(p_\tau\cdot q\right)C_1\\
&-BC_2F_A^*k^2\left(k\cdot p+p^2\right)\left({m_\ell}^2+p_+\cdot p_-\right)\left(p_\tau\cdot q\right)C_1\\
&+i(C_1C_2-1)(B^*F_A-BF_A^*)\epsilon^{kp-p_\tau q}k^2\left(p\cdot p_+\right)+i(C_1C_2-1)(B^*F_A-BF_A^*)M_\tau\epsilon^{kp-qs}k^2\left(p\cdot p_+\right)\\
&+i(C_1C_2-1)(B^*F_A-BF_A^*)\epsilon^{kp+p_\tau q}k^2\left(p\cdot p_-\right)+i(C_1C_2-1)(B^*F_A-BF_A^*)M_\tau\epsilon^{kp+qs}k^2\left(p\cdot p_-\right)\\
&+4|F_A|^2\left(k\cdot p_\tau\right)\left(k\cdot q\right)\left(p\cdot p_+\right)\left(p\cdot p_-\right)-4|F_A|^2M_\tau\left(k\cdot q\right)\left(k\cdot s\right)\left(p\cdot p_+\right)\left(p\cdot p_-\right)\\
&+2B^*C_2F_Ak^2\left(k\cdot q\right)\left(p\cdot p_+\right)\left(p\cdot p_-\right)\left(k\cdot p_\tau+p\cdot p_\tau\right)\\
&+2BC_2F_A^*k^2\left(k\cdot q\right)\left(p\cdot p_+\right)\left(p\cdot p_-\right)\left(k\cdot p_\tau+p\cdot p_\tau\right)\\
&+2B^*C_2F_Ak^2\left(k\cdot p_\tau\right)\left(p\cdot p_+\right)\left(p\cdot p_-\right)\left(k\cdot q+p\cdot q\right)\\
&+2BC_2F_A^*k^2\left(k\cdot p_\tau\right)\left(p\cdot p_+\right)\left(p\cdot p_-\right)\left(k\cdot q+p\cdot q\right)\\
&-2B^*C_2F_AM_\tau k^2\left(k\cdot s\right)\left(p\cdot p_+\right)\left(p\cdot p_-\right)\left(k\cdot q+p\cdot q\right)\\
&-2BC_2F_A^*M_\tau k^2\left(k\cdot s\right)\left(p\cdot p_+\right)\left(p\cdot p_-\right)\left(k\cdot q+p\cdot q\right)\\
&+4|B|^2{C_2}^2k^4\left(p\cdot p_+\right)\left(p\cdot p_-\right)\left(k\cdot p_\tau+p\cdot p_\tau\right)\left(k\cdot q+p\cdot q\right)\\
&-2B^*C_2F_AM_\tau k^2\left(k\cdot q\right)\left(p\cdot p_+\right)\left(p\cdot p_-\right)\left(k\cdot s+p\cdot s\right)\\
&-2BC_2F_A^*M_\tau k^2\left(k\cdot q\right)\left(p\cdot p_+\right)\left(p\cdot p_-\right)\left(k\cdot s+p\cdot s\right)\\
&-4|B|^2{C_2}^2M_\tau k^4\left(p\cdot p_+\right)\left(p\cdot p_-\right)\left(k\cdot q+p\cdot q\right)\left(k\cdot s+p\cdot s\right)\\
&-2|F_A|^2\left(k\cdot p_\tau\right)\left(k\cdot q\right)p^2\left({m_\ell}^2+p_+\cdot p_-\right)+2|F_A|^2M_\tau\left(k\cdot q\right)\left(k\cdot s\right)p^2\left({m_\ell}^2+p_+\cdot p_-\right)\\
&+B^*F_Ak^2\left(k\cdot q\right)\left(p\cdot p_\tau\right)\left({m_\ell}^2+p_+\cdot p_-\right)+BF_A^*k^2\left(k\cdot q\right)\left(p\cdot p_\tau\right)\left({m_\ell}^2+p_+\cdot p_-\right)\\
&-B^*C_2F_Ak^2\left(k\cdot q\right)p^2\left(k\cdot p_\tau+p\cdot p_\tau\right)\left({m_\ell}^2+p_+\cdot p_-\right)\\
&-BC_2F_A^*k^2\left(k\cdot q\right)p^2\left(k\cdot p_\tau+p\cdot p_\tau\right)\left({m_\ell}^2+p_+\cdot p_-\right)+B^*F_Ak^2\left(k\cdot p_\tau\right)\left(p\cdot q\right)\left({m_\ell}^2+p_+\cdot p_-\right)\\
&+BF_A^*k^2\left(k\cdot p_\tau\right)\left(p\cdot q\right)\left({m_\ell}^2+p_+\cdot p_-\right)-B^*F_AM_\tau k^2\left(k\cdot s\right)\left(p\cdot q\right)\left({m_\ell}^2+p_+\cdot p_-\right)\\
&-BF_A^*M_\tau k^2\left(k\cdot s\right)\left(p\cdot q\right)\left({m_\ell}^2+p_+\cdot p_-\right)+2|B|^2C_2k^4\left(k\cdot p_\tau+p\cdot p_\tau\right)\left(p\cdot q\right)\left({m_\ell}^2+p_+\cdot p_-\right)\\
&-B^*C_2F_Ak^2\left(k\cdot p_\tau\right)p^2\left(k\cdot q+p\cdot q\right)\left({m_\ell}^2+p_+\cdot p_-\right)\\
&-BC_2F_A^*k^2\left(k\cdot p_\tau\right)p^2\left(k\cdot q+p\cdot q\right)\left({m_\ell}^2+p_+\cdot p_-\right)\\
&+B^*C_2F_AM_\tau k^2\left(k\cdot s\right)p^2\left(k\cdot q+p\cdot q\right)\left({m_\ell}^2+p_+\cdot p_-\right)\\
&+BC_2F_A^*M_\tau k^2\left(k\cdot s\right)p^2\left(k\cdot q+p\cdot q\right)\left({m_\ell}^2+p_+\cdot p_-\right)\\
&+2|B|^2C_2k^4\left(p\cdot p_\tau\right)\left(k\cdot q+p\cdot q\right)\left({m_\ell}^2+p_+\cdot p_-\right)\\
&-2|B|^2{C_2}^2k^4p^2\left(k\cdot p_\tau+p\cdot p_\tau\right)\left(k\cdot q+p\cdot q\right)\left({m_\ell}^2+p_+\cdot p_-\right)
\end{aligned}
$$

$$- B^* F_A M_\tau k^2 \,(k\cdot q)\,(p\cdot s)\,\Big({m_\ell}^2 + p_+\cdot p_-\Big) - B F_A^* M_\tau k^2\,(k\cdot q)\,(p\cdot s)\,\Big({m_\ell}^2 + p_+\cdot p_-\Big)$$

$$- 2|B|^2 C_2 M_\tau k^4\,(k\cdot q + p\cdot q)\,(p\cdot s)\,\Big({m_\ell}^2 + p_+\cdot p_-\Big)$$

$$+ B^* C_2 F_A M_\tau k^2\,(k\cdot q)\,p^2\,(k\cdot s + p\cdot s)\,\Big({m_\ell}^2 + p_+\cdot p_-\Big)$$

$$+ B C_2 F_A^* M_\tau k^2\,(k\cdot q)\,p^2\,(k\cdot s + p\cdot s)\,\Big({m_\ell}^2 + p_+\cdot p_-\Big)$$

$$- 2|B|^2 C_2 M_\tau k^4\,(p\cdot q)\,(k\cdot s + p\cdot s)\,\Big({m_\ell}^2 + p_+\cdot p_-\Big)$$

$$+ 2|B|^2 {C_2}^2 M_\tau k^4 p^2\,(k\cdot q + p\cdot q)\,(k\cdot s + p\cdot s)\,\Big({m_\ell}^2 + p_+\cdot p_-\Big)$$

$$- i(B^* F_A - B F_A^*)\epsilon^{kpp_\tau q} k^2\Big(- 2C_2\,(p\cdot p_+)\,(p\cdot p_-) + (C_1 C_2 - 1)\,\Big({m_\ell}^2 + p_+\cdot p_-\Big)$$

$$+ C_2 p^2\,\Big({m_\ell}^2 + p_+\cdot p_-\Big)\Big) - i(B^* F_A - B F_A^*) M_\tau \epsilon^{kpqs} k^2\Big(- 2C_2\,(p\cdot p_+)\,(p\cdot p_-)$$

$$+ (C_1 C_2 - 1)\,\Big({m_\ell}^2 + p_+\cdot p_-\Big) + C_2 p^2\,\Big({m_\ell}^2 + p_+\cdot p_-\Big)\Big) - B^* F_A k^2\,(k\cdot q)\,(p\cdot p_-)\,(p_+\cdot p_\tau)$$

$$- B F_A^* k^2\,(k\cdot q)\,(p\cdot p_-)\,(p_+\cdot p_\tau) - 2|B|^2 C_2 k^4\,(p\cdot p_-)\,(k\cdot q + p\cdot q)\,(p_+\cdot p_\tau)$$

$$- B^* F_A k^2\,(k\cdot p_\tau)\,(p\cdot p_-)\,(p_+\cdot q) - B F_A^* k^2\,(k\cdot p_\tau)\,(p\cdot p_-)\,(p_+\cdot q)$$

$$+ B^* F_A M_\tau k^2\,(k\cdot s)\,(p\cdot p_-)\,(p_+\cdot q) + B F_A^* M_\tau k^2\,(k\cdot s)\,(p\cdot p_-)\,(p_+\cdot q)$$

$$- 2|B|^2 C_2 k^4\,(p\cdot p_-)\,(k\cdot p_\tau + p\cdot p_\tau)\,(p_+\cdot q) + 2|B|^2 C_2 M_\tau k^4\,(p\cdot p_-)\,(k\cdot s + p\cdot s)\,(p_+\cdot q)$$

$$+ B^* F_A M_\tau k^2\,(k\cdot q)\,(p\cdot p_-)\,(p_+\cdot s) + B F_A^* M_\tau k^2\,(k\cdot q)\,(p\cdot p_-)\,(p_+\cdot s)$$

$$+ 2|B|^2 C_2 M_\tau k^4\,(p\cdot p_-)\,(k\cdot q + p\cdot q)\,(p_+\cdot s) - B^* F_A k^2\,(k\cdot q)\,(p\cdot p_+)\,(p_-\cdot p_\tau)$$

$$- B F_A^* k^2\,(k\cdot q)\,(p\cdot p_+)\,(p_-\cdot p_\tau) - 2|B|^2 C_2 k^4\,(p\cdot p_+)\,(k\cdot q + p\cdot q)\,(p_-\cdot p_\tau)$$

$$+ 2|B|^2 k^4\,(p_+\cdot q)\,(p_-\cdot p_\tau) - B^* F_A k^2\,(k\cdot p_\tau)\,(p\cdot p_+)\,(p_-\cdot q) - B F_A^* k^2\,(k\cdot p_\tau)\,(p\cdot p_+)\,(p_-\cdot q)$$

$$+ B^* F_A M_\tau k^2\,(k\cdot s)\,(p\cdot p_+)\,(p_-\cdot q) + B F_A^* M_\tau k^2\,(k\cdot s)\,(p\cdot p_+)\,(p_-\cdot q)$$

$$- 2|B|^2 C_2 k^4\,(p\cdot p_+)\,(k\cdot p_\tau + p\cdot p_\tau)\,(p_-\cdot q) + 2|B|^2 C_2 M_\tau k^4\,(p\cdot p_+)\,(k\cdot s + p\cdot s)\,(p_-\cdot q)$$

$$+ 2|B|^2 k^4\,(p_+\cdot p_\tau)\,(p_-\cdot q) - 2|B|^2 M_\tau k^4\,(p_+\cdot s)\,(p_-\cdot q) + B^* F_A M_\tau k^2\,(k\cdot q)\,(p\cdot p_+)\,(p_-\cdot s)$$

$$+ B F_A^* M_\tau k^2\,(k\cdot q)\,(p\cdot p_+)\,(p_-\cdot s) + 2B^2 C_2 M_\tau k^4\,(p\cdot p_+)\,(k\cdot q + p\cdot q)\,(p_-\cdot s)$$

$$- 2|B|^2 M_\tau k^4\,(p_+\cdot q)\,(p_-\cdot s) + B^* F_A k^2\,(k\cdot p_-)\,(p\cdot p_+)\,(p_\tau\cdot q) + B F_A^* k^2\,(k\cdot p_-)\,(p\cdot p_+)\,(p_\tau\cdot q)$$

$$+ B^* F_A k^2\,(k\cdot p_+)\,(p\cdot p_-)\,(p_\tau\cdot q) + B F_A^* k^2\,(k\cdot p_+)\,(p\cdot p_-)\,(p_\tau\cdot q)$$

$$- 2|F_A|^2 k^2\,(p\cdot p_+)\,(p\cdot p_-)\,(p_\tau\cdot q) - 2B^* C_2 F_A k^2\,\Big(k^2 + k\cdot p\Big)\,(p\cdot p_+)\,(p\cdot p_-)\,(p_\tau\cdot q)$$

$$- 2B C_2 F_A^* k^2\,\Big(k^2 + k\cdot p\Big)\,(p\cdot p_+)\,(p\cdot p_-)\,(p_\tau\cdot q)$$

$$- 2|B|^2 {C_2}^2 k^4\,\Big(k^2 + 2\,(k\cdot p) + p^2\Big)\,(p\cdot p_+)\,(p\cdot p_-)\,(p_\tau\cdot q)$$

$$+ 2|B|^2 C_2 k^4\,(k\cdot p_+ + p\cdot p_+)\,(p\cdot p_-)\,(p_\tau\cdot q) + 2|B|^2 C_2 k^4\,(p\cdot p_+)\,(k\cdot p_- + p\cdot p_-)\,(p_\tau\cdot q)$$

$$- 2|B|^2 k^4\,(p_+\cdot p_-)\,(p_\tau\cdot q) + 2|B|^2 k^4\,\Big({m_\ell}^2 + p_+\cdot p_-\Big)\,(p_\tau\cdot q)$$

$$- B^* F_A k^2\,(k\cdot p)\,\Big({m_\ell}^2 + p_+\cdot p_-\Big)\,(p_\tau\cdot q) - B F_A^* k^2\,(k\cdot p)\,\Big({m_\ell}^2 + p_+\cdot p_-\Big)\,(p_\tau\cdot q)$$

$$+ |F_A|^2 k^2 p^2\,\Big({m_\ell}^2 + p_+\cdot p_-\Big)\,(p_\tau\cdot q) + B^* C_2 F_A k^2\,\Big(k^2 + k\cdot p\Big)\,p^2\,\Big({m_\ell}^2 + p_+\cdot p_-\Big)\,(p_\tau\cdot q)$$

$$+ B C_2 F_A^* k^2\,\Big(k^2 + k\cdot p\Big)\,p^2\,\Big({m_\ell}^2 + p_+\cdot p_-\Big)\,(p_\tau\cdot q) - 2|B|^2 C_2 k^4\,\Big(k\cdot p + p^2\Big)\,\Big({m_\ell}^2 + p_+\cdot p_-\Big)\,(p_\tau\cdot q$$

$$+ |B|^2 {C_2}^2 k^4 p^2\,\Big(k^2 + 2\,(k\cdot p) + p^2\Big)\,\Big({m_\ell}^2 + p_+\cdot p_-\Big)\,(p_\tau\cdot q)$$

$$\begin{aligned}
&+ M_\tau \Big(- |B|^2 {C_2}^2 \left(p^2 \left({m_\ell}^2 + p_+ \cdot p_-\right) - 2 \left(p \cdot p_+\right) \left(p \cdot p_-\right)\right) k^6 \\
&- \bigl(2|B|^2 {m_\ell}^2 + C_2 \bigl(|B|^2 C_2 \left({m_\ell}^2 + p_+ \cdot p_-\right) p^4 \\
&+ \left(-2|B|^2 C_2 \left(p \cdot p_+\right) \left(p \cdot p_-\right) - \left(2|B|^2 - F_A B^* - B F_A^*\right) \left({m_\ell}^2 + p_+ \cdot p_-\right)\right) p^2 \\
&+ 2|B|^2 \left(k \cdot p_-\right) \left(p \cdot p_+\right) + 2 \left(|B|^2 \left(k \cdot p_+\right) + \left(2|B|^2 - F_A B^* - B F_A^*\right) \left(p \cdot p_+\right)\right) \left(p \cdot p_-\right) \\
&+ 2|B|^2 \left(k \cdot p\right) \left(-{m_\ell}^2 - 2 C_2 \left(p \cdot p_+\right) \left(p \cdot p_-\right) - p_+ \cdot p_- + C_2 p^2 \left({m_\ell}^2 + p_+ \cdot p_-\right)\right) \bigr)\bigr) k^4 \\
&+ \Big(\left(B^* F_A + B F_A^*\right) \left(-2 C_1 {m_\ell}^2 - \left(C_1 C_2 + 1\right) \left(k \cdot p_-\right) \left(p \cdot p_+\right)\right) \\
&- \bigl((C_1 C_2 + 1)(B^* F_A + B F_A^*) \left(k \cdot p_+\right) \\
&+ 2 (B^* C_1 C_2 F_A - |F_A|^2 + B C_1 C_2 F_A^*) \left(p \cdot p_+\right) \bigr) \left(p \cdot p_-\right) \\
&+ (B^* C_1 C_2 F_A - |F_A|^2 + B C_1 C_2 F_A^*) p^2 \left({m_\ell}^2 + p_+ \cdot p_-\right) \\
&+ (B^* F_A + B F_A^*) \left(k \cdot p\right) \bigl(2 C_2 \left(p \cdot p_+\right) \left(p \cdot p_-\right) + \left(C_1 C_2 + 1\right) \left({m_\ell}^2 + p_+ \cdot p_-\right) \\
&- C_2 p^2 \left({m_\ell}^2 + p_+ \cdot p_-\right) \bigr) \Big) k^2 - 2 C_1 |F_A|^2 \bigl(C_1 {m_\ell}^2 + \left(k \cdot p_-\right) \left(p \cdot p_+\right) + \left(k \cdot p_+\right) \left(p \cdot p_-\right) \\
&- \left(k \cdot p\right) \left({m_\ell}^2 + p_+ \cdot p_-\right) \bigr) \Big) \left(q \cdot s\right) \Big]
\end{aligned} \tag{A.3}$$

$$\sum_{s_\nu, s_{\ell+}, s_{\ell-}} 2\Re e(\mathcal{M}_V {\mathcal{M}_A}^\dagger) = -64 G_F^2 |V_{ud}|^2 \frac{e^4}{k^4} \Im m \times$$

$$\begin{aligned}
&\Big[F_V(p\cdot k, k^2)(-2iB^* \big({m_l}^2 + p_+\cdot p_-\big)\left(k\cdot q p_\tau\cdot p - p\cdot q p_\tau\cdot k\right)k^2 \\
&+ iB^* C_2 p\cdot p_+\Big(\left(p_-\cdot q p_\tau\cdot p - p\cdot q p_\tau\cdot p_-\right)k^2 + k\cdot p_-\left(p\cdot q p_\tau\cdot k - k\cdot q p_\tau\cdot p\right) \\
&+ p\cdot k\left(k\cdot q p_\tau\cdot p_- - p_-\cdot q p_\tau\cdot k\right)\Big)k^2 + iB^*((p\cdot p_+ p_-\cdot q - p\cdot q p_+\cdot p_-)\, p_\tau\cdot k \\
&+ k\cdot p_+\left(p\cdot q p_\tau\cdot p_- - p_-\cdot q p_\tau\cdot p\right) + k\cdot q\left(p_+\cdot p_- p_\tau\cdot p - p\cdot p_+ p_\tau\cdot p_-\right))k^2 \\
&+ iB^* C_2 p\cdot p_-\Big(\left(p_+\cdot q p_\tau\cdot p - p\cdot q p_\tau\cdot p_+\right)k^2 + k\cdot p_+\left(p\cdot q p_\tau\cdot k - k\cdot q p_\tau\cdot p\right) \\
&+ p\cdot k\left(k\cdot q p_\tau\cdot p_+ - p_+\cdot q p_\tau\cdot k\right)\Big)k^2 + iB^*((p\cdot p_- p_+\cdot q - p\cdot q p_+\cdot p_-)\, p_\tau\cdot k \\
&+ k\cdot p_-\left(p\cdot q p_\tau\cdot p_+ - p_+\cdot q p_\tau\cdot p\right) + k\cdot q\left(p_+\cdot p_- p_\tau\cdot p - p\cdot p_- p_\tau\cdot p_+\right))k^2 \\
&- iB^* C_2 p\cdot p_+\big(p\cdot k\left(p\cdot q p_\tau\cdot p_- - p_-\cdot q p_\tau\cdot p\right) + p_\tau\cdot k\left(p_-\cdot q p^2 - p\cdot q p\cdot p_-\right) \\
&+ k\cdot q\left(p\cdot p_- p_\tau\cdot p - p_\tau\cdot p_- p^2\right)\big)k^2 - iB^* C_2 p\cdot p_-(p\cdot k\left(p\cdot q p_\tau\cdot p_+ - p_+\cdot q p_\tau\cdot p\right) \\
&+ p_\tau\cdot k\left(p_+\cdot q p^2 - p\cdot q p\cdot p_+\right) + k\cdot q\left(p\cdot p_+ p_\tau\cdot p - p_\tau\cdot p_+ p^2\right))k^2 \\
&+ iB^* C_2 p\cdot p_+\Big(\left(p\cdot q p_-\cdot s - p\cdot s p_-\cdot q\right)k^2 + k\cdot p_-(k\cdot q p\cdot s - k\cdot s p\cdot q) \\
&+ p\cdot k\left(k\cdot s p_-\cdot q - k\cdot q p_-\cdot s\right)\Big)M_\tau k^2 + iB^* C_2 p\cdot p_-\Big(\left(p\cdot q p_+\cdot s - p\cdot s p_+\cdot q\right)k^2 \\
&+ k\cdot p_+(k\cdot q p\cdot s - k\cdot s p\cdot q) + p\cdot k\left(k\cdot s p_+\cdot q - k\cdot q p_+\cdot s\right)\Big)M_\tau k^2 \\
&- 2iB^*(k\cdot s p\cdot q - k\cdot q p\cdot s)\left({m_l}^2 + p_+\cdot p_-\right)M_\tau k^2 + iB^*\big(k\cdot p_+\left(p\cdot s p_-\cdot q - p\cdot q p_-\cdot s\right) \\
&+ k\cdot s\left(p\cdot q p_+\cdot p_- - p\cdot p_+ p_-\cdot q\right) + k\cdot q\left(p\cdot p_+ p_-\cdot s - p\cdot s p_+\cdot p_-\right)\big)M_\tau k^2 \\
&+ iB^*\big(k\cdot p_-\left(p\cdot s p_+\cdot q - p\cdot q p_+\cdot s\right) + k\cdot s\left(p\cdot q p_+\cdot p_- - p\cdot p_- p_+\cdot q\right) \\
&+ k\cdot q\left(p\cdot p_- p_+\cdot s - p\cdot s p_+\cdot p_-\right)\big)M_\tau k^2 - iB^* C_2 p\cdot p_+\big(p\cdot k\left(p\cdot s p_-\cdot q - p\cdot q p_-\cdot s\right) \\
&+ k\cdot s\left(p\cdot q p\cdot p_- - p_-\cdot q p^2\right) + k\cdot q\left(p_-\cdot s\overline{p} - p\cdot s p\cdot p_-\right)\big)M_\tau k^2 \\
&- iB^* C_2 p\cdot p_-\big(p\cdot k\left(p\cdot s p_+\cdot q - p\cdot q p_+\cdot s\right) + k\cdot s\left(p\cdot q p\cdot p_+ - p_+\cdot q p^2\right) \\
&+ k\cdot q\left(p_+\cdot s\overline{p} - p\cdot s p\cdot p_+\right)\big)M_\tau k^2 - 2iC_1 F_A^*\left({m_l}^2 + p_+\cdot p_-\right)\left(k\cdot q p_\tau\cdot p - p\cdot q p_\tau\cdot k\right) \\
&- iF_A^* p\cdot p_+\Big(\left(p\cdot q p_\tau\cdot p_- - p_-\cdot q p_\tau\cdot p\right)k^2 + k\cdot p_-\left(k\cdot q p_\tau\cdot p - p\cdot q p_\tau\cdot k\right) \\
&+ p\cdot k\left(p_-\cdot q p_\tau\cdot k - k\cdot q p_\tau\cdot p_-\right)\Big) + iC_1 F_A^*\Big(\left(p\cdot p_+ p_-\cdot q - p\cdot q p_+\cdot p_-\right)p_\tau\cdot k \\
&+ k\cdot p_+\left(p\cdot q p_\tau\cdot p_- - p_-\cdot q p_\tau\cdot p\right) + k\cdot q\left(p_+\cdot p_- p_\tau\cdot p - p\cdot p_+ p_\tau\cdot p_-\right)\Big) \\
&- iF_A^* p\cdot p_-\Big(\left(p\cdot q p_\tau\cdot p_+ - p_+\cdot q p_\tau\cdot p\right)k^2 + k\cdot p_+\left(k\cdot q p_\tau\cdot p - p\cdot q p_\tau\cdot k\right) \\
&+ p\cdot k\left(p_+\cdot q p_\tau\cdot k - k\cdot q p_\tau\cdot p_+\right)\Big) + iC_1 F_A^*\Big(\left(p\cdot p_- p_+\cdot q - p\cdot q p_+\cdot p_-\right)p_\tau\cdot k \\
&+ k\cdot p_-\left(p\cdot q p_\tau\cdot p_+ - p_+\cdot q p_\tau\cdot p\right) + k\cdot q\left(p_+\cdot p_- p_\tau\cdot p - p\cdot p_- p_\tau\cdot p_+\right)\Big) \\
&+ \Big(B^* C_2 p\cdot q p\cdot p_+ k^2 + \left(B^* C_2 k^2 + F_A^*\right)k\cdot q p\cdot p_+ - \left(B^* k^2 + C_1 F_A^*\right)p_+\cdot q\Big)\,\epsilon^{\overline{k}\,\overline{p p_- - p_\tau}} \\
&+ \Big(B^* C_2 p\cdot q p\cdot p_- k^2 + \left(B^* C_2 k^2 + F_A^*\right)k\cdot q p\cdot p_- - \left(B^* k^2 + C_1 F_A^*\right)p_-\cdot q\Big)\,\epsilon^{\overline{k}\,\overline{p p_+ + p_\tau}}
\end{aligned}$$

$$
\begin{aligned}
&- iF_A^* p\cdot p_+ \big(\left(p\cdot s p_-\cdot q - p\cdot q p_-\cdot s\right) k^2 + k\cdot p_- (k\cdot s p\cdot q - k\cdot q p\cdot s) \\
&+ p\cdot k \left(k\cdot q p_-\cdot s - k\cdot s p_-\cdot q\right) \big) M_\tau \\
&- iF_A^* p\cdot p_- \big(\left(p\cdot s p_+\cdot q - p\cdot q p_+\cdot s\right) k^2 + k\cdot p_+ (k\cdot s p\cdot q - k\cdot q p\cdot s) \\
&+ p\cdot k \left(k\cdot q p_+\cdot s - k\cdot s p_+\cdot q\right) \big) M_\tau - 2iC_1 F_A^* (k\cdot s p\cdot q - k\cdot q p\cdot s) \left({m_l}^2 + p_+\cdot p_-\right) M_\tau \\
&+ iC_1 F_A^* \big(k\cdot p_+ \left(p\cdot s p_-\cdot q - p\cdot q p_-\cdot s\right) + k\cdot s \left(p\cdot q p_+\cdot p_- - p\cdot p_+ p_-\cdot q\right) \\
&+ k\cdot q \left(p\cdot p_+ p_-\cdot s - p\cdot s p_+\cdot p_-\right) \big) M_\tau + iC_1 F_A^* \big(k\cdot p_- \left(p\cdot s p_+\cdot q - p\cdot q p_+\cdot s\right) \\
&+ k\cdot s \left(p\cdot q p_+\cdot p_- - p\cdot p_- p_+\cdot q\right) + k\cdot q \left(p\cdot p_- p_+\cdot s - p\cdot s p_+\cdot p_-\right) \big) M_\tau \\
&- \left(B^* C_2 p\cdot q p\cdot p_+ k^2 + \left(B^* C_2 k^2 + F_A^*\right) k\cdot q p\cdot p_+ - \left(B^* k^2 + C_1 F_A^*\right) p_+\cdot q\right) \epsilon^{\overline{k}\overline{p}\overline{p_-}\overline{s}} M_\tau \\
&- \left(B^* C_2 p\cdot q p\cdot p_- k^2 + \left(B^* C_2 k^2 + F_A^*\right) k\cdot q p\cdot p_- - \left(B^* k^2 + C_1 F_A^*\right) p_-\cdot q\right) \epsilon^{\overline{k}\overline{p}\overline{p_+}\overline{s}} M_\tau \\
&+ \epsilon^{\overline{k}\overline{p}\overline{p_+}\overline{q}} \big(B^* \left(C_2 p\cdot p_- \left(p_\tau\cdot p - p\cdot s M_\tau\right) - p_\tau\cdot p_-\right) k^2 + \left(B^* C_2 k^2 + F_A^*\right) p\cdot p_- p_\tau\cdot k \\
&- C_1 F_A^* p_\tau\cdot p_- - \left(B^* C_2 k^2 + F_A^*\right) k\cdot s p\cdot p_- M_\tau + \left(B^* k^2 + C_1 F_A^*\right) p_-\cdot s M_\tau \big) \\
&+ \epsilon^{\overline{k}\overline{p}\overline{p_-}\overline{q}} \big(B^* \left(C_2 p\cdot p_+ \left(p_\tau\cdot p - p\cdot s M_\tau\right) - p_\tau\cdot p_+\right) k^2 + \left(B^* C_2 k^2 + F_A^*\right) p\cdot p_+ p_\tau\cdot k \\
&- C_1 F_A^* p_\tau\cdot p_+ - \left(B^* C_2 k^2 + F_A^*\right) k\cdot s p\cdot p_+ M_\tau \\
&+ \left(B^* k^2 + C_1 F_A^*\right) p_+\cdot s M_\tau \big) \big)
\end{aligned}
\tag{A.4}
$$

$$\sum_{s_\nu, s_{\ell+}, s_{\ell-}} 2\Re e(\mathcal{M}_{IB}\mathcal{M}_V{}^\dagger) = -\frac{32}{ab} G_F{}^2 |V_{ud}{}^2| \frac{e^4}{k^4} F_\pi M_\tau$$

$$\Im m \Bigg[F_V{}^*(p\cdot k, k^2) \Big(2a\left(m_l{}^2 + p_+ \cdot p_-\right)(M - p_\tau \cdot s)\, \epsilon^{\overline{k}\overline{p}\overline{p}_\tau\overline{q}}$$

$$- i\Big(2am_l{}^2 Mp\cdot qk^2 + aMp\cdot p_+p_-\cdot qk^2 + aMp\cdot p_-p_+\cdot qk^2 + 2ap_-\cdot sp_+\cdot qp_\tau\cdot pk^2$$

$$+ 2ap_-\cdot qp_+\cdot sp_\tau\cdot pk^2 - ap\cdot p_+p_-\cdot sp_\tau\cdot qk^2 - ap\cdot p_-p_+\cdot sp_\tau\cdot qk^2 + 2ap\cdot sp_+\cdot p_-p_\tau\cdot qk^2$$

$$- 2am_l{}^2 p\cdot qp_\tau\cdot sk^2 - ap\cdot p_+p_-\cdot qp_\tau\cdot sk^2 - ap\cdot p_-p_+\cdot qp_\tau\cdot sk^2 - 2ap\cdot sp_+\cdot qp_\tau\cdot p_-k^2$$

$$- 2ap\cdot sp_-\cdot qp_\tau\cdot p_+k^2 - 2am_l{}^2 Mk\cdot qp\cdot k - 2am_l{}^2 \mathrm{p}\tau^2 k\cdot sp\cdot q$$

$$+ 2aMk\cdot p_-k\cdot p_+p\cdot q + 2am_l{}^2 \mathrm{p}\tau^2 k\cdot qp\cdot s - aMk\cdot qk\cdot p_+p\cdot p_- - aMk\cdot qk\cdot p_-p\cdot p_+$$

$$- aMk\cdot p_+p\cdot kp_-\cdot q - aMk\cdot p_-p\cdot kp_+\cdot q - 2a\mathrm{p}\tau^2 k\cdot sp\cdot qp_+\cdot p_- + 2a\mathrm{p}\tau^2 k\cdot qp\cdot sp_+\cdot p_-$$

$$+ 2am_l{}^2 k\cdot qp\cdot sp_\tau\cdot k + ak\cdot p_+p\cdot sp_-\cdot qp_\tau\cdot k + 2bp\cdot sp\cdot p_+p_-\cdot qp_\tau\cdot k$$

$$+ ak\cdot p_+p\cdot qp_-\cdot sp_\tau\cdot k$$

$$+ 2ak\cdot qp\cdot p_+p_-\cdot sp_\tau\cdot k - 2bp\cdot qp\cdot p_+p_-\cdot sp_\tau\cdot k + ak\cdot p_-p\cdot sp_+\cdot qp_\tau\cdot k$$

$$+ 2bp\cdot sp\cdot p_-p_+\cdot qp_\tau\cdot k - 2ap\cdot kp_-\cdot sp_+\cdot qp_\tau\cdot k + ak\cdot p_-p\cdot qp_+\cdot sp_\tau\cdot k$$

$$+ 2ak\cdot qp\cdot p_-p_+\cdot sp_\tau\cdot k - 2bp\cdot qp\cdot p_-p_+\cdot sp_\tau\cdot k - 2ap\cdot kp_-\cdot qp_+\cdot sp_\tau\cdot k$$

$$- 2ak\cdot qp\cdot sp_+\cdot p_-p_\tau\cdot k - 2bk\cdot sp\cdot p_+p_-\cdot qp_\tau\cdot p$$

$$- 2am_l{}^2 k\cdot qk\cdot sp_\tau\cdot p - ak\cdot sk\cdot p_+p_-\cdot qp_\tau\cdot p$$

$$- 3ak\cdot qk\cdot p_+p_-\cdot sp_\tau\cdot p + 2bk\cdot qp\cdot p_+p_-\cdot sp_\tau\cdot p - ak\cdot sk\cdot p_-p_+\cdot qp_\tau\cdot p$$

$$- 2bk\cdot sp\cdot p_-p_+\cdot qp_\tau\cdot p - 3ak\cdot qk\cdot p_-p_+\cdot sp_\tau\cdot p + 2bk\cdot qp\cdot p_-p_+\cdot sp_\tau\cdot p$$

$$+ 2ak\cdot qk\cdot sp_+\cdot p_-p_\tau\cdot p - 2ak\cdot p_-k\cdot p_+p\cdot sp_\tau\cdot q + ak\cdot sk\cdot p_+p\cdot p_-p_\tau\cdot q$$

$$+ ak\cdot sk\cdot p_-p\cdot p_+p_\tau\cdot q + ak\cdot p_+p\cdot kp_-\cdot sp_\tau\cdot q + ak\cdot p_-p\cdot kp_+\cdot sp_\tau\cdot q$$

$$- 2ak\cdot sp\cdot kp_+\cdot p_-p_\tau\cdot q - 2am_l{}^2 p\cdot sp_\tau\cdot kp_\tau\cdot q$$

$$- 2ap\cdot sp_+\cdot p_-p_\tau\cdot kp_\tau\cdot q + 2am_l{}^2 k\cdot sp_\tau\cdot pp_\tau\cdot q$$

$$+ 2ak\cdot sp_+\cdot p_-p_\tau\cdot pp_\tau\cdot q + 2am_l{}^2 k\cdot qp\cdot kp_\tau\cdot s$$

$$- 2ak\cdot p_-k\cdot p_+p\cdot qp_\tau\cdot s + ak\cdot qk\cdot p_+p\cdot p_-p_\tau\cdot s$$

$$+ ak\cdot qk\cdot p_-p\cdot p_+p_\tau\cdot s + ak\cdot p_+p\cdot kp_-\cdot qp_\tau\cdot s$$

$$+ ak\cdot p_-p\cdot kp_+\cdot qp_\tau\cdot s + 2am_l{}^2 p\cdot qp_\tau\cdot kp_\tau\cdot s$$

$$+ 2ap\cdot qp_+\cdot p_-p_\tau\cdot kp_\tau\cdot s - 2am_l{}^2 k\cdot qp_\tau\cdot pp_\tau\cdot s - 2ak\cdot qp_+\cdot p_-p_\tau\cdot pp_\tau\cdot s$$

$$- ak\cdot sk\cdot p_+p\cdot qp_\tau\cdot p_- + 3ak\cdot qk\cdot p_+p\cdot sp_\tau\cdot p_- - 2ak\cdot qk\cdot sp\cdot p_+p_\tau\cdot p_-$$

$$+ 2bk\cdot sp\cdot qp\cdot p_+p_\tau\cdot p_- - 2bk\cdot qp\cdot sp\cdot p_+p_\tau\cdot p_- + 2ak\cdot sp\cdot kp_+\cdot qp_\tau\cdot p_-$$

$$+ 2ap\cdot sp_+\cdot qp_\tau\cdot kp_\tau\cdot p_- - 2ap\cdot qp_+\cdot sp_\tau\cdot kp_\tau\cdot p_- - 2ak\cdot sp_+\cdot qp_\tau\cdot pp_\tau\cdot p_-$$

$$+ 2ak\cdot qp_+\cdot sp_\tau\cdot pp_\tau\cdot p_- - ak\cdot sk\cdot p_-p\cdot qp_\tau\cdot p_+ + 3ak\cdot qk\cdot p_-p\cdot sp_\tau\cdot p_+$$

$$- 2ak\cdot qk\cdot sp\cdot p_-p_\tau\cdot p_+ + 2bk\cdot sp\cdot qp\cdot p_-p_\tau\cdot p_+ - 2bk\cdot qp\cdot sp\cdot p_-p_\tau\cdot p_+$$

$$
\begin{aligned}
&+2ak\cdot sp\cdot kp_-\cdot qp_\tau\cdot p_+ + 2ap\cdot sp_-\cdot qp_\tau\cdot kp_\tau\cdot p_+ - 2ap\cdot qp_-\cdot sp_\tau\cdot kp_\tau\cdot p_+\\
&-2ak\cdot sp_-\cdot qp_\tau\cdot pp_\tau\cdot p_+ + 2ak\cdot qp_-\cdot sp_\tau\cdot pp_\tau\cdot p_+ + 4ak\cdot sp\cdot qp_\tau\cdot p_-p_\tau\cdot p_+\\
&-4ak\cdot qp\cdot sp_\tau\cdot p_-p_\tau\cdot p_+ + i(ak\cdot p_+\left(M-p_\tau\cdot s\right) + 2bp\cdot p_+\left(p_\tau\cdot s-M\right)\\
&+a\left(p_+\cdot sp_\tau\cdot k-\left(2M+k\cdot s-2p_\tau\cdot s\right)p_\tau\cdot p_+\right))\epsilon^{\overline{k}\overline{pp_-q}}\\
&+iap_+\cdot qp_\tau\cdot k\epsilon^{\overline{k}\overline{pp_-s}} - iak\cdot p_+p_\tau\cdot q\epsilon^{\overline{k}\overline{pp_-s}}\\
&+2ibp\cdot p_+p_\tau\cdot q\epsilon^{\overline{k}\overline{pp_-s}} - iak\cdot qp_\tau\cdot p_+\epsilon^{\overline{k}\overline{pp_-s}}\\
&+2iap_\tau\cdot qp_\tau\cdot p_+\epsilon^{\overline{k}\overline{pp_-s}} - iak\cdot sp_+\cdot q\epsilon^{\overline{k}\overline{pp_-p_\tau}}\\
&+iak\cdot qp_+\cdot s\epsilon^{\overline{k}\overline{pp_-p_\tau}} + i(ak\cdot p_-\left(M-p_\tau\cdot s\right) + 2bp\cdot p_-\left(p_\tau\cdot s-M\right)\\
&+a\left(p_-\cdot sp_\tau\cdot k-\left(2M+k\cdot s-2p_\tau\cdot s\right)p_\tau\cdot p_-\right))\epsilon^{\overline{k}\overline{pp_+q}}\\
&+iap_-\cdot qp_\tau\cdot k\epsilon^{\overline{k}\overline{pp_+s}} - iak\cdot p_-p_\tau\cdot q\epsilon^{\overline{k}\overline{pp_+s}}\\
&+2ibp\cdot p_-p_\tau\cdot q\epsilon^{\overline{k}\overline{pp_+s}} - iak\cdot qp_\tau\cdot p_-\epsilon^{\overline{k}\overline{pp_+s}} + 2iap_\tau\cdot qp_\tau\cdot p_-\epsilon^{\overline{k}\overline{pp_+s}} - iak\cdot sp_-\cdot q\epsilon^{\overline{k}\overline{pp_+p_\tau}}\\
&+iak\cdot qp_-\cdot s\epsilon^{\overline{k}\overline{pp_+p_\tau}} + q\cdot s\Big(-a(2p_+\cdot p_-p_\tau\cdot pk^2 - p\cdot p_-p_\tau\cdot p_+k^2 + k\cdot p_+p\cdot p_-p_\tau\cdot k\\
&-2p\cdot kp_+\cdot p_-p_\tau\cdot k + \left(k\cdot p_+p\cdot k-k^2p\cdot p_+\right)p_\tau\cdot p_- + k\cdot p_-(p\cdot p_+p_\tau\cdot k-2k\cdot p_+p_\tau\cdot p+\\
&p\cdot kp_\tau\cdot p_+)) + i\left(ak\cdot p_+-2\left(bp\cdot p_++ap_\tau\cdot p_+\right)\right)\epsilon^{\overline{k}\overline{pp_-p_\tau}}\\
&+i\left(ak\cdot p_--2\left(bp\cdot p_-+ap_\tau\cdot p_-\right)\right)\epsilon^{\overline{k}\overline{pp_+p_\tau}}\Big)\\
&+2iam_l{}^2k\cdot q\epsilon^{\overline{k}\overline{pp}_\tau\overline{s}} + 2iak\cdot qp_+\cdot p_-\epsilon^{\overline{k}\overline{pp}_\tau\overline{s}}\\
&-2iam_l{}^2p_\tau\cdot q\epsilon^{\overline{k}\overline{pp}_\tau\overline{s}} - 2iap_+\cdot p_-p_\tau\cdot q\epsilon^{\overline{k}\overline{pp}_\tau\overline{s}}\Big)\Big)\Big]
\end{aligned}
\tag{A.5}
$$

$$
\begin{aligned}
&\sum_{s_\nu, s_{\ell+}, s_{\ell-}} 2\Re e(\mathcal{M}_{IB}\mathcal{M}_A{}^\dagger) = -64 G_F{}^2 |V_{ud}{}^2| \frac{e^4}{k^4} F_\pi M_\tau \Bigg(- \frac{B^* C_2 M p\cdot p_+ p_-\cdot q k^4}{b} \\
&- \frac{B^* C_2 M p\cdot p_- p_+\cdot q k^4}{b} + \frac{B^* C_2 M p\cdot q\left(m_l{}^2 + p_+\cdot p_-\right) k^4}{b} \\
&- \frac{B^* C_2 q\cdot s\left(m_l{}^2 + p_+\cdot p_-\right) p_\tau\cdot p k^4}{b} \\
&- \frac{B^* C_2 p\cdot p_+ p_-\cdot s p_\tau\cdot q k^4}{b} - \frac{B^* C_2 p\cdot p_- p_+\cdot s p_\tau\cdot q k^4}{b} + \frac{B^* C_2 p\cdot s\left(m_l{}^2 + p_+\cdot p_-\right) p_\tau\cdot q k^4}{b} \\
&+ \frac{B^* C_2 p\cdot p_+ p_-\cdot q p_\tau\cdot s k^4}{b} + \frac{B^* C_2 p\cdot p_- p_+\cdot q p_\tau\cdot s k^4}{b} - \frac{B^* C_2 p\cdot q\left(m_l{}^2 + p_+\cdot p_-\right) p_\tau\cdot s k^4}{b} \\
&+ \frac{B^* C_2 p\cdot p_+ q\cdot s p_\tau\cdot p_- k^4}{b} + \frac{B^* C_2 p\cdot p_- q\cdot s p_\tau\cdot p_+ k^4}{b} \\
&+ \frac{2 B^* C_2 q\cdot s\left(m_l{}^2 + p_+\cdot p_-\right)\left(p_\tau\cdot p\right)^2 k^2}{b} \\
&+ \frac{2 B^* C_2 M k\cdot q k\cdot p_+ p\cdot p_- k^2}{b} + \frac{B^* C_2 M k\cdot p_+ p\cdot q p\cdot p_- k^2}{b} + \frac{2 B^* C_2 M k\cdot q k\cdot p_- p\cdot p_+ k^2}{b} \\
&+ \frac{B^* C_2 M k\cdot p_- p\cdot q p\cdot p_+ k^2}{b} - \frac{4 B^* C_2 M k\cdot q p\cdot p_- p\cdot p_+ k^2}{a} + \frac{2 B^* C_2 M k\cdot q p\cdot p_- p\cdot p_+ k^2}{b} \\
&- \frac{4 B^* C_2 M p\cdot q p\cdot p_- p\cdot p_+ k^2}{a} + \frac{2 B^* M p\cdot p_+ p_-\cdot q k^2}{a} - \frac{F_A^* M p\cdot p_+ p_-\cdot q k^2}{b} \\
&- \frac{B^* C_2 M p\cdot k p\cdot p_+ p_-\cdot q k^2}{b} + \frac{2 B^* M p\cdot p_- p_+\cdot q k^2}{a} - \frac{F_A^* M p\cdot p_- p_+\cdot q k^2}{b} \\
&- \frac{B^* C_2 M p\cdot k p\cdot p_- p_+\cdot q k^2}{b} - \frac{2 B^* M k\cdot q p_+\cdot p_- k^2}{b} + \frac{2 B^* C_2 M p^2 k\cdot q\left(m_l{}^2 + p_+\cdot p_-\right) k^2}{a} \\
&- \frac{B^* C_2 M p^2 k\cdot q\left(m_l{}^2 + p_+\cdot p_-\right) k^2}{b} + \frac{4 B^* M k\cdot q\left(m_l{}^2 + p_+\cdot p_-\right) k^2}{b} \\
&- \frac{2 B^* C_2 M k\cdot q p\cdot k\left(m_l{}^2 + p_+\cdot p_-\right) k^2}{b} + \frac{2 B^* C_2 M p^2 p\cdot q\left(m_l{}^2 + p_+\cdot p_-\right) k^2}{a} \\
&- \frac{2 B^* M p\cdot q\left(m_l{}^2 + p_+\cdot p_-\right) k^2}{a} + \frac{F_A^* M p\cdot q\left(m_l{}^2 + p_+\cdot p_-\right) k^2}{b} \\
&- \frac{2 B^* \mathrm{p}\tau^2 q\cdot s\left(m_l{}^2 + p_+\cdot p_-\right) k^2}{b} \\
&- \frac{4 B^* C_2 p\cdot p_- p\cdot p_+ q\cdot s p_\tau\cdot k k^2}{a} - \frac{2 B^* C_2 p\cdot p_- p\cdot p_+ q\cdot s p_\tau\cdot k k^2}{b} \\
&+ \frac{B^* C_2 p\cdot s p\cdot p_+ p_-\cdot q p_\tau\cdot k k^2}{b} + \frac{2 B^* C_2 k\cdot q p\cdot p_+ p_-\cdot s p_\tau\cdot k k^2}{b} \\
&+ \frac{B^* C_2 p\cdot q p\cdot p_+ p_-\cdot s p_\tau\cdot k k^2}{b} + \frac{B^* C_2 p\cdot s p\cdot p_- p_+\cdot q p_\tau\cdot k k^2}{b}
\end{aligned}
$$

$$+\frac{B^*C_2p\cdot qp\cdot p_+p_-\cdot sp_\tau\cdot kk^2}{b}+\frac{B^*C_2p\cdot sp\cdot p_-p_+\cdot qp_\tau\cdot kk^2}{b}$$

$$-\frac{2B^*p_-\cdot sp_+\cdot qp_\tau\cdot kk^2}{b}+\frac{2B^*C_2k\cdot qp\cdot p_-p_+\cdot sp_\tau\cdot kk^2}{b}+\frac{B^*C_2p\cdot qp\cdot p_-p_+\cdot sp_\tau\cdot kk^2}{b}$$

$$-\frac{2B^*p_-\cdot qp_+\cdot sp_\tau\cdot kk^2}{b}+\frac{2B^*q\cdot sp_+\cdot p_-p_\tau\cdot kk^2}{b}$$

$$-\frac{2B^*C_2k\cdot qp\cdot s\left(m_l{}^2+p_+\cdot p_-\right)p_\tau\cdot kk^2}{b}$$

$$-\frac{2B^*C_2p\cdot qp\cdot s\left(m_l{}^2+p_+\cdot p_-\right)p_\tau\cdot kk^2}{b}+\frac{2B^*C_2p^2q\cdot s\left(m_l{}^2+p_+\cdot p_-\right)p_\tau\cdot kk^2}{a}$$

$$+\frac{B^*C_2p^2q\cdot s\left(m_l{}^2+p_+\cdot p_-\right)p_\tau\cdot kk^2}{b}+\frac{B^*C_2k\cdot p_+p\cdot p_-q\cdot sp_\tau\cdot pk^2}{b}$$

$$+\frac{B^*C_2k\cdot p_-p\cdot p_+q\cdot sp_\tau\cdot pk^2}{b}-\frac{4B^*C_2p\cdot p_-p\cdot p_+q\cdot sp_\tau\cdot pk^2}{a}-\frac{B^*C_2k\cdot sp\cdot p_+p_-\cdot qp_\tau\cdot pk^2}{b}$$

$$+\frac{B^*C_2k\cdot qp\cdot p_+p_-\cdot sp_\tau\cdot pk^2}{b}-\frac{B^*C_2k\cdot sp\cdot p_-p_+\cdot qp_\tau\cdot pk^2}{b}+\frac{B^*C_2k\cdot qp\cdot p_-p_+\cdot sp_\tau\cdot pk^2}{b}$$

$$+\frac{2B^*C_2Mk\cdot q\left(m_l{}^2+p_+\cdot p_-\right)p_\tau\cdot pk^2}{b}+\frac{2B^*C_2k\cdot qk\cdot s\left(m_l{}^2+p_+\cdot p_-\right)p_\tau\cdot pk^2}{b}$$

$$+\frac{2B^*C_2Mp\cdot q\left(m_l{}^2+p_+\cdot p_-\right)p_\tau\cdot pk^2}{b}+\frac{2B^*C_2k\cdot sp\cdot q\left(m_l{}^2+p_+\cdot p_-\right)p_\tau\cdot pk^2}{b}$$

$$+\frac{2B^*C_2p^2q\cdot s\left(m_l{}^2+p_+\cdot p_-\right)p_\tau\cdot pk^2}{a}-\frac{2B^*q\cdot s\left(m_l{}^2+p_+\cdot p_-\right)p_\tau\cdot pk^2}{a}$$

$$-\frac{F_A^*q\cdot s\left(m_l{}^2+p_+\cdot p_-\right)p_\tau\cdot pk^2}{b}-\frac{2B^*C_2p\cdot kq\cdot s\left(m_l{}^2+p_+\cdot p_-\right)p_\tau\cdot pk^2}{b}$$

$$+\frac{2B^*C_2q\cdot s\left(m_l{}^2+p_+\cdot p_-\right)p_\tau\cdot kp_\tau\cdot pk^2}{b}-\frac{B^*C_2k\cdot p_+p\cdot sp\cdot p_-p_\tau\cdot qk^2}{b}$$

$$-\frac{B^*C_2k\cdot p_-p\cdot sp\cdot p_+p_\tau\cdot qk^2}{b}$$

$$+\frac{4B^*C_2k\cdot sp\cdot p_-p\cdot p_+p_\tau\cdot qk^2}{a}+\frac{2B^*C_2k\cdot sp\cdot p_-p\cdot p_+p_\tau\cdot qk^2}{b}$$

$$+\frac{4B^*C_2p\cdot sp\cdot p_-p\cdot p_+p_\tau\cdot qk^2}{a}+\frac{2B^*k\cdot p_+p_-\cdot sp_\tau\cdot qk^2}{b}-\frac{2B^*p\cdot p_+p_-\cdot sp_\tau\cdot qk^2}{a}$$

$$-\frac{F_A^*p\cdot p_+p_-\cdot sp_\tau\cdot qk^2}{b}-\frac{B^*C_2p\cdot kp\cdot p_+p_-\cdot sp_\tau\cdot qk^2}{b}+\frac{2B^*k\cdot p_-p_+\cdot sp_\tau\cdot qk^2}{b}$$

$$-\frac{2B^*p\cdot p_-p_+\cdot sp_\tau\cdot qk^2}{a}-\frac{F_A^*p\cdot p_-p_+\cdot sp_\tau\cdot qk^2}{b}$$

$$-\frac{B^*C_2p\cdot kp\cdot p_-p_+\cdot sp_\tau\cdot qk^2}{b}-\frac{2B^*k\cdot sp_+\cdot p_-p_\tau\cdot qk^2}{b}$$

$$-\frac{2B^*M\left(m_l{}^2+p_+\cdot p_-\right)p_\tau\cdot qk^2}{b}-\frac{2B^*C_2p^2k\cdot s\left(m_l{}^2+p_+\cdot p_-\right)p_\tau\cdot qk^2}{a}$$

$$- \frac{2B^*C_2k\cdot qp\cdot p_-p\cdot p_+p_\tau\cdot sk^2}{b} + \frac{4B^*C_2p\cdot qp\cdot p_-p\cdot p_+p_\tau\cdot sk^2}{a}$$

$$- \frac{2B^*p\cdot p_+p_-\cdot qp_\tau\cdot sk^2}{a} + \frac{F_A^*p\cdot p_+p_-\cdot qp_\tau\cdot sk^2}{b} + \frac{B^*C_2p\cdot kp\cdot p_+p_-\cdot qp_\tau\cdot sk^2}{b}$$

$$- \frac{2B^*p\cdot p_-p_+\cdot qp_\tau\cdot sk^2}{a} + \frac{F_A^*p\cdot p_-p_+\cdot qp_\tau\cdot sk^2}{b} + \frac{B^*C_2p\cdot kp\cdot p_-p_+\cdot qp_\tau\cdot sk^2}{b}$$

$$+ \frac{2B^*k\cdot qp_+\cdot p_-p_\tau\cdot sk^2}{b} - \frac{2B^*C_2p^2k\cdot q\left({m_l}^2+p_+\cdot p_-\right)p_\tau\cdot sk^2}{a}$$

$$+ \frac{B^*C_2p^2k\cdot q\left({m_l}^2+p_+\cdot p_-\right)p_\tau\cdot sk^2}{b} - \frac{4B^*k\cdot q\left({m_l}^2+p_+\cdot p_-\right)p_\tau\cdot sk^2}{b}$$

$$+ \frac{2B^*C_2k\cdot qp\cdot k\left({m_l}^2+p_+\cdot p_-\right)p_\tau\cdot sk^2}{b} - \frac{2B^*C_2p^2p\cdot q\left({m_l}^2+p_+\cdot p_-\right)p_\tau\cdot sk^2}{a}$$

$$+ \frac{2B^*p\cdot q\left({m_l}^2+p_+\cdot p_-\right)p_\tau\cdot sk^2}{a} - \frac{F_A^*p\cdot q\left({m_l}^2+p_+\cdot p_-\right)p_\tau\cdot sk^2}{b}$$

$$- \frac{2B^*C_2k\cdot q\left({m_l}^2+p_+\cdot p_-\right)p_\tau\cdot pp_\tau\cdot sk^2}{b} - \frac{2B^*C_2p\cdot q\left({m_l}^2+p_+\cdot p_-\right)p_\tau\cdot pp_\tau\cdot sk^2}{b}$$

$$+ \frac{4B^*\left({m_l}^2+p_+\cdot p_-\right)p_\tau\cdot qp_\tau\cdot sk^2}{b} - \frac{2B^*C_2Mk\cdot qp\cdot p_+p_\tau\cdot p_-k^2}{b} - \frac{2B^*C_2k\cdot qk\cdot sp\cdot p_+p_\tau\cdot p_-k^2}{b}$$

$$- \frac{2B^*C_2Mp\cdot qp\cdot p_+p_\tau\cdot p_-k^2}{b} - \frac{B^*C_2k\cdot sp\cdot qp\cdot p_+p_\tau\cdot p_-k^2}{b} - \frac{B^*C_2k\cdot qp\cdot sp\cdot p_+p_\tau\cdot p_-k^2}{b}$$

$$- \frac{2B^*k\cdot p_+q\cdot sp_\tau\cdot p_-k^2}{b} + \frac{2B^*p\cdot p_+q\cdot sp_\tau\cdot p_-k^2}{a} + \frac{F_A^*p\cdot p_+q\cdot sp_\tau\cdot p_-k^2}{b}$$

$$+ \frac{B^*C_2p\cdot kp\cdot p_+q\cdot sp_\tau\cdot p_-k^2}{b} + \frac{2B^*Mp_+\cdot qp_\tau\cdot p_-k^2}{b} + \frac{2B^*k\cdot sp_+\cdot qp_\tau\cdot p_-k^2}{b}$$

$$- \frac{2B^*C_2p\cdot p_+q\cdot sp_\tau\cdot kp_\tau\cdot p_-k^2}{b} - \frac{2B^*C_2p\cdot p_+q\cdot sp_\tau\cdot pp_\tau\cdot p_-k^2}{b}$$

$$+ \frac{2B^*C_2k\cdot sp\cdot p_+p_\tau\cdot qp_\tau\cdot p_-k^2}{b} + \frac{2B^*C_2p\cdot sp\cdot p_+p_\tau\cdot qp_\tau\cdot p_-k^2}{b} - \frac{2B^*p_+\cdot sp_\tau\cdot qp_\tau\cdot p_-k^2}{b}$$

$$+ \frac{2B^*C_2k\cdot qp\cdot p_+p_\tau\cdot sp_\tau\cdot p_-k^2}{b} + \frac{2B^*C_2p\cdot qp\cdot p_+p_\tau\cdot sp_\tau\cdot p_-k^2}{b} - \frac{2B^*p_+\cdot qp_\tau\cdot sp_\tau\cdot p_-k^2}{b}$$

$$- \frac{2B^*C_2Mk\cdot qp\cdot p_-p_\tau\cdot p_+k^2}{b} - \frac{2B^*C_2k\cdot qk\cdot sp\cdot p_-p_\tau\cdot p_+k^2}{b} - \frac{2B^*C_2Mp\cdot qp\cdot p_-p_\tau\cdot p_+k^2}{b}$$

$$- \frac{B^*C_2k\cdot sp\cdot qp\cdot p_-p_\tau\cdot p_+k^2}{b} - \frac{B^*C_2k\cdot qp\cdot sp\cdot p_-p_\tau\cdot p_+k^2}{b} - \frac{2B^*k\cdot p_-q\cdot sp_\tau\cdot p_+k^2}{b}$$

$$+ \frac{2B^*p\cdot p_-q\cdot sp_\tau\cdot p_+k^2}{a} + \frac{F_A^*p\cdot p_-q\cdot sp_\tau\cdot p_+k^2}{b} + \frac{B^*C_2p\cdot kp\cdot p_-q\cdot sp_\tau\cdot p_+k^2}{b}$$

$$+ \frac{2B^*Mp_-\cdot qp_\tau\cdot p_+k^2}{b} + \frac{2B^*k\cdot sp_-\cdot qp_\tau\cdot p_+k^2}{b} - \frac{2B^*C_2p\cdot p_-q\cdot sp_\tau\cdot kp_\tau\cdot p_+k^2}{b}$$

$$- \frac{2B^*C_2p\cdot p_-q\cdot sp_\tau\cdot pp_\tau\cdot p_+k^2}{b} + \frac{2B^*C_2k\cdot sp\cdot p_-p_\tau\cdot qp_\tau\cdot p_+k^2}{b}$$

$$+\frac{2B^*C_2p\cdot sp\cdot p_-p_\tau\cdot qp_\tau\cdot p_+k^2}{b}-\frac{2B^*p_-\cdot sp_\tau\cdot qp_\tau\cdot p_+k^2}{b}+\frac{2B^*C_2k\cdot qp\cdot p_-p_\tau\cdot sp_\tau\cdot p_+k^2}{b}$$

$$+\frac{2B^*C_2p\cdot qp\cdot p_-p_\tau\cdot sp_\tau\cdot p_+k^2}{b}-\frac{2B^*p_-\cdot qp_\tau\cdot sp_\tau\cdot p_+k^2}{b}+\frac{4B^*q\cdot sp_\tau\cdot p_-p_\tau\cdot p_+k^2}{b}$$

$$+\frac{iB^*C_2Mp\cdot p_+\epsilon^{\overline{k}\overline{pp-q}}k^2}{b}-\frac{iB^*C_2p\cdot p_+q\cdot s\epsilon^{\overline{k}\overline{pp-p_\tau}}k^2}{b}+\frac{iB^*C_2Mp\cdot p_-\epsilon^{\overline{k}\overline{pp+q}}k^2}{b}$$

$$-\frac{iB^*C_2p\cdot p_-q\cdot s\epsilon^{\overline{k}\overline{pp+p_\tau}}k^2}{b}+\frac{iB^*C_2p\cdot p_+p_\tau\cdot k\epsilon^{\overline{pp-qs}}k^2}{b}+\frac{iB^*C_2p\cdot p_-p_\tau\cdot k\epsilon^{\overline{pp+qs}}k^2}{b}$$

$$+\frac{2F_A^*Mk\cdot qk\cdot p_+p\cdot p_-}{b}+\frac{2F_A^*Mk\cdot qk\cdot p_-p\cdot p_+}{b}-\frac{4F_A^*Mk\cdot qp\cdot p_-p\cdot p_+}{a}$$

$$+\frac{2C_1F_A^*Mp\cdot p_+p_-\cdot q}{a}+\frac{2C_1F_A^*Mp\cdot p_-p_+\cdot q}{a}-\frac{2C_1F_A^*Mk\cdot qp_+\cdot p_-}{b}$$

$$+\frac{2F_A^*Mp^2k\cdot q\left(m^2+p_+\cdot p_-\right)}{a}+\frac{4C_1F_A^*Mk\cdot q\left(m^2+p_+\cdot p_-\right)}{b}$$

$$-\frac{2F_A^*Mk\cdot qp\cdot k\left(m^2+p_+\cdot p_-\right)}{b}-\frac{2C_1F_A^*Mp\cdot q\left(m^2+p_+\cdot p_-\right)}{a}$$

$$-\frac{2C_1F_A^*\mathrm{p}\tau^2q\cdot s\left(m^2+p_+\cdot p_-\right)}{b}-\frac{4F_A^*p\cdot p_-p\cdot p_+q\cdot sp_\tau\cdot k}{a}+\frac{2F_A^*k\cdot qp\cdot p_+p_-\cdot sp_\tau\cdot k}{b}$$

$$-\frac{2C_1F_A^*p_-\cdot sp_+\cdot qp_\tau\cdot k}{b}+\frac{2F_A^*k\cdot qp\cdot p_-p_+\cdot sp_\tau\cdot k}{b}-\frac{2C_1F_A^*p_-\cdot qp_+\cdot sp_\tau\cdot k}{b}$$

$$+\frac{2C_1F_A^*q\cdot sp_+\cdot p_-p_\tau\cdot k}{b}-\frac{2F_A^*k\cdot qp\cdot s\left(m^2+p_+\cdot p_-\right)p_\tau\cdot k}{b}$$

$$+\frac{2F_A^*p^2q\cdot s\left(m^2+p_+\cdot p_-\right)p_\tau\cdot k}{a}+\frac{2F_A^*Mk\cdot q\left(m^2+p_+\cdot p_-\right)p_\tau\cdot p}{b}$$

$$+\frac{2F_A^*k\cdot qk\cdot s\left(m^2+p_+\cdot p_-\right)p_\tau\cdot p}{b}-\frac{2C_1F_A^*q\cdot s\left(m^2+p_+\cdot p_-\right)p_\tau\cdot p}{a}$$

$$+\frac{2F_A^*q\cdot s\left(m^2+p_+\cdot p_-\right)p_\tau\cdot kp_\tau\cdot p}{b}+\frac{4F_A^*k\cdot sp\cdot p_-p\cdot p_+p_\tau\cdot q}{a}+\frac{2C_1F_A^*k\cdot p_+p_-\cdot sp_\tau\cdot q}{b}$$

$$-\frac{2C_1F_A^*p\cdot p_+p_-\cdot sp_\tau\cdot q}{a}+\frac{2C_1F_A^*k\cdot p_-p_+\cdot sp_\tau\cdot q}{b}-\frac{2C_1F_A^*p\cdot p_-p_+\cdot sp_\tau\cdot q}{a}$$

$$-\frac{2C_1F_A^*k\cdot sp_+\cdot p_-p_\tau\cdot q}{b}-\frac{2C_1F_A^*M\left(m^2+p_+\cdot p_-\right)p_\tau\cdot q}{b}-\frac{2F_A^*p^2k\cdot s\left(m^2+p_+\cdot p_-\right)p_\tau\cdot q}{a}$$

$$+\frac{2C_1F_A^*p\cdot s\left(m^2+p_+\cdot p_-\right)p_\tau\cdot q}{a}-\frac{2F_A^*k\cdot s\left(m^2+p_+\cdot p_-\right)p_\tau\cdot pp_\tau\cdot q}{b}$$

$$-\frac{2F_A^*k\cdot qk\cdot p_+p\cdot p_-p_\tau\cdot s}{b}-\frac{2F_A^*k\cdot qk\cdot p_-p\cdot p_+p_\tau\cdot s}{b}+\frac{4F_A^*k\cdot qp\cdot p_-p\cdot p_+p_\tau\cdot s}{a}$$

$$-\frac{2C_1F_A^*p\cdot p_+p_-\cdot qp_\tau\cdot s}{a}-\frac{2C_1F_A^*p\cdot p_-p_+\cdot qp_\tau\cdot s}{a}+\frac{2C_1F_A^*k\cdot qp_+\cdot p_-p_\tau\cdot s}{b}$$

$$
\begin{aligned}
&+\frac{\left(B^*C_2k^2+F_A^*\right)\left(m^2+p_+\cdot p_-\right)p_\tau\cdot k\big)\epsilon^{\bar{k}\bar{p}\bar{q}\bar{s}}}{b}\\
&-\frac{i\left(B^*C_2p\cdot sk^2+\left(B^*C_2k^2+F_A^*\right)k\cdot s\right)\left(m^2+p_+\cdot p_-\right)\epsilon^{\bar{k}\bar{p}\overline{p_\tau}\bar{q}}}{b}\\
&-\frac{i\left(B^*C_2p\cdot qk^2+\left(B^*C_2k^2+F_A^*\right)k\cdot q\right)\left(m^2+p_+\cdot p_-\right)\epsilon^{\bar{k}\bar{p}\overline{p_\tau}\bar{s}}}{b}\\
&+\frac{i\left(\left(B^*C_2k^2+F_A^*\right)p\cdot p_+p_\tau\cdot k-\left(B^*k^2+C_1F_A^*\right)p_\tau\cdot p_+\right)\epsilon^{\bar{k}\overline{p_-}\bar{q}\bar{s}}}{b}\\
&+\frac{i\left(\left(B^*C_2p\cdot sk^2+\left(B^*C_2k^2+F_A^*\right)k\cdot s\right)p\cdot p_+-\left(B^*k^2+C_1F_A^*\right)p_+\cdot s\right)\epsilon^{\bar{k}\overline{p_-}\overline{p_\tau}\bar{q}}}{b}\\
&+\frac{i\left(\left(B^*C_2p\cdot qk^2+\left(B^*C_2k^2+F_A^*\right)k\cdot q\right)p\cdot p_+-\left(B^*k^2+C_1F_A^*\right)p_+\cdot q\right)\epsilon^{\bar{k}\overline{p_-}\overline{p_\tau}\bar{s}}}{b}\\
&+\frac{i\left(\left(B^*C_2k^2+F_A^*\right)p\cdot p_-p_\tau\cdot k-\left(B^*k^2+C_1F_A^*\right)p_\tau\cdot p_-\right)\epsilon^{\bar{k}\overline{p_+}\bar{q}\bar{s}}}{b}\\
&+\frac{i\left(\left(B^*C_2p\cdot sk^2+\left(B^*C_2k^2+F_A^*\right)k\cdot s\right)p\cdot p_--\left(B^*k^2+C_1F_A^*\right)p_-\cdot s\right)\epsilon^{\bar{k}\overline{p_+}\overline{p_\tau}\bar{q}}}{b}\\
&+\frac{i\left(\left(B^*C_2p\cdot qk^2+\left(B^*C_2k^2+F_A^*\right)k\cdot q\right)p\cdot p_--\left(B^*k^2+C_1F_A^*\right)p_-\cdot q\right)\epsilon^{\bar{k}\overline{p_+}\overline{p_\tau}\bar{s}}}{b}\\
&+\frac{i\left(B^*C_2k^2+F_A^*\right)}{ab}\Big(-a\left(k\cdot p_+p\cdot p_-+k\cdot p_-p\cdot p_+\right)+ap\cdot k\left(m^2+p_+\cdot p_-\right)\Big)\\
&-2\Big(b\left(m^2+p_+\cdot p_-\right)p^2+a\left(m^2+p_+\cdot p_-\right)p_\tau\cdot p-ap\cdot p_+p_\tau\cdot p_-
\end{aligned}
\tag{A.6}
$$

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