

Bounds from LFV processes on the Littlest Higgs Model with T-parity

Master Thesis



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Introducción

El Modelo Estándar de partículas es una de las teorías más exitosas creadas por el hombre, esta logra explicar, en su mayoría, los fenómenos, procesos, interacciones que ocurren a nivel fundamental entre partículas y ha sido comprobada con gran precisión experimentalmente. Por ello existe un gran interés en hacer extensiones del modelo para encontrar una explicación a aquellos problemas que no pueden ser resueltos de forma satisfactoria con dicha teoría.

Una prueba importante del Modelo Estándar fue el hallazgo en 2012 por el LHC, de una partícula propuesta por Peter Higgs en 1964 mediante el llamado *Mecanismo de Higgs*, dicha partícula es nombrada en su honor: Bosón de Higgs.

El Mecanismo de Higgs está estrechamente relacionado con el rompimiento de simetría electrodébil el cual se produce a una escala de energía $v = 246$ GeV, y se estima que la escala de energía de nueva física sea del orden de $\sim \mathcal{O}$ (TeV).

Se han propuesto diferentes modelos para tratar de explicar esta diferencia entre las escalas de energía llamado *problema de jerarquía* y estabilizar las contribuciones que existen de diagramas cuadráticamente divergentes a la masa del Higgs.

Little Higgs, es uno de los modelos desarrollados para abordar el problema descrito anteriormente. Modelos Little Higgs tienen como su principal característica la propuesta de que el Higgs es un pseudo-boson-Nambu-Goldstone originado por el rompimiento del grupo de simetría $SU(5)$ a una escala de energía $f \sim \mathcal{O}(\text{TeV})$. Dicha idea fue retomada por Arkani-Hamed, Cohen y Georgi quienes construyeron el primer modelo Little Higgs exitoso [24]. A partir de esto, se le han hecho varias extensiones al modelo Little Higgs y en esta tesis abordaremos una de ellas: *Modelo Littlest Higgs con T-paridad*.

Este modelo, abreviado LHT, tiene varios alcances y uno de los que nos propusimos a estudiar es que da nuevas cotas a procesos de violación de sabor leptónico (LFV) por medio de interacciones de partículas propuestas que aparecen a una escala de energía de TeV's. La estructura de la tesis se describe a continuación: en el Capítulo 1 nombrado *Standard Model* se explican los requisitos necesarios que debe cumplir la Lagrangiana del modelo estándar electrodébil tales como simetría de fase global y local. Sabemos que la Electrodinámica Cuántica es una teoría Abeliána, lo que implica que su mediador, el fotón, no interactúa consigo mismo, pero a diferencia de esto en la teoría Electrodébil sus bosones de gauge, W^\pm , Z^0 y A sí pueden interactuar entre ellos, lo que nos lleva a trabajar con una teoría no Abeliána por lo que es necesario conocer los campos de Yang-Mills, contenido del subcapítulo 1.1.3. Para la segunda parte del Capítulo 1 se detalla el Modelo Glashow-Weinberg-Salam introduciendo el Mecanismo de Higgs y cómo este le da origen a las masas de los bosones de gauge y a los fermiones. Para finalizar con este capítulo se muestra la forma de la Lagrangiana final del modelo dando un breve comentario de cada una de las partes que la constituyen.

Para el Capítulo 2 se han abordado temas que servirán para tener una mejor base teórica al momento de llegar al Modelo Littlest Higgs con T-paridad. Los temas incluidos en el capítulo son el Rompimiento Espontáneo de Simetría, Modelo Sigma Lineal, el Teorema de Goldstone y el Modelo

Sigma No Lineal. La elección de estos temas es debido a que en el modelo LHT aparecen bosones de Goldstone por el rompimiento de simetría además de estar cimentado en un modelo sigma no lineal. Como se dijo anteriormente, los alcances de LHT llegan a procesos de violación de sabor leptónico por lo que se incluye un breve subcapítulo dedicado a LFV.

El Capítulo 3 está dedicado completamente a la descripción de modelos Little Higgs, cómo es que abordan el llamado problema de jerarquía y las dos principales categorías en las que están divididos estos modelos.

En el Capítulo 4 se desarrolla detalladamente el Modelo de Little Higgs con T-paridad. Aquí se explica cómo es el rompimiento de simetría $SU(5)/SO(5)$ y la parametrización de la matriz de Goldstones. Se introduce la T-paridad, que es una simetría Z_2 , y la acción que tiene en los campos y lagrangianas para hacer el modelo consistente. Se muestra como el grupo de gauge $[SU(2) \times U(1)]^2$ que se encuentra dentro del grupo $SU(5)$ se "romperá" al grupo del Modelo Estándar $SU(2)_L \times U(1)_Y$ dando así lugar a la aparición a nuevas partículas en un sector pesado: W_H^\pm , Z_H y A_H y cómo después de considerar efectos de EWSB también recuperamos los bosones de gauge conocidos: W^\pm , Z^0 y A . También se incluye el sector de Fermiones, introduciendo los multipletes de $SU(5)$ que nos originarán el sector pesado de fermiones así como los fermiones del Modelo Estándar, y además se incluye el multiplete correspondiente a fermiones "espejo" los cuales son de vital importancia para el modelo. En el subcapítulo 4.5 se muestran las Reglas de Feynman de LHT más relevantes para los procesos que estamos interesados en estudiar. Y por último se agrega una sección dedicada a la mezcla de sabor T-impar, que trae consigo la aparición de nuevas matrices de mezcla de sabor $V_{Hu}, V_{Hd}, V_{H\nu}, V_{H\ell}$ que satisfacen la matriz CKM para quarks y la matriz PMNS para leptones, pero con un resultado bastante interesante: estas nuevas matrices además de contar con 3 ángulos de mezcla al igual que las matrices CKM y PMNS, no contienen solo una fase física sino contienen 3 fases, lo cual despierta un interés en estudiar que nuevos resultados pueden traer consigo.

Finalmente se incluyen 5 apéndices donde se muestran los cálculos explícitos realizados para obtener los resultados registrados en el Capítulo 4.

Abstract

The Standard Model is one of the most successful theories created by human, it manages to explain basically all the phenomena, processes, interactions that occur at a fundamental level between particles and verified by data experimentally obtained. Therefore, there is a great interest in making extensions of the model to find an explanation for those problems that cannot be solved satisfactorily with this theory.

An important test of SM was the discovery in 2012 by the LHC of a particle proposed by Peter Higgs in 1964 through the called *Higgs Mechanism*, that particle is named in his honor: Higgs Boson.

The Higgs Mechanism is closely related with the electroweak symmetry breaking (EWSB) which is produced at an energy scale $v = 246$ GeV, and it is estimated that the energy scale of new physics is of order $\sim \mathcal{O}(\text{TeV})$.

Different models have been proposed to attempt to explain this difference between the energy scales called *hierarchy problem* and to stabilize the contributions that exist from quadratically divergent diagrams to Higgs mass.

Little Higgs, in one of the models developed to address the problem described above. Little Higgs Models have as their main feature the proposal that the Higgs is a pseudo-Nambu-Goldstone boson caused by the breaking of $SU(5)$ symmetry group at an energy scale $f \sim \mathcal{O}(\text{TeV})$. This idea was taken up by Arkani-Hamed, Cohen and Georgi who built the first successful Little Higgs model [24]. From this, several extensions have been made to the Little Higgs model and in this thesis we will address one of them: Littlest Higgs Model with T-parity.

This model, abbreviated LHT, has several scopes and one of which we set to study is that this model gives new bounds to lepton flavor violation (LFV) processes through of interactions of particles that appear at energy scale of TeV's. The structure of the thesis is described below: in Chapter 1 named Standard Model, we explain the necessary requirements that the Lagrangian of the electroweak standard model must satisfy such as global and local phase symmetry. We know that Quantum Electrodynamics is an Abelian theory, which implies that the mediator, the photon, does not interact itself, but unlike this in the Electroweak theory their gauge bosons, W^{pm} , Z^0 and A can interact with each other, which leads us to work with a non-Abelian theory so it is necessary to know the fields of Yang-Mills, content of section 1.1.3. For the second part of Chapter 1 the Glashow-Weinberg-Salam Model is detailed by introducing the Higgs Mechanism and how it gives rise to the masses of gauge bosons and fermions. We show the shape of the final Lagrangian of this model with a brief comment of each of the parts that constitute it.

For Chapter 2, issues that we will serve to have a better theoretical basis at the time of arriving with the Littlest Higgs Model with T-parity have been addressed. The topics included in the chapter are the Spontaneous Symmetry Breaking, the Linear Sigma Model, the Goldstone Theorem and the non-linear Sigma Model. The choice of these topics is due to the fact that in the LHT model Goldstone bosons appear due to the symmetry breaking as well as being based on a non-linear sigma model.

As stated earlier, the scope of LHT reaches lepton flavor violation processes, so a brief subchapter dedicated to LFV is included.

Chapter 3 is dedicated entirely to the description of Little Higgs models, how they address the so-called hierarchy problem and the two main categories into which these models are divided.

In Chapter 4 the Little Higgs Model with T-parity is developed in detail. This explains how is the symmetry breaking of $SU(5)/SO(5)$ and the parameterization of the Goldstones matrix. The T-parity is introduced, which is a Z_2 symmetry, and the action it has in the Lagrangian and fields to make the model consistent. It shows how the gauge group $[SU(2) \times U(1)]^2$ that is within the $SU(5)$ group will "break" the Standard Model $SU(2)_L \times U(1)_Y$ group and giving place to the appearance of new particles in a heavy sector: W_H^\pm , Z_H and A_H and how after considering EWSB effects we also recover the known gauge bosons: W^\pm , Z^0 and A . The Fermion sector is also included, introducing the $SU(5)$ multiplets that will originate the heavy fermions sector as well as the Standard Model fermions, and also the multiplet corresponding to "mirror" fermions which are of vital importance is included for the model. Section 4.5 shows the Feynman Rules of LHT most relevant to the processes that we are interested in studying. And finally a subchapter dedicated to the T-odd flavor mixing is added, which brings with it the appearance of new V_{Hu} , V_{Hd} , $V_{H\nu}$, $V_{H\ell}$ flavor mixing matrices that satisfy the CKM matrix by quarks and the PMNS matrix by leptons, but with a very interesting result: these new matrices in addition to having 3 mixing angles just like the CKM and PMNS matrices, do not contain only one physical phase but contain 3 phases, which is interesting for the new possible results that could have.

Finally, 5 appendices are included showing the explicit calculations made to obtain the results quoted in Chapter 4.

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1 Standard Model

Particle physics is the branch of physics which studies elementary particles and their properties. We can say that the elementary particles are the fundamental constituents of all objects in the universe. We mean by *elementary particles*: these are particles which do not have any substructure.

It's not easy to tell which particles are elementary. It's possible to say which particles are not elementary, if one knows of some experiment that shows substructure of the object. But no experiment can guarantee that a given object does not have any substructure. A new experiment might show substructure in an object that was earlier considered elementary. For example, protons and neutrons came to be collectively called *nucleons* because they are the constituents of nuclei, and treated as elementary particles. A few decades later, new experiments indicated that the nucleons themselves have substructure. We now believe that they are made up of *quarks*, which are elementary particles like the electrons [1]. Thus the list of elementary particles changes with time. There is no guarantee that today's elementary particles would not turn out to be composite objects tomorrow.

The electron, the proton and the neutron are all examples of *fermions*, which have an intrinsic angular momentum, or *spin*, that is a half-integral multiple of the fundamental constant \hbar . In particular, all three of these have spin equal to $\frac{1}{2}\hbar$. The other alternative is to have spin in integral multiples of \hbar , and particles carrying such spin are called *bosons*.

The Standard Model (SM) constitutes one of the most successful achievements in modern physics. It provides a very elegant theoretical framework, which is able to describe the known experimental facts in particle physics with high precision [2].

According to this model, all matter is built from fermions: six *quarks* and six *leptons*. The leptons carry integral electric charge. The electron e with unit negative charge is familiar to everyone, and the other charged leptons are the *muon* μ and the *tau* τ , these are heavy versions of the electron. In modern terminology, we call the muon a particle of the second *generation*, and the tau belongs to the third generation.

In 1930, in order to explain the continuous spectrum of the electrons in nuclear beta decay, Pauli proposed that a neutral fermion was produced in such processes. Neutral fermions are collectively called *neutrinos*. A different flavour of charged lepton is paired with each flavour of neutrino, as indicated by the subscript: these are called the *electron-neutrino* (ν_e), *muon-neutrino* (ν_μ), and *tau-neutrino* (ν_τ). Summarizing, these neutrinos, along with the electron, the muon and the tau, form a class of elementary particles which are called *leptons*. There is another class of elementary particles which are called *quarks*. Unlike the leptons, no one has seen quarks in their free state. Quarks always appear in bound states. Bound states involving quarks are called *hadrons*. The quarks carry fractional charges. of $\frac{2}{3}e$ or $-\frac{1}{3}e$. For each of the various fundamental constituents, its symbol and the ratio of its electric charge Q to the elementary charge e of the electron are given in Table 1 [3].

Particle	Flavour			Q
leptons	e	μ	τ	-1
	ν_e	ν_μ	ν_τ	0
quarks	u	c	t	$\frac{2}{3}$
	d	s	b	$-\frac{1}{3}$

Table 1: The fundamental fermions.

We have looked at the particles but the SM also comprises their interactions. The different interactions are described in quantum language in terms of the exchange of characteristic *bosons* (particles of integral spin) between the fermions constituents. These boson mediators are listed in Table 2.

Interaction	Mediator	Symbol	Number	Spin (\hbar)/Parity
Strong	Gluon	g	8	1^-
Electromagnetic	Photon	γ	1	1^-
Weak	W and Z bosons	W^\pm, Z^0	3	$1^-, 1^+$
Gravity	Graviton	G	1	2^+

Table 2: The boson mediators.

There are four kinds of fundamental interactions. The oldest known one is the gravitational interaction, known since the time of Newton in the 17th century. It is supposedly mediated by exchange of a spin 2 boson called *graviton*. Electricity and magnetism were unified into the electromagnetic theory, and this is the second kind of interaction that we recognize to be fundamental and these interaction is mediated by *photon* exchange. The third one is the strong interaction, required to explain the stability of the atomic nuclei. The interquark force is mediated by a massless particle, the *gluon*. The last one is the weak interaction, needed to explain the phenomenon of beta radioactivity. The mediators of this interaction are the W^\pm and Z^0 bosons, with masses of order 100 times the proton mass.

To indicate the relative magnitudes of the four types of interaction, the comparative strengths of the force between two protons when just in contact are very roughly as follow. [3]

Strong	Electromagnetic	Weak	Gravity
1	10^{-2}	10^{-7}	10^{-39}

Table 3: Comparative strengths of the force between two protons.

In addition, the standard theory of electroweak interactions postulates that there is a spinless boson. It is called the *Higgs boson*. The discovery of the Higgs boson, announced on July 4th of 2012

by the CERN LHC collaborations, marked the completion of the SM. This event can be considered one of the greatest accomplishments of the High Energy Physics community. The Higgs mass value agrees quite well with the range preferred by the electroweak precision tests (EWPT), which confirms the success of the SM. Current measurements of its spin, parity, and couplings, also seem consistent with the SM. The fact that LHC has verified the linear realization of spontaneous symmetry breaking (SSB), as included in the SM, could also be taken as an indication that Nature likes scalars. [4].

The Standard Model was proposed in 1967, and the discovery of the Higgs boson was announced in 2012, i.e., a long 45 years later. It remained elusive for almost half a century. A big reason for the elusiveness of the Higgs boson is its coupling to fermions. The coupling of the Higgs boson to any fermion is proportional to the mass of the fermion, given by

$$h_f = \frac{\sqrt{2}m_f}{v} = \frac{gm_f}{\sqrt{2}M_W}, \quad (1)$$

with VEV $v = 246$ GeV. Our experimental detectors are made out particles in the first generation of fermions because they are constituents of stable material. The masses of the first generation fermions are much smaller compared to the masses of fermions in the other generations. For the electron, eq. (1) tell us that the coupling is of order 10^{-7} . With the up and the down quarks, the coupling are a little bigger, maybe an order to magnitude. This is the basic reason why the Higgs boson is hard to produce and to detect.

The mass of the Higgs boson is $M_H = 125.10 \pm 0.14$ GeV [5].

1.1 Gauge Invariance

1.1.1 Global phase symmetry

Consider the free Dirac Lagrangian

$$\mathcal{L}_0 = i\bar{\psi}\gamma^\mu\partial_\mu\psi - m\bar{\psi}\psi. \quad (2)$$

This Lagrangian is invariant under a change of phase of the field ψ . Suppose we change over to a new field

$$\psi'(x) = \exp(-ieQ\theta)\psi(x), \quad (3)$$

where e , Q and θ are all real numbers, with different interpretations. The quantity e represents a universal constant which sets up the scale of the phase, Q is a characteristic of the field ψ and θ is a variable which determines how large the phase is. Note that we have not changed the spacetime coordinates at all; the new field is defined in terms of the old field at the same spacetime point. Such symmetries are called *internal symmetries*.

Given the transformation (3), we show that

$$i\bar{\psi}'\gamma^\mu\partial_\mu\psi' - m\bar{\psi}'\psi' = i\bar{\psi}\gamma^\mu\partial_\mu\psi - m\bar{\psi}\psi. \quad (4)$$

Thus, the Lagrangian of eq. (2) is invariant under the transformation given in eq. (3). We can see that θ is independent of the spacetime coordinates, such as are called *global symmetry*.

1.1.2 Local symmetry

We now want to see what happens if the parameter $\theta = \theta(x)$, depends on spacetime coordinates. The derivative of $\psi'(x)$ is

$$\partial_\mu\psi'(x) = \exp(-ieQ\theta) [\partial_\mu\psi(x) - ieQ(\partial_\mu\theta)\psi(x)]. \quad (5)$$

There is an extra term, involving the derivatives of θ . The Lagrangian of eq. (2) is not invariant under this local transformation. We obtain then

$$\mathcal{L}'_0 - \mathcal{L}_0 = eQ(\partial_\mu\theta)\bar{\psi}(x)\gamma^\mu\psi(x). \quad (6)$$

Since we want the local symmetry, we would need to modify the original Lagrangian. So, instead of the Lagrangian of eq. (2), let us try

$$\mathcal{L} = i\bar{\psi}\gamma^\mu D_\mu\psi - m\bar{\psi}\psi, \quad (7)$$

where

$$D_\mu = \partial_\mu + ieQA_\mu, \quad (8)$$

bringing in a new A_μ . This D_μ is usually called the *covariant derivative*. The prescription of replacing ordinary derivatives by covariant derivatives is called *minimal substitution* [1].

The new Lagrangian must be invariant under the local symmetry, thus, under the local symmetry, A_μ changes to A'_μ such that

$$\mathcal{L}'_0 - eQ\bar{\psi}'\gamma^\mu\psi'A'_\mu = \mathcal{L}_0 - eQ\bar{\psi}\gamma^\mu\psi A_\mu, \quad (9)$$

or

$$A'_\mu = A_\mu + \partial_\mu\theta. \quad (10)$$

We can thus identify A_μ with the photon field and conclude that, in the urge for making the global phase symmetry local, we have introduced the photon field in a natural form. If we are to regard this new field as the physical photon field, we must add to the Lagrangian a term corresponding to its kinetic energy. Since the kinetic term must be invariant under (10), it can only involve the gauge invariant field strength tensor

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu. \quad (11)$$

We are thus led to the Lagrangian of QED [6]

$$\mathcal{L} = \bar{\psi}(i\gamma^\mu\partial_\mu - m)\psi + e\bar{\psi}\gamma^\mu A_\mu\psi - \frac{1}{4}F_{\mu\nu}F^{\mu\nu}. \quad (12)$$

Note that the addition of a mass term $\frac{1}{2}m^2 A_\mu A^\mu$ is prohibited by gauge invariance. The gauge particle, the photon, must be massless.

Local symmetries are also called *gauge symmetries*. Theories incorporating gauge symmetries are called *gauge theories*. The spin-1 particles which are necessary to keep the gauge invariance are called *gauge bosons*. QED is therefore a gauge theory, based on the gauge group $U(1)$. The gauge boson for QED is the photon.

1.1.3 Non-Abelian gauge symmetry: Yang-Mills fields

In 1954 Yang and Mills extended the gauge principle to non-Abelian symmetry. We are going to illustrate the construction for the simplest case of isospin $SU(2)$ [7].

Let the fermion field be an isospin doublet,

$$\psi = \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix}. \quad (13)$$

Under an $SU(2)$ transformation, we have

$$\psi(x) \rightarrow \psi'(x) = \exp\left(\frac{-i\vec{\tau} \cdot \vec{\theta}}{2}\right) \psi(x), \quad (14)$$

where $\vec{\tau} = (\tau_1, \tau_2, \tau_3)$ are the Pauli matrices, satisfying

$$\left[\frac{\tau_i}{2}, \frac{\tau_j}{2}\right] = i\epsilon_{ijk} \frac{\tau_k}{2} \quad i, j, k = 1, 2, 3, \quad (15)$$

and $\vec{\theta} = (\theta_1, \theta_2, \theta_3)$ are the $SU(2)$ transformation parameters. The free Lagrangian

$$\mathcal{L}_0 = \bar{\psi}(x)(i\gamma^\mu\partial_\mu - m)\psi(x), \quad (16)$$

is invariant under the global $SU(2)$ symmetry with $\{\theta_i\}$ being spacetime independent. However under the local symmetry transformation

$$\psi(x) \rightarrow \psi'(x) = U(\theta)\psi(x), \quad (17)$$

with

$$U(\theta) = \exp\left(\frac{-i\vec{\tau} \cdot \vec{\theta}(x)}{2}\right), \quad (18)$$

the free Lagrangian \mathcal{L}_0 is no longer invariant because the derivative term transforms as

$$\begin{aligned} \bar{\psi}(x)\partial_\mu\psi(x) &\rightarrow \bar{\psi}'(x)\partial_\mu\psi'(x) = \bar{\psi}(x)\partial_\mu\psi(x) \\ &\quad + \bar{\psi}(x)U^{-1}(\theta) [\partial_\mu U(\theta)] \psi(x). \end{aligned} \quad (19)$$

To construct a gauge invariant Lagrangian we follow a procedure similar to that of the Abelian case. We introduce the vector gauge field A_μ^i , $i = 1, 2, 3$ (one for each group generator) to form the gauge covariant derivative through the minimal coupling

$$D_\mu\psi = \left(\partial_\mu - ig\frac{\vec{\tau} \cdot \vec{A}_\mu}{2} \right) \psi, \quad (20)$$

where g is the coupling constant. We want that $D_\mu\psi$ have the same transformation property as ψ itself

$$D_\mu\psi \rightarrow (D_\mu\psi)' = U(\theta)D_\mu\psi. \quad (21)$$

This implies that

$$\begin{aligned} \left(\partial_\mu - ig\frac{\vec{\tau} \cdot \vec{A}'_\mu}{2} \right) (U(\theta)\psi) &= U(\theta) \left(\partial_\mu - ig\frac{\vec{\tau} \cdot \vec{A}_\mu}{2} \right) \psi, \\ \frac{\vec{\tau} \cdot \vec{A}'_\mu}{2} &= U(\theta)\frac{\vec{\tau} \cdot \vec{A}_\mu}{2}U^{-1}(x) - \frac{i}{g} [\partial_\mu U(\theta)] U^{-1}(\theta), \end{aligned} \quad (22)$$

which defines the transformation law for the gauge fields. For an infinitesimal change $\vec{\theta}(x) \ll 1$,

$$\begin{aligned} \frac{\vec{\tau} \cdot \vec{A}'_\mu}{2} &= \frac{\vec{\tau} \cdot \vec{A}_\mu}{2} - i\theta^j A_\mu^k \left[\frac{\tau_j}{2}, \frac{\tau_k}{2} \right] - \frac{1}{g} \left(\frac{\vec{\tau}}{2} \cdot \partial_\mu \vec{\theta} \right) \\ &= \frac{\vec{\tau} \cdot \vec{A}_\mu}{2} + \frac{1}{2} \epsilon^{ijk} \tau^i \theta^j A_\mu^k - \frac{1}{g} \left(\frac{\vec{\tau}}{2} \cdot \partial_\mu \vec{\theta} \right), \end{aligned}$$

or

$$A_\mu^i{}' = A_\mu^i + \epsilon^{ijk} \theta^j A_\mu^k - \frac{1}{g} \partial_\mu \theta^i. \quad (23)$$

The second term is the transformation for a triplet representation under $SU(2)$. Thus the A_μ^i s carry charge, in contrast to the Abelian gauge field.

To obtain the antisymmetric second-rank tensor of the gauge fields we can study the combination

$$[D_\mu, D_\nu] \psi \equiv ig \left(\frac{\vec{\tau}^i}{2} F_{\mu\nu}^i \right) \psi, \quad (24)$$

with

$$\frac{\vec{\tau} \cdot \vec{F}_{\mu\nu}}{2} = \partial_\mu \frac{\vec{\tau} \cdot \vec{A}_\nu}{2} - \partial_\nu \frac{\vec{\tau} \cdot \vec{A}_\mu}{2} - ig \left[\frac{\vec{\tau} \cdot \vec{A}_\mu}{2}, \frac{\vec{\tau} \cdot \vec{A}_\nu}{2} \right], \quad (25)$$

where

$$F_{\mu\nu}^i = \partial_\mu A_\nu^i - \partial_\nu A_\mu^i + g\epsilon^{ijk} A_\mu^j A_\nu^k. \quad (26)$$

From the fact that $D_\mu\psi$ has the same gauge transformation property as ψ , we see that

$$([D_\mu, D_\nu]\psi)' = U(\theta) [D_\mu, D_\nu]\psi, \quad (27)$$

substituting (24) on both sides of (27), we have

$$\vec{\tau} \cdot \vec{F}'_{\mu\nu} = U(\theta)(\vec{\tau} \cdot \vec{F}_{\mu\nu})U^{-1}(\theta). \quad (28)$$

For the infinitesimal transformation $\theta_i \ll 1$, this translates into

$$F_{\mu\nu}^{i'} = F_{\mu\nu}^i + \epsilon^{ijk}\theta^j F_{\mu\nu}^k. \quad (29)$$

The $F_{\mu\nu}^i$ transforms nontrivially, however, the product follow is gauge invariant

$$\text{tr} \left[(\vec{\tau} \cdot \vec{F}_{\mu\nu})(\vec{\tau} \cdot \vec{F}^{\mu\nu}) \right] \propto F_{\mu\nu}^i F^{i\mu\nu}. \quad (30)$$

The complete gauge invariant Lagrangian which describes the interaction between gauge fields A_μ^i and $SU(2)$ doublet fields

$$\mathcal{L} = \bar{\psi}i\gamma^\mu D_\mu\psi - m\bar{\psi}\psi - \frac{1}{4}F_{\mu\nu}^i F^{i\mu\nu}, \quad (31)$$

where $F_{\mu\nu}^i$ is given by (26) and $D_\mu\psi$ is given by (20). The pure Yang-Mills term, $-\frac{1}{4}F_{\mu\nu}^i F^{i\mu\nu}$, contains factors that are trilinear and quadrilinear in A_μ^i , which correspond to self-couplings of non-Abelian gauge fields.

1.2 Standard Electroweak Theory - The Weinberg-Salam Model

The SM is a gauge theory, based on the symmetry group $SU(3)_C \otimes SU(2)_L \otimes U(1)_Y$, which describes strong, weak and electromagnetic interactions, via the exchange of the corresponding spin-1 gauge fields: eight massless gluons and one massless photon, respectively, for the strong and electromagnetic interactions, and three massive bosons, W^\pm and Z^0 , for the weak interaction.

The gauge symmetry is broken by the vacuum, which triggers the SSB of the electroweak group to the electromagnetic subgroup

$$SU(3)_C \otimes SU(2)_L \otimes U(1)_Y \xrightarrow{SSB} SU(3)_C \otimes U(1)_{QED}. \quad (32)$$

The SSB mechanism generates the masses of the weak gauge bosons, and gives rise to the appearance of a physical scalar particle in the model, the so-called Higgs. The fermion masses and mixings are also generated through the SSB [2].

1.2.1 Higgs mechanism

We start with a Lagrangian with a local $U(1)$ symmetry

$$\mathcal{L} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + (D^\mu\phi)^\dagger(D_\mu\phi) - \mu^2\phi^\dagger\phi - \lambda(\phi^\dagger\phi)^2, \quad (33)$$

where $\mu^2 < 0$. The gauge covariant derivative D_μ is defined by

$$D_\mu = \partial_\mu + ieA_\mu. \quad (34)$$

The covariant derivative (34) is different to (8) one, the difference is a Q factor which is a characteristic of the field so we can omit it without to lose generality. Spontaneous symmetry breaking can occur if $\mu^2 < 0$, for which the minimum is obtained if

$$|\langle 0|\phi|0\rangle| = \frac{v}{\sqrt{2}}, \quad (35)$$

where

$$v = \sqrt{\frac{-\mu^2}{\lambda}}. \quad (36)$$

Only the magnitud of the VEV is determined by the minimization condition. The phase can be arbitrary. Let us assume that the vacuum is where the phase value is zero, the VEV of ϕ is $\frac{v}{\sqrt{2}}$. We can say that $\phi(x)$ can not be the quantum field in this case. We should write [8]

$$\phi(x) = \frac{1}{\sqrt{2}}(v + \eta(x) + i\zeta(x)), \quad (37)$$

where $\eta(x)$ and $\zeta(x)$ have zero VEVs. Using (37) in kinetic term of (33)

$$(D^\mu\phi)^\dagger(D_\mu\phi) = \frac{1}{2}(\partial^\mu\eta)(\partial_\mu\eta) + \frac{1}{2}(\partial^\mu\zeta)(\partial_\mu\zeta) + evA^\mu\partial_\mu\zeta + \frac{1}{2}e^2v^2A_\mu A^\mu + \dots. \quad (38)$$

We see that the $U(1)$ gauge boson has a mass term after the symmetry is broken, the mass is given by

$$M_A = ev. \quad (39)$$

But before we conclude that the gauge boson is massive, there is a problem to settle. There is a term $A^\mu\partial_\mu\zeta$ in (38). Since it is quadratic in the fields, so it should be part of the free Lagrangian of the system. However, it contains two different fields, A^μ and ζ , involving a derivative. If written in terms of creation and annihilation operators, this would imply that we can have Feynman diagrams in which an A^μ changes into a ζ , without any other particle interacting with them, and vice versa.

The eq. (37) is not the only way that ϕ can be written in terms of quantum fields. Alternatively we

could have written

$$\phi(x) = \frac{v + \eta(x)}{\sqrt{2}} e^{i\zeta(x)/v}. \quad (40)$$

If we keep only up to linear terms in quantum fields, this representation coincides with eq. (37). Here $\zeta(x)/v$ is the phase of the field ϕ , we know that any phase, even if it is a spacetime dependent one, is irrelevant because of the local $U(1)$ symmetry. Explicitly, the Lagrangian (33) is invariant under the transformations

$$\begin{aligned} \phi(x) &\rightarrow \phi'(x) = e^{-ie\theta(x)}\phi(x), \\ A_\mu(x) &\rightarrow A'_\mu(x) = A_\mu(x) + \partial_\mu\theta(x). \end{aligned} \quad (41)$$

Therefore, given the representation of the field $\phi(x)$ as in eq. (40), we can always choose a new gauge through eq. (41). In particular, if we choose $\theta(x) = \zeta(x)/ev$, the gauge transformed fields are given by

$$\begin{aligned} \phi'(x) &= \frac{v + \eta(x)}{\sqrt{2}}, \\ A'_\mu(x) &= A_\mu(x) + \frac{1}{ev}\partial_\mu\zeta(x). \end{aligned} \quad (42)$$

If we substitute these primed fields instead of the unprimed fields, the $\zeta(x)$ field disappears altogether from the Lagrangian. This Lagrangian then contains physical fields only, and the gauge in which this takes place is called the *unitary gauge*.

1.2.2 Higgs boson multiplet

We introduce a scalar multiplet which is a doublet under $SU(2)$ part of the gauge group, and write this as

$$\phi \begin{pmatrix} \phi^+ \\ \phi_0 \end{pmatrix} : (2, \frac{1}{2}). \quad (43)$$

The "2" indicates that it is a doublet of $SU(2)$, and we have normalized the $U(1)$ charge such that its value is $\frac{1}{2}$ for the multiplet ϕ .

The Lagrangian now contains terms involving ϕ as well. These are

$$\mathcal{L}_\phi = (D^\mu\phi)^\dagger(D_\mu\phi) - \mu^2\phi^\dagger\phi - \lambda(\phi^\dagger\phi)^2. \quad (44)$$

In general, when a multiplet transforms like an n -dimensional representation of $SU(2)$ and has a $U(1)$ quantum number Y , we should write

$$D_\mu = \partial_\mu + igT_a^{(n)}W_\mu^a + ig'YB_\mu, \quad (45)$$

where $T_a^{(n)}$ denote the generator of $SU(2)$ in the n -dimensional representation, and Y is the identity matrix times the hypercharge. For the doublet ϕ , the $SU(2)$ generators are $\tau_a/2$ (Pauli matrices), and we can write

$$\begin{aligned} D_\mu \phi &= (\partial_\mu + ig \frac{\tau_a}{2} W_\mu^a + i \frac{g'}{2} B_\mu) \phi \\ &= \partial_\mu \begin{pmatrix} \phi^+ \\ \phi_0 \end{pmatrix} + \frac{i}{2\sqrt{2}} \begin{pmatrix} gW_\mu^3 + g'B_\mu & g(W_\mu^1 - iW_\mu^2) \\ g(W_\mu^1 + iW_\mu^2) & -gW_\mu^3 + g'B_\mu \end{pmatrix} \begin{pmatrix} \phi^+ \\ \phi_0 \end{pmatrix}. \end{aligned} \quad (46)$$

Let us now suppose that μ^2 , the gauge symmetry will be spontaneously broken. The minimum for the scalar potential will be obtained for scalar field configurations given by

$$\phi_0 = \frac{v}{\sqrt{2}} \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \quad (47)$$

and $v = \sqrt{-\mu^2/\lambda}$. Any other vacuum satisfying eq. (47) can be reached from this by a global $SU(2) \times U(1)$ transformation. In this vacuum state, the generators T_1 and T_2 are no more part of the symmetry, since $\tau_{x,y} \phi_0 = (T_1 \pm iT_2) \phi_0 \neq 0$. The diagonal generators T_3 and the $U(1)$ generator Y also do not annihilate the vacuum state. However, notice that

$$\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 0 \\ v/\sqrt{2} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}. \quad (48)$$

Thus there is one diagonal generator which annihilates the vacuum. This is a linear combination of T_3 and Y , given by

$$Q = T_3 + Y : Q\phi_0 = 0. \quad (49)$$

The original gauge symmetry is therefore broken down to a $U(1)$ symmetry generated by Q . This is not the original $U(1)$ part of the symmetry group. As we have shown, its generator is actually a combination of an $SU(2)$ generator and the original $U(1)$ generator. To make this distinction, we will write the original symmetry from now as $SU(2) \times U(1)_Y$, whereas the remnant symmetry will be called $U(1)_Q$. This latter is in fact the electromagnetic gauge symmetry.

1.2.3 Gauge boson masses

Since ϕ_0 cannot be described by creation and annihilation operators, we can write

$$\phi(x) = \begin{pmatrix} \phi^+ \\ \frac{v+H(x)+i\zeta(x)}{\sqrt{2}} \end{pmatrix}, \quad (50)$$

where ϕ^+ , H and ζ are all quantum fields with vanishing expectation values in the vacuum state. The gauge boson masses are identified by substituting the vacuum expectation value ϕ_0 for $\phi(x)$ in the

Lagrangian (44). The relevant term in (44) is

$$\begin{aligned}
 \left| \left(ig \frac{\tau_a}{2} W_\mu^a + i \frac{g'}{2} B_\mu \right) \phi_0 \right|^2 &= \frac{1}{8} \left| \begin{pmatrix} gW_\mu^3 + g'B_\mu & g(W_\mu^1 - iW_\mu^2) \\ g(W_\mu^1 + iW_\mu^2) & -gW_\mu^3 + g'B_\mu \end{pmatrix} \begin{pmatrix} 0 \\ v \end{pmatrix} \right|^2 \\
 &= \frac{1}{8} v^2 g^2 [(W_\mu^1)^2 + (W_\mu^2)^2] + \frac{1}{8} v^2 (g'B_\mu - gW_\mu^3)(g'B_\mu - gW_\mu^3) \\
 &= \left(\frac{1}{2} v g \right)^2 W_\mu^+ W_\mu^{-\mu} + \frac{1}{8} v^2 \begin{pmatrix} W_\mu^3 & B_\mu \end{pmatrix} \begin{pmatrix} g^2 & -gg' \\ -gg' & g'^2 \end{pmatrix} \begin{pmatrix} W_\mu^3 \\ B_\mu \end{pmatrix},
 \end{aligned} \tag{51}$$

since $W^\pm = (W^1 \mp iW^2)/\sqrt{2}$. For any charged spin-1 field V_μ of mass M , the mass term in the Lagrangian is $MV_\mu^\dagger V^\mu$. Thus, the mass of the charged W boson is

$$M_W = \frac{1}{2} v g. \tag{52}$$

The remaining term is off-diagonal in the W_μ^3 and B_μ basis

$$\begin{aligned}
 \frac{1}{8} v^2 [g^2 (W_\mu^3)^2 - 2gg' W_\mu^3 B_\mu + g'^2 B_\mu^2] &= \frac{1}{8} v^2 [gW_\mu^3 - g'B_\mu]^2 \\
 &\quad + 0 [gW_\mu^3 + g'B_\mu]^2.
 \end{aligned} \tag{53}$$

One of the eigenvalues of the 2×2 matrix in (51) is zero, we have included this term in (53) with a combination of fields that is orthogonal to the combination given in the first term. Now, the physical fields Z_μ and A_μ diagonalize the mass matrix so that (53) must be identified with

$$\frac{1}{2} M_Z^2 Z_\mu^2 + \frac{1}{2} M_A^2 A_\mu^2. \tag{54}$$

So, on normalizing the fields, we have [6]

$$\begin{aligned}
 A_\mu &= \frac{g'W_\mu^3 + gB_\mu}{\sqrt{g^2 + g'^2}} && \text{with } M_A = 0, \\
 Z_\mu &= \frac{gW_\mu^3 - g'B_\mu}{\sqrt{g^2 + g'^2}} && \text{with } M_Z = \frac{1}{2} v \sqrt{g^2 + g'^2}.
 \end{aligned} \tag{55}$$

We can define the combinations

$$\begin{aligned}
 Z_\mu &= \cos\theta_W W_\mu^3 - \sin\theta_W B_\mu, \\
 A_\mu &= \sin\theta_W W_\mu^3 + \cos\theta_W B_\mu,
 \end{aligned} \tag{56}$$

where θ_W is called the *Weinberg angle*, defined by

$$\frac{M_W}{M_Z} = \cos\theta_W \quad \text{or} \quad \frac{g'}{g} = \tan\theta_W. \quad (57)$$

The inequality $M_Z \neq M_W$ is due to the mixing between the W_μ^3 and B_μ fields. The mass eigenstates are then automatically a massless photon (A_μ) and a massive (Z_μ) field with $M_Z > M_W$.

1.2.4 Masses of the Fermions

An attractive feature of the SM is that the same Higgs doublet which generates W^\pm and Z masses is also sufficient to give masses to the leptons and quarks. For example, to generate the electron mass, we include the following $SU(2) \times U(1)$ gauge invariant term in the Lagrangian

$$\mathcal{L} = -G_e \left[(\bar{\nu}_e \quad \bar{e})_L \begin{pmatrix} \phi^+ \\ \phi^0 \end{pmatrix} e_R + \bar{e}_R (\phi^- \quad \bar{\phi}^0)_L \begin{pmatrix} \nu_e \\ e \end{pmatrix}_L \right]. \quad (58)$$

The Higgs doublet has exactly the required $SU(2) \times U(1)$ quantum numbers to couple to $\bar{e}_L e_R$. We spontaneously break the symmetry and substitute

$$\phi = \sqrt{\frac{1}{2}} \begin{pmatrix} 0 \\ v + H(x) \end{pmatrix} \quad (59)$$

into (58). The neutral Higgs field $H(x)$ is the only remnant of the Higgs doublet, after the spontaneous breaking has taken place. On substitution of ϕ , the Lagrangian becomes

$$\mathcal{L} = -\frac{G_e}{\sqrt{2}} v (\bar{e}_L e_R + \bar{e}_R e_L) - \frac{G_e}{\sqrt{2}} (\bar{e}_L e_R + \bar{e}_R e_L) h. \quad (60)$$

We now choose G_e so that

$$m_e = \frac{G_e v}{\sqrt{2}}, \quad (61)$$

and hence generate the required electron mass,

$$\mathcal{L} = -m_e \bar{e} e - \frac{m_e}{v} \bar{e} e h. \quad (62)$$

Note however that, since G_e is arbitrary, the actual mass of the electron is not predicted.

The quark masses are generated in the same way. The only novel feature is that to generate a mass for the upper member of a quark doublet, we must construct a new Higgs doublet from ϕ

$$\phi_c = i\tau_2 \phi^* = \begin{pmatrix} -\bar{\phi}^0 \\ \phi^- \end{pmatrix} \xrightarrow{\text{breaking}} \sqrt{\frac{1}{2}} \begin{pmatrix} v + H(x) \\ 0 \end{pmatrix}. \quad (63)$$

Due to the special properties of $SU(2)$, ϕ_c transforms identically to ϕ . It can therefore be used to construct a gauge invariant contribution to the Lagrangian

$$\begin{aligned}\mathcal{L} &= -G_d \begin{pmatrix} \bar{u} & \bar{d} \end{pmatrix}_L \begin{pmatrix} \phi^+ \\ \phi^0 \end{pmatrix} d_R - G_u \begin{pmatrix} \bar{u} & \bar{d} \end{pmatrix}_L \begin{pmatrix} -\bar{\phi}^0 \\ \phi^- \end{pmatrix} u_R + h.c. \\ &= -m_d \bar{d}d - m_u \bar{u}u - \frac{m_d}{v} \bar{d}dh - \frac{m_u}{v} \bar{u}uh.\end{aligned}\quad (64)$$

Here, we have just considered the $(u \ d)_L$ quark doublet. However, weak interactions operate on $(u \ d')_L, (c \ s')_L, \dots$ doublets, where the primed states are linear combinations of the flavor eigenstates. The quark Lagrangian is therefore of the form

$$\mathcal{L} = -G_d^{ij} \begin{pmatrix} \bar{u}_i & \bar{d}'_i \end{pmatrix}_L \begin{pmatrix} \phi^+ \\ \phi^0 \end{pmatrix} d_{jR} - G_u^{ij} \begin{pmatrix} \bar{u}_i & \bar{d}'_i \end{pmatrix}_L \begin{pmatrix} -\bar{\phi}^0 \\ \phi^- \end{pmatrix} u_{jR} + h.c., \quad (65)$$

with $i, j = 1, \dots, N$, where N is the number of quarks doublets. We can rewrite the quark Lagrangian in diagonal form

$$\mathcal{L} = -m_d^i \bar{d}'_i d_i \left(1 + \frac{h}{v}\right) - m_u^i \bar{u}_i u_i \left(1 + \frac{h}{v}\right). \quad (66)$$

1.2.5 The Final Lagrangian

To summarize the standard (Weinberg-Salam) model, we gather together all the ingredients of the Lagrangians. The complete Lagrangian is given by Figure 1 [6].

$$\begin{aligned}\mathcal{L} &= -\frac{1}{4} \mathbf{W}_{\mu\nu} \cdot \mathbf{W}^{\mu\nu} - \frac{1}{4} B_{\mu\nu} B^{\mu\nu} && \left\{ \begin{array}{l} \mathbf{W}^\pm, \mathbf{Z}, \gamma \text{ kinetic} \\ \text{energies and} \\ \text{self-interactions} \end{array} \right. \\ &+ \bar{L} \gamma^\mu \left(i \partial_\mu - g \frac{1}{2} \boldsymbol{\tau} \cdot \mathbf{W}_\mu - g' \frac{Y}{2} B_\mu \right) L && \left\{ \begin{array}{l} \text{lepton and quark} \\ \text{kinetic energies} \\ \text{and their} \\ \text{interactions with} \\ \mathbf{W}^\pm, \mathbf{Z}, \gamma \end{array} \right. \\ &+ \bar{R} \gamma^\mu \left(i \partial_\mu - g' \frac{Y}{2} B_\mu \right) R && \\ &+ \left| \left(i \partial_\mu - g \frac{1}{2} \boldsymbol{\tau} \cdot \mathbf{W}_\mu - g' \frac{Y}{2} B_\mu \right) \phi \right|^2 - V(\phi) && \left\{ \begin{array}{l} \mathbf{W}^\pm, \mathbf{Z}, \gamma, \text{ and Higgs} \\ \text{masses and} \\ \text{couplings} \end{array} \right. \\ &- (G_1 \bar{L} \phi R + G_2 \bar{L} \phi_c R + \text{hermitian conjugate}). && \left\{ \begin{array}{l} \text{lepton and quark} \\ \text{masses and} \\ \text{coupling to Higgs} \end{array} \right.\end{aligned}$$

Figure 1: The complete Lagrangian of the SM.

L denotes a left-handed fermion (lepton or quark) doublet, and R denotes a right-handed fermion singlet.

2 Background

We briefly review some topics related to spontaneous symmetry breaking in this chapter, as this will be needed in this Thesis.

2.1 Spontaneous Symmetry Breaking

We can do a general comment before starting to describe the SSB: if SSB occurs due to the expectation value of a vector field, the ground state must prefer a certain direction in spacetime, which would mean a breaking of Lorentz invariance as well. The same is true if, instead of a vector field, any non-trivial representation of the Lorentz group is used. In this case, we can consider non-trivial ground state configuration for scalar fields only [1], because we always want to deal with Lorentz invariant theories. We begin first with a SSB in classical field theory. Consider a Lagrangian with only one real scalar field as follows

$$\mathcal{L} = \frac{1}{2} (\partial^\mu \phi) (\partial_\mu \phi) - \frac{1}{2} \mu^2 \phi^2 - \frac{\lambda}{4} \phi^4, \quad (67)$$

We can see clearly that this Lagrangian is Z_2 -symmetric

$$\phi \rightarrow -\phi. \quad (68)$$

This Lagrangian has two parameters: μ^2 and λ , which need to be real in order for the Lagrangian to be hermitian. We can write the Lagrangian above as

$$\mathcal{L} = \frac{1}{2} (\partial^\mu \phi) (\partial_\mu \phi) - V(\phi), \quad (69)$$

where

$$V(\phi) = \frac{1}{2} \mu^2 \phi^2 + \frac{\lambda}{4} \phi^4, \quad (70)$$

$V(\phi)$ is called the *potential* of the theory, and the derivative terms are sometimes called the *kinetic* terms.

If λ were negative, the Hamiltonian would become more and more negative with increasing values of ϕ . This is unphysical because it would not have any ground state. Thus, for a physically viable system, we must have $\lambda > 0$.

Now, we ask us, what happens with μ^2 ?. If μ^2 is positive, it can be interpreted as the square of the mass μ of the particle whose field is ϕ . For answering what happens when $\mu^2 < 0$ first we are going to find the value ϕ in the ground state. The minimum of $V(\phi)$ should occur where

$$\frac{\partial V}{\partial \phi} = \phi (\mu^2 + \lambda \phi^2) = 0, \quad (71)$$

the solutions of this equation are $\langle\phi\rangle = 0$ and $\pm v$, where

$$v = \sqrt{\frac{-\mu^2}{\lambda}}. \quad (72)$$

If $\mu^2 > 0$, the only real solution is $\langle\phi\rangle = 0$. When $\mu^2 < 0$, all three are real solutions, as seen from Figure 2, but the solution at vanishing field value is really a local maximum. The minima of the system occur for $\langle\phi\rangle = \pm v$, which means that there are two degenerate minima, and either of them can correspond to the ground state of the system. The constant v is called the *vacuum expectation value (VEV)* of ϕ . The field ϕ cannot be treated as a quantum field.

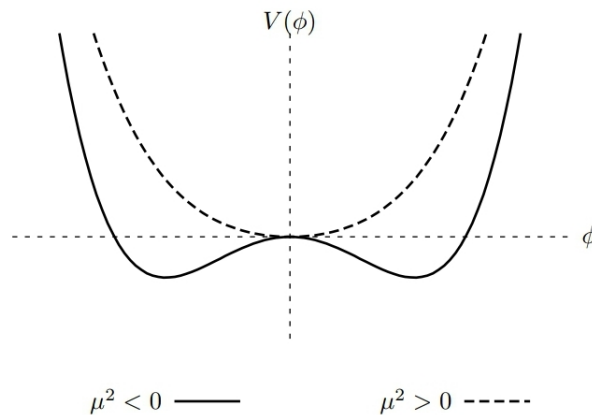


Figure 2: Shape of the potential of eq. (70).

There is an easy way out of this problem. To interpret this theory, suppose that the system is near one of the minima, say the $+v$, we can define

$$\phi(x) = v + \sigma(x), \quad (73)$$

where $\langle\sigma\rangle = 0$, so that it is a quantum field. We say that it represents the fluctuation of the field $\phi(x)$ around its minima denoted by v . We should rewrite the Lagrangian in terms of $\sigma(x)$ field. Eliminating the parameter μ^2 with the use of the definition of v , we obtain

$$\mathcal{L} = \frac{1}{2} (\partial_\mu \sigma) (\partial^\mu \sigma) - \lambda v^2 \sigma^2 - \lambda v \sigma^3 - \frac{\lambda}{4} \sigma^4, \quad (74)$$

apart from a constant proportional to v^4 which is of no importance. The coefficient of σ^2 term indicates that the σ field has a mass M_ϕ given by

$$M_\phi^2 = 2\lambda v^2. \quad (75)$$

In the new Lagrangian there is a cubic term, which means that the Lagrangian is not invariant under the transformation

$$\sigma \rightarrow -\sigma. \quad (76)$$

Because we selected one of the two minima we lose the symmetry. This is the essence of the phenomenon called *spontaneous symmetry breaking*.

2.2 The Linear Sigma Model

A more interesting theory arises when the broken symmetry is continuous, rather than discrete. The most important example is a generalization of the preceding theory called the *linear sigma model* [9]. The Lagrangian of the linear sigma model involves a set of N real scalar fields $\phi^i(x)$

$$\mathcal{L} = \frac{1}{2} (\partial_\mu \phi^i)^2 + \frac{1}{2} \mu^2 (\phi^i)^2 - \frac{\lambda}{4} [(\phi^i)^2]^2, \quad (77)$$

with an implicit sum over i in each factor $(\phi^i)^2$. The above Lagrangian is invariant under the symmetry

$$\phi^i \rightarrow R^{ij} \phi^j, \quad (78)$$

for any $N \times N$ orthogonal matrix R , this group of transformations is just the rotation group N -dimensional orthogonal group, or simply $O(N)$.

The ground state is a constant field ϕ_0^i , whose value is chosen to minimize the potential

$$V(\phi^i) = -\frac{1}{2} \mu^2 (\phi^i)^2 + \frac{\lambda}{4} [(\phi^i)^2]^2. \quad (79)$$

This potential is minimized for any ϕ_0^i that satisfies

$$(\phi_0^i)^2 = \frac{\mu^2}{\lambda}, \quad (80)$$

whose potential is represented in Figure 3.

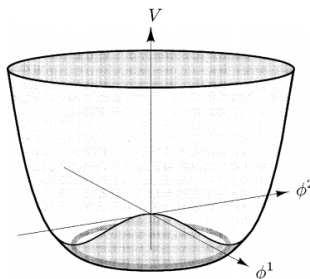


Figure 3: Potential for SSB of a $O(N)$ symmetry for the case $N = 2$. Oscillations along the trough in the potential correspond to the massless ϕ fields.

The ϕ_0^i points in the Nth direction

$$\phi_0^i = (0, 0, \dots, 0, v), \quad (81)$$

where $v = \frac{\mu}{\sqrt{\lambda}}$. Then, we can define a set of shifted fields as follow

$$\phi^i(x) = (\pi^k(x), v + \sigma(x)), \quad k = 1, \dots, N - 1. \quad (82)$$

Rewriting the Lagrangian (77) in terms of the π and σ fields

$$\begin{aligned} \mathcal{L} = & \frac{1}{2}(\partial_\mu \pi^k)^2 + \frac{1}{2}(\partial_\mu \sigma)^2 - \frac{1}{2}(2\mu^2)\sigma^2 \\ & - \sqrt{\lambda}\mu\sigma^3 - \sqrt{\lambda}\mu(\pi^k)^2\sigma - \frac{\lambda}{4}\sigma^4 - \frac{\lambda}{2}(\pi^k)^2\sigma^2 - \frac{\lambda}{4}[(\pi^k)^2]^2. \end{aligned} \quad (83)$$

In the last Lagrangian we get a massive σ field, and also a set of $N - 1$ massless π fields. The original $O(N)$ symmetry is hidden, leaving only the subgroup $O(N - 1)$, which rotates the π fields among themselves. For the massive σ field describes oscillations of ϕ^i in the radial direction, in which the potential has a nonvanishing second derivative. The massless π fields describe oscillations of ϕ^i in the tangential directions, along the trough of the potential. This is represented in a cartoon in the Figures 4 and 5.

2.3 Goldstone's Theorem

A feature of the above Lagrangian after SSM, it has a mass term for the σ field, but none for the π fields, it's an example of *Goldstone theorem*.

The appearance of massless particles when a continuous symmetry is spontaneously broken is a general result, know as *Goldstone theorem*. The theorem states:

"If a Lagrangian is invariant under a continuous symmetry group that has n generators, and if its ground state is symmetric under a continuous group containing n' generators, there should be $n - n'$ massless states in the spectrum of the theory [1]"

The massless state is called a *Nambu-Goldstone boson* associated with the symmetry breaking. This boson need not be an observable state [8].

Goldstone theorem predicts that if a continuous symmetry is spontaneously broken, there will be states whose energies go to zero when the 3-momentum goes to zero.

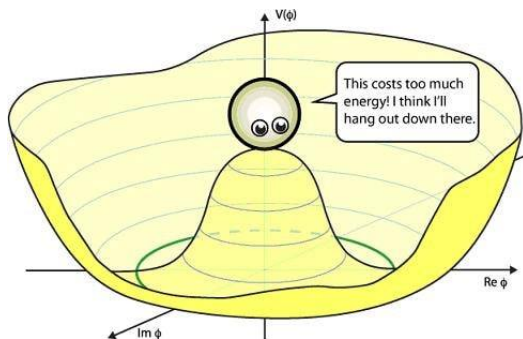


Figure 4: Cartoon that represented a massive σ field after SSB.

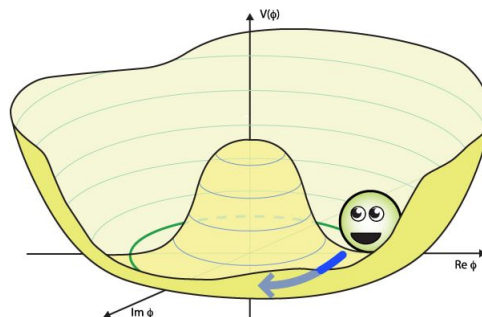


Figure 5: Cartoon that represented a Nambu-Goldstone boson after SSB.

2.4 The Nonlinear Sigma Model

The nonlinear sigma model theory was first proposed as an alternative description of SSB. It will be interesting to us for three reasons [9]:

1. It provides a simple explicit example of an asymptotically free theory.
2. It will give us a second dimensional expansion with which we can study the Wilson-Fisher fixed point. Then we can see where the Wilson-Fisher fixed point goes in the space of Lagrangians for dimensions d well below 4.
3. We will show that the nonlinear sigma model is exactly solvable in a limit that is different from the standard weak-coupling limit.

2.4.1 The $d=2$ Nonlinear Sigma Model

In $d = 2$, a scalar field is dimensionless; thus, any theory of scalar fields ϕ^i with a Lagrangian of the form

$$\mathcal{L} = f_{ij}(\{\phi^i\}) \partial_\mu \phi^i \partial^\mu \phi^j, \quad (84)$$

has dimensionless couplings and so is renormalizable. Since any function $f(\{\phi^i\})$ leads to a renormalizable theory, this class of scalar field theories contains an infinite number of marginal parameters, so we must impose some symmetries on the theory.

A simple choice is to take the scalar fields ϕ^i to form an N -component vector field $n^i(x)$, constrained to satisfy

$$\sum_{i=1}^N |n^i(x)|^2 = f^2, \quad (85)$$

where f is the lowest energy of the system.

If the theory has $O(N)$ symmetry, the function f in eq. (84) can depend only on the invariant length of $\vec{n}(x)$, which is constrained by eq. (85). Then the most general possible choice for f is a constant.

Similarly, the only possible nonderivative interaction $g(\{n^i\})$ that one might add to eq. (84) is a constant. With these restrictions, the most general Lagrangian one can build from $\vec{n}(x)$ with two derivatives and $O(N)$ symmetry is [9]

$$\mathcal{L} = \frac{1}{2g^2} |\partial_\mu \vec{n}|^2. \quad (86)$$

The physical interpretation that we can give it's a phenomenological description of a system with $O(N)$ symmetry spontaneously broken by the vev of a field that transforms as a vector of $O(N)$.

Consider the case in N -component ϕ^4 theory in its spontaneously broken phase. The field ϕ^i acquires a vev, which we can write in terms of a magnitude and a direction parameterized by

$$\langle \phi^i \rangle = \phi_0 n^i(x), \quad (87)$$

the fluctuations of ϕ_0 correspond to a massive field, the field called σ . The fluctuation of the direction of the vector $\vec{n}(x)$ correspond to the $N - 1$ Goldstone bosons, so contains $N - 1$ degrees of freedom. Formally, the nonlinear sigma model is the limit of ϕ^4 theory as the mass of the σ field is taken to infinity while ϕ_0 is held constant.

It's convenient to solve the constraint and parametrize \vec{n} by $N - 1$ Goldstone boson fields π^k

$$n^i = (\pi^1, \dots, \pi^{N-1}, \sigma), \quad (88)$$

where, by definition in (85)

$$\sigma = \sqrt{f^2 - \pi^2}. \quad (89)$$

The configuration $\pi^k = 0$ corresponds to a uniform state of spontaneous symmetry breaking, oriented in the N direction. The representation (88) implies that

$$|\partial_\mu n^i|^2 = |\partial_\mu \vec{\pi}|^2 + \frac{(\vec{\pi} \cdot \partial_\mu \vec{\pi})^2}{f^2 - \pi^2}. \quad (90)$$

Thus, the Lagrangian (86) takes the form

$$\mathbf{L} = \frac{1}{2g^2} \left[|\partial_\mu \vec{\pi}|^2 + \frac{(\vec{\pi} \cdot \partial_\mu \vec{\pi})^2}{f^2 - \pi^2} \right]. \quad (91)$$

It's important to see that there is no mass term for the field $\vec{\pi}$, as required by Goldstone's theorem. The perturbation theory for the π^k field can be read off straightforwardly by expanding the Lagrangian in powers of π^k

$$\mathcal{L} = \frac{1}{2g^2} \left[|\partial_\mu \vec{\pi}|^2 + \frac{(\vec{\pi} \cdot \partial_\mu \vec{\pi})^2}{f^2} + \dots \right]. \quad (92)$$

Recall the constrain (85), we can rewrite it as

$$\sigma^2 + \pi^2 = f^2, \quad (93)$$

then, the ϕ^i field

$$\phi^i = \sigma + i\vec{\tau} \cdot \vec{\pi}, \quad (94)$$

with $\vec{\tau}$ are the generators of $O(N-1)$. From (93), the eq. (94) takes the form

$$\phi^i = \sqrt{f^2 - \pi^2} + i\vec{\tau} \cdot \vec{\pi} = f \left(\sqrt{1 - \frac{\pi^2}{f^2}} + i \frac{\vec{\tau} \cdot \vec{\pi}}{f} \right), \quad (95)$$

we can write

$$\sin \left(\frac{\vec{\tau} \cdot \vec{\pi}}{f} \right) \sim \frac{\vec{\tau} \cdot \vec{\pi}}{f}, \quad (96)$$

so, the ϕ^i field looks like

$$\phi^i = f \exp \left(i \frac{\vec{\tau} \cdot \vec{\pi}}{f} \right). \quad (97)$$

We can define as

$$U(x) \equiv \exp \left(i \frac{\vec{\tau} \cdot \vec{\pi}}{f} \right). \quad (98)$$

$U(x)$ is a way to parametrize of Goldstone boson fields. Finally, the Lagrangian (86), from (98), in general takes the form

$$\mathcal{L} = \frac{f^2}{2g^2} \text{Tr} \left(\partial_\mu U(x) \partial^\mu U^\dagger(x) \right). \quad (99)$$

2.5 Lepton Flavor Violation (LFV)

Lepton mixing implies that the generational lepton numbers are not conserved. The most easily observable would be processes in which the initial and final states do not contain any neutrino. No such process has been observed so far.

Neutrino oscillations provide a clear evidence for lepton flavor violation (LFV) in the neutral sector, pointing out to physics beyond the Standard Model. However, no evidence of lepton flavor violating processes in the charged sector has been found despite the great experimental effort on searching for that violation [10].

LFV is absent in the SM with massless neutrinos. A minimal extension that includes right handed neutrinos allowing for small neutrino Dirac masses might allow LFV reactions such as $\mu \rightarrow e\gamma$. In the SM, particle masses are proportional to the strengths of the interactions between the particles and the Higgs bosons. Thus, the Dirac mass term like for example ν_e must be multiplied by some very small coupling strength such that his mass is at least 50,000 times smaller than the mass of the electron. But the electron and the ν_e are part of the same weak doublet, and there seems to be no reason why they should have such enormously different interaction strengths to the Higgs bosons [11].

We discuss some processes which are expected, and discuss what sort of rates to expect for them.

2.5.1 Radiative decays

The muon might decay into the electron with the emission of a photon

$$\mu \rightarrow e + \gamma. \quad (100)$$

It cannot happen at the tree level. Figure 6 shows how the process might occur at the one-loop level, this diagram is one of three diagrams possible that contributes to the process.

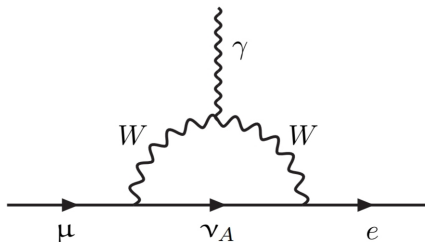


Figure 6: One-loop diagram for the process $\mu \rightarrow e + \gamma$.

Experimentally, only an upper bound is known for the branching ratio [5] of the process in eq. (100)

$$\mathcal{B}(\mu \rightarrow e + \gamma) < 4.2 \times 10^{-13}. \quad (101)$$

For any acceptable value of neutrino masses, this branching ratio is well below 10^{-20} .

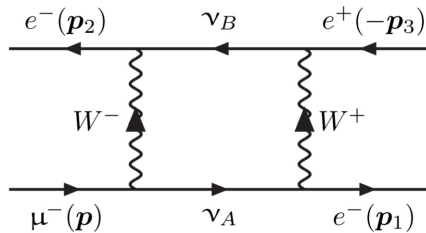
Processes like $\tau \rightarrow \mu + \gamma$ and $\tau \rightarrow e + \gamma$ are also expected to occur because of neutrino mixing [13,14,47]. They are also expected to be very suppressed for exactly the same reason.

2.5.2 Purely leptonic decays

We can also contemplate decays processes involving charged leptons and antileptons only

$$\mu^- \rightarrow e^- e^- e^+, \quad (102)$$

and its charge conjugate, as well as similar decays of the τ lepton where the final state can contain both muons and electrons. This process can occur at the one-loop level. One possible diagram is obtained by attaching an outgoing e^+e^- pair to the photon line of Figure 6 whereby the photon line itself becomes an internal virtual line. There are other possibilities shown in Figure 7.

Figure 7: One-loop box diagram for the process $\mu \rightarrow 3e$.

The branching ratio for this process is given by [5]

$$\mathcal{B}(\mu \rightarrow 3e) < 1.0 \times 10^{-12}. \quad (103)$$

2.5.3 $\mu - e$ conversion in nuclei $\mu N \rightarrow eN$

Other experiments look for flavour transitions of charged leptons either in decays, like $\mu \rightarrow e\gamma$, or in bound states of a nucleus and a captured muon, usually referred to as $\mu - e$ conversion. Considerable theoretical and experimental effort has been dedicated to $\mu - e$ conversion [15].

Muon-to-electron conversion is the spontaneous decay of a muon to an electron without the emission of neutrinos, within the Coulomb potential of an atomic nucleus: it is therefore only possible for negative muons.

Since no neutrinos are produced, muon-to-electron conversion is not a weak interaction: thus an observation of the process can only come from new physics.

The constraint of unchanged nucleus means that all the energy of the muon goes into the kinetic energy of the electron and the recoil of the parent nucleus, hence the signature of such a process is the presence of a monochromatic electron at an energy which is essentially the muon mass, corrected for the binding energy and nuclear recoil

$$E_e \approx m_\mu - B_\mu - E_r, \quad (104)$$

where $B_\mu \approx Z^2 \alpha^2 m_\mu / 2$ is the muon binding energy and $E_r \approx m_\mu^2 / 2m_N$ the nuclear recoil energy.

There are many excellent articles motivating the search, such as Calibbi and Signorelli [16], de Gouvêa and Vogel [17], or Marciano et al. [18].

The Mu2e experiment will search for the charged-lepton flavor violating (CLFV) neutrino-less conversion of a negative muon into an electron in the field of a nucleus. The goal of the experiment is to improve the previous upper limit by four orders of magnitude and reach a SES (single event sensitivity) of 3×10^{-17} on the conversion rate, a 90% CL of 8×10^{-17} , and a 5σ discovery reach at 2×10^{-16} . We can see that this bound is more restrictive than the eqs. (101) and (103). The experiment will begin operations in 2022 [19].

3 A Description of Little Higgs Models

The SM is a theory of electromagnetic, weak and strong interactions, whose predictions are in excellent agreement with the results of all particle physics experiments performed to date. Theorists, however, regard the SM as an effective theory, which is adequate at the presently explored energy scales but must become inadequate at a certain higher energy scale Λ . Let us assume that precision electroweak data are indeed telling us that there are no new particles beyond the Standard Model with masses at or below the weak scale. Let us assume that the SM is valid up to a cut-off scale of $\Lambda = 10$ TeV. At even higher energies new physics takes over, which implies that we do not know how to compute loop diagrams with momenta larger than Λ , thus we will cut such loops off at Λ . The hierarchy problem arises from the fact that there are quadratically divergent loop contributions to the Higgs mass which drive the Higgs mass to unacceptably large values unless the tree level mass parameter is finely tuned to cancel the large quantum corrections [20]. The question then becomes if it is possible to add new physics at the TeV scale to the SM which stabilizes the Higgs mass but does not violate the bounds obtained. To understand the requirements on this new physics better we must look at the source of the Higgs mass instability. The three most dangerous radiative corrections to the Higgs mass in the SM come from one-loop diagrams with top quarks, $SU(2)$ gauge bosons, and the Higgs itself running in the loop, how we can see in the Figure 8 [21].

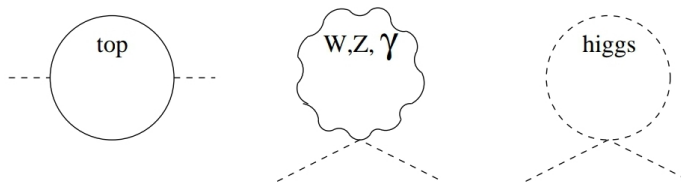


Figure 8: The most significant quadratically divergent contributions to the Higgs mass in the SM.

Assuming that the SM remains valid up to a cut-off of order the LHC center-of-mass energy, $\Lambda \sim 10$ TeV, the three diagrams give [20, 21]

$$\begin{array}{lll}
 \text{top loop} & -\frac{3}{8\pi^2}\lambda_t^2\Lambda^2 & \sim -(2 \text{ TeV})^2 \\
 \text{Higgs loop} & \frac{1}{16\pi^2}\lambda^2\Lambda^2 & \sim (500 \text{ GeV})^2 \\
 SU(2) \text{ gauge boson loops} & \frac{9}{64\pi^2}g^2\Lambda^2 & \sim (700 \text{ GeV})^2.
 \end{array}$$

The total Higgs mass-squared includes the sum of these loop contributions and a tree-level mass-squared parameter. This means that they are bigger than the expected physical Higgs mass at around 150 GeV. This is the hierarchy problem, which is demonstrated graphically in Figure 9.

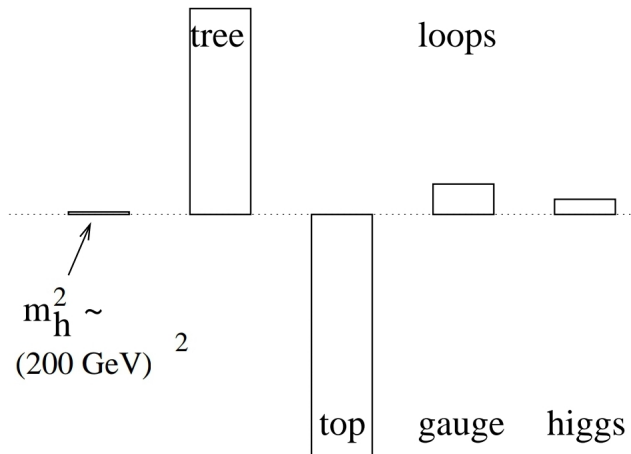


Figure 9: Size of the Higgs mass corrections. The Higgs mass is taken at about $\sim v$.

We expect new physics that cuts off the divergent top loop at or below 2 TeV. In a weakly coupled theory this implies that there are new particles with masses at or below 2 TeV. These particles must couple to the Higgs, giving rise to a new loop diagram that cancels the quadratically divergent contribution from the top loop. The new particles must be related to the top quark by some symmetry, implying that the new particles have similar quantum numbers to top quarks.

Little Higgs models offer an explanation to the little hierarchy between the Higgs mass, assumed to be near the electroweak scale $v = 246$ GeV and the new physics scale f , whose natural value is expected to be ~ 1 TeV.

Little Higgs theories are realizations of an old idea to stabilize the Higgs mass by making the Higgs a pseudo-Goldstone boson resulting from a spontaneously broken approximate symmetry. Early attempts at constructing such models [22, 23] were not entirely successful and quadratic divergences to the Higgs mass remained. The first successful model which canceled all relevant quadratic divergences based on the pseudo-Goldstone idea was constructed by Arkani-Hamed, Cohen and Georgi in [24].

The generic structure of Little Higgs models, is a global symmetry broken at a scale f which around a TeV. At the scale f there are new gauge bosons, scalars, and fermions responsible for cancelling the one loop quadratic divergences to the Higgs mass from SM particles. The spectrum of a generic theory is summarized in Figure 10 [20].

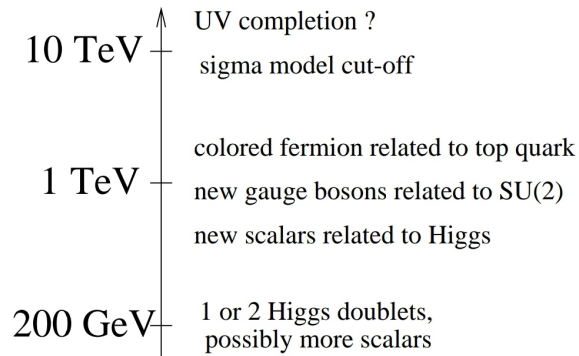


Figure 10: Generic Little Higgs Spectrum.

We can divide Little Higgs models into two categories

- Product group models,
- Simple group models.

3.1 Product Group Models

The majority of little Higgs models are product group models. The product group models have the following generic features. First, the models all contain a set of $SU(2)$ gauge bosons at the TeV scale, obtained from the diagonal breaking of two or more gauge groups down to SM $SU(2)_L$, and thus contain free parameters in the gauge sector from the independent gauge couplings. Second, since the collective symmetry breaking in the gauge sector is achieved by multiple gauged subgroups of the global symmetry, models can be built in which the SM Higgs doublet is embedded within a single non-linear sigma model field; many product group models make this simple choice. Third, the fermion sector of this class of models can usually be chosen to be very simple, involving only a single new vector-like quark [25].

The Littlest Higgs model belongs to the product group class, is the simplest model of the product group class [26]. Also include the theory space models: the Big Moose [24] and the Minimal Moose [27], the $SU(6)/Sp(6)$ model [28], and two extensions of the Littlest Higgs with built in custodial $SU(2)$ symmetry [29,31]. There are also product group models with a discrete symmetry called T-parity. This parity explicitly forbids any tree-level contribution from heavy gauge bosons to observables involving only SM particles as external states, this implies that in T-parity symmetric, corrections to precision electroweak observables are generated exclusively at loop level. Under this symmetry the new particles are odd and the SM ones even.

3.2 Simple Group Models

The simple group models share two features that distinguish them from the product group models. First, the simple group models all contain an $SU(N) \times U(1)$ gauge symmetry that is broken down

to $SU(2)_L \times U(1)_Y$, yielding a set of TeV scale gauge bosons. The two gauge couplings of the $SU(N) \times U(1)$ are fixed in terms of the two SM $SU(2)_L \times U(1)_Y$ gauge couplings, leaving no free parameters in the gauge sector once the symmetry breaking scale is fixed. This gauge structure also forbids mixing between the SM W^\pm bosons and the TeV scale gauge bosons. Second, in order to implement the collective symmetry breaking, simple group models require at least two sigma-model multiplets. The SM Higgs doublet is embedded as a linear combination of the Goldstone bosons from these multiplets [25].

Some examples about simple group models are: the $SU(4)$ simple group model [32], the $SU(9)/SU(8)$ model [30], and Simplest Little Higgs Model [33].

4 Littlest Higgs Model with T-parity

4.1 The Model

The Littlest Higgs with T-parity (*LHT*) is a non-linear σ model based on the coset space $SU(5)/SO(5)$, where $SU(5)$ is the global symmetry, so guarantees 14 Nambu-Goldstone bosons. The global symmetry is broken by the VEV of a 5×5 symmetric tensor. It is convenient to choose a basis in which the symmetric tensor is proportional to

$$\Sigma_0 = \begin{pmatrix} 0 & 0 & \mathbb{I} \\ 0 & 1 & 0 \\ \mathbb{I} & 0 & 0 \end{pmatrix}, \quad (105)$$

where \mathbb{I} represents a unit 2×2 matrix. Here $\Sigma_0 = \langle \Sigma \rangle$ is the VEV of gauge field Σ . It transforms as $\Sigma \rightarrow U\Sigma U^T$ for $U \in SU(5)$.

The original $SU(5)$ generators are shown in the Appendix A, but we are going to use a different representation of the Lie algebra of $SU(5)$. We introduce the matrix A [34]

$$A = \begin{pmatrix} \frac{1}{2} + \frac{i}{2} & 0 & 0 & \frac{1}{2} - \frac{i}{2} & 0 \\ 0 & \frac{1}{2} + \frac{i}{2} & 0 & 0 & \frac{1}{2} - \frac{i}{2} \\ 0 & 0 & 1 & 0 & 0 \\ \frac{1}{2} - \frac{i}{2} & 0 & 0 & \frac{1}{2} + \frac{i}{2} & 0 \\ 0 & \frac{1}{2} - \frac{i}{2} & 0 & 0 & \frac{1}{2} + \frac{i}{2} \end{pmatrix}, \quad (106)$$

where A satisfies $A^2 = \Sigma_0$ and $A^T = A$. We transform the usual matrix representation of $SU(5)$ via the unitary transformation

$$\hat{\lambda}_a = A\lambda_a A^{-1} = \begin{pmatrix} \frac{1}{2} + \frac{i}{2} & 0 & 0 & \frac{1}{2} - \frac{i}{2} & 0 \\ 0 & \frac{1}{2} + \frac{i}{2} & 0 & 0 & \frac{1}{2} - \frac{i}{2} \\ 0 & 0 & 1 & 0 & 0 \\ \frac{1}{2} - \frac{i}{2} & 0 & 0 & \frac{1}{2} + \frac{i}{2} & 0 \\ 0 & \frac{1}{2} - \frac{i}{2} & 0 & 0 & \frac{1}{2} + \frac{i}{2} \end{pmatrix} \lambda_a \begin{pmatrix} \frac{1}{2} - \frac{i}{2} & 0 & 0 & \frac{1}{2} + \frac{i}{2} & 0 \\ 0 & \frac{1}{2} - \frac{i}{2} & 0 & 0 & \frac{1}{2} + \frac{i}{2} \\ 0 & 0 & 1 & 0 & 0 \\ \frac{1}{2} + \frac{i}{2} & 0 & 0 & \frac{1}{2} - \frac{i}{2} & 0 \\ 0 & \frac{1}{2} + \frac{i}{2} & 0 & 0 & \frac{1}{2} - \frac{i}{2} \end{pmatrix}, \quad (107)$$

with λ_a the generators of $SU(5)$.

With the new representation of $SU(5)$, the unbroken $SO(5)$ generators satisfy

$$\hat{T}_a \Sigma_0 + \Sigma_0 \hat{T}_a^T = 0, \quad (108)$$

while the broken generators obey

$$\hat{X}_a \Sigma_0 - \Sigma_0 \hat{X}_a^T = 0. \quad (109)$$

As usual, the Goldstone bosons are fluctuations about this background in the broken directions $\Pi \equiv \pi^a \hat{X}^a$, and can be parameterized by the non-linear sigma model field

$$\Sigma(x) = e^{i\Pi/f} \Sigma_0 e^{i\Pi^T/f} = e^{2i\Pi/f} \Sigma_0, \quad (110)$$

where f is the effective NP scale. Thus the matrix of NGB may be written as

$$\Pi = \begin{pmatrix} \chi + \frac{\eta}{2\sqrt{5}} & \frac{h}{\sqrt{2}} & \phi \\ \frac{h^\dagger}{\sqrt{2}} & \frac{2\eta}{\sqrt{5}} & \frac{h^T}{\sqrt{2}} \\ \phi^\dagger & \frac{h^*}{\sqrt{2}} & \chi^T + \frac{\eta}{2\sqrt{5}} \end{pmatrix}. \quad (111)$$

Here $\chi = \chi^a \sigma^a / 2$ with σ^a the three Pauli matrices, η is a real singlet, and they are the Goldstone bosons which are eaten to become the longitudinal modes of the partners of the SM gauge fields, $h = \begin{pmatrix} h^+ \\ h^0 \end{pmatrix}$ is the SM Higgs doublet and finally $\phi = \begin{pmatrix} -i\Phi^{++} & -i\frac{\Phi^+}{\sqrt{2}} \\ -i\frac{\Phi^+}{\sqrt{2}} & \frac{-\Phi^0 + \Phi^P}{\sqrt{2}} \end{pmatrix}$ [44] is a complex $SU(2)_L$ triplet. The parametrization of the eq. (111) is computed in the Appendix B.

As χ is given by $\chi^a \sigma^a / 2$ it has 3 degrees of freedom, in the eq. (B.7) we can see that η has only 1 degree of freedom, h has 4 as shown in eq. (B.10), and finally ϕ has 6 given by the eq. (B.14). So together they add up to the desired 14 Goldstone bosons.

4.2 Gauge and Scalar Sector

We now gauge a subgroup of the global symmetry, and we do this in such a way that each gauge coupling by itself preserves enough of the global symmetry to ensure that the Higgs doublet remains an exact NGB. The gauge group is taken to be $G_1 \times G_2 = [SU(2) \times U(1)]^2$, subgroup of the $SU(5)$ global symmetry.

$$\begin{array}{ccc} SU(5) & \xrightarrow{f} & SO(5) \\ \cup & & \cup \\ [SU(2) \times U(1)]^2 & \xrightarrow{f} & SU(2)_w \times U(1)_Y \xrightarrow{v_h} U(1)_{EW} \end{array}$$

Figure 11: The global $SU(5)$ contains two copies of local $[SU(2) \times U(1)]^2$ that are diagonally broken to one copy $SU(2) \times U(1)$ contained in $SO(5)$.

As we can see in *Figure 11* [34], f is the energy scale where the symmetry breaking $[SU(2) \times U(1)]^2 \rightarrow SU(2)_W \times U(1)_Y$ occurs. The global symmetry breaking scale, f , is constrained on the order of a TeV.

The generators of the first $G_1 = SU(2) \times U(1)$ are embedded into $SU(5)$ as

$$Q_1^a = \begin{pmatrix} \sigma^a/2 & 0 \\ 0 & 0_{3 \times 3} \end{pmatrix}, \quad Y_1 = \text{diag}(3, 3, -2, -2, -2)/10, \quad (112)$$

while the generators of the second $SU(2) \times U(1)$ are given by

$$Q_2^a = \begin{pmatrix} 0_{3 \times 3} & 0 \\ 0 & -\sigma^{a*}/2 \end{pmatrix}, \quad Y_2 = \text{diag}(2, 2, 2, -3, -3)/10. \quad (113)$$

The linear combination $Q_1^a + Q_2^a$ satisfies eq. (108), and generates the unbroken symmetry that we identify as $SU(2)_W$ of the SM. Now, the linear combination $Y_1 + Y_2$ satisfies eq. (108) too, and generates the unbroken symmetry that we identify as $U(1)_Y$. The orthogonal combinations are a subset of the broken generator, i.e., $\{Q_1^a - Q_2^a, Y_1 - Y_2\} \subset \{\hat{X}_a\}$. The vacuum breaks the $[SU(2) \times U(1)]^2$ gauge symmetry down to the diagonal subgroup, giving one set of $[SU(2) \times U(1)]$ gauge bosons masses of order of f , while the other set members are left massless, and are identified as the $[SU(2)_L \times U(1)_Y]$ gauge fields of the SM.

T-parity is a natural symmetry of most little Higgs models where SM particles are even under this symmetry (T-even), while the new particles at the TeV scale are odd (T-odd). T-parity explicitly forbids any tree-level contributions from the heavy gauge bosons to the observables involving only SM particles as external states, as a result, the corrections to EWPO are generated exclusively at loop level. This implies that the constraints are generically weaker than in the Little Higgs models without this symmetry since the most serious constraints resulted from the tree-level corrections to EWPO through the exchange of heavy gauge bosons. In other words, when we introduced T-parity, eliminate the tree-level electroweak precision constraints that the Littlest Higgs model has: because the external states in all experimentally tested processes are T-even, there is no T-odd state that can contribute to such processes at tree-level.

Using the action of the eqs. (108) and (109), a natural action of T-parity on the gauge fields is defined as

$$G_1 \leftrightarrow G_2, \quad (114)$$

its action on the gauge fields G_i exchanges the two gauge groups $SU(2)_i \times U(1)_i$. Then, T invariance requires that the gauge couplings associated to both factors are equal, leading to the Gauge Lagrangian

$$\mathcal{L}_G = \sum_{j=1}^2 \left[-\frac{1}{2} \text{Tr} \left(\widetilde{W}_{j\mu\nu} \widetilde{W}_j^{\mu\nu} \right) - \frac{1}{4} B_{j\mu\nu} B_j^{\mu\nu} \right], \quad (115)$$

where

$$\widetilde{W}_{j\mu} = W_{j\mu}^a Q_j^a, \quad \widetilde{W}_{j\mu\nu} = \partial_\mu \widetilde{W}_{j\nu} - \partial_\nu \widetilde{W}_{j\mu} - ig \left[\widetilde{W}_{j\mu}, \widetilde{W}_{j\nu} \right], \quad B_{j\mu\nu} = \partial_\mu B_{j\nu} - \partial_\nu B_{j\mu}. \quad (116)$$

As mentioned earlier, the combination $\{Q_1^a + Q_2^a, Y_1 + Y_2\}$ generates the SM gauge group. The SM gauge bosons are the T-even combinations multiplying the unbroken gauge generators,

$$W^\pm = \frac{1}{2} [(W_1^1 + W_2^1) \mp i(W_1^2 + W_2^2)], \quad W^3 = \frac{W_1^3 + W_2^3}{\sqrt{2}}, \quad B = \frac{B_1 + B_2}{\sqrt{2}}, \quad (117)$$

whereas the heavy gauge bosons are the T-odd combinations

$$W_H^\pm = \frac{1}{2} [(W_1^1 - W_2^1) \mp i(W_1^2 - W_2^2)], \quad W_H^3 = \frac{W_1^3 - W_2^3}{\sqrt{2}}, \quad B_H = \frac{B_1 - B_2}{\sqrt{2}}. \quad (118)$$

The eqs. (117) and (118) are computed in Appendix C.

Recalling that the heavy particles are T-odd and all SM particles are T-even, we have to impose an extra transformation rule for the scalar sector. In order to ensure that the SM Higgs doublet is T-even and the remaining Goldstone fields are T-odd, the T action on the scalar fields is defined as follows,

$$\Pi \xrightarrow{T} \hat{\Pi} = -\Omega \Pi \Omega, \quad \Omega = \text{diag}(-1, -1, 1, -1, -1), \quad (119)$$

where Ω is an element of the center of the gauge group which commutes with Σ_0 .

$$\hat{\Pi} = \begin{pmatrix} -\chi - \frac{\eta}{2\sqrt{5}} & \frac{h}{\sqrt{2}} & -\phi \\ \frac{h^\dagger}{\sqrt{2}} & -\frac{2\eta}{\sqrt{5}} & \frac{h^T}{\sqrt{2}} \\ -\phi^\dagger & \frac{h^*}{\sqrt{2}} & -\chi^T - \frac{\eta}{2\sqrt{5}} \end{pmatrix}, \quad (120)$$

if we compared the eq. (120) with the eq. (111), we can see that the SM Higgs doublet keeps its parity.

Then,

$$\Sigma \xrightarrow{T} \tilde{\Sigma} = \Sigma_0 \Omega \Sigma^\dagger \Omega \Sigma_0. \quad (121)$$

We know that $\Sigma = e^{2i\Pi/f} \Sigma_0$, and since χ and η are the Goldstone bosons which are eaten to become the longitudinal modes of the partners of the SM gauge fields, we have

$$\Pi = \begin{pmatrix} 0 & \frac{h}{\sqrt{2}} & \phi \\ \frac{h^\dagger}{\sqrt{2}} & 0 & \frac{h^T}{\sqrt{2}} \\ \phi^\dagger & \frac{h^*}{\sqrt{2}} & 0 \end{pmatrix}, \quad (122)$$

expanding Σ until order of $\frac{1}{f^2}$

$$\begin{aligned}\Sigma &= \left(\mathbb{I} + \frac{2i}{f}\Pi - \frac{2}{f^2}\Pi^2 + \dots \right) \Sigma_0 \\ &= \Sigma_0 + \frac{2i}{f} \begin{pmatrix} \phi & \frac{h}{\sqrt{2}} & 0 \\ \frac{h^\dagger}{\sqrt{2}} & 0 & \frac{h^T}{\sqrt{2}} \\ 0 & \frac{h^*}{\sqrt{2}} & \phi^\dagger \end{pmatrix} - \frac{1}{f^2} \begin{pmatrix} h^T h & \sqrt{2}\phi h^* & h^\dagger h + 2\phi\phi^\dagger \\ \sqrt{2}h^\dagger\phi & 2h^T h^* & \sqrt{2}h^T\phi^\dagger \\ 2\phi^\dagger\phi + h^T h^* & \sqrt{2}\phi^\dagger h & h^\dagger h^* \end{pmatrix} + \mathcal{O}\left(\frac{1}{f^3}\right).\end{aligned}\quad (123)$$

Now, including the EWSB, the vacuum expectation values of h and ϕ are [34]

$$\langle h \rangle_0 = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 \\ v_h \end{pmatrix}, \quad \langle \phi \rangle_0 = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, \quad (124)$$

where $v_h = 246$ GeV is the EWSB scale and h is the physical Higgs field. Thus, considering the eq. (124), the eq. (122) transforms as

$$\Pi = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & v_h & 0 & 0 \\ 0 & v_h & 0 & 0 & v_h \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & v_h & 0 & 0 \end{pmatrix}. \quad (125)$$

Then, including the EWSB effects, the vev of the Σ field has the form [36]

$$\Sigma = \exp \left\{ \frac{i\sqrt{2}}{f} \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & v_h & 0 & 0 \\ 0 & v_h & 0 & 0 & v_h \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & v_h & 0 & 0 \end{pmatrix} \right\} \Sigma_0 = \begin{pmatrix} 0 & 0 & 0 & 1 & 0 \\ 0 & -\frac{1}{2}(1 - c_v) & \frac{i}{\sqrt{2}}s_v & 0 & \frac{1}{2}(1 + c_v) \\ 0 & \frac{i}{\sqrt{2}}s_v & c_v & 0 & \frac{i}{\sqrt{2}}s_v \\ 1 & 0 & 0 & 0 & 0 \\ 0 & \frac{1}{2}(1 + c_v) & \frac{i}{\sqrt{2}}s_v & 0 & -\frac{1}{2}(1 - c_v) \end{pmatrix}, \quad (126)$$

where

$$s_{v_h} = \sin\left(\frac{\sqrt{2}v_h}{f}\right), \quad c_{v_h} = \cos\left(\frac{\sqrt{2}v_h}{f}\right). \quad (127)$$

The scalar Lagrangian of the gauged theory is obtained from the non-linear σ model field Σ in the Littlest Higgs

$$\mathcal{L}_S = \frac{f^2}{8} \text{Tr}[(\mathcal{D}_\mu \Sigma)(\mathcal{D}^\mu \Sigma)^\dagger], \quad (128)$$

where

$$\mathcal{D}_\mu \Sigma = \partial_\mu \Sigma - i \sum_{j=1}^2 [g_j W_j^a (Q_j^a \Sigma + \Sigma Q_j^{aT}) + g'_j B_j (Y_j \Sigma + \Sigma Y_j^T)]. \quad (129)$$

In the above equation, the Q_j and Y_j are the gauged generators, W_j^a and B_j are the $SU(2)_j$ and $U(1)_j$ gauge fields, respectively, and g_j and g'_j are the corresponding coupling constants. We know that the T-parity exchanges the two gauge groups as we can see from eq. (114), i.e., under this T-parity the gauge bosons and the $[SU(2) \times U(1)]_j$ generators change as

$$W_1^a \leftrightarrow W_2^a, \quad B_1 \leftrightarrow B_2, \quad Q_1^a \leftrightarrow Q_2^a, \quad Y_1 \leftrightarrow Y_2. \quad (130)$$

In Appendix D, we verify the calculation of $Q_1^a \leftrightarrow Q_2^a$ and $Y_1 \leftrightarrow Y_2$.

Because the Lagrangian \mathcal{L}_S is T-even, from the eqs. (125) and (126) is easy to see that

$$g_1 = g_2 = \sqrt{2}g_W, \quad g'_1 = g'_2 = \sqrt{2}g', \quad (131)$$

where g_W is the $SU(2)_W$ coupling constant and g' is the $U(1)_Y$ coupling constant.

Then, the eq. (129) looks as

$$\mathcal{D}_\mu \Sigma = \partial_\mu \Sigma - i\sqrt{2} \sum_{j=1}^2 [g_W W_j^a (Q_j^a \Sigma + \Sigma Q_j^{aT}) + g' B_j (Y_j \Sigma + \Sigma Y_j^T)]. \quad (132)$$

Under T-parity, the relevant term for to calculate the mass of gauge bosons is

$$\left| -i\sqrt{2} \sum_{j=1}^2 [g_W W_j^a (Q_j^a \tilde{\Sigma} + \tilde{\Sigma} Q_j^{aT}) + g' B_j (Y_j \tilde{\Sigma} + \tilde{\Sigma} Y_j^T)] \right|^2. \quad (133)$$

We have to lower order $\tilde{\Sigma} = \Sigma_0$, in the gauge sector before EWSB, there is a linear combination of gauge bosons that acquire a mass of order f , from (133)

$$\begin{aligned} \begin{pmatrix} W_L^a \\ W_H^a \end{pmatrix} &= \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} W_1^a \\ W_2^a \end{pmatrix}, \\ \begin{pmatrix} B_L \\ B_H \end{pmatrix} &= \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} B_1 \\ B_2 \end{pmatrix}. \end{aligned} \quad (134)$$

From (134) we see the SM light gauge bosons are W_L^a and B_L , they are massless and T-even, while the heavy gauge bosons are W_H^a and B_H , they are T-odd and massives. After high energy symmetry breaking, their masses are [37]

$$M_{W_H^a} = g_W f, \quad M_{B_H} = \frac{g' f}{\sqrt{5}}. \quad (135)$$

In Appendix D, we computed the masses of gauge bosons.

Now, considering the EWSB effects of order $(v_h/f)^2$, the Σ field according to eq. (126) is

$$\Sigma = \begin{pmatrix} 0 & 0 & 0 & 1 & 0 \\ 0 & -\epsilon^2/2 & -i\epsilon & 0 & 1 - \epsilon^2/2 \\ 0 & i\epsilon & 1 - \epsilon^2 & 0 & i\epsilon \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 1 - \epsilon^2/2 & i\epsilon & 0 & -\epsilon^2/2 \end{pmatrix}, \quad (136)$$

with $\epsilon = \frac{v_h}{f}$.

When we included the EWSB effects, the LHT model must reproduce the SM gauge bosons, i.e., W^\pm , Z and A bosons have to appear moreover the masses of heavy sector will have corrections of order $\frac{v^2}{f^2}$.

We are separating explicitly the third component W_1^3 and W_2^3 , which is convenient because in the Scalar Lagrangian of order $\frac{v^2}{f^2}$ this component has other transformation. Now in W_1^a and W_2^a , $a = 1, 2$.

The Scalar Lagrangian, with EWSB effects is explicitly

$$\begin{aligned} \mathcal{L}_S = & \frac{f^2 g_W^2}{4} \left[-\frac{\epsilon^4}{6} W_1^a W_2^a + \epsilon^2 W_1^a W_2^a + (W_1^a - W_2^a)^2 \right] \\ & + \frac{f^2 g_W^2}{4} \left[\frac{\epsilon^4}{8} \left((W_1^3 - W_2^3)^2 - \frac{1}{4} W_1^3 W_2^3 \right) + \epsilon^2 W_1^3 W_2^3 + (W_1^3 - W_2^3)^2 \right] \\ & + \frac{f^2 g'^2}{4} \left[\epsilon^4 \left(\frac{1}{8} (B_1^2 + B_2^2) - \frac{5}{12} B_1 B_2 \right) + \epsilon^2 B_1 B_2 + \frac{1}{5} (B_1 - B_2)^2 \right] \\ & + \frac{f^2 g_W g'}{4} \left[\frac{\epsilon^4}{4} \left(- (W_1^3 - W_2^3) (B_1 - B_2) + \frac{2}{3} (B_1 W_2^3 + B_2 W_1^3) \right) - \epsilon^2 (B_1 W_2^3 + B_2 W_1^3) \right]. \end{aligned} \quad (137)$$

Also we are considering the crossed terms of W_j and B_j in the Scalar Lagrangian. We can see in the Scalar Lagrangian, the terms W_j^3 are separated explicitly of the terms W_1^1 , W_1^2 , W_2^1 and W_2^2 , moreover W_j^3 is the only component of W_1^a and W_2^a that mixes with the B_j bosons.

Finally, the Scalar Lagrangian for the light and heavy gauge bosons sector is

$$\begin{aligned} \mathcal{L}_S = & \frac{1}{2} \left[\frac{g_W^2 v_h^2}{4} \left(1 - \frac{v_h^2}{6f^2} \right) \right] W_L^+ W_L^- \\ & + \frac{1}{2} \left[f^2 g_W^2 \left(1 - \frac{v_h^2}{4f^2} \right) \right] W_H^+ W_H^- \\ & + \frac{1}{2} \left[f^2 g_W^2 \left(1 - \frac{v_h^2}{4f^2} \right) \right] (Z_H)^2 \\ & + \frac{1}{2} \left[\frac{f^2 g'^2}{5} \left(1 - \frac{5v_h^2}{4f^2} \right) \right] (A_H)^2 \\ & + \frac{1}{2} \left[\frac{g_W^2 v_h^2}{4 \cos^2 \theta_W} \left(1 - \frac{v_h^2}{6f^2} \right) \right] Z_L^2. \end{aligned} \quad (138)$$

The light gauge sector includes the W_L^\pm , Z_L , and A_L bosons, that we identify as the SM gauge bosons with masses

$$\begin{aligned} M_{W_L^\pm} &= \frac{g_W v_h}{2} \left(1 - \frac{v_h^2}{6f^2}\right)^{1/2} \approx \frac{g_W v_h}{2} \left(1 - \frac{v_h^2}{12f^2}\right), \\ M_{Z_L} &= \frac{g_W v_h}{2 \cos \theta_H} \left(1 - \frac{v_h^2}{6f^2}\right)^{1/2} = \frac{M_{W_L^\pm}}{\cos \theta_W}, \\ M_{A_L} &= 0, \end{aligned} \tag{139}$$

and the mass of the heavy bosons are

$$\begin{aligned} M_{W_H^\pm} &= M_{Z_H} = fg_W \left(1 - \frac{v_h^2}{4f^2}\right)^{1/2} \approx fg_W \left(1 - \frac{v_h^2}{8f^2}\right), \\ M_{A_H} &= \frac{fg'}{\sqrt{5}} \left(1 - \frac{5v_h^2}{4f^2}\right)^{1/2} \approx \frac{fg'}{\sqrt{5}} \left(1 - \frac{5v_h^2}{8f^2}\right). \end{aligned} \tag{140}$$

In Appendix D, we show the process to obtain their masses.

4.3 Fermion Sector

The LHT must implemented the fermion sector too. We would like to introduce SM fermions that transform linearly under the gauge symmetries to avoid large contributions to four fermion operators that would require the scale f to be large. As we have seen before, T-parity exchanges $SU(2)_1$ and $SU(2)_2$ one must introduce two doublets ψ_1 , and ψ_2 , which transform linearly under $SU(2)_1$ and $SU(2)_2$ respectively, i.e., for each SM $SU(2)_L$ doublet, a doublet under $SU(2)_1$ and one under $SU(2)_2$ are introduced. The T-even combination is associated with the SM $SU(2)_L$ doublet while the T-odd combination is given a mass of order the breaking scale f .

For each lepton/quark doublet, these can be embedded in incomplete representation Ψ_1 , Ψ_2 of the global $SU(5)$ symmetry. An additional set of fermions forming a T-odd $SO(5)$ multiplet Ψ_R^c , which is right-handed and transforms non-linearly under the full $SU(5)$, is introduced to give mass to the extra fermions; the field content can be expressed as follow [36]

$$\Psi_1 = \begin{pmatrix} \psi_1 \\ 0 \\ 0 \end{pmatrix}, \quad \Psi_2 = \begin{pmatrix} 0 \\ 0 \\ \psi_2 \end{pmatrix}, \quad \Psi_R^c = \begin{pmatrix} \psi_R^c \\ \chi_R^c \\ \tilde{\psi}_R^c \end{pmatrix}, \tag{141}$$

with ψ_i for each SM left-handed lepton doublet is [38]

$$\psi_i = -i\sigma^2 l_{iL} = -i\sigma^2 \begin{pmatrix} \nu_{iL} \\ \ell_{iL} \end{pmatrix}, \quad i = 1, 2, \tag{142}$$

and for left-handed quarks doublet [39]

$$\psi_i = -i\sigma^2 q_i = -i\sigma^2 \begin{pmatrix} u_{iL} \\ d_{iL} \end{pmatrix}, \quad i = 1, 2. \quad (143)$$

Also ψ_R^c has the form as the eqs. (142) and (143), with the difference that there are right-handed and heavy fermions: ℓ_{HR} , u_{HR} , and d_{HR} . The extra doublet $\tilde{\psi}_R^c$, is assumed to be heavy enough to agree with EWPD, is T-odd as desired, and the gauge singlet χ_R^c completes the $SO(5)$ representation.

These fields transform under $SU(5)$ as follow

$$\Psi_1 \rightarrow V^* \Psi_1, \quad \Psi_2 \rightarrow V \Psi_2, \quad \Psi_R^c \rightarrow U \Psi_R^c, \quad (144)$$

where $U = \exp(iu^a (\Pi, V) T^a)$ [40] belongs to the unbroken $SO(5)$ and is a non-linear representation of the $SU(5)$, the function u^a depends on the Goldstone fields Π and the $SU(5)$ rotation V in a non-linear way.

Under the action of T-parity on the multiplets takes

$$\Psi_1 \leftrightarrow -\Sigma_0 \Psi_2, \quad \Psi_R^c \rightarrow -\Psi_R^c. \quad (145)$$

We desire that Ψ^c interacted with other fields which obey linear transformations laws, so we introduce a field $\xi = \exp(i\Pi/f)$. In terms of ξ the field Σ can be expressed as $\Sigma = \xi^2 \Sigma_0$, and from the linear transformation of Σ we have to by action of T-parity

$$\xi \xrightarrow{T} \Omega \xi^\dagger \Omega, \quad (146)$$

and invariant under global transformation,

$$\Sigma = \xi^2 \Sigma_0 \rightarrow V \Sigma V^T, \quad \xi \rightarrow V \xi U^\dagger = U \xi \Sigma_0 V^T \Sigma_0. \quad (147)$$

Recalling that the T-even combination of ψ_1 and ψ_2 are the SM electroweak leptons and quarks doublets, while T-odd combination is given a Dirac mass of order $\mathcal{O}(f)$ with the ψ^c through the following non-linear Yukawa Lagrangian

$$\mathcal{L}_{YH} = \kappa_1 f \left(\bar{\Psi}_2 \xi + \bar{\Psi}_1 \Sigma_0 \Omega \xi^\dagger \Omega \right) \Psi_R^c + \kappa_2 \Psi_R^T \Psi_R^c + h.c. \quad (148)$$

where we included another field, defined as [41]

$$\Psi_R = \begin{pmatrix} 0 \\ \chi_R \\ \tilde{\psi}_R \end{pmatrix}, \quad \Psi_R \xrightarrow{T} -\Psi_R. \quad (149)$$

Since we want to calculate first the mass of order $\mathcal{O}(f)$ we can approach $\xi = \exp(i\Pi/f) \approx \mathbb{I}$, then

$$\begin{aligned}
 \mathcal{L}_{Y_H} &= \kappa_1 f \left(\bar{\Psi}_2 \xi + \bar{\Psi}_1 \Sigma_0 \Omega \xi^\dagger \Omega \right) \Psi_R^c \\
 &= \kappa_1 f \left(\bar{\Psi}_2 + \bar{\Psi}_1 \Sigma_0 \Omega \Omega \right) \Psi_R^c \\
 &= \kappa_1 f \left(\bar{\Psi}_2 + \bar{\Psi}_1 \Sigma_0 \right) \Psi_R^c \\
 &= \kappa_1 f \left(\begin{pmatrix} 0 & 0 & \bar{\psi}_2 \end{pmatrix} + \begin{pmatrix} \bar{\psi}_1 & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix} \right) \Psi_R^c \\
 &= \kappa_1 f \begin{pmatrix} 0 & 0 & \bar{\psi}_2 + \bar{\psi}_1 \end{pmatrix} \begin{pmatrix} \psi_R^c \\ \chi_R^c \\ \tilde{\psi}_R^c \end{pmatrix} \\
 &= \kappa_1 f \left(\bar{\psi}_1 + \bar{\psi}_2 \right) \tilde{\psi}_R^c,
 \end{aligned} \tag{150}$$

we defined the T-odd combination as

$$\psi_H = \frac{1}{\sqrt{2}} (\psi_1 + \psi_2), \tag{151}$$

and the T-even combination given by

$$\psi_{SM} = \frac{1}{\sqrt{2}} (\psi_1 - \psi_2). \tag{152}$$

Then,

$$\begin{aligned}
 \mathcal{L}_{Y_H} &= \sqrt{2} \kappa_1 f \bar{\psi}_H \tilde{\psi}_R^c + \kappa_2 \bar{\Psi}_R^T \Psi_R^c + h.c. \\
 &= \sqrt{2} \kappa_1 f \bar{\psi}_H \tilde{\psi}_R^c + \kappa_2 \begin{pmatrix} 0 & \chi_R & \tilde{\psi}_R \end{pmatrix} \begin{pmatrix} \psi_R^c \\ \chi_R^c \\ \tilde{\psi}_R^c \end{pmatrix} + h.c. \\
 &= \sqrt{2} \kappa_1 f \bar{\psi}_H \tilde{\psi}_R^c + \kappa_2 \chi_R \chi_R^c + \kappa_2 \tilde{\psi}_R \tilde{\psi}_R^c + h.c.
 \end{aligned} \tag{153}$$

As we can see from the eq. (153), the Yukawa Lagrangian gives a Dirac mass $M_- = \sqrt{2} \kappa_1 f$ to the T-odd combination ψ_H , together with $\tilde{\psi}_R^c$, and a mass $\kappa_2 f$ to the Dirac pair $\chi_R \chi_R^c$, and $\tilde{\psi}_R \tilde{\psi}_R^c$. The T-even combination ψ_{SM} remains massless and is identified with the SM lepton or quark doublet.

If now we consider EWSM effects of order $(v_h/f)^2$ the field ξ is

$$\xi \approx \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 - \frac{v_h^2}{8f^2} & \frac{iv_h}{2f} & 0 & -\frac{v_h^2}{8f^2} \\ 0 & \frac{iv_h}{2f} & 1 - \frac{v_h^2}{4f^2} & 0 & \frac{iv_h}{2f} \\ 0 & 0 & 0 & 1 & 0 \\ 0 & -\frac{v_h^2}{8f^2} & \frac{iv_h}{2f} & 0 & 1 - \frac{v_h^2}{8f^2} \end{pmatrix}, \quad (154)$$

and considering as $\tilde{\psi}_R^c = -i\sigma^2(\tilde{q}_L)^c$ and $(\tilde{q}_L)^c = \begin{pmatrix} \tilde{u}_L^c & \tilde{d}_L^c \end{pmatrix}^T$, then

$$\Psi_1 = \begin{pmatrix} -d_{1L} \\ u_{1L} \\ 0 \\ 0 \\ 0 \end{pmatrix}, \quad \Psi_2 = \begin{pmatrix} 0 \\ 0 \\ 0 \\ -d_{2L} \\ u_{2L} \end{pmatrix}, \quad \Psi_R^c = \begin{pmatrix} -d_R^c \\ u_R^c \\ \chi_R^c \\ -\tilde{d}_L^c \\ \tilde{u}_L^c \end{pmatrix}, \quad (155)$$

thus, introducing the above matrix and these multiplets in the Yukawa Lagrangian we have among other terms

$$\mathcal{L}_{Y_H} = \kappa_1 f \tilde{d}_L^c (\bar{d}_{1L} + \bar{d}_{2L}) + \kappa_1 f \tilde{u}_L^c (\bar{u}_{1L} + \bar{u}_{2L}) \left(1 - \frac{v_h^2}{8f^2}\right) - \kappa_1 f u_R^c (\bar{u}_{1L} + \bar{u}_{2L}) + \dots, \quad (156)$$

defining the T-odd combinations as

$$u_H = \frac{1}{\sqrt{2}}(u_{1L} + u_{2L}), \quad d_H = \frac{1}{\sqrt{2}}(d_{1L} + d_{2L}), \quad (157)$$

the Yukawa Lagrangian is written as

$$\mathcal{L}_{Y_H} = \sqrt{2}\kappa_1 f \bar{d}_H \tilde{d}_L^c + \sqrt{2}\kappa_1 f \bar{u}_H \tilde{u}_L^c \left(1 - \frac{v_h^2}{8f^2}\right) - \sqrt{2}\kappa_1 f u_R^c u_H + \dots. \quad (158)$$

After EWSB, a small mass splitting between the T-odd up and down-type quarks is induced, and their masses are [42]

$$m_{u_H} = \sqrt{2}\kappa_1 f \left(1 - \frac{v_h^2}{8f^2}\right), \quad (159)$$

$$m_{d_H} = \sqrt{2}\kappa_1 f.$$

The Yukawa Lagrangian \mathcal{L}_{Y_H} fixes the transformation properties of the heavy fermions including their gauge couplings, even more, the non-linear couplings of the right-handed heavy fermions are fixed to be [43] (we change the signs of the second line of eq. (160) and the covariant derivative (161) in order

to obtain the SM couplings)

$$\begin{aligned}
 \mathcal{L}_{Y_H} &= i\bar{\Psi}_1\gamma^\mu D_\mu^*\Psi_1 + i\bar{\Psi}_2\gamma^\mu D_\mu\Psi_2 \\
 &+ i\bar{\Psi}_R^c\gamma^\mu\left(\partial_\mu - \frac{1}{2}\xi^\dagger(D_\mu\xi) - \frac{1}{2}\xi\left(\Sigma_0 D_\mu^*\Sigma_0\xi^\dagger\right)\right)\Psi_R^c \\
 &+ \Psi_R^c \rightarrow \Psi_L^c,
 \end{aligned} \tag{160}$$

with the covariant derivative defined as

$$D_\mu = \partial_\mu - \sqrt{2}ig_W(W_{1\mu}^a Q_1^a + W_{2\mu}^a Q_2^a) - \sqrt{2}ig'(Y_1 B_{1\mu} + Y_2 B_{2\mu}). \tag{161}$$

In order to assign the proper SM hypercharge $y = -1$ to the charged right-handed leptons ℓ_R , which are $SU(5)$ singlets and T-even, the corresponding gauge and T-invariant Lagrangian is similar to the SM one

$$\mathcal{L}'_F = i\bar{\ell}_R\gamma^\mu(\partial_\mu + ig'yB_\mu)\ell_R. \tag{162}$$

4.4 Top Sector

In order to avoid dangerous contributions to the Higgs mass from one loop quadratic divergences, the third generation Yukawa sector must be modified. The Ψ_1 and Ψ_2 multiplets for the third generation must be completed to representations of the $SU(3)_1$ (upper-left corner) and $SU(3)_2$ (lower-right corner) subgroups of $SU(5)$. These multiplets are (we change the notation $U_{Li} \rightarrow t'_{Li}$ and $U_{Ri} \rightarrow t'_{Ri}$) [36]

$$Q_1 = \begin{pmatrix} q_1 \\ t'_{L1} \\ 0 \end{pmatrix}, \quad Q_2 = \begin{pmatrix} 0 \\ t'_{L2} \\ q_2 \end{pmatrix}, \tag{163}$$

they obey the same transformation laws under T-parity and the $SU(5)$ symmetry as Ψ_1 and Ψ_2 . The quark doublets are embedded such that

$$q_i = -\sigma^2 \begin{pmatrix} u_{Li} \\ b_{Li} \end{pmatrix}. \tag{164}$$

In addition to the SM right-handed top quark field u_{3R} , which is assumed to be T-even, the model contains two $SU(2)_L$ -singlet fermions t'_{R1} and t'_{R2} of hypercharge $2/3$, which transform under T-parity as

$$t'_{R1} \leftrightarrow -t'_{R2}. \tag{165}$$

The top Yukawa couplings arise from the Lagrangian of the form

$$\begin{aligned} \mathcal{L}_t = & \frac{1}{2\sqrt{2}} \lambda_1 f \epsilon_{ijk} \epsilon_{xy} \left[(\bar{Q}_1)_i (\Sigma)_{jx} (\Sigma)_{ky} - (\bar{Q}_2 \Sigma_0)_i (\tilde{\Sigma})_{jx} (\tilde{\Sigma})_{ky} \right] u_{3R} \\ & + \lambda_2 f \left(\bar{t}'_{L1} t'_{R1} + \bar{t}'_{L2} t'_{R2} \right) + h.c. \end{aligned} \quad (166)$$

where $\tilde{\Sigma} = \Sigma_0 \Omega \Sigma^\dagger \Omega \Sigma_0$, is the image of Σ under T-parity and the indices $i, j = 1, 2, 3$ and $x, y = 4, 5$. In Appendix E, we calculated the T-parity eigenstates

$$q_\pm = \frac{1}{\sqrt{2}} (q_1 \mp q_2), \quad t'_{L\pm} = \frac{1}{\sqrt{2}} (t'_{L1} \mp t'_{L2}), \quad t'_{R\pm} = \frac{1}{\sqrt{2}} (t'_{R1} \mp t'_{R2}). \quad (167)$$

Thus, the Lagrangian is

$$\mathcal{L}_t = \lambda_1 f \left(\frac{1}{2} (1 + c_v) \bar{t}'_{L+} + \frac{s_v}{\sqrt{2}} \bar{u}_{L+} \right) u_{3R} + \lambda_2 f \left(\bar{t}'_{L+} t'_{R+} + \bar{t}'_{L-} t'_{R-} \right) + h.c. \quad (168)$$

The T-odd states t'_{L-} and t'_{R-} combine to form a Dirac fermion T_- , with mass $M_{T_-} = \lambda_2 f$.

We can see that the T-odd top sector doesn't mix with the T-even heavy and the T-even SM top quark. The mass terms for the T-even states are diagonalized by [36]

$$t_L = \cos \beta u_{L+} - \sin \beta t'_{L+}, \quad t_R = \cos \alpha u_{3R} - \sin \alpha t'_{R+}, \quad (169)$$

$$T_{L+} = \sin \beta u_{L+} + \cos \beta t'_{L+}, \quad T_{R+} = \sin \alpha u_{3R} + \cos \alpha t'_{R+}, \quad (170)$$

where t is identified with the SM top and T_+ is its T-even heavy partner.

The masses of the two T-even Dirac fermions to leading order in (v_h/f) [36]

$$m_t = \frac{\lambda_1 \lambda_2 v_h}{\sqrt{\lambda_1^2 + \lambda_2^2}}, \quad m_{T_+} = \sqrt{\lambda_1^2 + \lambda_2^2} f. \quad (171)$$

As the Yukawa couplings of the other SM quarks are small, there is no need to introduce additional heavy partners to cancel their quadratically divergent contribution of the Higgs mass. Then the Yukawa coupling for the other up-type and down-type fermions is given by

$$\begin{aligned} \mathcal{L}_{up} = & -\frac{1}{2\sqrt{2}} \lambda_u f \epsilon_{ijk} \epsilon_{xy} \left[(\bar{Q}_1)_i (\Sigma)_{jx} (\Sigma)_{ky} - (\bar{Q}_2 \Sigma_0)_i (\tilde{\Sigma})_{jx} (\tilde{\Sigma})_{ky} \right] u_R + h.c., \\ \mathcal{L}_{down} = & \frac{i\lambda_d}{2\sqrt{2}} f \epsilon_{ij\epsilon_{xyz}} \left[(\bar{\Psi}_2)_x (\Sigma)_{iy} (\Sigma)_{jz} X - (\bar{\Psi}_1 \Sigma_0)_x (\tilde{\Sigma})_{iy} (\tilde{\Sigma})_{jz} \tilde{X} \right] d_R + h.c., \end{aligned} \quad (172)$$

and their masses are [39]

$$\begin{aligned} m_u^i &= \lambda_u^i v_h \left(1 - \frac{v_h^2}{3f^2} \right), \\ m_d^j &= \lambda_d^j v_h \left(1 - \frac{v_h^2}{12f^2} \right), \end{aligned} \quad (173)$$

with $i = 1, 2$ and $j = 1, 2, 3$. In Appendix E we show the process for obtained the above masses.

4.5 Feynman Rules for the LHT

In this section we show a list of Feynman rules of the T-parity, for this we need to defined the mass eigenstates.

As shown above, the the matrix of NGB is

$$\Pi = \begin{pmatrix} \chi + \frac{\eta}{2\sqrt{5}} & \frac{h}{\sqrt{2}} & \phi \\ \frac{h^\dagger}{\sqrt{2}} & \frac{2\eta}{\sqrt{5}} & \frac{h^T}{\sqrt{2}} \\ \phi^\dagger & \frac{h^*}{\sqrt{2}} & \chi^T + \frac{\eta}{2\sqrt{5}} \end{pmatrix},$$

where $\chi = \chi^a \sigma^a / 2$ with σ^a the three Pauli matrices, η is a real singlet, and $\phi = \begin{pmatrix} -i\Phi^{++} & -i\frac{\Phi^+}{\sqrt{2}} \\ -i\frac{\Phi^+}{\sqrt{2}} & \frac{-\Phi^0 + \Phi^P}{\sqrt{2}} \end{pmatrix}$ [44]. Since $\chi = \chi^a \sigma^a / 2$ has 3 free parameters, for convenience we can write it as

$$\chi = \begin{pmatrix} -\omega^0/2 - \eta/\sqrt{20} & -\omega^+/\sqrt{2} \\ -\omega^-/\sqrt{2} & \omega^0/2 - \eta/\sqrt{20} \end{pmatrix}, \quad (174)$$

and the Higgs written with the NGB is

$$h = \begin{pmatrix} -i\pi^+/\sqrt{2} \\ \frac{v_h + h + i\pi^0}{2} \end{pmatrix}, \quad (175)$$

thus, writing explicitly all the Goldstone bosons the matrix Π is [36]

$$\Pi = \begin{pmatrix} -\omega^0/2 - \eta/\sqrt{20} & -\omega^+/\sqrt{2} & -i\pi^+/\sqrt{2} & -i\Phi^{++} & -i\frac{\Phi^+}{\sqrt{2}} \\ -\omega^-/2 & \omega^0/2 - \eta/\sqrt{20} & \frac{v_h + h + i\pi^0}{2} & -i\frac{\Phi^+}{\sqrt{2}} & \frac{-i\Phi^0 + \Phi^P}{\sqrt{2}} \\ -i\pi^-/\sqrt{2} & (v_h + h - i\pi^0)/2 & \sqrt{\frac{4}{5}}\eta & -i\pi^+/\sqrt{2} & (v_h + h + i\pi^0)/2 \\ i\Phi^{--} & i\frac{\Phi^-}{\sqrt{2}} & i\pi^-/\sqrt{2} & -\omega^0/2 - \eta/\sqrt{20} & -\omega^-/\sqrt{2} \\ i\frac{\Phi^-}{\sqrt{2}} & \frac{i\Phi^0 + \Phi^P}{\sqrt{2}} & \frac{v_h + h - i\pi^0}{2} & -\omega^+/\sqrt{2} & \omega^0/2 - \eta/\sqrt{20} \end{pmatrix}, \quad (176)$$

where the superscripts indicate the electric charge. The fields ω and η are eaten when the extended gauge group is broken down to $SU(2)_L \times U(1)_Y$, whereas the π fields are absorbed by the SM W/Z

bosons after EWSB. The fields h and Φ remain in the spectrum.

Taking the Scalar Lagrangian in the mass eigenbasis, the mixing terms have the form

$$\mathcal{L}_S = M_{W_H^-} W_H^{\mu-} \left[\partial_\mu \omega^+ \left(1 - \frac{1}{24} \epsilon^2 \right) - \frac{i}{6} \epsilon^2 \partial_\mu \Phi^+ \right] + M_{W_L^-} W_L^{\mu-} \partial_\mu \pi^+ \left(1 - \frac{1}{12} \epsilon^2 \right) + h.c., \quad (177)$$

with $\epsilon = \frac{v_h}{f}$. It is necessary to extend until fifth order the Σ field for obtained the precise $(1 - \frac{1}{12} \epsilon^2)$. We can see that the Goldstone bosons fields in the Matrix Π are not canonically normalized. So we normalize the Goldstone bosons fields the following way

$$\begin{aligned} \pi^\pm &\rightarrow \pi^\pm \left(1 + \frac{1}{12} \epsilon^2 \right), \\ \omega^\pm &\rightarrow \omega^\pm \left(1 + \frac{1}{24} \epsilon^2 \right), \\ \Phi^\pm &\rightarrow \Phi^\pm \left(1 + \frac{1}{24} \epsilon^2 \right). \end{aligned} \quad (178)$$

After this rescaling, there are still kinetic mixing terms involving the ω and ϕ fields

$$\mathcal{L}_S = \partial_\mu \omega^+ \partial^\mu \omega^- + \partial_\mu \Phi^+ \partial^\mu \Phi^- + \frac{i}{12} \epsilon^2 (\partial_\mu \omega^+ \partial^\mu \Phi^- - \partial_\mu \Phi^+ \partial^\mu \omega^-), \quad (179)$$

we need to diagonalize the matrix

$$\partial_\mu \begin{pmatrix} \omega^+ & \Phi^+ \end{pmatrix} \begin{pmatrix} 1 & \frac{i}{12} \epsilon^2 \\ -\frac{i}{12} \epsilon^2 & 1 \end{pmatrix} \partial^\mu \begin{pmatrix} \omega^- \\ \Phi^- \end{pmatrix}. \quad (180)$$

thus, we have as result

$$\begin{aligned} \omega'^\pm &= \omega^\pm \mp \frac{i}{24} \epsilon^2 \Phi^\pm, \\ \Phi'^\pm &= \Phi^\pm \pm \frac{i}{24} \epsilon^2 \omega^\pm. \end{aligned} \quad (181)$$

Introducing these new states, the Lagrangian is given by

$$\mathcal{L}_S = M_{W_H^-} W_H^{\mu-} \left[\partial_\mu \omega'^+ - \frac{i}{8} \epsilon^2 \partial_\mu \Phi'^+ \right] + M_{W_L^-} W_L^{\mu-} \partial_\mu \pi^+ + h.c. \quad (182)$$

Finally, a last rotation given by

$$\begin{aligned} \omega''^\pm &= \omega'^\pm \left(1 + \frac{v_h^2}{24f^2} \right) \mp i \Phi'^\pm \left(1 + \frac{v_h^2}{24f^2} \right) \frac{v_h^2}{6f^2}, \\ \Phi''^\pm &= \Phi'^\pm \left(1 + \frac{v_h^2}{24f^2} \right) \mp i \omega'^\pm \left(1 + \frac{v_h^2}{24f^2} \right) \frac{v_h^2}{12f^2}, \end{aligned} \quad (183)$$

ω''^\pm is the combination of Goldstones eaten by the heavy gauge bosons, and Φ''^\pm is the uneaten combination.

A similar process but a heavier algebra, we can obtain the other mass eigenstates [43]

$$\begin{aligned}
 \pi^0 &\rightarrow \pi^0 \left(1 + \frac{v_h^2}{12f^2} \right), \\
 \pi^\pm &\rightarrow \pi^\pm \left(1 + \frac{v_h^2}{12f^2} \right), \\
 h &\rightarrow h, \\
 \Phi^0 &\rightarrow \Phi^0 \left(1 + \frac{v_h^2}{12f^2} \right), \\
 \Phi^P &\rightarrow \Phi^P + \left(\sqrt{10}\eta - \sqrt{2}\omega^0 + \Phi^P \right) \frac{v_h^2}{12f^2}, \\
 \Phi^{++} &\rightarrow \Phi^{++}, \\
 \eta &\rightarrow \eta + \frac{5g'\eta - 4\sqrt{5} [g'(\omega^0 + \sqrt{2}\Phi^P) - 6g_W x_H \omega^0] v_h^2}{24g'}, \\
 \omega^0 &\rightarrow \omega^0 + \frac{5g_W(\omega^0 + 4\sqrt{2}\Phi^P) - 4\sqrt{5}\eta(5g_W + 6g'x_H) v_h^2}{120g_W f^2}.
 \end{aligned} \tag{184}$$

The other mass eigenstates that were seen before are W_L^\pm , Z_L , A_L , W_H^\pm , and A_H .

Now, we show the Feynman rules for the LHT in the mass eigenstates basis, they are given in terms of generic couplings for the following general vertices involving scalars (S), fermions (F), and/or gauge bosons (V)

$$\begin{aligned}
 [SFF] &= i(c_L P_L + c_R P_R), \\
 [SV_\mu V_\nu] &= iK g^{\mu\nu}, \\
 [V_\mu FF] &= i\gamma^\mu (g_L P_L + g_R P_R), \\
 [SSFF] &= i(f_L P_L + f_R P_R), \\
 [S(p_1)S(p_2)V_\mu] &= iG(p_1 - p_2)^\mu, \\
 [SS(p_1)S(p_2)] &= iJ(p_1^2 + p_2^2 + 4p_1 \cdot p_2),
 \end{aligned} \tag{185}$$

where all momenta are assumed incoming. The conjugate vertices are obtained replacing

$$c_{L,R} \leftrightarrow c_{R,L}^*, \quad K \leftrightarrow K^*, \quad g_{L,R} \leftrightarrow g_{L,R}^*, \quad f_{L,R} \leftrightarrow f_{R,L}^*, \quad G \leftrightarrow G^*, \quad J \leftrightarrow J^*. \tag{186}$$

We adopt the convention Feynman rule = $i\mathcal{L}$.

The Higgs couplings with the gauge bosons are given in the Table 4.

Moreover, we show a list $[SV_\mu V_\nu]$ and $[S(p_1)S(p_2)V_\mu]$ couplings at $\mathcal{O}(v_h^2/f^2)$ in the Table 5 and Table 6, necessary to calculate LFV amplitudes mediated by gauge bosons.

$[SV_\mu V_\nu]$	Vertices (K)	$[SV_\mu V_\nu]$	Vertices (K)
$hW_{L\mu}^+ W_{L\nu}^-$	$i\frac{g_W^2 v_h}{2} \left(1 - \frac{v_h^2}{3f^2}\right)$	$hW_{H\mu}^+ W_{H\nu}^-$	$-i\frac{g_W^2 v_h}{2} \left(1 - \frac{v_h^2}{3f^2}\right)$
$hZ_{L\mu} Z_{L\nu}$	$i\frac{v_h g_W^2}{4 \cos^2 \theta_W} \left(1 - \frac{v_h^2}{3f^2}\right)$	$hZ_{H\mu} Z_{H\nu}$	$-i\frac{g_W^2 v_h}{4} \left(1 - \frac{4v_h^2}{3f^2}\right)$
$hA_{L\mu} A_{L\nu}$	0	$hA_{H\mu} A_{H\nu}$	$-i\frac{g'^2 v_h}{4} \left(1 - \frac{4v_h^2}{3f^2}\right)$
-	-	$hZ_H A_H$	$-i\frac{g_W g' v_h}{2} \left(1 - \frac{4v_h^2}{3f^2}\right)$

 Table 4: Higgs couplings with gauge bosons at $\mathcal{O}(v_h^2/f^2)$.

$[SV_\mu V_\nu]$	Vertices (K)	$[SV_\mu V_\nu]$	Vertices (K)
$\Phi^0 Z_{H\mu} Z_{L\nu}$	$-i\frac{g_W^2}{2\sqrt{2}c_W} \frac{v_h^2}{f}$	$\omega^- W_{H\mu}^+ A_{L\nu}$	$g_W s_W M_{W_H}$
$\Phi^0 A_{H\mu} Z_{L\nu}$	$i\frac{g'^2}{2\sqrt{2}s_W} \frac{v_h^2}{f}$	$\omega^+ W_{H\mu}^- Z_{L\nu}$	$g_W c_W M_{W_H} \left(1 - \frac{v_h^2}{4c_W^2 f^2}\right)$
$\Phi^+ W_{H\mu}^- Z_{L\nu}$	$i\frac{g_W^2}{c_W} \frac{v_h^2}{4f}$	$\omega^- W_{H\mu}^+ Z_{L\nu}$	$-g_W c_W M_{W_H} \left(1 - \frac{v_h^2}{4c_W^2 f^2}\right)$
$\omega^+ W_{H\mu}^- A_{L\nu}$	$-g_W s_W M_{W_H}$	-	-

Table 5: Gauge boson-Goldstone boson interactions.

$[S(p_1)S(p_2)V_\mu]$	Vertices (G)	$[S(p_1)S(p_2)V_\mu]$	Vertices (G)
$\Phi^0 \Phi^P Z_{L\mu}$	$-\frac{g_W}{c_W} \left(1 - \frac{v_h^2}{4f^2}\right)$	$\Phi^0 \omega^0 Z_{L\mu}$	$\frac{g_W}{\sqrt{2}c_W} \frac{v_h^2}{4f^2}$
$\Phi^+ \Phi^- Z_{L\mu}$	$-ig' s_W \left(1 - \frac{v_h^2}{8s_W^2 f^2}\right)$	$\Phi^+ \Phi^- A_{L\mu}$	$-ie$
$\Phi^+ \omega^- Z_{L\mu}$	$-\frac{g_W}{c_W} \frac{v_h^2}{8f^2}$	$\Phi^{++} \Phi^{--} Z_{L\mu}$	$i\frac{g_W}{c_W} \left(1 - 2s_W^2\right)$
$\Phi^{++} \Phi^{--} A_{L\mu}$	$-2ie$	$\omega^+ \omega^- A_{L\mu}$	$-igs_W$
$\omega^+ \omega^- Z_{L\mu}$	$ig_W c_W \left(1 - \frac{v_h^2}{8c_W^2 f^2}\right)$	-	-

 Table 6: Non-vanishing Scalar-Scalar-Vector couplings [45]. We adopt the the convention Feynman rule = $i\mathcal{L}$.

The Fermion couplings must also be included in the Littlest Higgs with T-parity, this sector is too different than the original Littlest Higgs model. The SM fermions of the first two generation of quarks and all three generations of leptons have their usual SM couplings. Nevertheless, in the third generation the new Yukawa interactions required to cancel the quadratic divergences of the top quark will shift some SM couplings at $\mathcal{O}(v_h^2/f^2)$.

We need to introduce kinetic terms for the fermions. Recall Ψ_1 and Ψ_2 , from the eq. (141), these

transform linearly, their kinetic terms are

$$\begin{aligned}
 \mathcal{L}_{Y_H} &= i\bar{\Psi}_1\gamma^\mu D_\mu^*\Psi_1 + i\bar{\Psi}_2\gamma^\mu D_\mu\Psi_2 \\
 &+ i\bar{\Psi}_R^c\gamma^\mu \left(\partial_\mu - \frac{1}{2}\xi^\dagger (D_\mu\xi) - \frac{1}{2}\xi \left(\Sigma_0 D_\mu^* \Sigma_0 \xi^\dagger \right) \right) \Psi_R^c \\
 &+ \Psi_R^c \rightarrow \Psi_L^c,
 \end{aligned} \tag{187}$$

with the covariant derivative defined as

$$D_\mu = \partial_\mu - \sqrt{2}ig_W (W_{1\mu}^a Q_1^a + W_{2\mu}^a Q_2^a) - \sqrt{2}ig' (Y_1 B_{1\mu} + Y_2 B_{2\mu}), \tag{188}$$

and

$$\Psi_L^c = \begin{pmatrix} 0 \\ 0 \\ -i\sigma^2 \begin{pmatrix} \bar{\nu}_R^c \\ \bar{\ell}_R^c \end{pmatrix} \end{pmatrix}. \tag{189}$$

The $U(1)$ charges $Y_i^{(\Psi_j)}$ are given in Table 7 [37] (in this Table we change the signs of q_1 and ℓ_1 in order to obtain the SM results) which can be obtained from gauge invariance of the Yukawa couplings and T-parity. The fermion-gauge boson couplings can be obtained from the kinetic term by inserting the mass eigenstates of all particles involved.

q_1	$(\mathbf{2}, -1/30; \mathbf{1}, -2/15)$	q_2	$(\mathbf{1}, 2/15; \mathbf{2}, 1/30)$
t'_1	$(\mathbf{1}, 8/15; \mathbf{1}, 2/15)$	t'_2	$(\mathbf{1}, 2/15; \mathbf{1}, 8/15)$
t'_{1R}	$(\mathbf{1}, 8/15; \mathbf{1}, 2/15)$	t'_{2R}	$(\mathbf{1}, 2/15; \mathbf{1}, 8/15)$
u_{3R}	$(\mathbf{1}, 1/3; \mathbf{1}, 1/3)$	d_R	$(\mathbf{1}, -1/6; \mathbf{1}, -1/6)$
ℓ_1	$(\mathbf{2}, 1/5; \mathbf{1}, 3/10)$	ℓ_2	$(\mathbf{1}, -3/10; \mathbf{1}, -1/5)$
e_R	$(\mathbf{1}, -1/2; \mathbf{1}, -1/2)$	-	-

Table 7: The $[SU(2)_1 \times U(1)_1] \times [SU(2)_2 \times U(1)_2]$ quantum numbers of the fermion fields.

In order to assign the proper SM hypercharge $y = -1$ to the charged right-handed leptons ℓ_R , which are $SU(5)$ singlets and T-even, the corresponding gauge and T-invariant Lagrangian is similar to the SM one

$$\mathcal{L}'_F = i\bar{\ell}_R\gamma^\mu (\partial_\mu + ig'yB_\mu) \ell_R. \tag{190}$$

From the Lagrangians (187) and (189) we obtain the Feynman rules that we show on the Table 8.

$[V_\mu FF]$	g_L	g_R
$\bar{\ell}_i A_{L\mu} \ell_j$	$-\delta_{ij} e$	$-\delta_{ij} e$
$\bar{\ell}_i Z_{L\mu} \ell_j$	$\delta_{ij} \frac{g_W}{2c_W} (-1 + 2s_W^2)$	$\delta_{ij} g' s_W$
$\bar{\nu}_i Z_{L\mu} \nu_j$	$-\delta_{ij} \frac{g_W}{2c_W}$	0
$\bar{\nu}_i W_{L\mu}^+ \ell_j$	$\frac{g_W}{\sqrt{2}} (V_{PMNS})_{ij}$	0
$\bar{\ell}_i W_{L\mu}^- \nu_j$	$\frac{g_W}{\sqrt{2}} (V_{PMNS})_{ij}$	0
$\bar{\ell}_i^c A_{L\mu} \ell_j^c$	$\delta_{ij} e$	$\delta_{ij} e$
$\bar{\ell}_i^c Z_{L\mu} \ell_j^c$	$-\delta_{ij} \frac{g_W}{2c_W} (-1 + 2s_W^2)$	$-\delta_{ij} \frac{g_W}{2c_W} (-1 + 2s_W^2)$
$\bar{\nu}_i^c Z_{L\mu} \nu_j^c$	$-\delta_{ij} \frac{g_W}{2c_W} \left(1 - \frac{v_h^2}{4f^2}\right)$	$-\delta_{ij} \frac{g_W}{2c_W} \left(1 - \frac{v_h^2}{4f^2}\right)$
$\bar{\ell}_i^c A_{L\mu} \ell_j^c$	$-\delta_{ij} e$	$-\delta_{ij} e$
$\bar{\ell}_i^c Z_{L\mu} \ell_j^c$	$\delta_{ij} \frac{g_W}{2c_W} (-1 + 2s_W^2)$	$\delta_{ij} \frac{g_W}{2c_W} (-1 + 2s_W^2)$
$\bar{\nu}_i^c Z_{L\mu} \nu_j^c$	$\delta_{ij} \frac{g_W}{2c_W}$	$\delta_{ij} \frac{g_W}{2c_W} \left(1 - \frac{v_h^2}{4f^2}\right)$
$\bar{u}_i A_{L\mu} u_j$	$\delta_{ij} \frac{2}{3} e$	$\delta_{ij} \frac{2}{3} e$
$\bar{u}_i Z_{L\mu} u_j$	$\delta_{ij} \frac{g_W}{2c_W} \left(1 - \frac{4}{3} s_W^2\right)$	$-\delta_{ij} \frac{2}{3} g' s_W$
$\bar{d}_i A_{L\mu} d_j$	$-\delta_{ij} \frac{1}{3} e$	$-\delta_{ij} \frac{1}{3} e$
$\bar{d}_i Z_{L\mu} d_j$	$\delta_{ij} \frac{g_W}{2c_W} \left(-1 + \frac{2}{3} s_W^2\right)$	$\delta_{ij} \frac{1}{3} g' s_W$
$\bar{u}_i W_{L\mu}^+ d_j$	$\frac{g_W}{\sqrt{2}} (V_{CKM})_{ij}$	0
$\bar{d}_i W_{L\mu}^- u_j$	$\frac{g_W}{\sqrt{2}} (V_{CKM})_{ij}$	0
$\bar{u}_{Hi} Z_{L\mu} u_{Hj}$	$\delta_{ij} \frac{g_W}{2c_W} \left(1 - \frac{4}{3} s_W^2\right)$	$\delta_{ij} \frac{g_W}{2c_W} \left(1 - \frac{4}{3} s_W^2\right)$
$\bar{d}_{Hi} Z_{L\mu} d_{Hj}$	$\delta_{ij} \frac{g_W}{2c_W} \left(-1 + \frac{2}{3} s_W^2\right)$	$\delta_{ij} \frac{g_W}{2c_W} \left(-1 + \frac{2}{3} s_W^2\right)$

Table 8: Fermion couplings to SM gauge bosons.

In [38], [39], [45] the notation changes to $\nu_i^c \rightarrow \nu_{Hi}$ and $\ell_i^c \rightarrow \ell_{Hi}$. Now, in the Table 9 we show the fermion couplings to heavy gauge bosons [46].

$[V_\mu FF]$	g_L	g_R
$\bar{\ell}_{Hi} A_{H\mu} \ell_j$	$\frac{g_W}{2} \left(\frac{t_W}{5} - x_H \frac{v_h^2}{f^2}\right) V_{ij}$	0
$\bar{\ell}_{Hi} Z_{H\mu} \ell_j$	$-\frac{g_W}{2} \left(1 + x_H \frac{t_W}{5} \frac{v_h^2}{f^2}\right) V_{ij}$	0
$\bar{\nu}_{Hi} W_{H\mu}^+ \ell_j$	$\frac{g_W}{\sqrt{2}} V_{ij}$	0
$\bar{u}_{Hi} A_{H\mu} u_j$	$\frac{g_W}{2} \left(\frac{t_W}{5} + x_H \frac{v_h^2}{f^2}\right) V_{ij}^u$	0
$\bar{d}_{Hi} A_{H\mu} d_j$	$\frac{g_W}{2} \left(\frac{t_W}{5} - x_H \frac{v_h^2}{f^2}\right) V_{ij}^d$	0
$\bar{u}_{Hi} Z_{H\mu} u_j$	$\frac{g_W}{2} \left(1 - x_H \frac{t_W}{5} \frac{v_h^2}{f^2}\right) V_{ij}^u$	0
$\bar{d}_{Hi} Z_{H\mu} d_j$	$-\frac{g_W}{2} \left(1 + x_H \frac{t_W}{5} \frac{v_h^2}{f^2}\right) V_{ij}^d$	0
$\bar{u}_{Hi} W_{H\mu}^+ d_j$	$\frac{g_W}{\sqrt{2}} V_{ij}^d$	0
$\bar{d}_{Hi} W_{H\mu}^- u_j$	$\frac{g_W}{\sqrt{2}} V_{ij}^u$	0

Table 9: Fermion couplings to heavy gauge bosons.

In Table 9, V_{ij} are the matrix elements of the 3×3 unitary mixing matrix parametrizing the misalignment between the SM left-handed charged leptons ℓ with the heavy mirror ones ℓ_H .

As well as in the lepton sector, the misalignment between the partner quark mass eigenstates and the mirror quarks as well as those between the mirror and SM ones are parametrized by the corresponding 3×3 unitary matrices [46]

$$V^u = V_L^{qH\dagger} V_L^u, \quad V^d = V_L^{qH\dagger} V_L^d, \quad W^q = \tilde{V}_R^{q\dagger} V_R^{qH} \quad (191)$$

where the products are of the unitary matrices rotating left and right handed fields in order to diagonalize the various mass matrices. The only physical combination is the Cabbibo-Kobayashi-Maskawa matrix $V_{CKM} = V^{u\dagger} V^d$ relating the SM up and down quark sectors.

Finally we written in the Table 10 and Table 11 [39] the Feynman rules for the fermion couplings to SM and heavy Goldstone bosons obtained from Lagrangians (148) and (172).

$[SFF]$	c_L	c_R
$\bar{u}^i \pi^+ d^j$	$\frac{g_W}{\sqrt{2}M_{W_L}} m_u^i (V_{CKM})_{ij}$	$-\frac{g_W}{\sqrt{2}M_{W_L}} m_d^j (V_{CKM})_{ij}$
$\bar{u}^i \pi^0 u^j$	$-\frac{g_W m_u^i}{2M_{Z_L} \cos \theta_W} \delta_{ij}$	$\frac{g_W m_u^i}{2M_{Z_L} \cos \theta_W} \delta_{ij}$
$\bar{d}^i \pi^0 d^j$	$\frac{g_W m_d^i}{2M_{Z_L} \cos \theta_W} \delta_{ij}$	$-\frac{g_W m_d^i}{2M_{Z_L} \cos \theta_W} \delta_{ij}$
$\bar{u}_H^i \pi^+ d_H^j$	$-\frac{g_W}{8\sqrt{2}M_{W_L}} \frac{v_h^2}{f^2} m_{Hi}^d \delta_{ij}$	$-\frac{g_W}{8\sqrt{2}M_{W_L}} \frac{v_h^2}{f^2} m_{Hi}^d \delta_{ij}$
$\bar{\nu}^i \pi^+ \ell^j$	$\frac{g_W}{\sqrt{2}M_{W_L}} m_\nu^i (V_{PMNS})_{ij}$	$-\frac{g_W}{\sqrt{2}M_{W_L}} m_\ell^j (V_{PMNS})_{ij}$
$\bar{\nu}^i \pi^0 \nu^j$	$-\frac{g_W m_\nu^i}{2M_{Z_L} \cos \theta_W} \delta_{ij}$	$\frac{g_W m_\nu^i}{2M_{Z_L} \cos \theta_W} \delta_{ij}$
$\bar{\ell}^i \pi^0 \ell^j$	$\frac{g_W m_\ell^i}{2M_{Z_L} \cos \theta_W} \delta_{ij}$	$-\frac{g_W m_\ell^i}{2M_{Z_L} \cos \theta_W} \delta_{ij}$
$\bar{\nu}_H^i \pi^+ \ell_H^j$	$-\frac{g_W}{8\sqrt{2}M_{W_L}} \frac{v_h^2}{f^2} m_{Hi}^\ell \delta_{ij}$	$-\frac{g_W}{8\sqrt{2}M_{W_L}} \frac{v_h^2}{f^2} m_{Hi}^\ell \delta_{ij}$

Table 10: Fermion couplings to SM Goldstone bosons.

Since the lepton structure is the same for the quarks, we can take the results already obtained above, lepton masses are generated in a completely analogous way. After EWSB, the masses of ν_H and ℓ_H are similar to the T-odd up and down-type quarks from eq. (159)

$$m_{\nu_H} = \sqrt{2}\kappa_1 f \left(1 - \frac{v_h^2}{8f^2}\right), \quad m_{\ell_H} = \sqrt{2}\kappa_1 f. \quad (192)$$

The masses from Yukawa couplings is given by eq. (173),

$$m_\nu^i = \lambda_\nu^i v_h \left(1 - \frac{v_h^2}{3f^2}\right), \quad m_\ell^j = \lambda_\ell^j v_h \left(1 - \frac{v_h^2}{12f^2}\right), \quad (193)$$

with $i = 1, 2$ and $j = 1, 2, 3$.

$[SFF]$	c_L	c_R
$\bar{u}_H^i \omega^+ d^j$	$\frac{g_W}{\sqrt{2}M_{W_H}} m_{H_i}^u (V_{Hd})_{ij}$	$-\frac{g_W}{\sqrt{2}M_{W_H}} m_d^j (V_{Hd})_{ij}$
$\bar{u}_H^i \omega^0 u^j$	$\frac{g_W}{2M_{Z_H}} m_{H_i}^u \left(1 + \frac{v_h^2}{f^2} \left(\frac{1}{8} - \frac{x_H}{\tan \theta_W}\right)\right) (V_{Hu})_{ij}$	$-\frac{g_W}{2M_{Z_H}} m_u^j (V_{Hu})_{ij}$
$\bar{u}_H^i \eta u^j$	$-\frac{g'}{10M_{A_H}} m_{H_i}^u \left(1 + \frac{v_h^2}{f^2} \left(\frac{5}{8} + x_H \tan \theta_W\right)\right) (V_{Hu})_{ij}$	$\frac{g'}{10M_{A_H}} m_u^j (V_{Hu})_{ij}$
$\bar{d}_H^i \omega^- u^j$	$\frac{g_W}{\sqrt{2}M_{W_H}} m_{H_i}^d \left(1 - \frac{v_h^2}{8f^2}\right) (V_{Hu})_{ij}$	$-\frac{g_W}{\sqrt{2}M_{W_H}} m_u^j (V_{Hu})_{ij}$
$\bar{d}_H^i \omega^0 d^j$	$-\frac{g_W}{2M_{Z_H}} m_{H_i}^d \left(1 + \frac{v_h^2}{f^2} \left(-\frac{1}{4} + \frac{x_H}{\tan \theta_W}\right)\right) (V_{Hd})_{ij}$	$\frac{g_W}{2M_{Z_H}} m_d^j (V_{Hd})_{ij}$
$\bar{d}_H^i \eta d^j$	$-\frac{g'}{10M_{A_H}} m_{H_i}^d \left(1 - \frac{v_h^2}{f^2} \left(\frac{5}{4} + x_H \tan \theta_W\right)\right) (V_{Hd})_{ij}$	$\frac{g'}{10M_{A_H}} m_d^j (V_{Hd})_{ij}$
$\bar{\nu}_H^i \omega^+ \ell^j$	$\frac{g_W}{\sqrt{2}M_{W_H}} m_{H_i}^\nu (V_{H\ell})_{ij}$	$-\frac{g_W}{\sqrt{2}M_{W_H}} m_\ell^j (V_{H\ell})_{ij}$
$\bar{\nu}_H^i \omega^0 \nu^j$	$\frac{g_W}{2M_{Z_H}} m_{H_i}^\nu \left(1 + \frac{v_h^2}{f^2} \left(\frac{1}{8} - \frac{x_H}{\tan \theta_W}\right)\right) (V_{H\nu})_{ij}$	$-\frac{g_W}{2M_{Z_H}} m_\nu^j (V_{H\nu})_{ij}$
$\bar{\nu}_H^i \eta \nu^j$	$-\frac{g'}{10M_{A_H}} m_{H_i}^\nu \left(1 + \frac{v_h^2}{f^2} \left(\frac{5}{8} + x_H \tan \theta_W\right)\right) (V_{H\nu})_{ij}$	$\frac{g'}{10M_{A_H}} m_\nu^j (V_{H\nu})_{ij}$
$\bar{\ell}_H^i \omega^- \nu^j$	$\frac{g_W}{\sqrt{2}M_{W_H}} m_{H_i}^\ell \left(1 - \frac{v_h^2}{8f^2}\right) (V_{H\nu})_{ij}$	$-\frac{g_W}{\sqrt{2}M_{W_H}} m_\nu^j (V_{H\nu})_{ij}$
$\bar{\ell}_H^i \omega^0 \ell^j$	$-\frac{g_W}{2M_{Z_H}} m_{H_i}^\ell \left(1 + \frac{v_h^2}{f^2} \left(-\frac{1}{4} + \frac{x_H}{\tan \theta_W}\right)\right) (V_{H\ell})_{ij}$	$\frac{g_W}{2M_{Z_H}} m_\ell^j (V_{H\ell})_{ij}$
$\bar{\ell}_H^i \eta \ell^j$	$-\frac{g'}{10M_{A_H}} m_{H_i}^\ell \left(1 - \frac{v_h^2}{f^2} \left(\frac{5}{4} + x_H \tan \theta_W\right)\right) (V_{H\ell})_{ij}$	$\frac{g'}{10M_{A_H}} m_\ell^j (V_{H\ell})_{ij}$

Table 11: Fermion couplings to heavy Goldstone bosons.

4.6 T-odd Flavor Mixing

We start reviewing briefly the process like in the SM.

The Yukawa sector generates mass matrices for the three up-type quarks given by $M_{u_j}^i$ after EWSB which is diagonalized by 2×2 matrices, U and V [42]

$$(M_u)_j^i = (V_u)_k^i (M_u^D)_l^k (U_u^\dagger)_j^l. \quad (194)$$

The gauge eigenstates are then expressed in terms of (the primed) mass eigenstates by

$$u_L^i = (V_u)_j^i u_L'^j, \quad u_R^i = (U_u)_j^i u_R'^j. \quad (195)$$

A similar procedure applies to the down-type quark mass matrix. The change to the mass eigenbasis leaves most of the gauge interaction portion of the Lagrangian invariant.

$$\frac{g}{\sqrt{2}} [\bar{u}^i \gamma^\mu W_\mu^+ P_L d^i + \bar{d}^i \gamma^\mu W_\mu^- P_L u^i] = \frac{g}{\sqrt{2}} [\bar{u}'_i (V_u^\dagger)_j^i \gamma^\mu W_\mu^+ P_L (V_d)_k^j d'^k + \bar{d}'_i (V_d^\dagger)_j^i \gamma^\mu W_\mu^- P_L (V_u)_k^j u'^k], \quad (196)$$

where $g = g_W$. In the SM, the only observable rotation is the combination

$$(V_u^\dagger)_k^i (V_d)_j^k \equiv (V_{CKM})_j^i. \quad (197)$$

Now, in the lepton sector, if neutrinos are not massless, their mass matrix, just as in the case for quarks, will be nondiagonal and complex. Thus the mass eigenstates are different from gauge eigenstates [7]

$$\nu_{\ell L} = \sum_i U_{\ell i} \nu_{iL}, \quad (198)$$

where $\nu_{\ell L} = \nu_{eL}, \nu_{\mu L}, \nu_{\tau L}$ are weak eigenstates and $\nu_{iL} = \nu_{1L}, \nu_{2L}, \nu_{3L}$ are mass eigenstates with mass eigenvalues $m_1, m_2,$ and m_3 . Since $\nu_{\ell L}$ and ℓ_L form a doublet, the interaction of three W bosons with the leptons can be written as [1]

$$\mathcal{L}_{cc} = -\frac{g}{\sqrt{2}} \sum_{\ell} \bar{\ell}_L \gamma^{\mu} W_{\mu}^{-} \nu_{\ell L} + h.c. \quad (199)$$

Using (198) in (199), we obtain

$$\mathcal{L}_{cc} = -\frac{g}{\sqrt{2}} \sum_{\ell, i} (\bar{\ell}_L \gamma^{\mu} W_{\mu}^{-} U_{\ell i} \nu_{iL} + \bar{\nu}_{iL} \gamma^{\mu} W_{\mu}^{+} U_{\ell i}^{*} \ell_L). \quad (200)$$

U is called the *PMNS matrix*.

We recall that an $N \times N$ unitary matrix has $N(N-1)/2$ real parameters and $N(N+1)/2$ complex phases, thus, if we consider the CKM matrix V_{CKM} , this matrix has in principle 3 mixing angles and 6 phases. Now, not all of these phases are physical, so we can eliminate some phases varying the phase of each quark state independently. The number of phases that can be eliminated is $2N-1=5$. Finally, the CKM matrix has 4 independent parameters: 3 mixing angles and 1 phase.

However in the LHT there are new interactions mediated by the heavy gauge bosons $W_H^{\pm}, Z_H,$ and A_H , involving a SM and a mirror quark, as a result, there appear two 3×3 unitary mixing matrices V_{Hd} and V_{Hu} , which describe the interactions of mirror quarks with ordinary SM ones.

The V_{Hd} matrix parameterizes the interactions of light d^j -quarks with heavy mirrors u_H^i that are mediated by W_H . It also parameterizes the flavour interactions between d^j and d_H^i mediated by Z_H and A_H . The V_{Hu} matrix parameterizes, on the other hand, the interactions of the light u-type quarks with the mirror fermions.

In the lepton sector something similar occur, thus, the above comments apply to $V_{H\nu}$ and $V_{H\ell}$.

We are going to show how the V_{Hd} and V_{Hu} matrices appear.

The mass eigenbasis in the T-odd fermion sector is not necessarily aligned with the SM fermion sector [42]. These additional mixings are a source of flavor changing processes. The interaction that gives the T-odd doublets their mass, eq. (148), can be extended to include generational mixing

$$\kappa_j^i f \left(\bar{\Psi}_{2i} \xi + \bar{\Psi}_{1i} \Sigma_0 \Omega \xi^{\dagger} \Omega \right) \Psi_R^{cj} + h.c. \quad (201)$$

The resulting mass matrix $\sqrt{2}f\kappa_j^i$ is diagonalized by two $U(3)$ matrices

$$\kappa_j^i = (V_H)_k^i (\kappa_D)_l^k (U_H^\dagger)_j^l. \quad (202)$$

We can see that V_H acts on the left handed fields while U_H acts on the right handed fields. We note that these matrices are identical for the up and down-type T-odd fermions, since the resulting Dirac mass terms are $SU(2)$ symmetric.

The kinetic terms of the gauge interaction in the T-parity eigenbasis are given qualitatively by [42]

$$g\bar{Q}_{-i}\gamma^\mu A_{\mu-}Q_+^i + g\bar{Q}_{+i}\gamma^\mu A_{\mu-}Q_-^i, \quad (203)$$

where the $A_{\mu-}$ and Q_- are the T-odd gauge bosons and fermions with mass $\sim f$. The Q_+ are the T-even eigenstates. Identifying the mass eigenstates with a H and L index for heavy and light, respectively, these interactions can be re-expressed as

$$g\bar{Q}_{Hi}V_{Hj}^\dagger\gamma^\mu A_{\mu H} \begin{pmatrix} (V_u)_k^j u_L^k \\ (V_d)_k^j d_L^k \end{pmatrix} + g \begin{pmatrix} \bar{u}_{Lk}(V_u^\dagger)_i^k \\ \bar{d}_{Lk}(V_d^\dagger)_i^k \end{pmatrix} \gamma^\mu A_{\mu H} V_{Hj}^i Q_H^j, \quad (204)$$

where

$$Q_H^i = \begin{pmatrix} u_H^i \\ d_H^i \end{pmatrix}. \quad (205)$$

The rotation matrix V_H is in $U(3)$, and operates on the flavor indices of the left handed T-odd fermions. In analogy with the CKM matrix, the rotations relevant to flavor physics are

$$(V_H^\dagger)_k^i (V_u)_j^k \equiv (V_{Hu})_j^i, \quad (V_H^\dagger)_k^i (V_d)_j^k \equiv (V_{Hd})_j^i. \quad (206)$$

The two matrices are related through the SM CKM matrix

$$V_{Hu}^\dagger V_{Hd} = (V_u^\dagger)_k^j (V_H)_i^k (V_H^\dagger)_k^i (V_d)_j^k = V_u^\dagger V_d = V_{CKM}. \quad (207)$$

This is an important result. The matrix V_{Hu} can be calculated from the relation $V_{Hu} = V_{Hd}V_{CKM}^\dagger$. Since the LHT also introduces mirror leptons there appear two new mixing matrices $V_{H\nu}$ and $V_{H\ell}$, in addition to V_{PMNS} describing the SM lepton flavour violating interactions. $V_{H\nu}$ is involved in the interactions of an ordinary neutrino and a mirror lepton (charged or neutral), while $V_{H\ell}$ appears in the interactions of an ordinary charged lepton with a mirror lepton. Similarly as in the case of the matrices V_{Hu} and V_{Hd} , the new mixing matrices satisfy

$$V_{H\nu}^\dagger V_{H\ell} = V_{PMNS}. \quad (208)$$

We know that the mixing matrix V_{Hd} is unitary so that this matrix has 3 mixing angles and 6 complex phases. Analogously to V_{CKM} , we should eliminate from V_{Hd} some of the phases by rotating the interacting states. In this case, however, we have less freedom. The phases of the standard fields, have been already chosen as to eliminate the maximum number of phases from V_{CKM} . Acting on the mirror states only three phases can be still rotated away from V_{Hd} . Thus this matrix has 3 mixing angles and 3 phases. It follows, then, that V_{PMNS} contains 3 mixing angles and 1 phase like V_{CKM} , while $V_{H\nu}$ is described by 3 mixing angles and 3 phases like V_{Hd} .

We can parameterize the mixing matrix V_{Hd} as a product of three rotations and introducing a complex phase in each of them [47]

$$V_{Hd} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & c_{23}^d & s_{23}^d e^{-i\delta_{23}^d} \\ 0 & -s_{23}^d e^{i\delta_{23}^d} & c_{23}^d \end{pmatrix} \begin{pmatrix} c_{13}^d & 0 & s_{13}^d e^{-i\delta_{13}^d} \\ 0 & 1 & 0 \\ -s_{13}^d e^{i\delta_{13}^d} & 0 & c_{13}^d \end{pmatrix} \begin{pmatrix} c_{12}^d & s_{12}^d e^{-i\delta_{12}^d} & 0 \\ -s_{12}^d e^{i\delta_{12}^d} & c_{12}^d & 0 \\ 0 & 0 & 1 \end{pmatrix}. \quad (209)$$

Performing the product one obtains the expression

$$V_{Hd} = \begin{pmatrix} c_{12}^d c_{13}^d & s_{12}^d c_{13}^d e^{-i\delta_{12}^d} & s_{13}^d e^{-i\delta_{13}^d} \\ -s_{12}^d c_{23}^d e^{i\delta_{12}^d} - c_{12}^d s_{23}^d e^{i(\delta_{13}^d - \delta_{23}^d)} & c_{12}^d c_{23}^d - s_{12}^d s_{23}^d s_{13}^d e^{i(\delta_{13}^d - \delta_{12}^d - \delta_{23}^d)} & s_{23}^d c_{13}^d e^{-i\delta_{23}^d} \\ s_{12}^d s_{23}^d e^{i(\delta_{12}^d + \delta_{23}^d)} - c_{12}^d c_{23}^d s_{13}^d e^{i\delta_{13}^d} & -c_{12}^d s_{23}^d e^{i\delta_{23}^d} - s_{12}^d c_{23}^d s_{13}^d e^{i(\delta_{13}^d - \delta_{12}^d)} & c_{23}^d c_{13}^d \end{pmatrix}. \quad (210)$$

As in the case of the CKM matrix the angles θ_{ij}^d can all be made to lie in the first quadrant with $0 \leq \delta_{12}^d, \delta_{23}^d, \delta_{13}^d \leq 2\pi$.

5 Prospects

There are many models that attempt to explain the hierarchy problem between the VEV of the EWSB $v = 246$ GeV and the energy scale of new physics (NP) $\gtrsim \mathcal{O}(\text{TeV})$. Little Higgs is one of these models whose main feature is that the Higgs boson appears as a Nambu-Goldstone boson associated with a symmetry breaking at a scale $f \sim \mathcal{O}(\text{TeV})$.

In this thesis we studied the Littlest Higgs Model with T-parity. LHT is a model based on symmetry breaking of $SU(5)/SO(5)$ where 14 Goldstone bosons appear. In the new T-parity Z_2 discrete symmetry the particles of the heavy sector are T-odd while the SM ones are T-even, which ensures that tree-level contributions to EWPO are forbidden. The only contributions allowed are at loop-level.

LHT has two energy scales, the first scale is the energy scale of new physics where the heavy sector with heavy gauge bosons W_H^\pm , Z_H and A_H appears, and the second scale is the energy scale of EWSB, here we recovered the SM gauge bosons W^\pm , Z^0 and A and the heavy gauge bosons have corrections at order of $\mathcal{O}(\frac{v}{f})^2$ in their masses.

In the Fermion sector of LHT we have a heavy mirror sector to quarks and leptons to SM and because there appear new interactions mediated by W_H^\pm , Z_H and A_H involving a SM and a mirror quark/lepton, appear new mixing matrices V_{Hd} , V_{Hu} or $V_{H\nu}$, $V_{H\ell}$, respectively. An important result of these new mixing matrices is that they have 3 mixing angles and 3 physical phases unlike the CKM and PMNS matrices that only have one physical phase. One prospect is to study these "heavy phases" and the new information they could have in physical observables as

$$A_{CP} = \frac{\Gamma(\ell^- \rightarrow 3\ell) - \Gamma(\ell^+ \rightarrow 3\bar{\ell})}{\Gamma(\ell^- \rightarrow 3\ell) + \Gamma(\ell^+ \rightarrow 3\bar{\ell})}.$$

With LHT also we will study LFV processes. We expect that the new interactions of the heavy sector particles with the SM ones and the new source of LFV that LHT induces by LFV Higgs decays provide new bounds to the processes called "wrong sign".

For example, in [46] there appears a table with these measurements

Branching Ratio	90% C.L. Bound
$\tau \rightarrow \mu\bar{e}\mu$	1.7×10^{-8}
$\tau \rightarrow e\bar{\mu}e$	1.5×10^{-8}
$\tau \rightarrow \mu\bar{e}e$	1.8×10^{-8}
$\tau \rightarrow e\bar{\mu}\mu$	2.7×10^{-8}
$\tau \rightarrow e\bar{e}e$	2.7×10^{-8}
$\tau \rightarrow \mu\bar{\mu}\mu$	2.1×10^{-8}

Table 12: Limits from lepton flavor violating processes mediated by electroweak gauge and Higgs bosons.

But when LHT contributions are included the branching ratios of each process are given by [46]

Process	Branching Ratio
$\tau \rightarrow \mu \bar{e} \mu$	0
$\tau \rightarrow e \bar{\mu} e$	0
$\tau \rightarrow \mu \bar{e} e$	8.2×10^{-12}
$\tau \rightarrow e \bar{\mu} \mu$	2.2×10^{-12}
$\tau \rightarrow e \bar{e} e$	7.4×10^{-12}
$\tau \rightarrow \mu \bar{\mu} \mu$	1.4×10^{-12}

Table 13: LHT contributions mediated by T-odd (non-singlet) leptons to LFV processes.

As we can see the wrong sign processes vanish. We will recalculate these processes using other hypotheses on parameter space of LHT to obtain a result different to zero.

A Appendix: $SU(5)$ generators

We begin by enumerating the generalized generators of the $SU(5)$ [35].

$SU(3)$ Generators

The $SU(3)$ generators are embedded in $SU(5)$

$$\lambda_1 = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \quad \lambda_2 = \begin{pmatrix} 0 & -i & 0 & 0 & 0 \\ i & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \quad \lambda_3 = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix},$$

$$\lambda_4 = \begin{pmatrix} 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \quad \lambda_5 = \begin{pmatrix} 0 & 0 & -i & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ i & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \quad \lambda_6 = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix},$$

$$\lambda_7 = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -i & 0 & 0 \\ 0 & i & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \quad \lambda_8 = \frac{1}{\sqrt{3}} \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & -2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

Mixed Quantum Numbers Generators

The generators with non vanishing $SU(3)$ and $SU(2)$ quantum numbers are not present in the SM. They can be computed through the commutator relations between the generators of $SU(3)$ and the generators of $SU(2)$

$$\lambda_9 = \begin{pmatrix} 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \quad \lambda_{10} = \begin{pmatrix} 0 & 0 & 0 & i & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ i & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \quad \lambda_{11} = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix},$$

$$\lambda_{12} = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -i & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & i & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \quad \lambda_{13} = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \quad \lambda_{14} = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -i & 0 \\ 0 & 0 & i & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix},$$

$$\lambda_{15} = \begin{pmatrix} 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \end{pmatrix}, \quad \lambda_{16} = \begin{pmatrix} 0 & 0 & 0 & 0 & -i \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ i & 0 & 0 & 0 & 0 \end{pmatrix}, \quad \lambda_{17} = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \end{pmatrix},$$

$$\lambda_{18} = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -i \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & i & 0 & 0 & 0 \end{pmatrix}, \quad \lambda_{19} = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \end{pmatrix}, \quad \lambda_{20} = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -i \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & i & 0 & 0 \end{pmatrix}.$$

SU(2) Generators

The SU(2) generators are embedded in SU(5)

$$\lambda_{21} = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 \end{pmatrix}, \quad \lambda_{22} = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -i \\ 0 & 0 & 0 & i & 0 \end{pmatrix}, \quad \lambda_{23} = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & -1 \end{pmatrix}.$$

Diagonal Generator

We have another diagonal generator that we will identify eventually with the SM hypercharge

$$\lambda_{24} = \frac{1}{\sqrt{15}} \begin{pmatrix} -2 & 0 & 0 & 0 & 0 \\ 0 & -2 & 0 & 0 & 0 \\ 0 & 0 & -2 & 0 & 0 \\ 0 & 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & 0 & 3 \end{pmatrix},$$

and

$$-\frac{2(5e + \sqrt{5}r)}{5\sqrt{3}} = \frac{2\eta}{\sqrt{5}}. \quad (\text{B.7})$$

Thus, we have

$$\Pi^1 = \begin{pmatrix} \chi + \frac{\eta}{2\sqrt{5}} & & \\ & \frac{2\eta}{\sqrt{5}} & \\ & & \chi^T + \frac{\eta}{2\sqrt{5}} \end{pmatrix}. \quad (\text{B.8})$$

The second matrix by blocks is

$$\Pi^2 = \begin{pmatrix} & & \left(\frac{1+i}{2}\right)(c-im) & & \\ & & \left(\frac{1+i}{2}\right)(d-in) & & \\ \left(\frac{1-i}{2}\right)(c+im) & \left(\frac{1-i}{2}\right)(d+in) & 0 & \left(\frac{1+i}{2}\right)(c-im) & \left(\frac{1+i}{2}\right)(d-in) \\ & & \left(\frac{1-i}{2}\right)(c+im) & & \\ & & \left(\frac{1-i}{2}\right)(d+in) & & \end{pmatrix}. \quad (\text{B.9})$$

If

$$h/\sqrt{2} = \frac{1}{\sqrt{2}} = \begin{pmatrix} h^+ \\ h^0 \end{pmatrix} = \begin{pmatrix} \frac{1-i}{2}(c+im) \\ \frac{1-i}{2}(d+in) \end{pmatrix}, \quad (\text{B.10})$$

then

$$\Pi^2 = \begin{pmatrix} & h/\sqrt{2} & \\ h^\dagger/\sqrt{2} & & h^T/\sqrt{2} \\ & h^*/\sqrt{2} & \end{pmatrix}, \quad (\text{B.11})$$

where $h/\sqrt{2}$ is the SM Higgs doublet. Finally, the third matrix by blocks is

$$\Pi^3 = \begin{pmatrix} & & \frac{1}{6}[6f + \mu^*] & \frac{1}{2}[\omega + \rho^*] \\ & & \frac{1}{2}[\omega + \rho^*] & \frac{1}{6}[6k + \nu^*] \\ & 0 & & \\ \frac{1}{6}[6f + \mu] & \frac{1}{2}[\omega + \rho] & & \\ \frac{1}{2}[\omega + \rho] & \frac{1}{6}[6k + \nu] & & \end{pmatrix} \quad (\text{B.12})$$

with

$$\mu = i(-3b - \sqrt{3}e + 3q + \sqrt{15}r), \quad \nu = i(3b - \sqrt{3}e - 3q + \sqrt{15}r), \quad \omega = g + h, \quad \rho = i(p - a). \quad (\text{B.13})$$

If we choose as notation the physical mass eigenstates [44]

$$\phi = \begin{pmatrix} \phi^{++} & \frac{\phi^+}{\sqrt{2}} \\ \frac{\phi^+}{\sqrt{2}} & \phi^0 \end{pmatrix} = \begin{pmatrix} -i\Phi^{++} & -i\frac{\Phi^+}{\sqrt{2}} \\ -i\frac{\Phi^+}{\sqrt{2}} & \frac{-\Phi^0 + \Phi^P}{\sqrt{2}} \end{pmatrix} = \begin{pmatrix} \frac{1}{6}[6f + \mu] & \frac{1}{2}[\omega + \rho] \\ \frac{1}{2}[\omega + \rho] & \frac{1}{6}[6k + \nu] \end{pmatrix}, \quad (\text{B.14})$$

and

$$B = B_1 Y_1 + B_2 Y_2 = \frac{1}{5} \begin{pmatrix} \frac{3B_1}{2} + B_2 & & & & & \\ & \frac{3B_1}{2} + B_2 & & & & \\ & & B_2 - B_1 & & & \\ & & & -B_1 - \frac{3B_2}{2} & & \\ & & & & -B_1 - \frac{3B_2}{2} & \\ & & & & & -B_1 - \frac{3B_2}{2} \end{pmatrix}. \quad (\text{C.4})$$

Since SM gauge bosons are T-even, it satisfies

$$W\Sigma_0 - \Sigma_0 W^T = \begin{pmatrix} & & \frac{W_1^3 + W_2^3}{2} & & \frac{(W_1^1 + W_2^1) - i(W_1^2 + W_2^2)}{2} & \\ & & \frac{(W_1^1 + W_2^1) + i(W_1^2 + W_2^2)}{2} & & -\frac{W_1^3 + W_2^3}{2} & \\ & & 0 & & & \\ -\frac{W_1^3 + W_2^3}{2} & & & & & \\ -\frac{(W_1^1 + W_2^1) - i(W_1^2 + W_2^2)}{2} & & -\frac{(W_1^1 + W_2^1) + i(W_1^2 + W_2^2)}{2} & & & \\ & & \frac{W_1^3 + W_2^3}{2} & & & \end{pmatrix}, \quad (\text{C.5})$$

and

$$B\Sigma_0 - \Sigma_0 B^T = \begin{pmatrix} & & \frac{B_1 + B_2}{2} & & & \\ & & & & \frac{B_1 + B_2}{2} & \\ & & 0 & & & \\ -\frac{B_1 + B_2}{2} & & & & & \\ & & -\frac{B_1 + B_2}{2} & & & \end{pmatrix}. \quad (\text{C.6})$$

Thus, from eqs. (C.5) and (C.6), we have respectively

$$W^\pm = \frac{1}{2}[(W_1^1 + W_2^1) \mp i(W_1^2 + W_2^2)], \quad W^3 = \frac{W_1^3 + W_2^3}{\sqrt{2}}, \quad B = \frac{B_1 + B_2}{\sqrt{2}}. \quad (\text{C.7})$$

And now, as $\{Q_1^a - Q_2^a, Y_1 - Y_2\}$ is broken, then

$$W_H = W_1^a Q_1^a - W_2^a Q_2^a = W_1^1 Q_1^1 + W_1^2 Q_1^2 + W_1^3 Q_1^3 - W_2^1 Q_2^1 - W_2^2 Q_2^2 - W_2^3 Q_2^3 = \begin{pmatrix} \frac{W_1^3}{2} & \frac{W_1^1 - iW_1^2}{2} & & & & \\ \frac{W_1^1 + iW_1^2}{2} & -\frac{W_1^3}{2} & & & & \\ & & 0 & & & \\ & & & \frac{W_2^3}{2} & & \frac{W_2^1 + iW_2^2}{2} \\ & & & \frac{W_2^1 - iW_2^2}{2} & & -\frac{W_2^3}{2} \end{pmatrix}, \quad (\text{C.8})$$

and

$$B_H = B_1 Y_1 - B_2 Y_2 = \frac{1}{5} \begin{pmatrix} \frac{3B_1}{2} - B_2 & & & & \\ & \frac{3B_1}{2} - B_2 & & & \\ & & -(B_1 + B_2) & & \\ & & & \frac{3B_2}{2} - B_1 & \\ & & & & \frac{3B_2}{2} - B_1 \end{pmatrix}. \quad (\text{C.9})$$

So

$$W_H \Sigma_0 - \Sigma_0 W_H^T = \begin{pmatrix} & & \frac{W_1^3 - W_2^3}{2} & \frac{(W_1^1 - W_2^1) - i(W_1^2 - W_2^2)}{2} & \\ & & \frac{(W_1^1 - W_2^1) + i(W_1^2 - W_2^2)}{2} & -\frac{W_1^3 - W_2^3}{2} & \\ & & 0 & & \\ -\frac{W_1^3 - W_2^3}{2} & -\frac{(W_1^1 - W_2^1) + i(W_1^2 - W_2^2)}{2} & & & \\ -\frac{(W_1^1 - W_2^1) - i(W_1^2 - W_2^2)}{2} & \frac{W_1^3 - W_2^3}{2} & & & \end{pmatrix}, \quad (\text{C.10})$$

and

$$B_H \Sigma_0 - \Sigma_0 B_H^T = \begin{pmatrix} & & \frac{B_1 - B_2}{2} & & \\ & & & \frac{B_1 - B_2}{2} & \\ & & 0 & & \\ -\frac{B_1 - B_2}{2} & & & & \\ & -\frac{B_1 - B_2}{2} & & & \end{pmatrix}. \quad (\text{C.11})$$

Finally the T-odd combinations expand the heavy gauge sector

$$W_H^\pm = \frac{1}{2} [(W_1^1 - W_2^1) \mp i(W_1^2 - W_2^2)], \quad W_H^3 = \frac{W_1^3 - W_2^3}{\sqrt{2}}, \quad B_H = \frac{B_1 - B_2}{\sqrt{2}}. \quad (\text{C.12})$$

D Appendix: Scalar Sector

The action of T-parity on the $[SU(5) \times U(1)]_j$ gauge groups is defined as

$$A_1 \leftrightarrow A_2, \quad (\text{D.1})$$

and their generators change as

$$Q_1^a \xrightarrow{T} -\Sigma_0 (Q_1^a)^T \Sigma_0. \quad (\text{D.2})$$

The $SU(2)_1$ generators are

$$Q_1^1 = \begin{pmatrix} 0 & 1/2 & 0 & 0 & 0 \\ 1/2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \quad Q_1^2 = \begin{pmatrix} 0 & -i/2 & 0 & 0 & 0 \\ i/2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \quad Q_1^3 = \begin{pmatrix} 1/2 & 0 & 0 & 0 & 0 \\ 0 & -1/2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \quad (D.3)$$

and the $U(1)_1$ generator is

$$Y_1 = \frac{1}{10} \begin{pmatrix} 3 & & & & \\ & 3 & & & \\ & & -2 & & \\ & & & -2 & \\ & & & & -2 \end{pmatrix}. \quad (D.4)$$

Now, the $SU(2)_2$ generators are

$$Q_2^1 = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1/2 \\ 0 & 0 & 0 & -1/2 & 0 \end{pmatrix}, \quad Q_2^2 = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -i/2 \\ 0 & 0 & 0 & i/2 & 0 \end{pmatrix}, \quad Q_2^3 = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1/2 & 0 \\ 0 & 0 & 0 & 0 & 1/2 \end{pmatrix}, \quad (D.5)$$

and the $U(1)_2$ generator is

$$Y_2 = \frac{1}{10} \begin{pmatrix} 2 & & & & \\ & 2 & & & \\ & & 2 & & \\ & & & -3 & \\ & & & & -3 \end{pmatrix}. \quad (D.6)$$

From the eq. (D.2)

$$\begin{aligned}
 Q_1^1 \xrightarrow{T} -\Sigma_0 (Q_1^1)^T \Sigma_0 &= - \begin{pmatrix} 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 1/2 & 0 & 0 & 0 \\ 1/2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \end{pmatrix} \\
 &= \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1/2 \\ 0 & 0 & 0 & -1/2 & 0 \end{pmatrix} = Q_2^1,
 \end{aligned} \tag{D.7}$$

$$\begin{aligned}
 Q_1^2 \xrightarrow{T} -\Sigma_0 (Q_1^2)^T \Sigma_0 &= - \begin{pmatrix} 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & i/2 & 0 & 0 & 0 \\ -i/2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \end{pmatrix} \\
 &= \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -i/2 \\ 0 & 0 & 0 & i/2 & 0 \end{pmatrix} = Q_2^2,
 \end{aligned} \tag{D.8}$$

$$\begin{aligned}
 Q_1^3 \xrightarrow{T} -\Sigma_0 (Q_1^3)^T \Sigma_0 &= - \begin{pmatrix} 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 1/2 & 0 & 0 & 0 & 0 \\ 0 & -1/2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \end{pmatrix} \\
 &= \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1/2 & 0 \\ 0 & 0 & 0 & 0 & 1/2 \end{pmatrix} = Q_2^3,
 \end{aligned} \tag{D.9}$$

$$\begin{aligned}
 Y_1 \xrightarrow{T} -\Sigma_0 (Y_1)^T \Sigma_0 &= -\frac{1}{10} \begin{pmatrix} 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 3 & & & & \\ & 3 & & & \\ & & -2 & & \\ & & & -2 & \\ & & & & -2 \end{pmatrix} \begin{pmatrix} 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \end{pmatrix} \\
 &= \frac{1}{10} \begin{pmatrix} 2 & & & & \\ & 2 & & & \\ & & 2 & & \\ & & & -3 & \\ & & & & -3 \end{pmatrix} = Y_2,
 \end{aligned} \tag{D.10}$$

$$\begin{aligned}
 Q_2^1 \xrightarrow{T} -\Sigma_0 (Q_2^1)^T \Sigma_0 &= - \begin{pmatrix} 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1/2 \\ 0 & 0 & 0 & -1/2 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \end{pmatrix} \\
 &= \begin{pmatrix} 0 & 1/2 & 0 & 0 & 0 \\ 1/2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} = Q_1^1,
 \end{aligned} \tag{D.11}$$

$$\begin{aligned}
 Q_2^2 \xrightarrow{T} -\Sigma_0 (Q_2^2)^T \Sigma_0 &= - \begin{pmatrix} 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -i/2 \\ 0 & 0 & 0 & i/2 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \end{pmatrix} \\
 &= \begin{pmatrix} 0 & -i/2 & 0 & 0 & 0 \\ i/2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} = Q_1^2,
 \end{aligned} \tag{D.12}$$

$$\begin{aligned}
 Q_2^3 \xrightarrow{T} -\Sigma_0 (Q_2^3)^T \Sigma_0 &= - \begin{pmatrix} 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1/2 & 0 \\ 0 & 0 & 0 & 0 & 1/2 \end{pmatrix} \begin{pmatrix} 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \end{pmatrix} \\
 &= \begin{pmatrix} 1/2 & 0 & 0 & 0 & 0 \\ 0 & -1/2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} = Q_1^3,
 \end{aligned} \tag{D.13}$$

$$\begin{aligned}
 Y_2 \xrightarrow{T} -\Sigma_0 (Y_2)^T \Sigma_0 &= -\frac{1}{10} \begin{pmatrix} 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 2 & & & & \\ & 2 & & & \\ & & 2 & & \\ & & & -3 & \\ & & & & -3 \end{pmatrix} \begin{pmatrix} 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \end{pmatrix} \\
 &= \frac{1}{10} \begin{pmatrix} 3 & & & & \\ & 3 & & & \\ & & -2 & & \\ & & & -2 & \\ & & & & -2 \end{pmatrix} = Y_1.
 \end{aligned} \tag{D.14}$$

From the eqs. (D.7)-(D.14), we conclude

$$Q_1^a \leftrightarrow Q_2^a, \quad \text{and} \quad Y_1 \leftrightarrow Y_2. \quad (\text{D.15})$$

Now, we know that the Scalar Lagrangian in Littlest Higgs is

$$\mathcal{L}_S = \frac{f^2}{8} \text{Tr}[(\mathcal{D}_\mu \Sigma)(\mathcal{D}^\mu \Sigma)^\dagger], \quad (\text{D.16})$$

where

$$\mathcal{D}_\mu \Sigma = \partial_\mu \Sigma - i \sum_{j=1}^2 [g_j W_j^a (Q_j^a \Sigma + \Sigma Q_j^{aT}) + g'_j B_j (Y_j \Sigma + \Sigma Y_j^T)], \quad (\text{D.17})$$

when we introduce the T-parity, the Scalar Lagrangian must to be T-even, thus, from the eqs. (D.1) and (D.15) we have that $g_1 = g_2 = \sqrt{2}g_W$ and $g'_1 = g'_2 = \sqrt{2}g'$, so the eq. (D.17) is written as

$$\mathcal{D}_\mu \Sigma = \partial_\mu \Sigma - i\sqrt{2} \sum_{j=1}^2 [g_W W_j^a (Q_j^a \Sigma + \Sigma Q_j^{aT}) + g' B_j (Y_j \Sigma + \Sigma Y_j^T)]. \quad (\text{D.17})$$

Recall that the action of T-parity in the scalar sector is defined as

$$\Pi \rightarrow -\Omega \Pi \Omega, \quad \text{and} \quad \Sigma \rightarrow \tilde{\Sigma} = \Sigma_0 \Omega \Sigma^\dagger \Omega \Sigma_0, \quad (\text{D.15})$$

with $\Omega = \text{diag}(-1, -1, 1, -1, -1)$.

The term relevant for to calculate the mass of gauge bosons, under the T-parity, is

$$\left| -i\sqrt{2} \sum_{j=1}^2 [g_W W_j^a (Q_j^a \tilde{\Sigma} + \tilde{\Sigma} Q_j^{aT}) + g' B_j (Y_j \tilde{\Sigma} + \tilde{\Sigma} Y_j^T)] \right|^2. \quad (\text{D.16})$$

Calculating before introducing EWSM effects and taking at first order $\Sigma = \Sigma_0$, then

$$\tilde{\Sigma} = \Sigma_0 \Omega \Sigma^\dagger \Omega \Sigma_0 = \Sigma_0 \Omega \Sigma_0 \Omega \Sigma_0 = \Sigma_0. \quad (\text{D.17})$$

Developing the eq. (D.16) by parts

$$g_W W_1^a (Q_1^a \Sigma_0 + \Sigma_0 Q_1^{aT}) = \begin{pmatrix} 0 & 0 & g_W W_1^a / 2 \\ 0 & 0 & 0 \\ g_W W_1^a / 2 & 0 & 0 \end{pmatrix}, \quad (\text{D.18})$$

where $W_1^a = W_1^a \sigma^a$,

$$g' B_1 (Y_1 \Sigma_0 + \Sigma_0 Y_1^T) = \begin{pmatrix} 0 & 0 & g' B_1 / 10 \\ 0 & -2g' B_1 / 5 & 0 \\ g' B_1 / 10 & 0 & 0 \end{pmatrix}, \quad (\text{D.19})$$

$$g_W W_2^a (Q_2^a \Sigma_0 + \Sigma_0 Q_2^{aT}) = \begin{pmatrix} 0 & 0 & -g_W W_2^a / 2 \\ 0 & 0 & 0 \\ -g_W W_2^a / 2 & 0 & 0 \end{pmatrix}, \quad (\text{D.20})$$

where $W_2^a = W_2^a \sigma^{a*}$,

$$g' B_2 (Y_2 \Sigma_0 + \Sigma_0 Y_2^T) = \begin{pmatrix} 0 & 0 & -g' B_2 / 10 \\ 0 & 2g' B_2 / 5 & 0 \\ -g' B_2 / 10 & 0 & 0 \end{pmatrix}. \quad (\text{D.21})$$

Summing the eqs. (D.18), (D.19), (D.20), and (D.21)

$$\begin{pmatrix} 0 & 0 & \frac{g_W}{2} (W_1^a - W_2^a) + \frac{g'}{10} (B_1 - B_2) \\ 0 & \frac{2g'}{5} (B_2 - B_1) & 0 \\ \frac{g_W}{2} (W_1^a - W_2^a) + \frac{g'}{10} (B_1 - B_2) & 0 & 0 \end{pmatrix}, \quad (\text{D.22})$$

substituting (D.22) in (D.16)

$$2 \left| \begin{pmatrix} 0 & 0 & \left(\frac{g_W}{2} (W_1^a - W_2^a) + \frac{g'}{10} (B_1 - B_2) \right) \mathbb{I}_{2 \times 2} \\ 0 & \frac{2g'}{5} (B_2 - B_1) & 0 \\ \left(\frac{g_W}{2} (W_1^a - W_2^a) + \frac{g'}{10} (B_1 - B_2) \right) \mathbb{I}_{2 \times 2} & 0 & 0 \end{pmatrix} \right|^2, \quad (\text{D.23})$$

where $\mathbb{I}_{2 \times 2}$ is a identity matrix 2×2 , and omitting all crossed terms of W_j^a and B_j , we have

$$\begin{pmatrix} \left(\frac{g_W^2}{2} (W_1^a - W_2^a)^2 + \frac{g'^2}{50} (B_1 - B_2)^2 \right)_{2 \times 2} & 0 & 0 \\ 0 & \frac{8g'^2}{25} (B_1 - B_2)^2 & 0 \\ 0 & 0 & \left(\frac{g_W^2}{2} (W_1^a - W_2^a)^2 + \frac{g'^2}{50} (B_1 - B_2)^2 \right)_{2 \times 2} \end{pmatrix}, \quad (\text{D.24})$$

since $\mathcal{L}_S = \frac{f^2}{8} \text{Tr}[(\mathcal{D}_\mu \Sigma)(\mathcal{D}^\mu \Sigma)^\dagger]$, we have that

$$\begin{aligned}
 \mathcal{L}_S &= \frac{f^2}{8} \left(2g_W^2 (W_1^a - W_2^a)^2 + \frac{2}{5} (B_1 - B_2)^2 \right) \\
 &= \frac{f^2}{8} \left(2g_W^2 (W_1^{a2} + W_2^{a2} - 2W_1^a W_2^a) + \frac{2}{5} (B_1^2 + B_2^2 - 2B_1 B_2) \right) \\
 &= \frac{f^2}{4} g_W^2 \begin{pmatrix} W_1^a & W_2^a \end{pmatrix} \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix} \begin{pmatrix} W_1^a \\ W_2^a \end{pmatrix} \\
 &\quad + \frac{f^2}{20} g'^2 \begin{pmatrix} B_1 & B_2 \end{pmatrix} \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix} \begin{pmatrix} B_1 \\ B_2 \end{pmatrix}.
 \end{aligned} \tag{D.25}$$

We defined gauge bosons mass eigenstates

$$\begin{aligned}
 \begin{pmatrix} W_L^a \\ W_H^a \end{pmatrix} &= \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} W_1^a \\ W_2^a \end{pmatrix}, \\
 \begin{pmatrix} B_L \\ B_H \end{pmatrix} &= \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} B_1 \\ B_2 \end{pmatrix}.
 \end{aligned} \tag{D.26}$$

We can see that W_L^a and B_L are T-even, while W_H^a and B_H are T-odd. Also we can write this as

$$W_L^a = \cos \psi W_1^a + \sin \psi W_2^a, \quad W_H^a = \sin \psi W_1^a - \cos \psi W_2^a, \tag{D.27}$$

with the mixing angle is given by $\sin \psi = \cos \psi = 1/\sqrt{2}$. Thus, diagonalizing the matrix of (D.25) and from (D.26), we have

$$\begin{aligned}
 \mathcal{L}_S &= \frac{f^2}{4} g_W^2 \begin{pmatrix} W_1^a & W_2^a \end{pmatrix} \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix} \begin{pmatrix} W_1^a \\ W_2^a \end{pmatrix} \\
 &\quad + \frac{f^2}{20} g'^2 \begin{pmatrix} B_1 & B_2 \end{pmatrix} \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix} \begin{pmatrix} B_1 \\ B_2 \end{pmatrix} \\
 &= \frac{f^2}{4} g_W^2 \begin{pmatrix} W_L^a & W_H^a \end{pmatrix} \begin{pmatrix} 0 & 0 \\ 0 & 2 \end{pmatrix} \begin{pmatrix} W_L^a \\ W_H^a \end{pmatrix} \\
 &\quad + \frac{f^2}{20} g'^2 \begin{pmatrix} B_L & B_H \end{pmatrix} \begin{pmatrix} 0 & 0 \\ 0 & 2 \end{pmatrix} \begin{pmatrix} B_L \\ B_H \end{pmatrix} + \dots \\
 &= \frac{f^2}{2} g_W^2 W_H^{a2} + \frac{f^2}{10} g'^2 B_H^2 \\
 &= \frac{1}{2} \left(f^2 g_W^2 W_H^{a2} + \frac{f^2 g'^2}{5} B_H^2 \right)
 \end{aligned} \tag{D.28}$$

Thus, the masses of gauge bosons are

$$M_{W_H^a} = g_W f, \quad M_{B_H} = \frac{g' f}{\sqrt{5}}. \quad (\text{D.29})$$

If we consider EWSB effects of order $(v_h/f)^2$, the Σ field is

$$\Sigma = \begin{pmatrix} 0 & 0 & 0 & 1 & 0 \\ 0 & \epsilon^4/12 - \epsilon^2/2 & i(\epsilon - \epsilon^3/3) & 0 & 1 - \epsilon^2/2 + \epsilon^4/12 \\ 0 & i(\epsilon - \epsilon^3/3) & 1 - \epsilon^2 + \epsilon^4/6 & 0 & i(\epsilon - \epsilon^3/3) \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 1 - \epsilon^2/2 + \epsilon^4/12 & i(\epsilon - \epsilon^3/3) & 0 & \epsilon^4/12 - \epsilon^2/2 \end{pmatrix}, \quad (\text{D.30})$$

with $\epsilon = v_h/f$.

Substituting (D.30) in (D.16), we can recalculate the masses of gauge bosons as above, now we don't omit the crossed terms of W_j^a and B_j , where $a = 1, 2$. We have that

$$\begin{aligned} \mathcal{L}_S = & \frac{f^2 g_W^2}{4} \left[-\frac{\epsilon^4}{6} W_1 W_2 + \epsilon^2 W_1 W_2 + (W_1 - W_2)^2 \right] \\ & + \frac{f^2 g_W^2}{4} \left[\frac{\epsilon^4}{8} \left((W_1^3 - W_2^3)^2 - \frac{1}{4} W_1^3 W_2^3 \right) + \epsilon^2 W_1^3 W_2^3 + (W_1^3 - W_2^3)^2 \right] \\ & + \frac{f^2 g'^2}{4} \left[\epsilon^4 \left(\frac{1}{8} (B_1^2 + B_2^2) - \frac{5}{12} B_1 B_2 \right) + \epsilon^2 B_1 B_2 + \frac{1}{5} (B_1 - B_2)^2 \right] \\ & + \frac{f^2 g_W g'}{4} \left[\frac{\epsilon^4}{4} \left(- (W_1^3 - W_2^3) (B_1 - B_2) + \frac{2}{3} (B_1 W_2^3 + B_2 W_1^3) \right) - \epsilon^2 (B_1 W_2^3 + B_2 W_1^3) \right]. \end{aligned} \quad (\text{D.31})$$

Analyzing the Scalar lagrangian by parts, taking the eigenstates defined in (D.26), and diagonalizing the matrices of (D.31)

$$\begin{aligned} -\frac{\epsilon^4}{6} W_1^a W_2^a + \epsilon^2 W_1^a W_2^a + (W_1^a - W_2^a)^2 &= \begin{pmatrix} W_L^a & W_H^a \end{pmatrix} \begin{pmatrix} \frac{\epsilon^2}{2} (1 - \epsilon^2/6) & 0 \\ 0 & 2(1 - \epsilon^2/4) \end{pmatrix} \begin{pmatrix} W_L^a \\ W_H^a \end{pmatrix} \\ &= \frac{\epsilon^2}{2} \left(1 - \frac{\epsilon^2}{6} \right) W_L^{a2} + 2 \left(1 - \frac{\epsilon^2}{4} \right) W_H^{a2}. \end{aligned} \quad (\text{D.32})$$

If we defined as [38]

$$W_H^\pm = \frac{1}{\sqrt{2}} (W_H^1 \mp i W_H^2), \quad W_L^\pm = \frac{1}{\sqrt{2}} (W_L^1 \mp i W_L^2), \quad (\text{D.33})$$

then

$$-\frac{\epsilon^4}{6}W_1^aW_2^a + \epsilon^2W_1^aW_2^a + (W_1^a - W_2^a)^2 = \frac{\epsilon^2}{2}\left(1 - \frac{\epsilon^2}{6}\right)W_L^+W_L^- + 2\left(1 - \frac{\epsilon^2}{4}\right)W_H^+W_H^-. \quad (\text{D.34})$$

Now, the second line of Lagrangian

$$\begin{aligned} \frac{\epsilon^4}{8}\left((W_1^3 - W_2^3)^2 - \frac{1}{4}W_1^3W_2^3\right) + \epsilon^2W_1^3W_2^3 + (W_1^3 - W_2^3)^2 &= \frac{\epsilon^4}{8}\left((W_1^3)^2 + (W_2^3)^2 - \frac{10}{3}W_1^3W_2^3\right) \\ &+ \epsilon^2W_1^3W_2^3 + (W_1^3 - W_2^3)^2, \end{aligned} \quad (\text{D.35})$$

$$\begin{pmatrix} W_L^3 & W_H^3 \end{pmatrix} \begin{pmatrix} \frac{\epsilon^2}{2}\left(1 - \frac{\epsilon^2}{6}\right) & 0 \\ 0 & 2\left(1 - \frac{\epsilon^2}{4}\right) \end{pmatrix} \begin{pmatrix} W_L^3 \\ W_H^3 \end{pmatrix} = \frac{\epsilon^2}{2}\left(1 - \frac{\epsilon^2}{6}\right)(W_L^3)^2 + 2\left(1 - \frac{\epsilon^2}{4}\right)(W_H^3)^2. \quad (\text{D.36})$$

For the third line of the Lagrangian

$$\begin{aligned} \epsilon^4\left(\frac{1}{8}(B_1^2 + B_2^2) - \frac{5}{12}B_1B_2\right) + \epsilon^2B_1B_2 + \frac{1}{5}(B_1 - B_2)^2 \\ = \begin{pmatrix} B_L & B_H \end{pmatrix} \begin{pmatrix} \frac{\epsilon^2}{2}\left(1 - \frac{\epsilon^2}{6}\right) & 0 \\ 0 & \frac{2}{5}\left(1 - \frac{5\epsilon^2}{4}\right) \end{pmatrix} \begin{pmatrix} B_L \\ B_H \end{pmatrix} = \frac{\epsilon^2}{2}\left(1 - \frac{\epsilon^2}{6}\right)(B_L)^2 + \frac{2}{5}\left(1 - \frac{5\epsilon^2}{4}\right)(B_H)^2. \end{aligned} \quad (\text{D.37})$$

Finally

$$\frac{\epsilon^4}{4}\left(- (W_1^3 - W_2^3)(B_1 - B_2) + \frac{2}{3}(B_1W_2^3 + B_2W_1^3)\right) - \epsilon^2(B_1W_2^3 + B_2W_1^3), \quad (\text{D.38})$$

with aid of

$$\begin{aligned} B_1W_2^3 + B_2W_1^3 &= \frac{1}{2}[(W_1^3 + W_2^3)(B_1 + B_2) - (W_1^3 - W_2^3)(B_1 - B_2)] \\ &= W_L^3B_L - W_H^3B_H, \end{aligned} \quad (\text{D.39})$$

substituting (D.39) in (D.38)

$$\frac{\epsilon^4}{4}\left(-\frac{8}{3}W_H^3B_H + \frac{2}{3}W_L^3B_L\right) - \epsilon^2(W_L^3B_L - W_H^3B_H) = -\epsilon^2\left(1 - \frac{\epsilon^2}{6}\right)W_L^3B_L + \epsilon^2\left(1 - \frac{2\epsilon^2}{3}\right)W_H^3B_H. \quad (\text{D.40})$$

Substituting (D.34), (D.36), (D.37), and (D.40) in (D.31), we have

$$\begin{aligned}
\mathcal{L}_S = & \frac{1}{2} \left[\frac{g_W^2 v_h^2}{4} \left(1 - \frac{v_h^2}{6f^2} \right) \right] W_L^+ W_L^- \\
& + \frac{1}{2} \left[f^2 g_W^2 \left(1 - \frac{v_h^2}{4f^2} \right) \right] W_H^+ W_H^- \\
& + \frac{1}{2} \left[\frac{g_W^2 v_h^2}{4} \left(1 - \frac{v_h^2}{6f^2} \right) \right] (W_L^3)^2 \\
& + \frac{1}{2} \left[f^2 g_W^2 \left(1 - \frac{v_h^2}{4f^2} \right) \right] (W_H^3)^2 \\
& + \frac{1}{2} \left[\frac{g'^2 v_h^2}{4} \left(1 - \frac{v_h^2}{6f^2} \right) \right] (B_L)^2 \\
& + \frac{1}{2} \left[\frac{f^2 g'^2}{5} \left(1 - \frac{5v_h^2}{4f^2} \right) \right] (B_H)^2 \\
& + \frac{1}{2} \left[-\frac{g_W g' v_h^2}{2} \left(1 - \frac{v_h^2}{6f^2} \right) \right] W_L^3 B_L \\
& + \frac{1}{2} \left[\frac{g_W g' v_h^2}{2} \left(1 - \frac{2v_h^2}{3f^2} \right) \right] W_H^3 B_H.
\end{aligned} \tag{D.41}$$

As we can see in the Lagrangian above the neutral light sector is given by W_L^3 , B_L , and $W_L^3 B_L$

$$\frac{v_h^2}{4} \left(1 - \frac{v_h^2}{6f^2} \right) \left(g_W^2 (W_L^3)^2 - 2g_W g' W_L^3 B_L + g'^2 (B_L)^2 \right). \tag{D.42}$$

The eq. (D.42) has the same form that the terms off-diagonal of the standard model for electroweak interactions, so from this sector we will get the SM neutral gauge bosons Z_L and A_L . We need to diagonalize the matrix

$$\begin{pmatrix} W_L^3 & B_L \end{pmatrix} \begin{pmatrix} g_W^2 & -g_W g' \\ -g_W g' & g'^2 \end{pmatrix} \begin{pmatrix} W_L^3 \\ B_L \end{pmatrix}, \tag{D.43}$$

the physical fields Z_L and A_L are [6]

$$\begin{pmatrix} Z_L \\ A_L \end{pmatrix} = \frac{1}{\sqrt{g_W^2 + g'^2}} \begin{pmatrix} g_W & -g' \\ g' & g_W \end{pmatrix} \begin{pmatrix} W_L^3 \\ B_L \end{pmatrix}, \tag{D.44}$$

For the light sector the mixing angle is just the SM Weinberg angle

$$\frac{g'}{g_W} = \tan \theta_W, \tag{D.45}$$

in terms of θ_W

$$A_L = \cos \theta_W B_L + \sin \theta_W W_L^3, \quad Z_L = -\sin \theta_W B_L + \cos \theta_W W_L^3. \tag{D.46}$$

Thus,

$$\begin{pmatrix} W_L^3 & B_L \end{pmatrix} \begin{pmatrix} g_W^2 & -g_w g' \\ -g_w g' & g'^2 \end{pmatrix} \begin{pmatrix} W_L^3 \\ B_L \end{pmatrix} = \begin{pmatrix} Z_L & A_L \end{pmatrix} \begin{pmatrix} g_W^2 + g'^2 & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} Z_L \\ A_L \end{pmatrix}, \quad (\text{D.47})$$

so the eq. (D.42) looks like

$$\frac{v_h^2}{4} \left(1 - \frac{v_h^2}{6f^2}\right) \left(g_W^2 (W_L^3)^2 - 2g_w g' W_L^3 B_L + g'^2 (B_L)^2\right) = \frac{v_h^2}{4} \left(1 - \frac{v_h^2}{6f^2}\right) (g_W^2 + g'^2) Z_L^2. \quad (\text{D.48})$$

then

$$\frac{v_h^2}{4} \left(1 - \frac{v_h^2}{6f^2}\right) (g_W^2 + g'^2) Z_L^2 = \frac{g_W^2 v_h^2}{4 \cos^2 \theta_W} \left(1 - \frac{v_h^2}{6f^2}\right) Z_L^2. \quad (\text{D.49})$$

Replacing this in the Lagrangian

$$\begin{aligned} \mathcal{L}_S = & \frac{1}{2} \left[\frac{g_W^2 v_h^2}{4} \left(1 - \frac{v_h^2}{6f^2}\right) \right] W_L^+ W_L^- \\ & + \frac{1}{2} \left[f^2 g_W^2 \left(1 - \frac{v_h^2}{4f^2}\right) \right] W_H^+ W_H^- \\ & + \frac{1}{2} \left[f^2 g_W^2 \left(1 - \frac{v_h^2}{4f^2}\right) \right] (W_H^3)^2 \\ & + \frac{1}{2} \left[\frac{f^2 g'^2}{5} \left(1 - \frac{5v_h^2}{4f^2}\right) \right] (B_H)^2 \\ & + \frac{1}{2} \left[\frac{g_w g' v_h^2}{2} \left(1 - \frac{2v_h^2}{3f^2}\right) \right] W_H^3 B_H \\ & + \frac{1}{2} \left[\frac{g_W^2 v_h^2}{4 \cos^2 \theta_W} \left(1 - \frac{v_h^2}{6f^2}\right) \right] Z_L^2. \end{aligned} \quad (\text{D.50})$$

Now, the heavy gauge sector is

$$f^2 g_W^2 \left(1 - \frac{v_h^2}{4f^2}\right) (W_H^3)^2 + \frac{g_w g' v_h^2}{2} \left(1 - \frac{2v_h^2}{3f^2}\right) W_H^3 B_H + \frac{f^2 g'^2}{5} \left(1 - \frac{5v_h^2}{4f^2}\right) (B_H)^2. \quad (\text{D.51})$$

The new mass eigenstates in the neutral heavy sector will be a linear combination of the W_H^3 and the B_H gauge bosons, producing an A_H and a Z_H . The mixing angle introduced into the neutral heavy sector by EWSB will be of order (v_h^2/f^2) [37]

$$\sin \theta_H \approx x_H \frac{v_h^2}{f^2}, \quad x_H = \frac{5g_w g'}{4(5g_W^2 - g'^2)}. \quad (\text{D.52})$$

The new heavy neutral mass eigenstates are given by

$$Z_H = \sin \theta_H B_H + \cos \theta_H W_H^3, \quad A_H = \cos \theta_H B_H - \sin \theta_H W_H^3. \quad (\text{D.53})$$

So we have

$$\begin{pmatrix} Z_H \\ A_H \end{pmatrix} = \begin{pmatrix} 1 & x_H \frac{v_h^2}{f^2} \\ -x_H \frac{v_h^2}{f^2} & 1 \end{pmatrix} \begin{pmatrix} W_H^3 \\ B_H \end{pmatrix}, \quad (\text{D.54})$$

and their masses are [37]

$$M_{Z_H}^2 = g_W^2 f^2 \left(1 - \frac{v_h^2}{4f^2}\right), \quad M_{A_H}^2 = \frac{g'^2 f^2}{5} \left(1 - \frac{5v_h^2}{4f^2}\right). \quad (\text{D.55})$$

Finally, the Scalar Lagrangian is

$$\begin{aligned} \mathcal{L}_S &= \frac{1}{2} \left[\frac{g_W^2 v_h^2}{4} \left(1 - \frac{v_h^2}{6f^2}\right) \right] W_L^+ W_L^- \\ &+ \frac{1}{2} \left[f^2 g_W^2 \left(1 - \frac{v_h^2}{4f^2}\right) \right] W_H^+ W_H^- \\ &+ \frac{1}{2} \left[f^2 g_W^2 \left(1 - \frac{v_h^2}{4f^2}\right) \right] (Z_H)^2 \quad . \\ &+ \frac{1}{2} \left[\frac{f^2 g'^2}{5} \left(1 - \frac{5v_h^2}{4f^2}\right) \right] (A_H)^2 \\ &+ \frac{1}{2} \left[\frac{g_W^2 v_h^2}{4 \cos^2 \theta_W} \left(1 - \frac{v_h^2}{6f^2}\right) \right] Z_L^2. \end{aligned} \quad (\text{D.50})$$

The light gauge sector includes W_L^\pm , Z_L , and A_L bosons, that we identify as the SM gauge bosons with masses

$$\begin{aligned} M_{W_L^\pm} &= \frac{g_W v_h}{2} \left(1 - \frac{v_h^2}{6f^2}\right)^{1/2} \approx \frac{g_W v_h}{2} \left(1 - \frac{v_h^2}{12f^2}\right), \\ M_{Z_L} &= \frac{g_W v_h}{2 \cos \theta_H} \left(1 - \frac{v_h^2}{6f^2}\right)^{1/2} = \frac{M_{W_L^\pm}}{\cos \theta_W}, \\ M_{A_L} &= 0, \end{aligned} \quad (\text{D.51})$$

and the mass of the heavy bosons are

$$\begin{aligned} M_{W_H^\pm} &= M_{Z_H} = f g_W \left(1 - \frac{v_h^2}{4f^2}\right)^{1/2} \approx f g_W \left(1 - \frac{v_h^2}{8f^2}\right), \\ M_{A_H} &= \frac{f g'}{\sqrt{5}} \left(1 - \frac{5v_h^2}{4f^2}\right)^{1/2} \approx \frac{f g'}{\sqrt{5}} \left(1 - \frac{5v_h^2}{8f^2}\right). \end{aligned} \quad (\text{D.52})$$

E Appendix: Top Sector

In the Top Sector we introduced these multiplets [36]

$$Q_1 = \begin{pmatrix} -\sigma^2 \begin{pmatrix} u_{L1} \\ b_{L1} \end{pmatrix} \\ t'_{L1} \\ 0 \end{pmatrix}, \quad Q_2 = \begin{pmatrix} 0 \\ t'_{L2} \\ -\sigma^2 \begin{pmatrix} u_{L2} \\ b_{L2} \end{pmatrix} \end{pmatrix}, \quad (\text{E.1})$$

thus,

$$Q_1 = \begin{pmatrix} ib_{L1} \\ -iu_{L1} \\ t'_{L1} \\ 0 \\ 0 \end{pmatrix}, \quad Q_2 = \begin{pmatrix} 0 \\ 0 \\ t'_{L2} \\ ib_{L2} \\ -iu_{L2} \end{pmatrix}. \quad (\text{E.2})$$

Now, we take the Σ field from the eq. (126), and the transformation $\tilde{\Sigma} = \Sigma_0 \Omega \Sigma^\dagger \Omega \Sigma_0$, thus, $\Sigma = \tilde{\Sigma}$. The top Yukawa couplings arise from the Lagrangian of the form

$$\begin{aligned} \mathcal{L}_t = & \frac{1}{2\sqrt{2}} \lambda_1 f \epsilon_{ijk} \epsilon_{xy} \left[(\overline{Q}_1)_i (\Sigma)_{jx} (\Sigma)_{ky} - (\overline{Q}_2 \Sigma_0)_i (\tilde{\Sigma})_{jx} (\tilde{\Sigma})_{ky} \right] u_{3R} \\ & + \lambda_2 f \left(\bar{t}'_{L1} t'_{R1} + \bar{t}'_{L2} t'_{R2} \right) + h.c. \end{aligned} \quad (\text{E.3})$$

Because the indices $i, j = 1, 2, 3$ and $x, y = 4, 5$, we have only the terms

$$\epsilon_{123} = \epsilon_{231} = \epsilon_{312} = 1, \quad \epsilon_{213} = \epsilon_{321} = \epsilon_{132} = -1. \quad (\text{E.4})$$

Then,

$$\begin{aligned} & + \epsilon_{xy} \left((\overline{Q}_1)_1 (\Sigma)_{2x} (\Sigma)_{3y} - (\overline{Q}_2 \Sigma_0)_1 (\tilde{\Sigma})_{2x} (\tilde{\Sigma})_{3y} \right) \\ & - \epsilon_{xy} \left((\overline{Q}_1)_2 (\Sigma)_{1x} (\Sigma)_{3y} - (\overline{Q}_2 \Sigma_0)_2 (\tilde{\Sigma})_{1x} (\tilde{\Sigma})_{3y} \right) \\ & + \epsilon_{xy} \left((\overline{Q}_1)_2 (\Sigma)_{3x} (\Sigma)_{1y} - (\overline{Q}_2 \Sigma_0)_2 (\tilde{\Sigma})_{3x} (\tilde{\Sigma})_{1y} \right) \\ & - \epsilon_{xy} \left((\overline{Q}_1)_3 (\Sigma)_{2x} (\Sigma)_{1y} - (\overline{Q}_2 \Sigma_0)_3 (\tilde{\Sigma})_{2x} (\tilde{\Sigma})_{1y} \right) \\ & + \epsilon_{xy} \left((\overline{Q}_1)_3 (\Sigma)_{1x} (\Sigma)_{2y} - (\overline{Q}_2 \Sigma_0)_3 (\tilde{\Sigma})_{1x} (\tilde{\Sigma})_{2y} \right) \\ & - \epsilon_{xy} \left((\overline{Q}_1)_1 (\Sigma)_{3x} (\Sigma)_{2y} - (\overline{Q}_2 \Sigma_0)_1 (\tilde{\Sigma})_{3x} (\tilde{\Sigma})_{2y} \right), \end{aligned} \quad (\text{E.5})$$

the terms $\epsilon_{45} = -\epsilon_{54} = 1$ while the others terms vanish.

We have that

$$\bar{Q}_1 = \begin{pmatrix} -i\bar{b}_{L1} & i\bar{u}_{L1} & \bar{t}'_{L1} & 0 & 0 \end{pmatrix}, \quad \bar{Q}_2\Sigma_0 = \begin{pmatrix} -i\bar{b}_{L2} & i\bar{u}_{L2} & \bar{t}'_{L2} & 0 & 0 \end{pmatrix}, \quad (\text{E.6})$$

and

$$\Sigma = \tilde{\Sigma} = \begin{pmatrix} 0 & 0 & 0 & 1 & 0 \\ 0 & -\frac{1}{2}(1-c_v) & \frac{i}{\sqrt{2}}s_v & 0 & \frac{1}{2}(1+c_v) \\ 0 & \frac{i}{\sqrt{2}}s_v & c_v & 0 & \frac{i}{\sqrt{2}}s_v \\ 1 & 0 & 0 & 0 & 0 \\ 0 & \frac{1}{2}(1+c_v) & \frac{i}{\sqrt{2}}s_v & 0 & -\frac{1}{2}(1-c_v) \end{pmatrix}. \quad (\text{E.7})$$

From eqs. (E.5), (E.6) and (E.7), we have that

$$\mathcal{L}_t = \frac{1}{2\sqrt{2}}\lambda_1 f \left((1+c_v) (\bar{t}'_{L1} - \bar{t}'_{L2}) + \frac{2s_v}{\sqrt{2}} (\bar{u}_{L1} - \bar{u}_{L2}) \right) u_{3R} + \lambda_2 f (\bar{t}'_{L1} t'_{R1} + \bar{t}'_{L2} t'_{R2}) + h.c. \quad (\text{E.8})$$

The T-parity eigenstates are given by [36]

$$q_{\pm} = \frac{1}{\sqrt{2}} (q_1 \mp q_2), \quad t'_{L\pm} = \frac{1}{\sqrt{2}} (t'_{L1} \mp t'_{L2}), \quad t'_{R\pm} = \frac{1}{\sqrt{2}} (t'_{R1} \mp t'_{R2}). \quad (\text{E.9})$$

Substituting (E.9) in (E.8)

$$\begin{aligned} \mathcal{L}_t &= \lambda_1 f \left(\frac{1}{2} (1+c_v) \bar{t}'_{L+} + \frac{s_v}{\sqrt{2}} \bar{u}_{L+} \right) u_{3R} + \lambda_2 f (\bar{t}'_{L+} t'_{R+} + \bar{t}'_{L-} t'_{R-}) + h.c. \\ \mathcal{L}_t &= \left(\frac{\lambda_1 f}{2} (1+c_v) \bar{t}'_{L+} u_{3R} + \frac{\lambda_1 f s_v}{\sqrt{2}} \bar{u}_{L+} u_{3R} + \lambda_2 f \bar{t}'_{L+} t'_{R+} \right) + \lambda_2 f (\bar{t}'_{L-} t'_{R-}) + h.c. \end{aligned} \quad (\text{E.10})$$

From the Lagrangian (E.10) we can see that the first part contain only T-even states, while the second part contain only T-odd ones, it's important to note that T-even and T-odd states don't mix, so the T-even sector must diagonalize to find the T-eigenstates given by eqs. (169-170).

The T-odd states t'_{L-} and t'_{R-} combine to form a Dirac fermion T_- , with mass $M_{T_-} = \lambda_2 f$ [36].

The Yukawa coupling for the other up-type is given by

$$\mathcal{L}_{up} = -\frac{1}{2\sqrt{2}}\lambda_u f \epsilon_{ijk} \epsilon_{xy} \left[(\bar{Q}_1)_i (\Sigma)_{jx} (\Sigma)_{ky} - (\bar{Q}_2\Sigma_0)_i (\tilde{\Sigma})_{jx} (\tilde{\Sigma})_{ky} \right] u_R + h.c., \quad (\text{E.11})$$

where, now multiplets Q_1 and Q_2 are

$$Q_1 = \begin{pmatrix} -d_{1L} \\ u_{1L} \\ 0 \\ 0 \\ 0 \end{pmatrix}, \quad Q_2 = \begin{pmatrix} 0 \\ 0 \\ 0 \\ -d_{2L} \\ u_{2L} \end{pmatrix}, \quad (\text{E.12})$$

thus,

$$\bar{Q}_1 = \begin{pmatrix} -\bar{d}_{1L} & \bar{u}_{1L} & 0 & 0 & 0 \end{pmatrix}, \quad \bar{Q}_2 \Sigma_0 = \begin{pmatrix} -\bar{d}_{2L} & \bar{u}_{2L} & 0 & 0 & 0 \end{pmatrix}, \quad (\text{E.13})$$

the terms that contribute are the same of the eq. (E.5), omitting only the terms $(\bar{Q}_1)_3$ and $(\bar{Q}_2 \Sigma_0)_3$,

$$\begin{aligned} & + \epsilon_{xy} \left((\bar{Q}_1)_1 (\Sigma)_{2x} (\Sigma)_{3y} - (\bar{Q}_2 \Sigma_0)_1 (\tilde{\Sigma})_{2x} (\tilde{\Sigma})_{3y} \right) \\ & - \epsilon_{xy} \left((\bar{Q}_1)_2 (\Sigma)_{1x} (\Sigma)_{3y} - (\bar{Q}_2 \Sigma_0)_2 (\tilde{\Sigma})_{1x} (\tilde{\Sigma})_{3y} \right) \\ & + \epsilon_{xy} \left((\bar{Q}_1)_2 (\Sigma)_{3x} (\Sigma)_{1y} - (\bar{Q}_2 \Sigma_0)_2 (\tilde{\Sigma})_{3x} (\tilde{\Sigma})_{1y} \right) \\ & - \epsilon_{xy} \left((\bar{Q}_1)_1 (\Sigma)_{3x} (\Sigma)_{2y} - (\bar{Q}_2 \Sigma_0)_1 (\tilde{\Sigma})_{3x} (\tilde{\Sigma})_{2y} \right). \end{aligned} \quad (\text{E.14})$$

So, the \mathcal{L}_{up} is

$$\mathcal{L}_{up} = -\frac{1}{2\sqrt{2}} \lambda_u f \frac{2s_{v_h}}{\sqrt{2}} (\bar{u}_{1L} - \bar{u}_{2L}) u_R + h.c. = -\frac{s_{v_h}}{\sqrt{2}} \lambda_u f \bar{u}_L u_R + h.c., \quad (\text{E.15})$$

expanding to order of $\mathcal{O}(v/f)^2$, the up-type quark masses are [39]

$$m_u^i = \lambda_u^i v_h \left(1 - \frac{v_h^2}{3f^2} \right), \quad (i = 1, 2). \quad (\text{E.16})$$

For the down-type quark Yukawa coupling, we take [39]

$$\mathcal{L}_{down} = \frac{i\lambda_d}{2\sqrt{2}} f \epsilon_{ij} \epsilon_{xyz} \left[(\bar{\Psi}_2)_x (\Sigma)_{iy} (\Sigma)_{jz} X - (\bar{\Psi}_1 \Sigma_0)_x (\tilde{\Sigma})_{iy} (\tilde{\Sigma})_{jz} \tilde{X} \right] d_R + h.c., \quad (\text{E.17})$$

where we sum over $i, j = 1, 2$ and $x, y, z = 3, 4, 5$, and $X = (\Sigma_{33})^{-1/4} = c_v^{-1/4}$. Here, the multiplets Ψ_1 and Ψ_2 are

$$\Psi_1 = \begin{pmatrix} u_{1L} \\ d_{1L} \\ 0 \\ 0 \\ 0 \end{pmatrix}, \quad \Psi_2 = \begin{pmatrix} 0 \\ 0 \\ 0 \\ u_{2L} \\ d_{2L} \end{pmatrix}, \quad (\text{E.18})$$

we have that $\epsilon_{12} = -\epsilon_{21} = 1$, then

$$\epsilon_{xyz} \left[(\bar{\Psi}_2)_x (\Sigma)_{1y} (\Sigma)_{2z} X - (\bar{\Psi}_1 \Sigma_0)_x (\tilde{\Sigma})_{1y} (\tilde{\Sigma})_{2z} \tilde{X} - (\bar{\Psi}_2)_x (\Sigma)_{2y} (\Sigma)_{1z} X + (\bar{\Psi}_1 \Sigma_0)_x (\tilde{\Sigma})_{2y} (\tilde{\Sigma})_{1z} \tilde{X} \right] d_R. \quad (\text{E.19})$$

Since $(\bar{\Psi}_1 \Sigma_0)_3 = (\bar{\Psi}_2)_3 = 0$, then all the terms ϵ_{3yz} vanish. The remaining terms are

$$\begin{aligned} & \epsilon_{453} \left[(\bar{\Psi}_2)_4 (\Sigma)_{15} (\Sigma)_{23} X - (\bar{\Psi}_1 \Sigma_0)_4 (\tilde{\Sigma})_{15} (\tilde{\Sigma})_{23} \tilde{X} - (\bar{\Psi}_2)_4 (\Sigma)_{25} (\Sigma)_{13} X + (\bar{\Psi}_1 \Sigma_0)_4 (\tilde{\Sigma})_{25} (\tilde{\Sigma})_{13} \tilde{X} \right] \\ & + \epsilon_{435} \left[(\bar{\Psi}_2)_4 (\Sigma)_{13} (\Sigma)_{25} X - (\bar{\Psi}_1 \Sigma_0)_4 (\tilde{\Sigma})_{13} (\tilde{\Sigma})_{25} \tilde{X} - (\bar{\Psi}_2)_4 (\Sigma)_{23} (\Sigma)_{15} X + (\bar{\Psi}_1 \Sigma_0)_4 (\tilde{\Sigma})_{23} (\tilde{\Sigma})_{15} \tilde{X} \right] \\ & + \epsilon_{534} \left[(\bar{\Psi}_2)_5 (\Sigma)_{13} (\Sigma)_{24} X - (\bar{\Psi}_1 \Sigma_0)_5 (\tilde{\Sigma})_{13} (\tilde{\Sigma})_{24} \tilde{X} - (\bar{\Psi}_2)_5 (\Sigma)_{23} (\Sigma)_{14} X + (\bar{\Psi}_1 \Sigma_0)_5 (\tilde{\Sigma})_{23} (\tilde{\Sigma})_{14} \tilde{X} \right] \\ & + \epsilon_{543} \left[(\bar{\Psi}_2)_5 (\Sigma)_{14} (\Sigma)_{23} X - (\bar{\Psi}_1 \Sigma_0)_5 (\tilde{\Sigma})_{14} (\tilde{\Sigma})_{23} \tilde{X} - (\bar{\Psi}_2)_5 (\Sigma)_{24} (\Sigma)_{13} X + (\bar{\Psi}_1 \Sigma_0)_5 (\tilde{\Sigma})_{24} (\tilde{\Sigma})_{13} \tilde{X} \right], \end{aligned} \quad (\text{E.20})$$

so we have

$$\mathcal{L}_{down} = -\frac{\lambda_d f}{2} s_{v_h} c_{v_h}^{-1/4} (\bar{d}_{1L} - \bar{d}_{2L}) d_R = -\frac{\lambda_d f}{\sqrt{2}} s_{v_h} c_{v_h}^{-1/4} \bar{d}_L d_R, \quad (\text{E.21})$$

taking only the terms until $\mathcal{O}(v_h/f)^2$

$$\begin{aligned} s_{v_h} & \approx \frac{\sqrt{2} v_h}{f} - \frac{\sqrt{2} v_h^3}{3 f^3}, \\ c_{v_h}^{-1/4} & \approx 1 + \frac{v_h^2}{4 f^2}, \end{aligned}$$

thus, the down-type quark masses are [39]

$$m_d^j = \lambda_d^j v_h \left(1 - \frac{v_h^2}{12 f^2} \right), \quad (j = 1, 2, 3). \quad (\text{E.22})$$

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