Covariant Extension of Generalized Parton Distributions using Artificial Neural Networks

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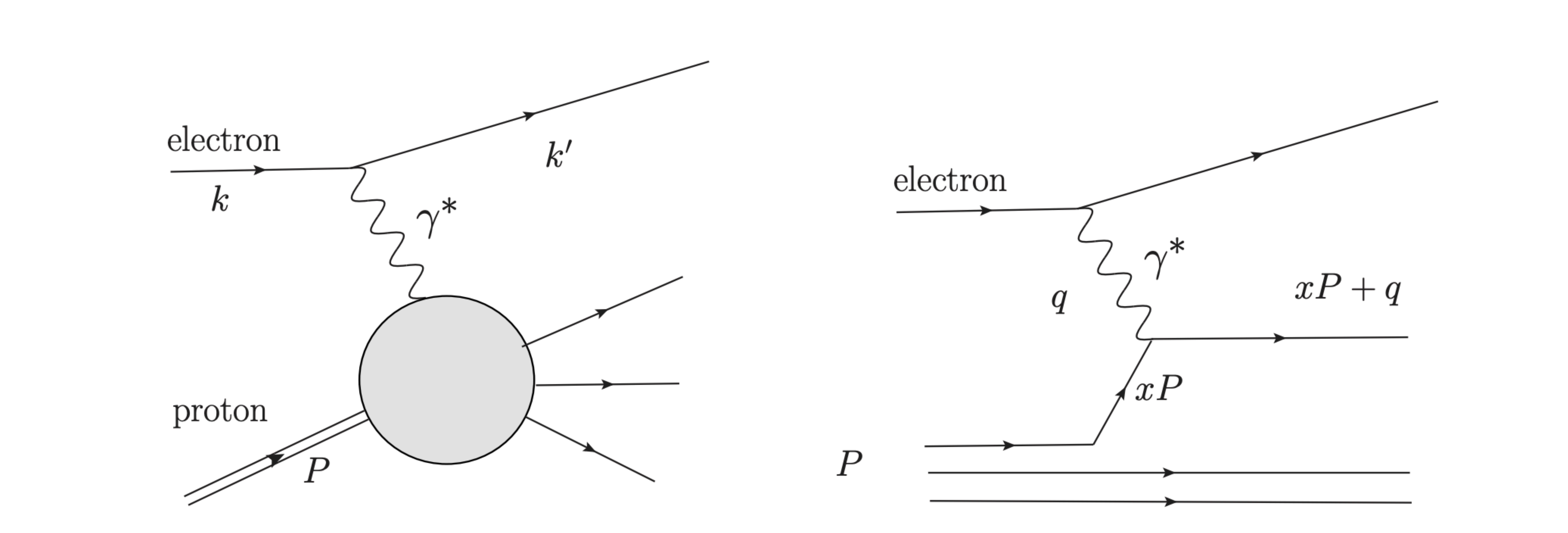
Non-Perturbative Physics: Tools and Applications 06 Sept 2023

Outline

- (Generalized) Parton Distribution Functions
- Covariant Extension via Radon Transform (RT) inversion
- RT inversion using Artificial Neural Networks
- Results on analytical models
- Conclusions and outlook

Parton Distribution Function (PDF)

Inclusive processes (DIS)



$$Q^2 = -q^2$$

$$\frac{d^2\sigma}{d\Omega dE} = \frac{\alpha^2}{Q^4}\frac{E'}{E} l_{\mu\nu} W^{\mu\nu}$$

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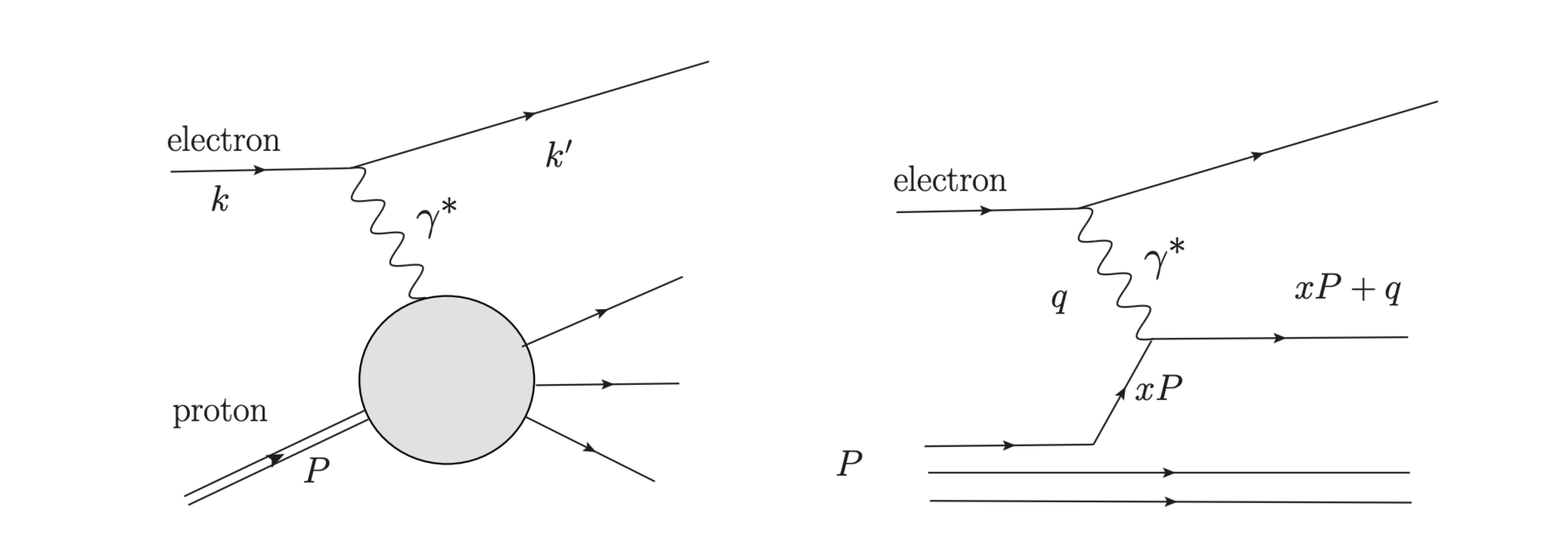

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leptonic tensor　　hadronic tensor

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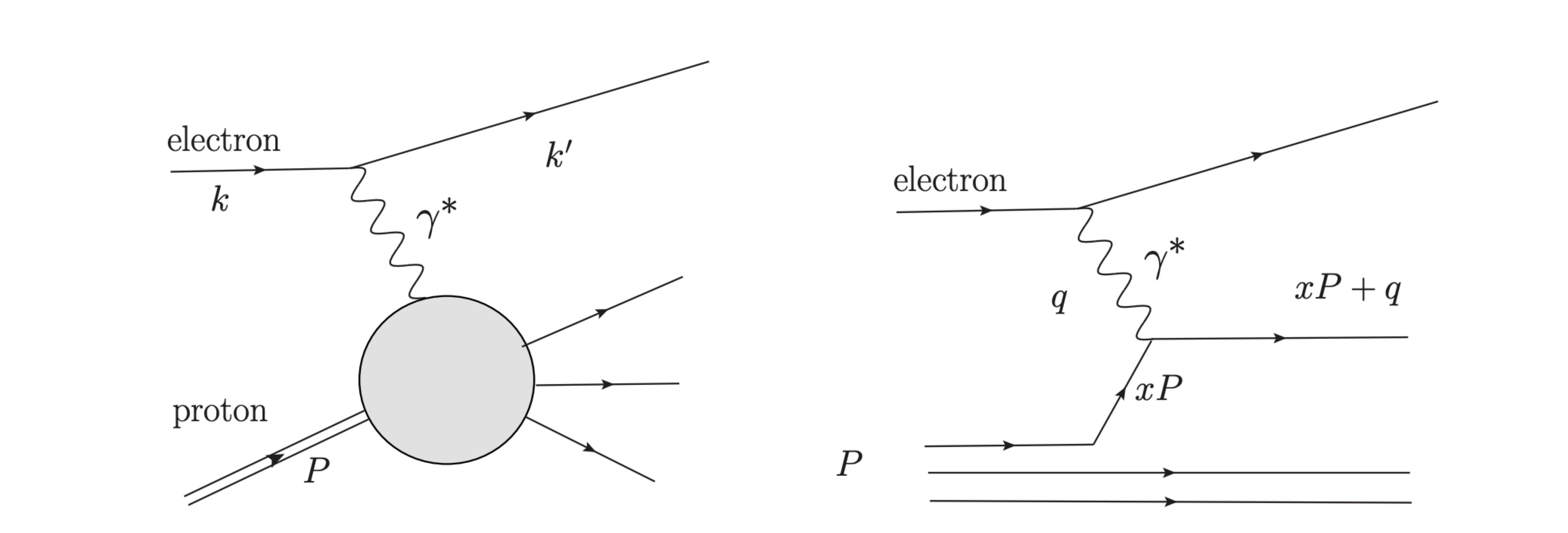

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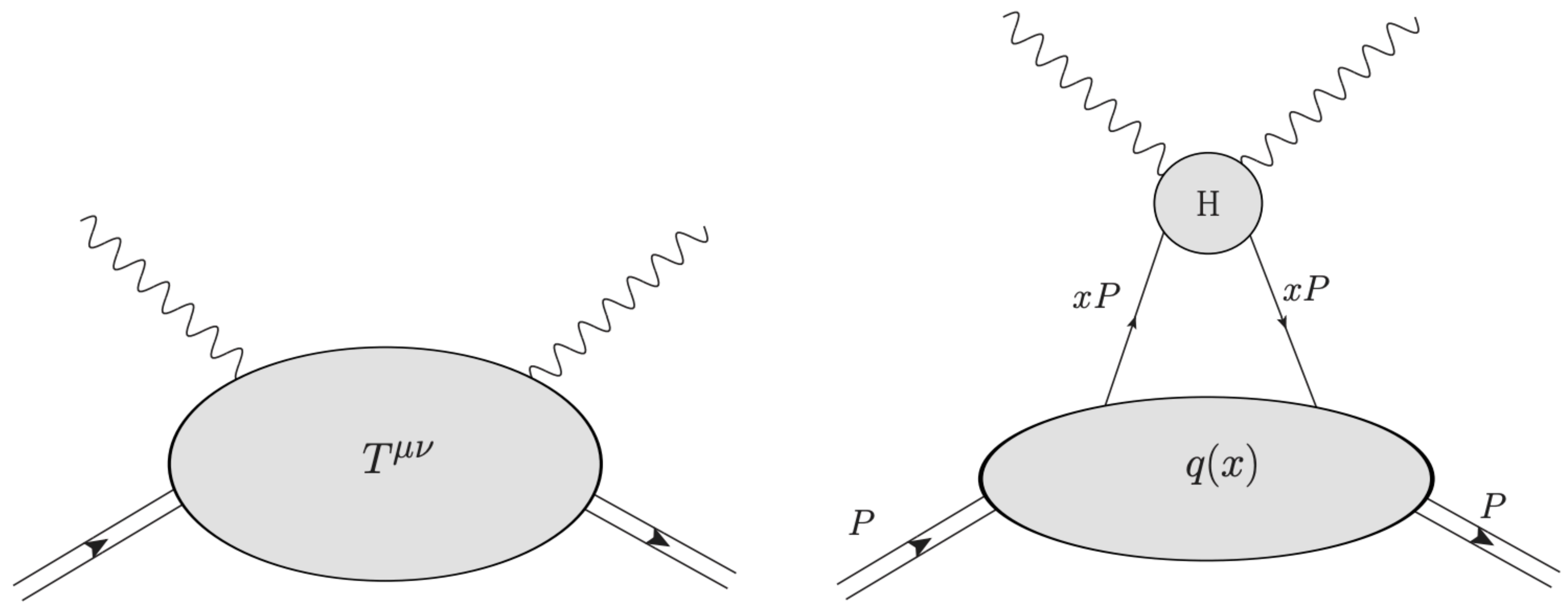
leptonic tensor hadronic tensor

Optical theorem: $W^{\mu\nu} = \Im T^{\mu\nu}$ Compton tensor

$$T^{\mu\nu}(q,P) = \frac{i}{4\pi}\int d^4x\, e^{iq\cdot x}\sum_\sigma \langle P,\sigma\,|\,T J^\mu(x) J^\nu(0)\,|\,P,\sigma\rangle$$

$$W^{\mu\nu}(q,P) = -W_1\left(g^{\mu\nu} - \frac{q^\mu q^\nu}{q^2}\right) + W_2\frac{1}{M^2}\left(P^\mu - \frac{P\cdot q}{q^2}q^\mu\right)\left(P^\nu - \frac{P\cdot q}{q^2}q^\nu\right)$$



Factorization

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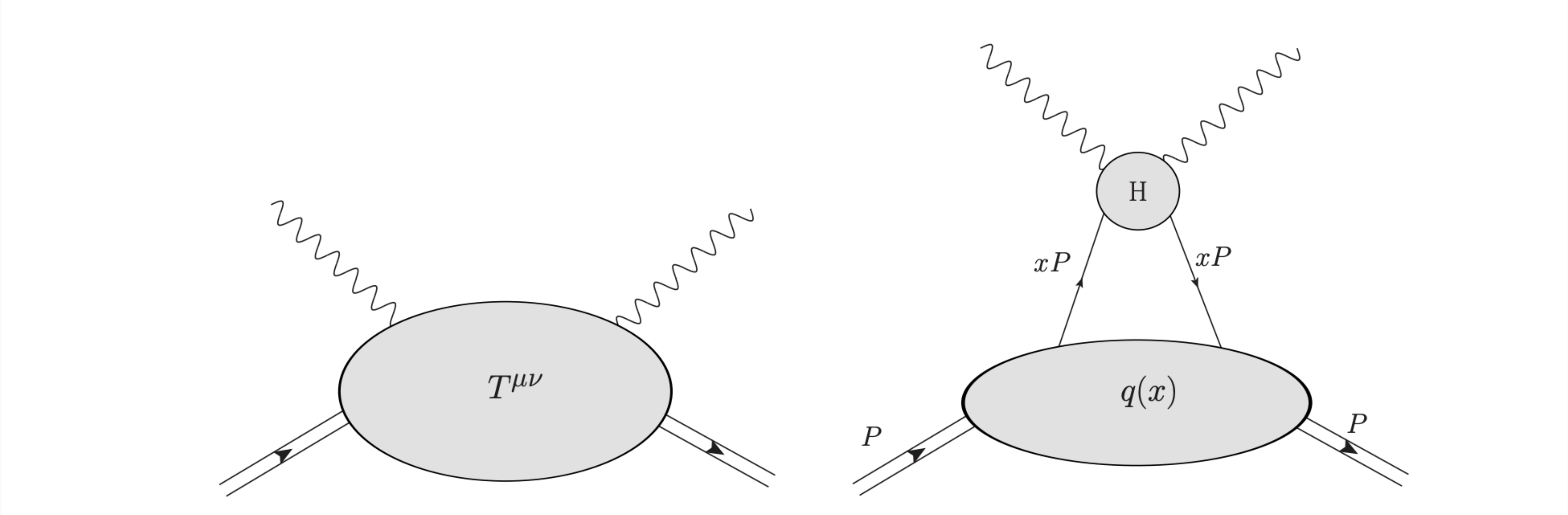

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$$W_k(x_B, Q^2) = \sum_{p=q,g}\int_0^1 dx\, C_k^{(i)}(x, x_B, Q^2, \mu^2)\, q_p(x,\mu^2) + O\left(1/Q^2\right)$$

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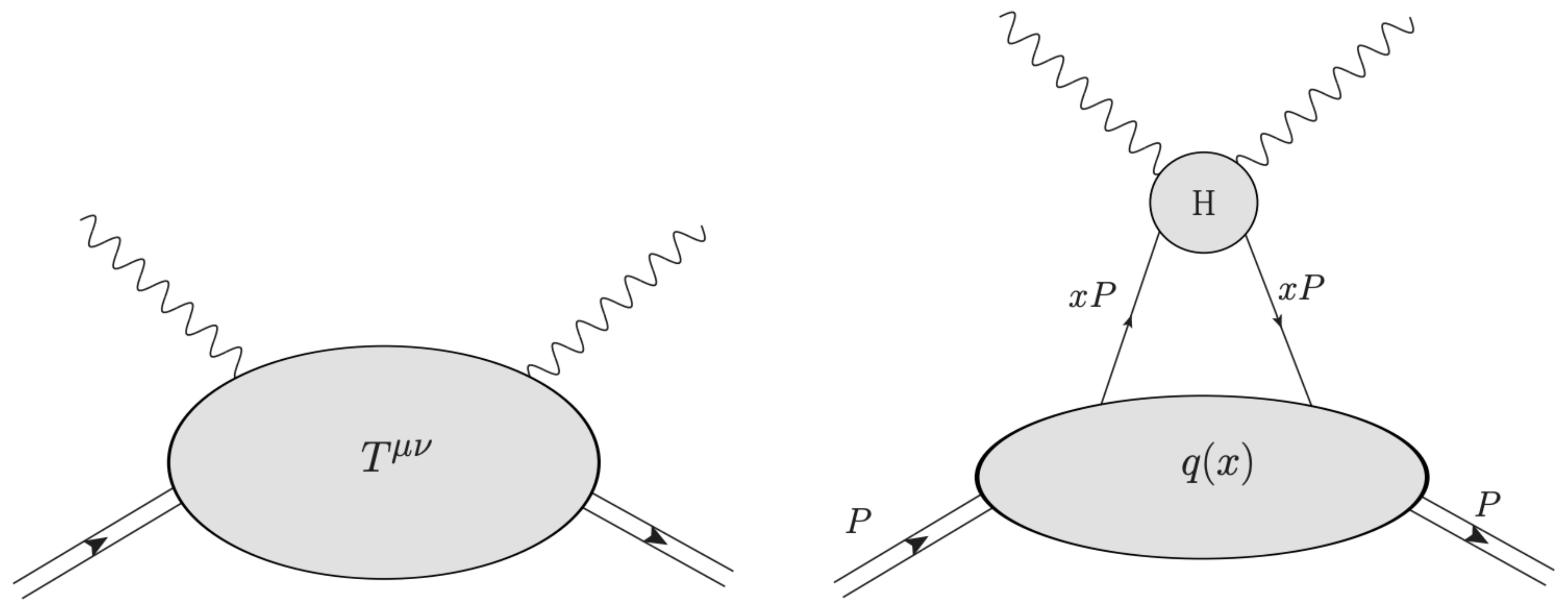

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Probability density of finding a parton with longitudinal momentum fraction x

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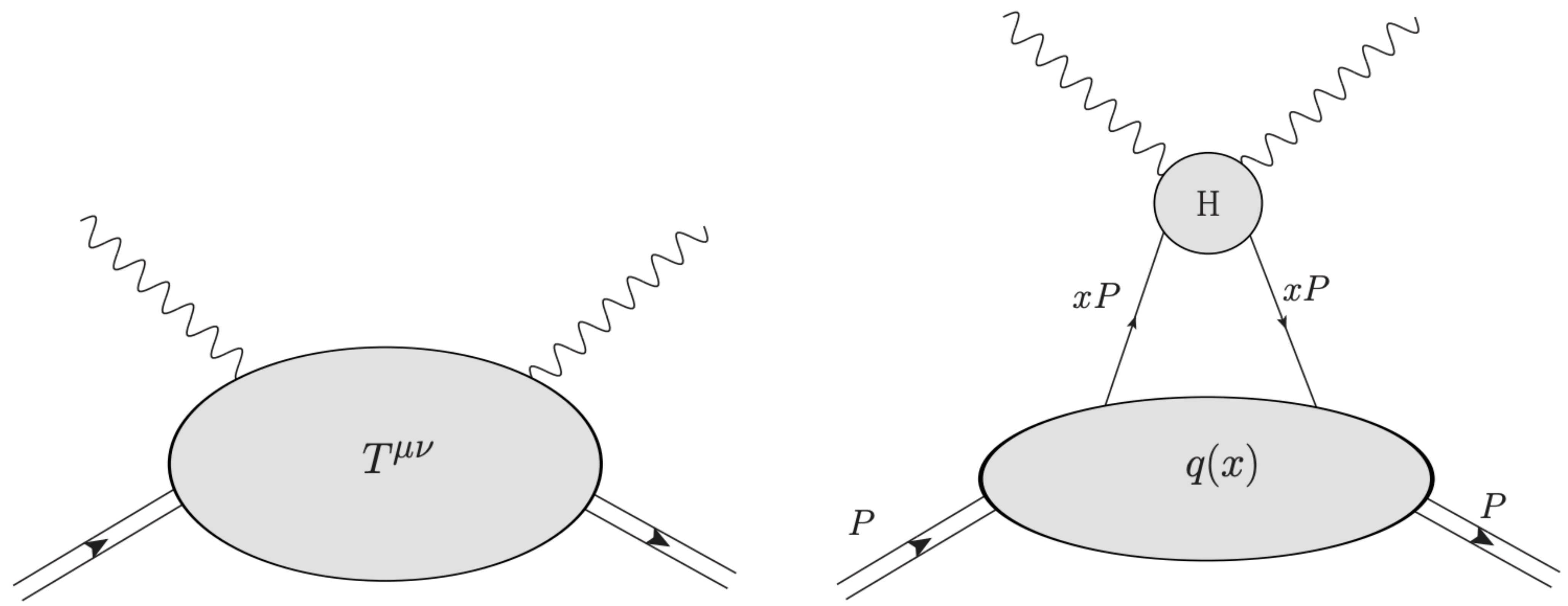

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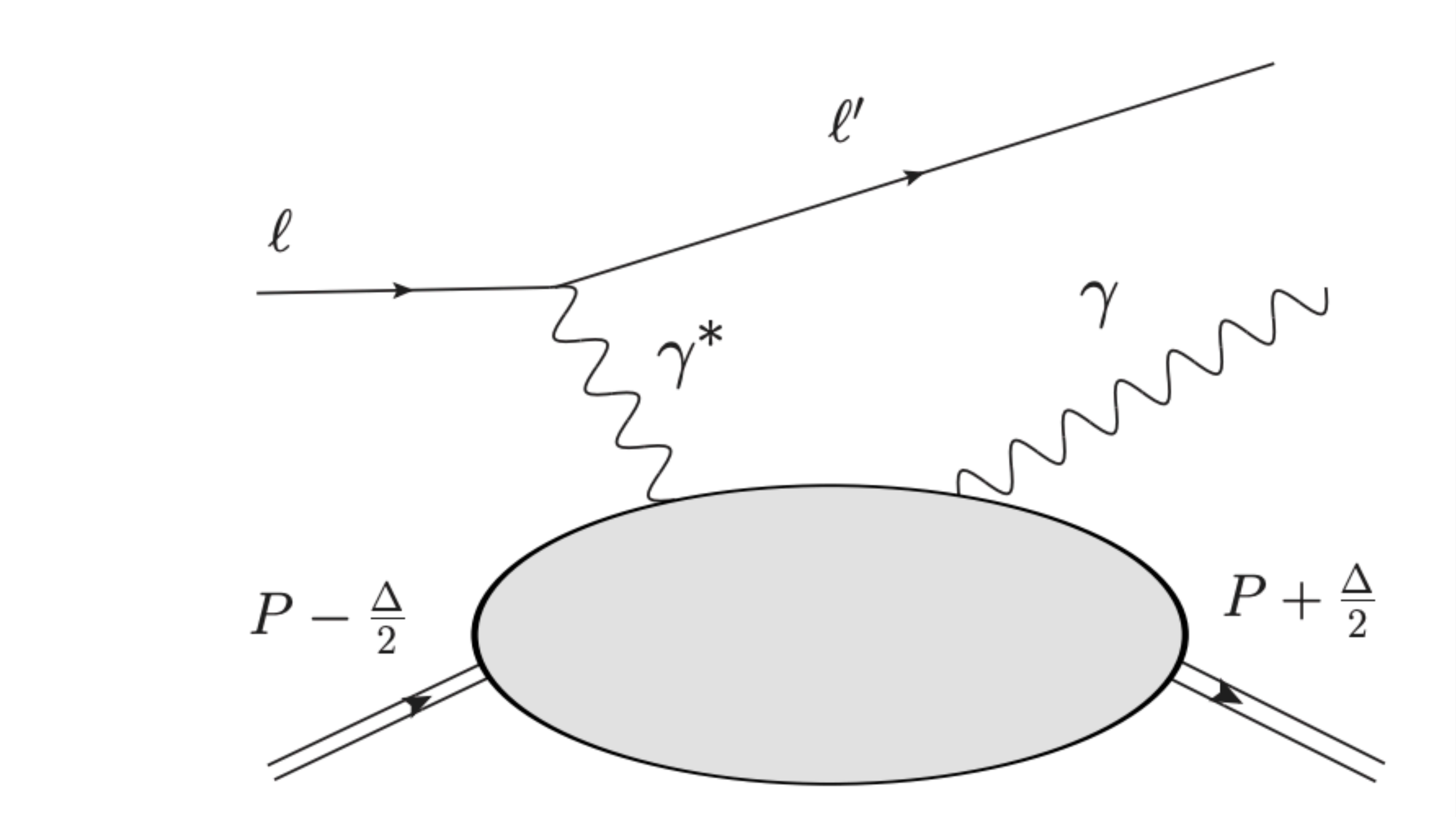
Probability density of finding a parton with longitudinal momentum fraction x

No Wilson line in light-cone gauge $A^+ = 0$

Generalized Parton Distribution (GPD)

Exclusive processes (DVCS)

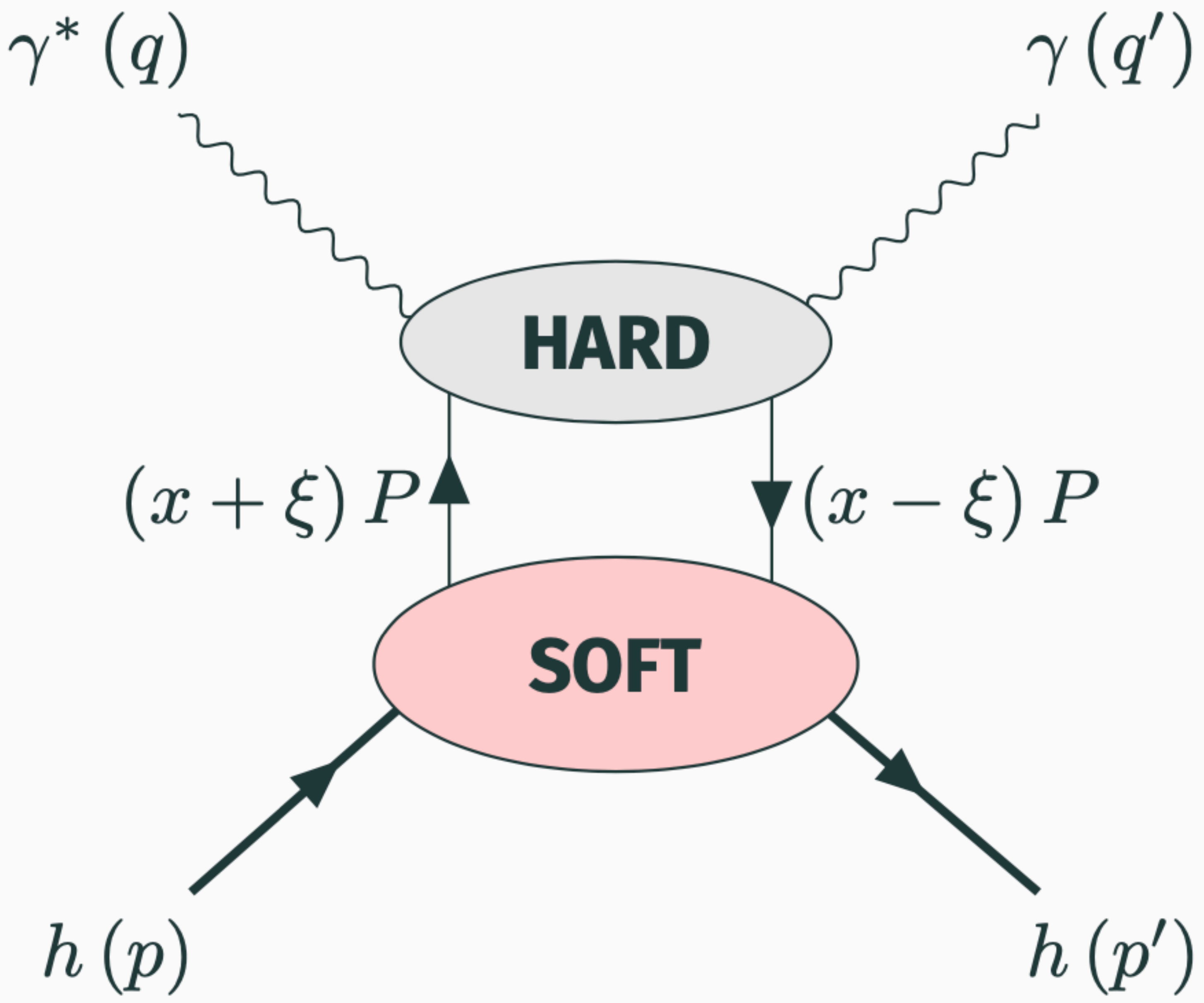


$Q^2 \to \infty$

$$P = \frac{p+p'}{2}, \quad \Delta = p' - p$$

$$\xi = \frac{(p'-p)^+}{2P^+} = -\frac{\Delta^+}{2P^+}$$

$$t = \Delta^2$$



Factorization of the amplitude

$$\mathscr{M}(\xi, t; Q^2) = \sum_{p=q,g} \int_{-1}^{1} \frac{dx}{\xi} K^p\left(\frac{x}{\xi}, \frac{Q^2}{\mu_F^2}, \alpha_s(\mu_F)\right) \mathscr{H}^p\left(x, \xi, t; \mu_F\right) + O(1/Q^2)$$

hard / perturbative

soft / non perturbative GPD

Ex: Chiral even twist-2 operators

Nucleon quark GPD

$$\mathcal{H}^q(x,\xi,t) = \frac{1}{2}\int dz^- \, e^{ixP^+z^-} \langle P+\Delta/2 \,|\, \bar{\psi}^q(-z/2)\gamma^+\psi^q(z/2) \,|\, P-\Delta/2\rangle \Big|_{z^+=z_\perp=0}$$

$$= \frac{1}{2P^+}\left[H^q(x,\xi,t)\bar{u}(p')\gamma^+u(p) + E^q(x,\xi,t)\bar{u}(p')\frac{i\sigma^{+\mu}\Delta_\mu}{2M}u(p)\right]$$

$$\tilde{\mathcal{H}}^q(x,\xi,t) = \frac{1}{2}\int dz^- \, e^{ixP^+z^-} \langle P+\Delta/2 \,|\, \bar{\psi}^q(-z/2)\gamma^+\gamma^5\psi^q(z/2) \,|\, P-\Delta/2\rangle \Big|_{z^+=z_\perp=0}$$

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Nucleon gluon GPD

$$\mathcal{H}^g(x,\xi,t) = \frac{1}{2}\int dz^- \, e^{ixP^+z^-} \langle P+\Delta/2 \,|\, G^{\mu+}(-z/2)G^+_\mu(z/2) \,|\, P-\Delta/2\rangle \Big|_{z^+=z_\perp=0}$$

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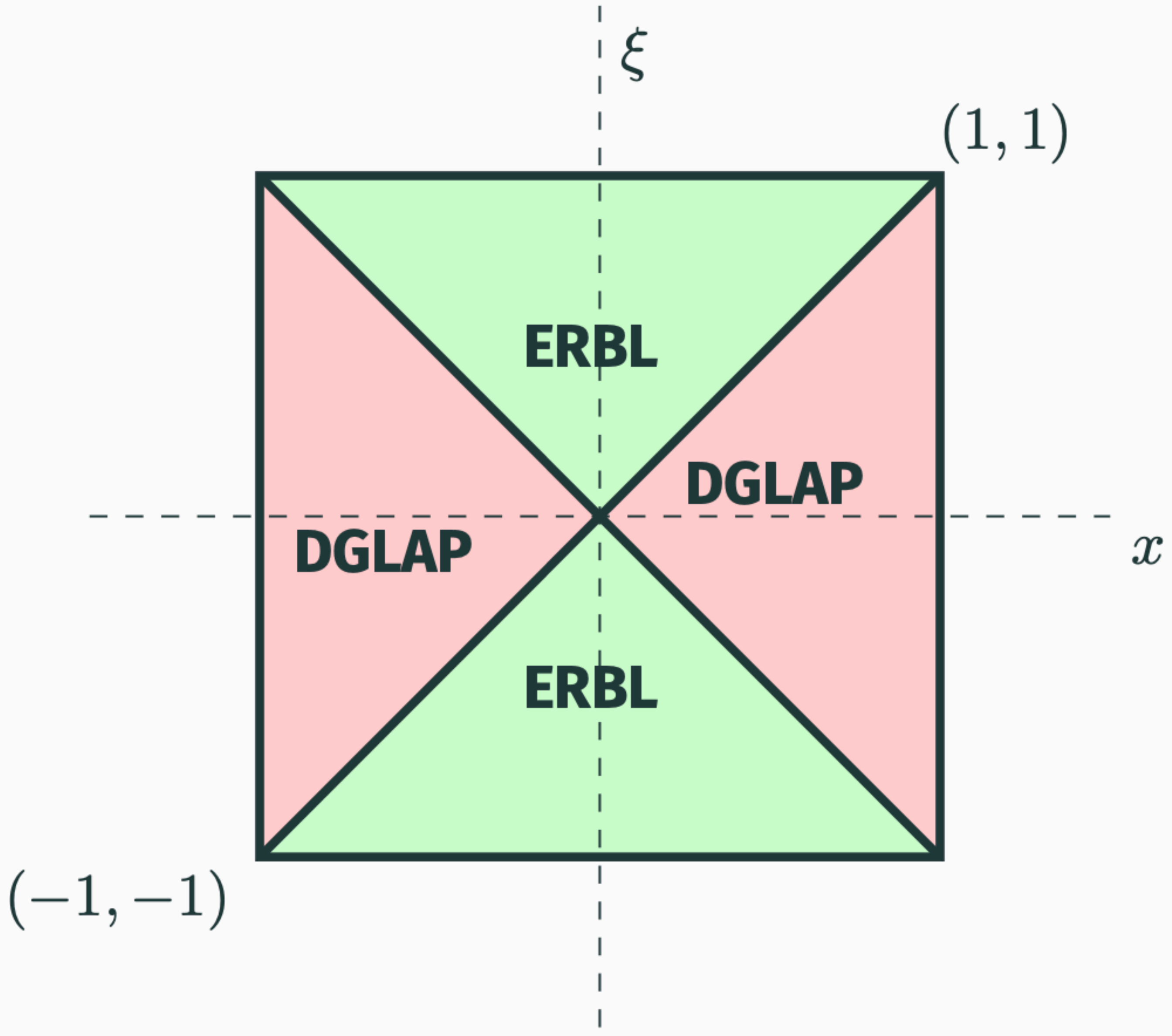
quark GPD of spin zero hadron

$$H^q_\pi(x,\xi,t) = \frac{1}{2}\int \frac{dz^-}{2\pi} e^{ixP^+z^-} \langle \pi(P+\Delta/2) \,|\, \bar{\psi}^q(-z/2)\gamma^+\psi^q(z/2) \,|\, \pi(P-\Delta/2)\rangle \Big|_{z^+=0,\; z^\perp=0}$$

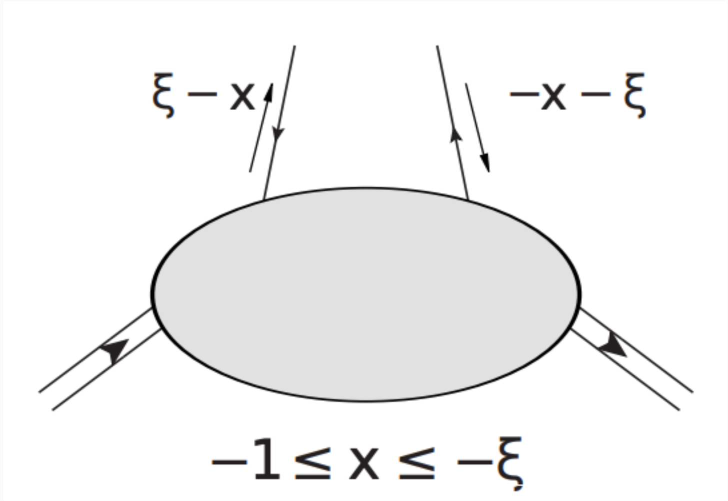
GPD properties

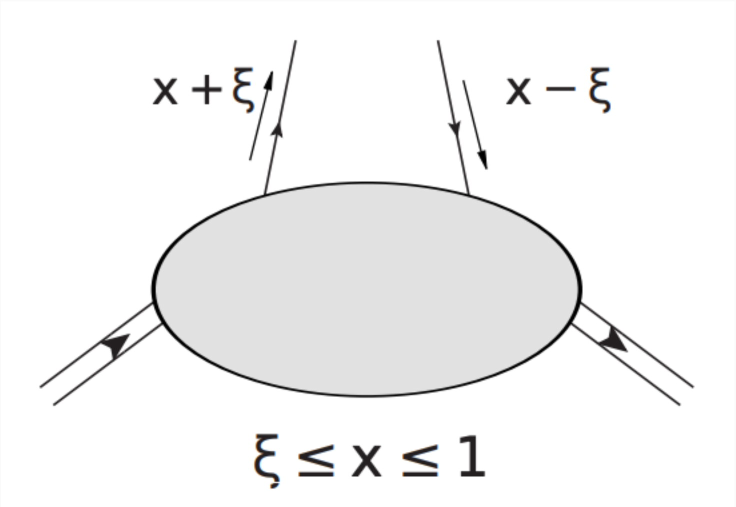
- **Support** $(x, \xi) \in [-1,1] \otimes [-1,1]$



DGLAP $|x| > |\xi|$

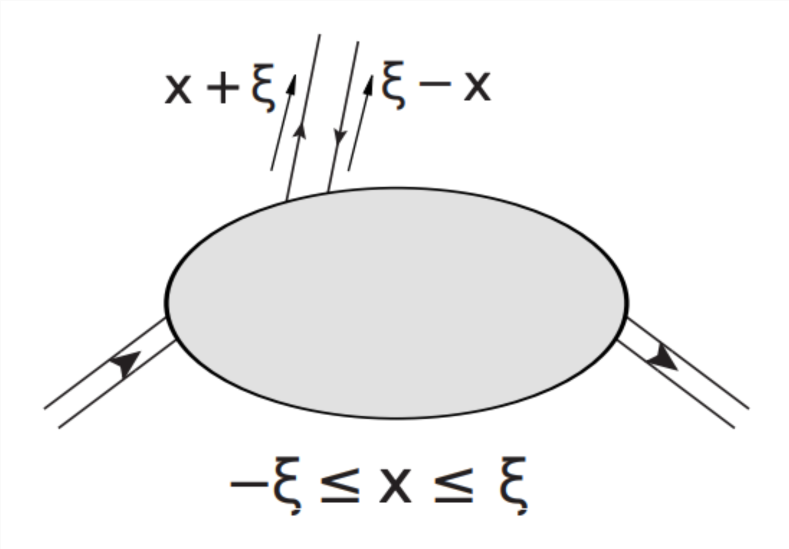
Emission/absorption quark ($x > 0$) or antiquark ($x < 0$)





ERBL $|x| < |\xi|$

Emission of quark/antiquark pair



- $\xi-$parity

$$H^q(x, -\xi, t) = H^q(x, \xi, t)$$ Time inversion symmetry

- Polynomiality

$$\mathcal{A}_m(\xi, t) = \int_{-1}^{1} dx\, x^m H^q(x, \xi, t) = \sum_{\substack{k=0 \\ k\, even}}^{m+1} C_{k,m}(t)\, \xi^k$$ Lorentz symmetry

- Positivity

$$\left| H^q(x, \xi, t=0) \right| \leq \sqrt{q\left(\frac{x+\xi}{1+\xi}\right) q\left(\frac{x-\xi}{1-\xi}\right)}, \quad |x| > |\xi|$$ Hilbert space norm

- Form factors

$$\int_{-1}^{1} dx\, H^q(x, \xi, t) = F^q(t), \quad \int_{-1}^{1} dx\, x\, H^q(x, \xi, t) = A^q(t) + \xi^2 C^q(t)$$

- **Forward limit**

$$H^q(x,0,0) = q(x)\theta(x) - \bar{q}(-x)\theta(-x)$$

- **Hadron 3D tomography**

$$\rho^q(x, b_\perp) = \int \frac{d^2\Delta_\perp}{(2\pi)^2} e^{-ib_\perp \cdot \Delta_\perp} H^q(x,0,-\Delta_\perp^2)$$

Probability density of finding a parton with longitudinal momentum fraction x and position $b_\perp$ in the transverse plane.

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Truncation in Overlap representation of LCWF is consistent only in DGLAP region

$$H^q(x,\xi,t)\Big|_{|x|>|\xi|} = \sum_{N,\beta} \sqrt{1-\xi^2}^{\,2-N} \int [d\bar{x}]_N [d^2\bar{\mathbf{k}}_\perp]_N \, \delta(x-\bar{x}_i)\, \psi^*_{N,\beta}(x_i^{out}, \mathbf{k}_{i\perp}^{out})\, \psi_{N,\beta}(x_i^{in}, \mathbf{k}_{i\perp}^{in})$$

Positivity is built in, but for polynomiality GPD must be extended to ERBL region

Covariant extension via Radon transform inversion

Is it possible to extend the GPD from DGLAP to ERBL preserving polynomiality?

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$$\langle P + \Delta/2 \,|\, \bar{\psi}(-z/2)\gamma_\mu \psi(z/2) \,|\, P - \Delta/2 \rangle \Big|_{z^2=0} = \int d\beta \, d\alpha e^{-i\beta(P\cdot z) + i\alpha \frac{\Delta \cdot z}{2}} \left[2P_\mu \, f(\beta, \alpha, t) - \Delta_\mu \, g(\beta, \alpha, t) \right]$$

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$$= \int_\Omega d\beta\, d\alpha\,\left(f(\beta,\alpha,t)+\xi\, g(\beta,\alpha,t)\right)\delta(x-\beta-\xi\alpha)$$

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Radon transform

- Integral over line parametrized by x, ξ
- It guarantees polynomiality

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Double Distributions (DD)

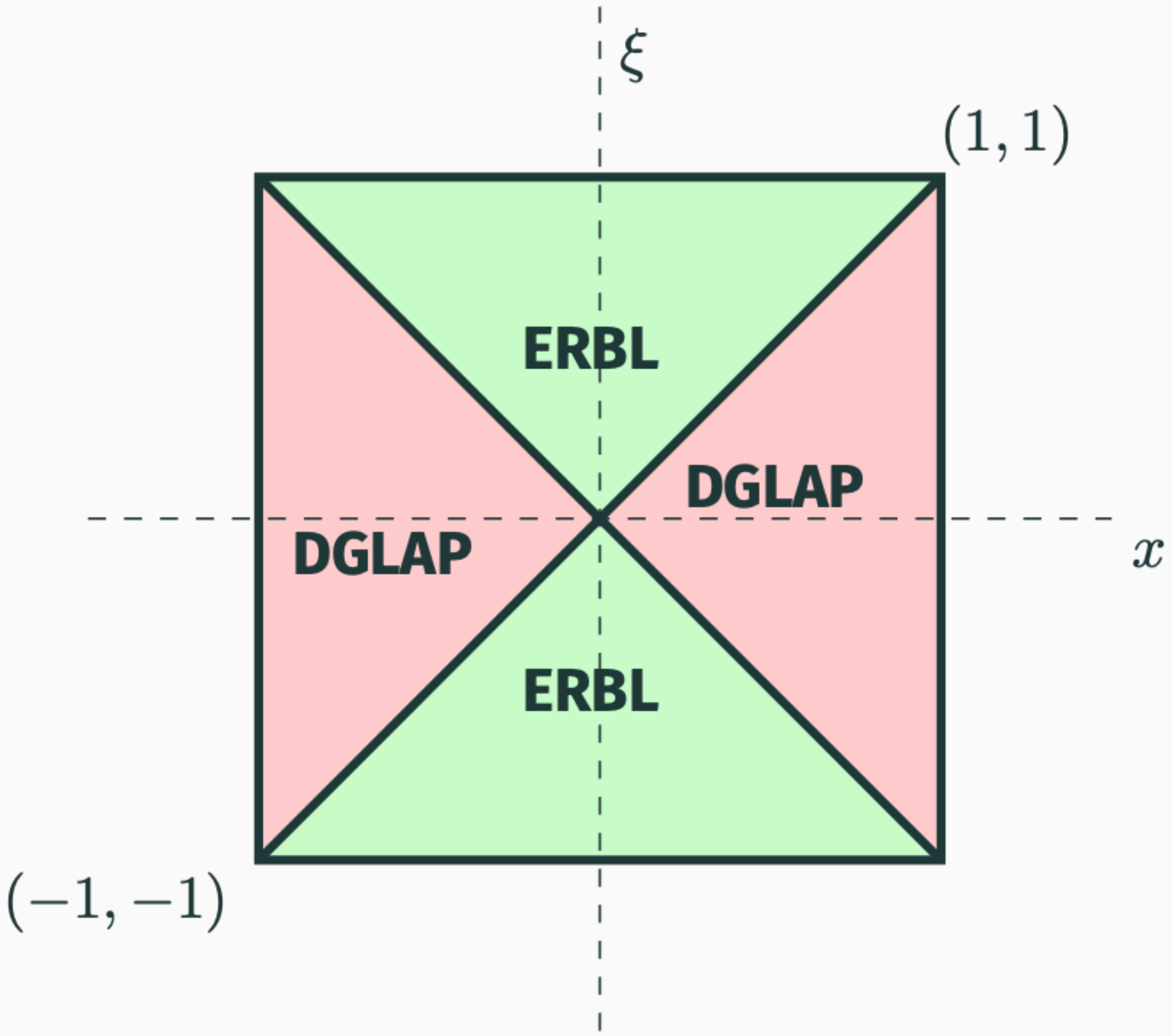
Not uniquely fixed

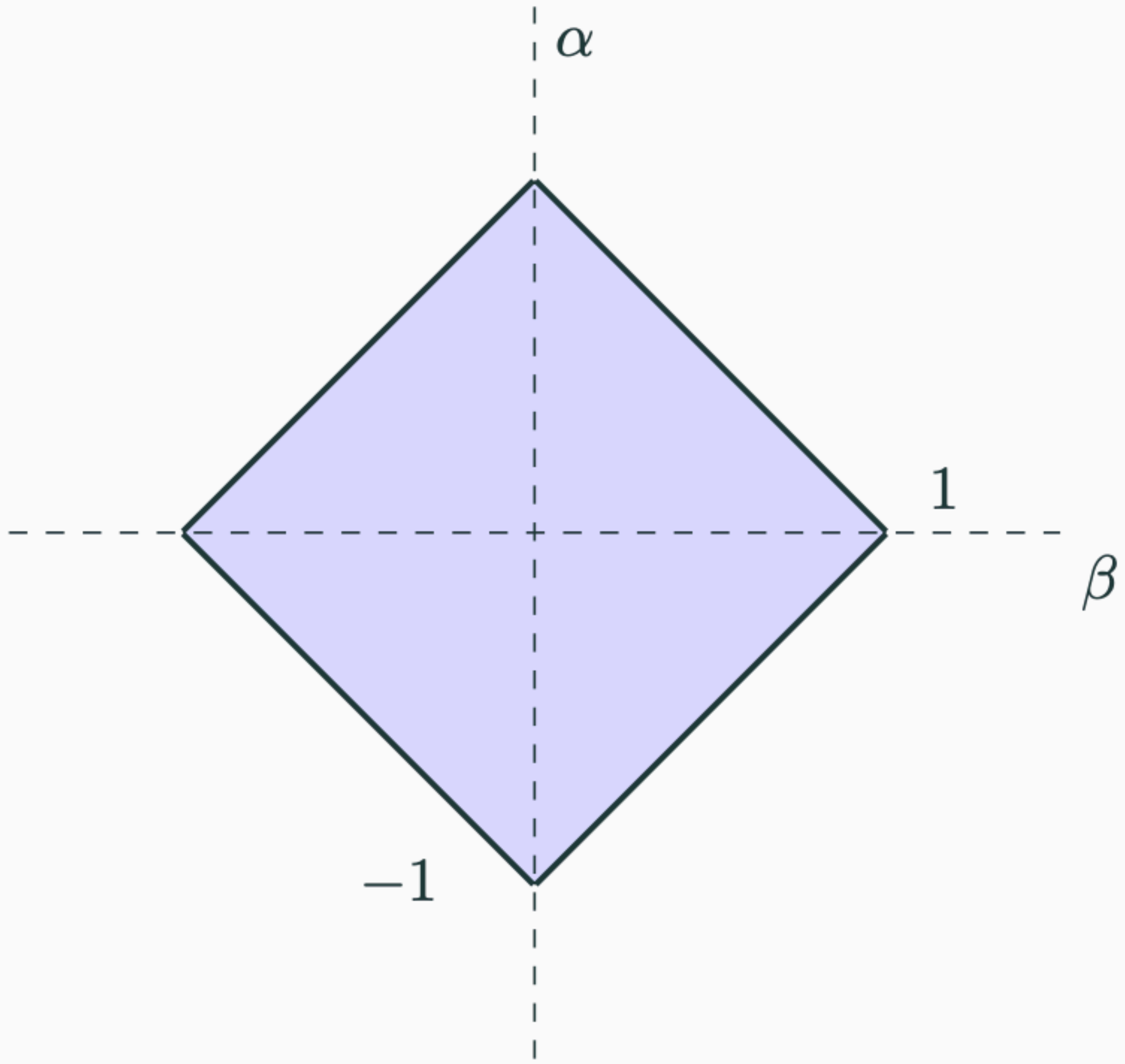
Radon transform

- Integral over line parametrized by x, ξ
- It guarantees polynomiality

$$H^q(x,\xi) = \mathscr{R}[h] = \int_\Omega d\beta d\alpha\, \delta\left(x - \beta - \alpha\xi\right) h(\beta,\alpha) \quad + \quad \theta(|\xi| - |x|)\ \text{D-terms}$$

Support: $\Omega = \{(\beta,\alpha) \,|\, |\beta| + |\alpha| \le 1\} = \Omega^+\theta(\beta) + \Omega^-\theta(-\beta)$





line $\alpha = -\dfrac{\beta}{\xi} + \dfrac{x}{\xi}$

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- Cannot be fixed by DGLAP data.
- Its DD has support only in $\beta = 0$
- Contribute only to $C_{m,m+1}(t)$

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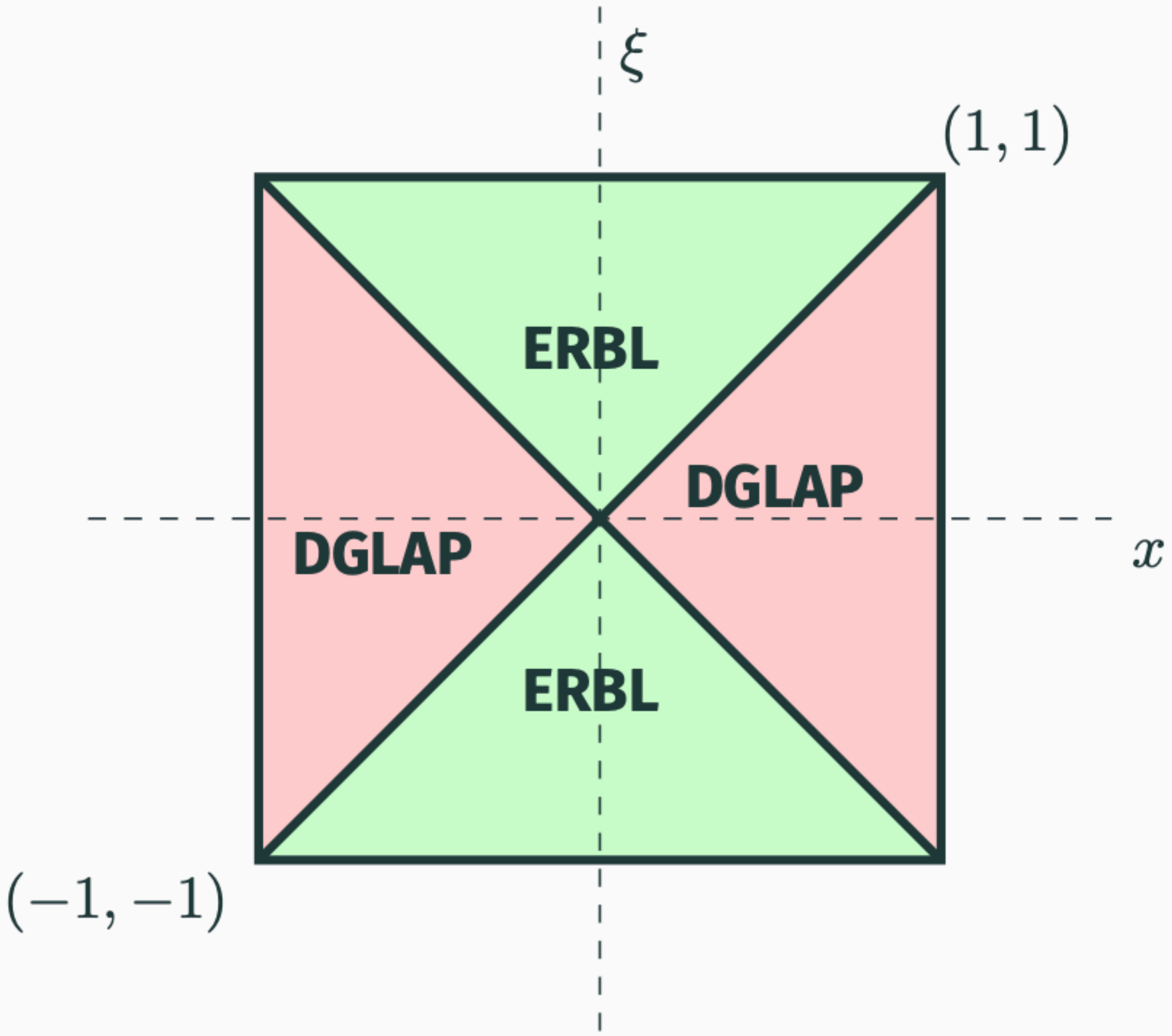

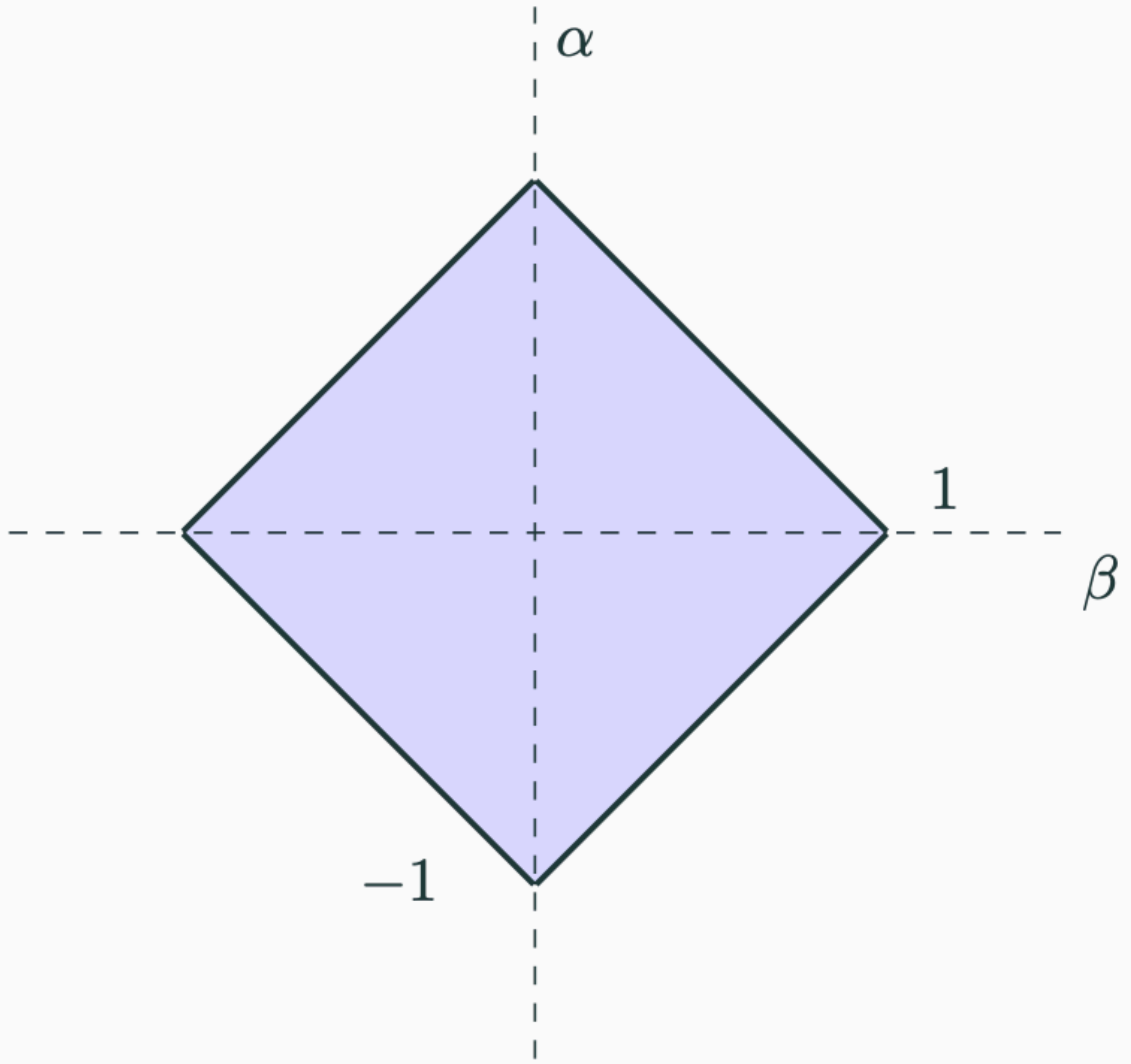

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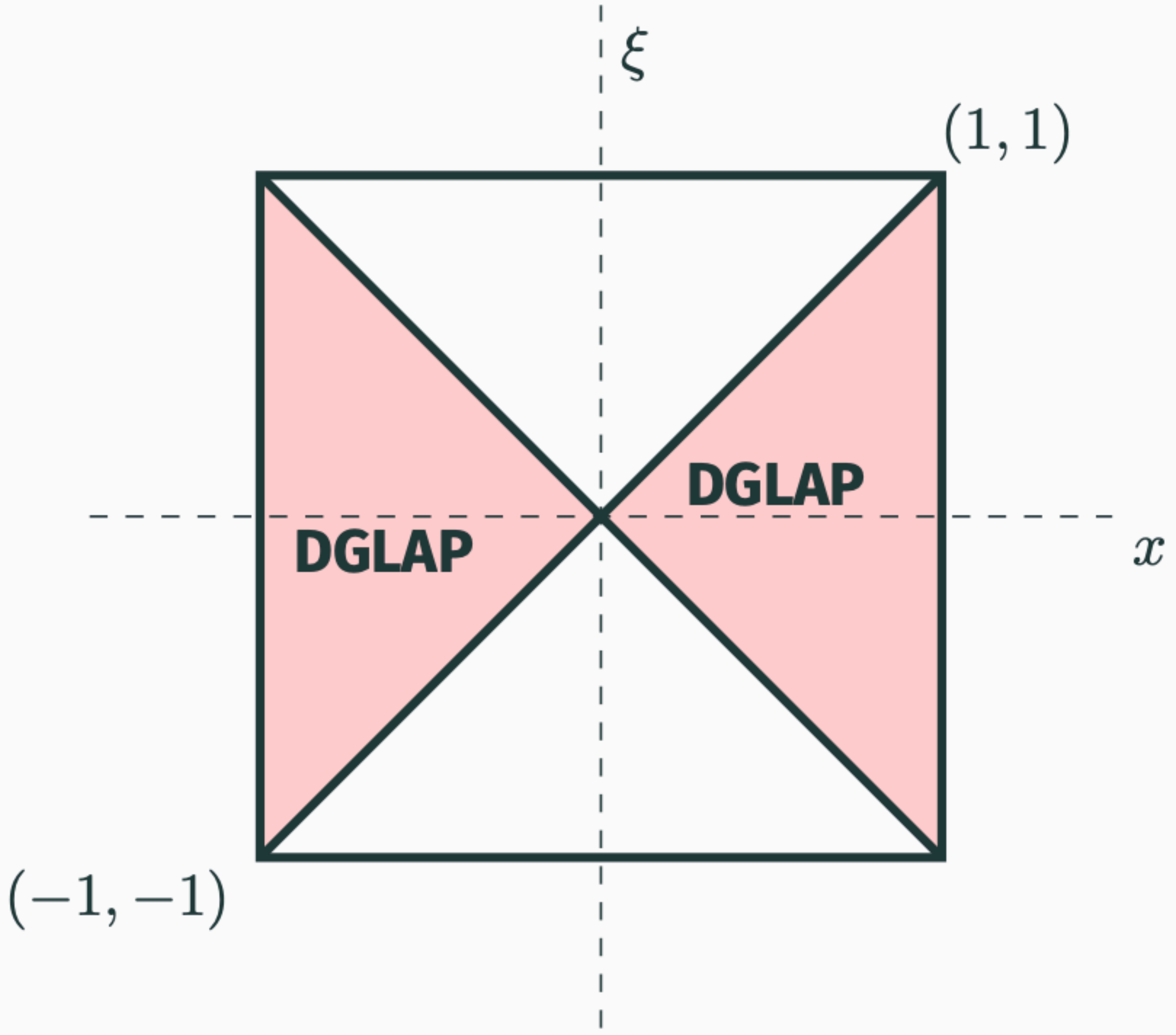

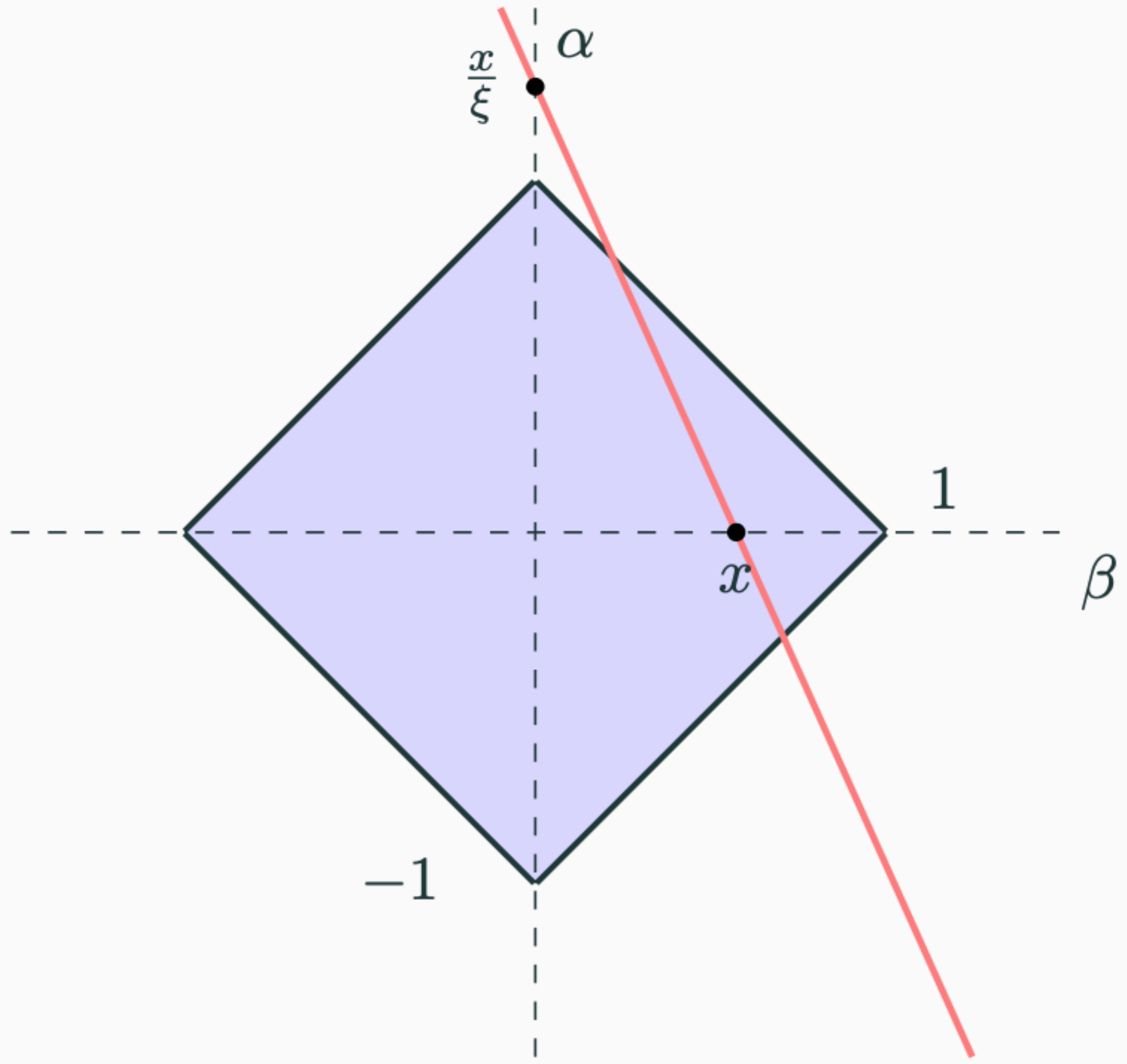

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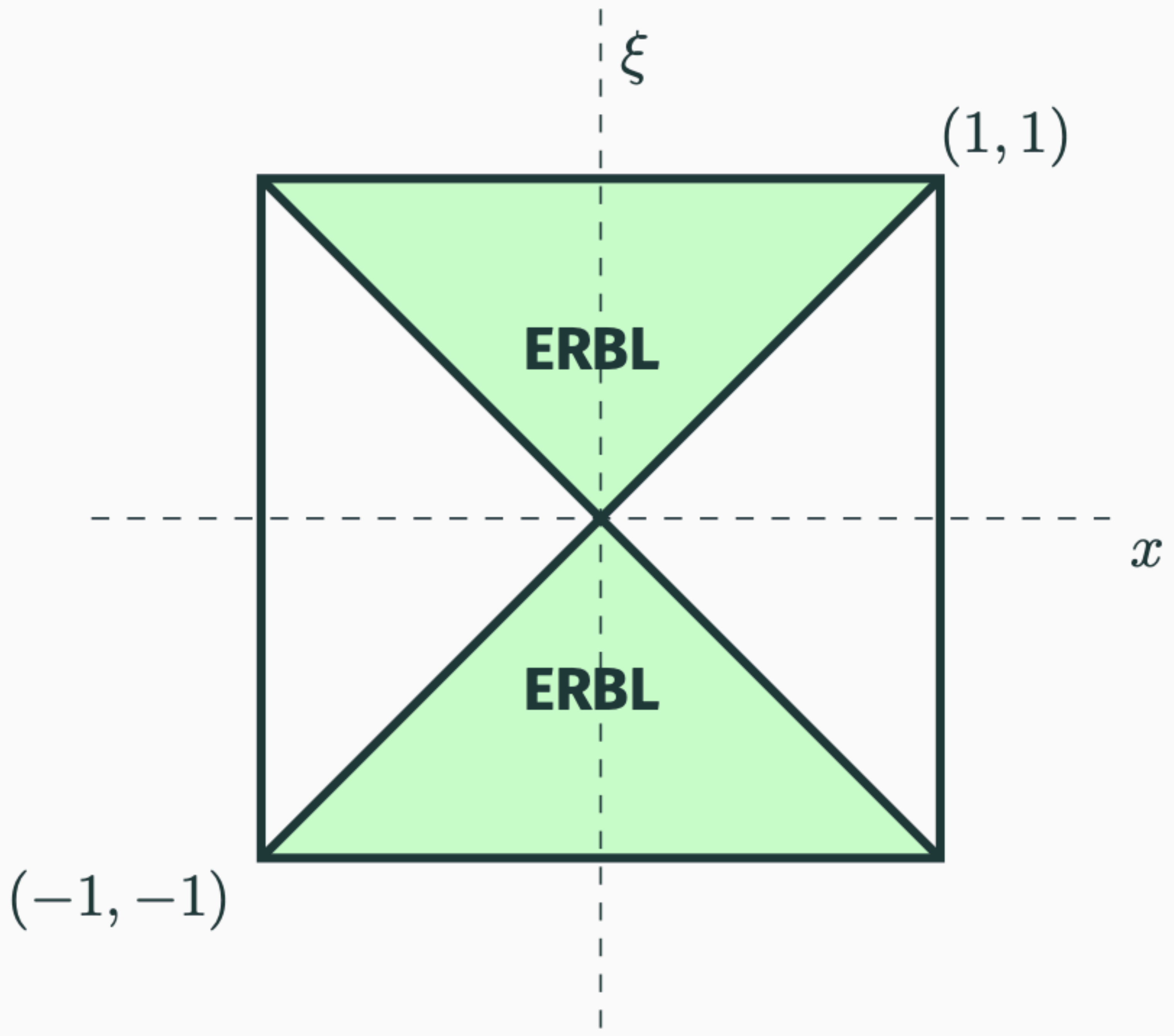

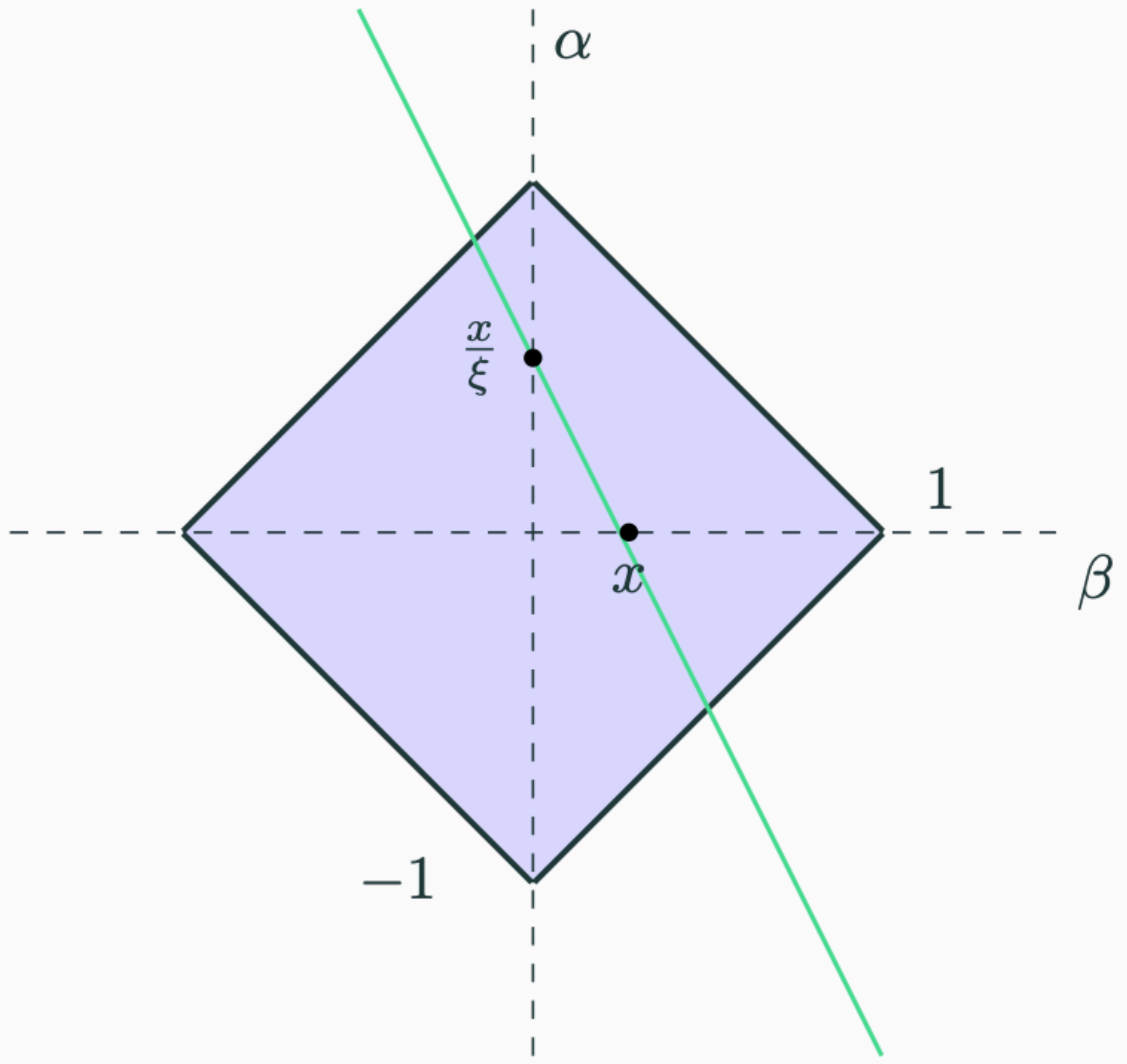

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Covariant Extension ⟷ Radon transform inversion

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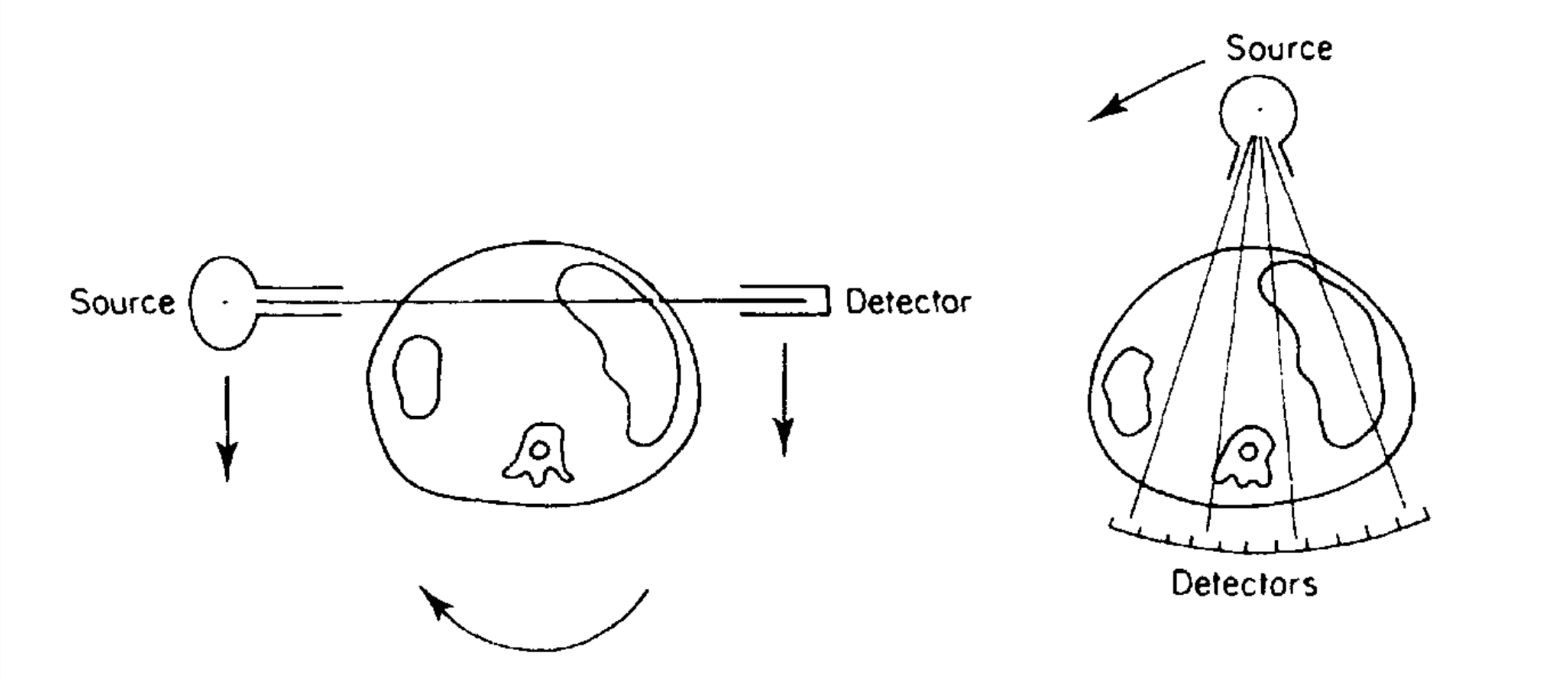
Is it possible to invert the Radon transform knowing the GPD only in DGLAP?

Answer from Computerized Tomography



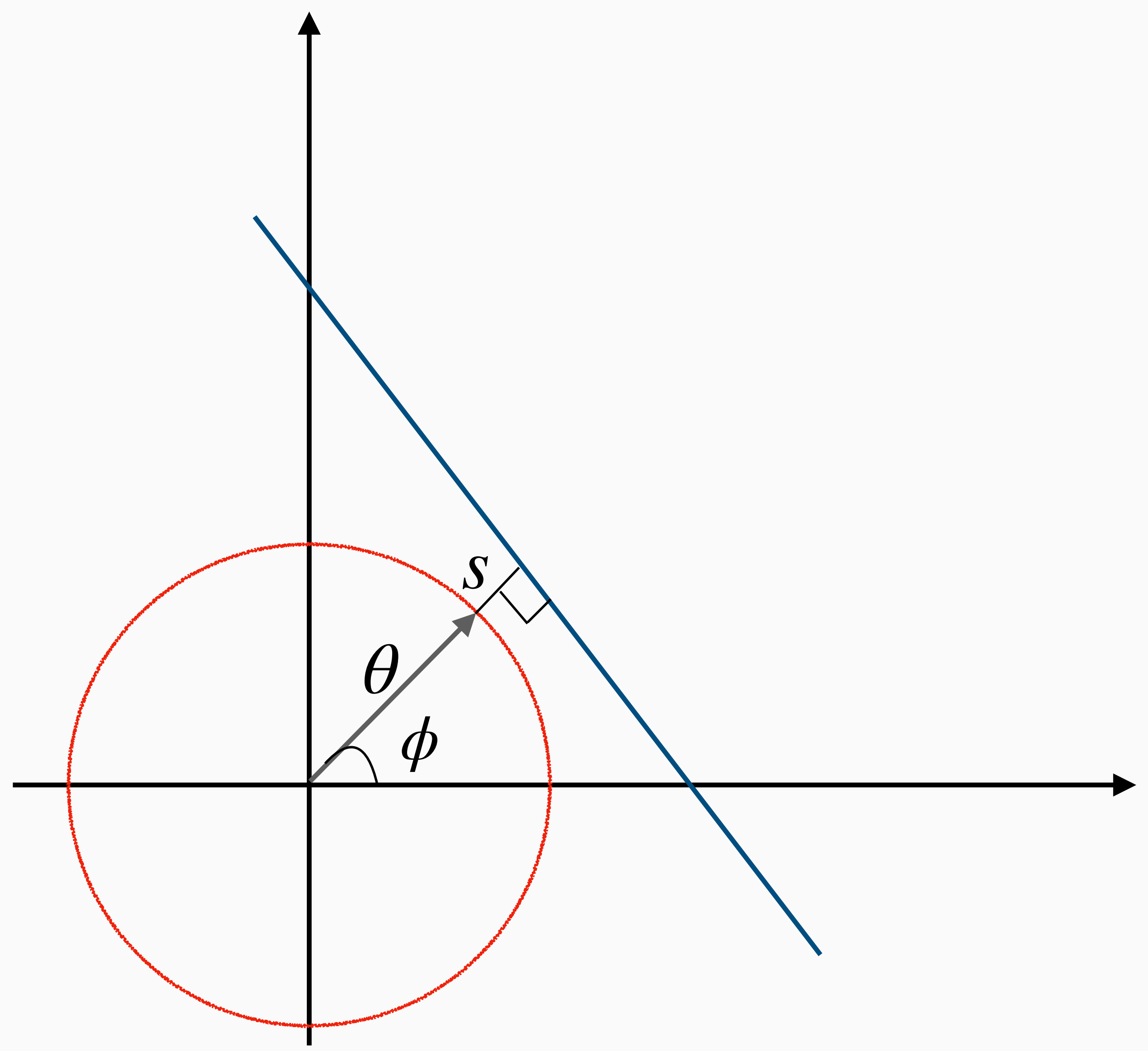
Parallel scanning

Fan-beam scanning

$$\frac{\Delta I}{I} = f(x)\, dx$$

$$\frac{I_{out}}{I_{in}}[L] = \exp\left\{ -\int_L f(x)\, dx \right\}$$

$$\mathscr{R}f(\theta, s) \equiv \int_{z\cdot\theta=s} dz\, f(z) \qquad f \in \mathcal{S}(\mathbb{R}^n)\,, \quad \theta \in S^{n-1}\,, \quad s \in \mathbb{R}$$



GPD: $z \to (\beta, \alpha)\,, \quad \theta \to (\cos(\phi), \sin(\phi))\,, \quad x = \dfrac{s}{\cos(\phi)}\,, \quad \xi = \tan(\phi)$

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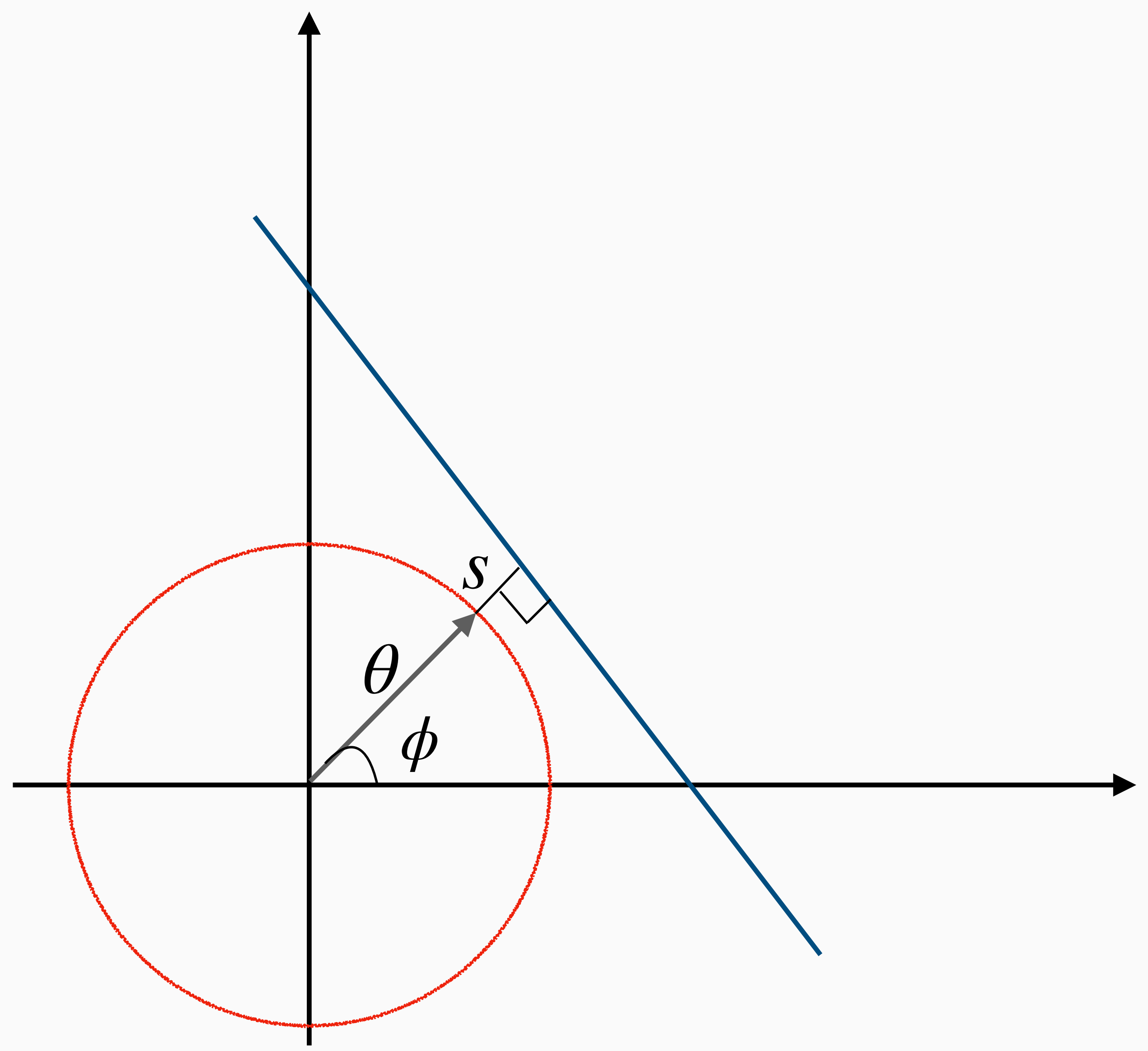

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Uniqueness theorems:

[F. Natterer, *The Mathematics of Computerized Tomography*]

Given f $\in \mathcal{S}(\mathbb{R}^n)$, if $\mathcal{R}f(\theta, s) = 0$ $\forall$ hyperplane $z \cdot \theta = s$ that does not intersect a compact convex set K $\subset \mathbb{R}^n \Rightarrow f(z) = 0$ outside K

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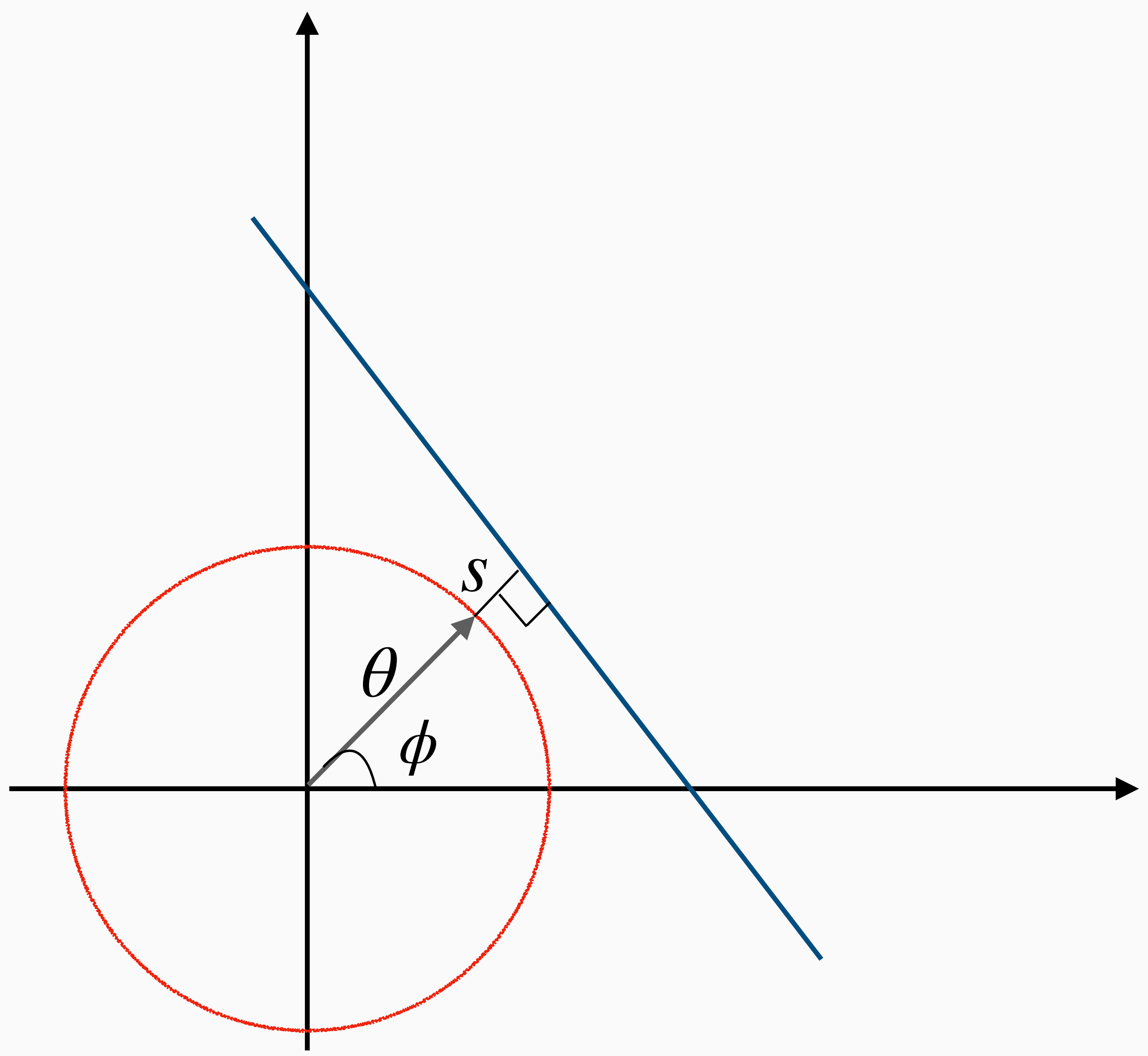

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→ DD is uniquely fixed by GPD in DGLAP (except for $\beta = 0$)

J. Boman and E.T. Quinto, Mathematical Journal 55 (1987)

Given f in $\mathbb{R}^2$ compactly-supported and locally summable, $(\theta_0, s_0) \in S^1 \times \mathbb{R}$, U_0 open neighborhood of θ_0 :

$$\mathscr{R}f(\theta, s) = 0 \ \ \forall s > s_0, \ \ \theta \in U_0 \quad \Rightarrow \quad f(z) = 0 \quad if \ z \cdot \theta_0 > s_0$$

J. Boman and E.T. Quinto, Mathematical Journal 55 (1987)

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→ DD is uniquely fixed (except for $\beta = 0$) by knowing GPD in:

$x \in [-1,1]\,, \quad \xi \in [0, \lambda x]\,,\;\; 0 < \lambda \leq 1$ (H. Moutarde)

In experiments $\xi \lesssim 0.2$

Numerical Inversion of Radon Transform

Artificial Neural Networks

H. Dutrieux et al. *Eur.Phys.J.C*. 82 (2022)

Finite Elements Methods

N. Chouika et al. *Eur.Phys.J.C*. 77 (2017)

J.M. Morgado et al. *Phys.Rev.D* 105 (2022)

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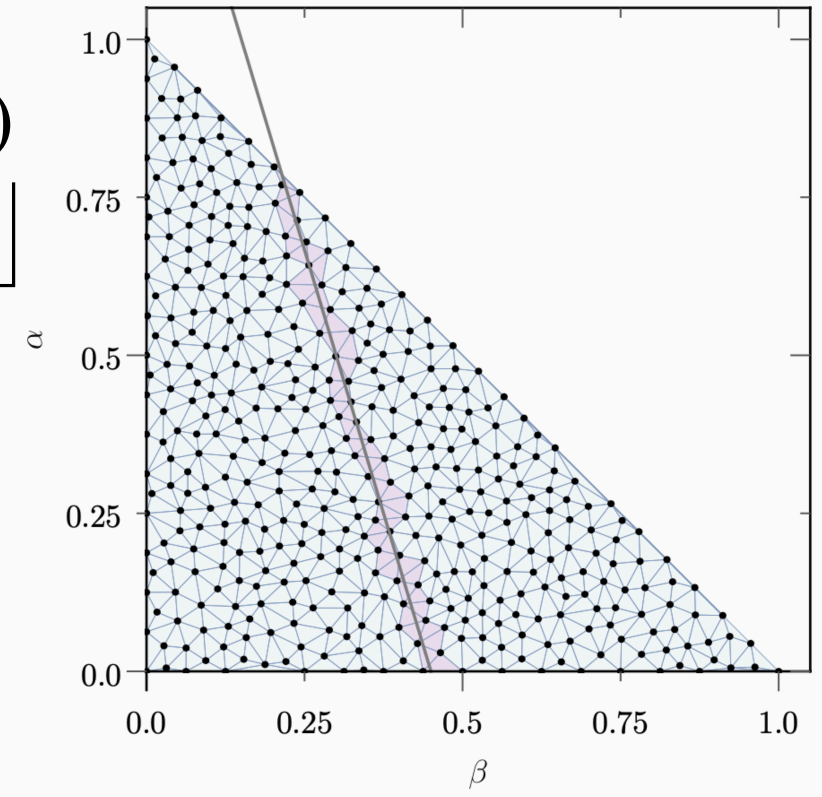
Lagrange Polynomial

$$H_i \equiv H(x_i, \xi_i) = \sum_j h_j \int_{\Omega^+} d\beta d\alpha \delta(x_i - \beta - \alpha\xi_i)\, v_j(\beta, \alpha)$$

R_{ij}

$$h = \left(R^T R\right)^{-1} R^T H$$



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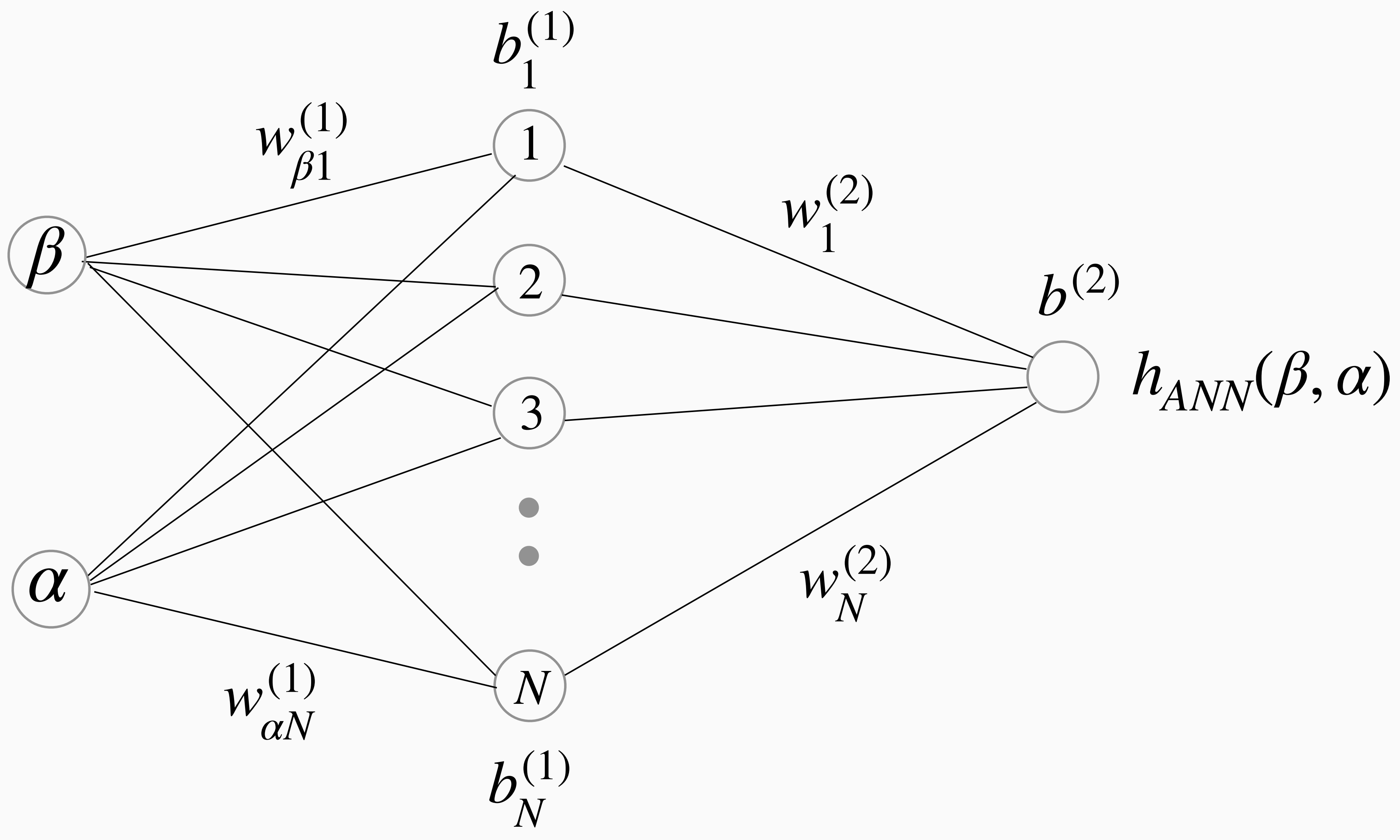

Unbiased parametrization of DD

Finite Elements Methods

N. Chouika et al. *Eur.Phys.J.C.* 77 (2017)

J.M. Morgado et al. *Phys.Rev.D* 105 (2022)

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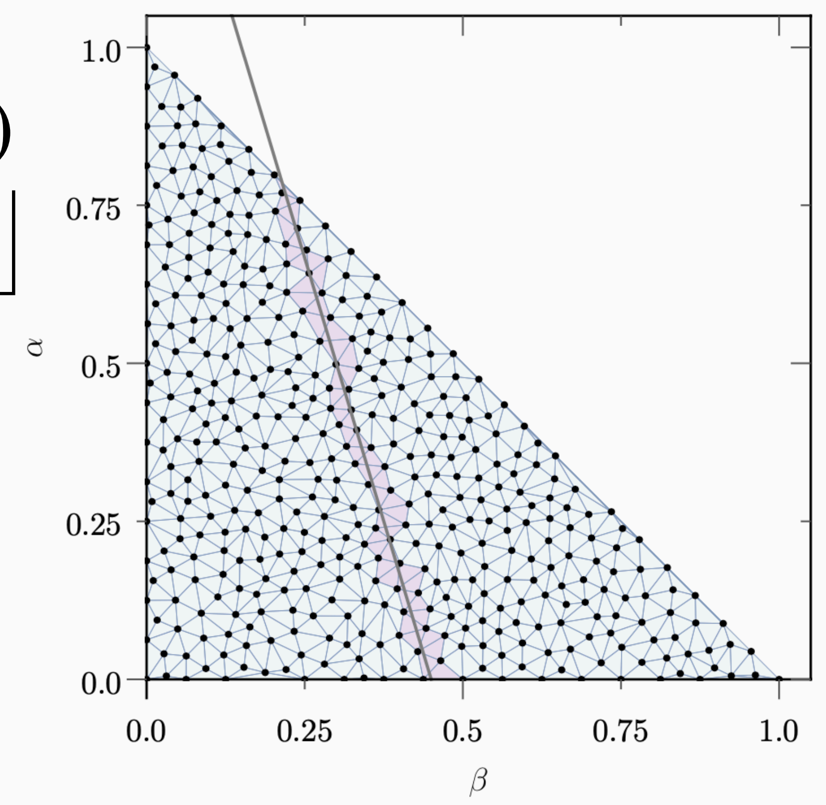
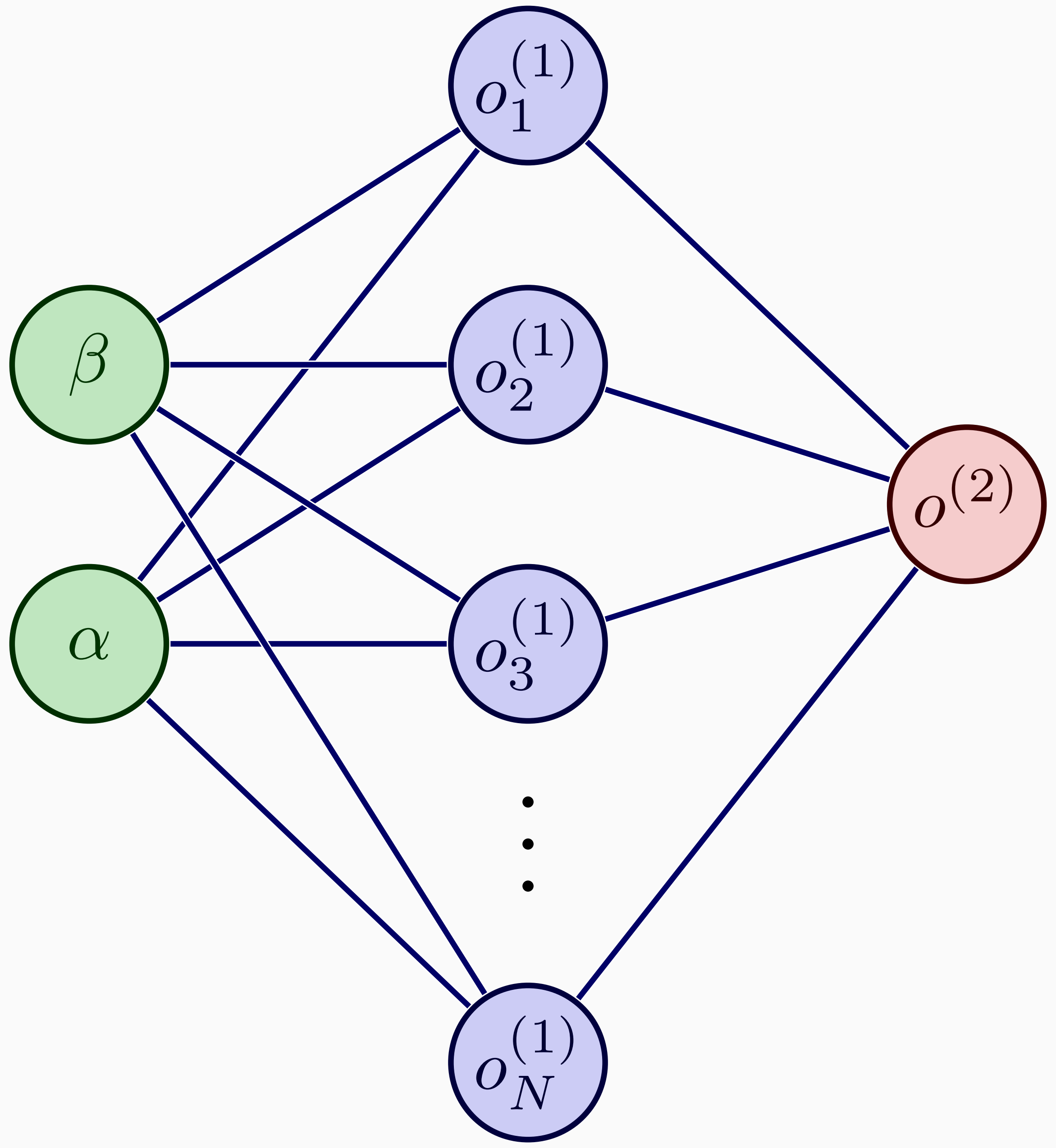


Parametrizing $h(\beta, \alpha)$ using **Artificial Neural Networks (ANN)**



$$= h_{ANN}(\beta, \alpha) = \sum_{i=1}^{N} w_i^{(2)} o_i^{(1)} + b^{(2)}$$

$$= \sum_{i=1}^{N} w_i^{(2)} \left[\sigma\left(w_{\beta i}^{(1)} \beta + w_{\alpha i}^{(1)} \alpha + b_i^{(1)} \right) + \sigma\left(w_{\beta i}^{(1)} \beta - w_{\alpha i}^{(1)} \alpha + b_i^{(1)} \right) \right] + b^{(2)}$$

- Adam optimizer (gradient descent)
- Dropout regularization

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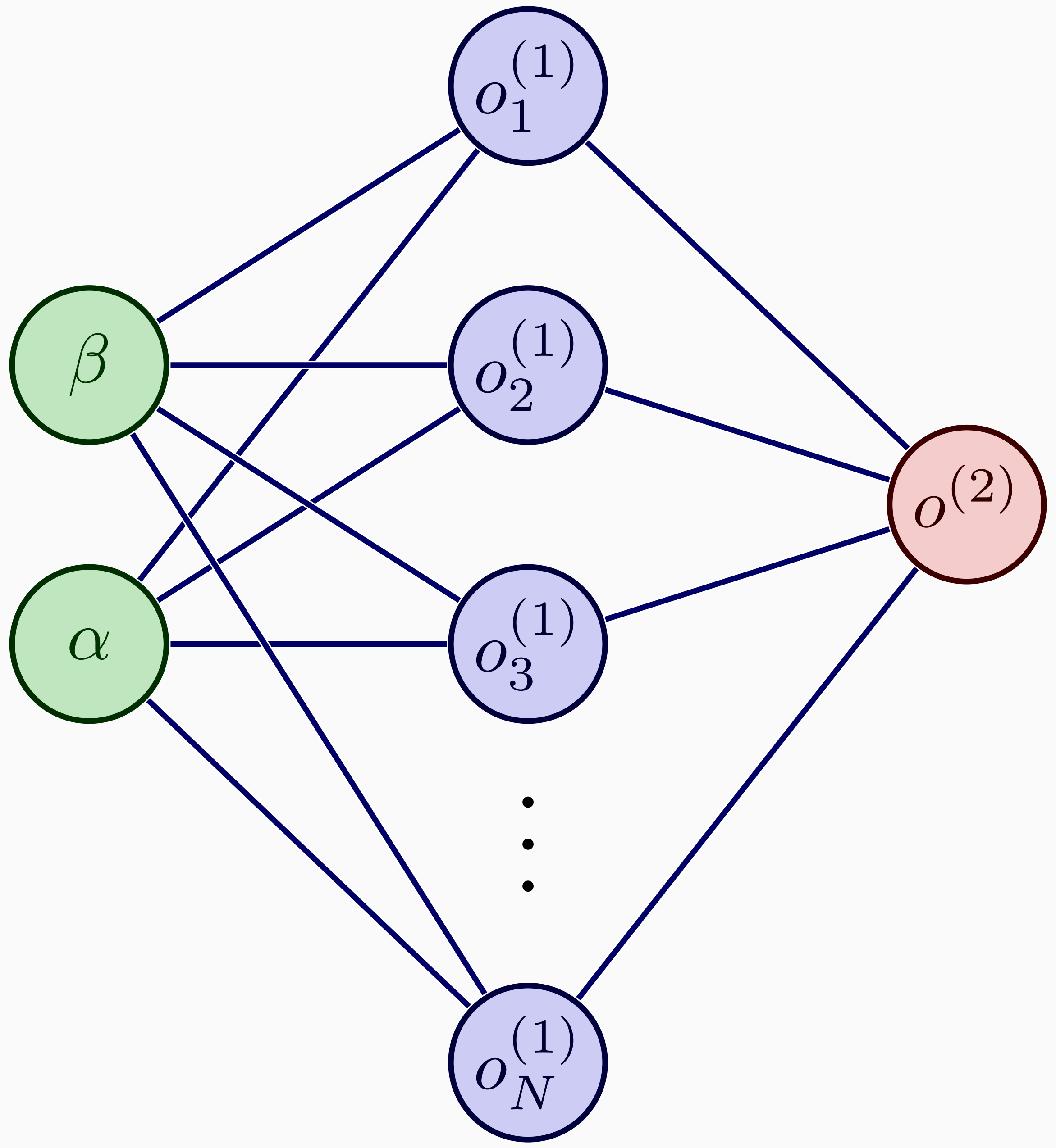

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$$h(\beta, -\alpha) = h(\beta, \alpha) \leftrightarrow H(x, -\xi) = H(x, \xi)$$

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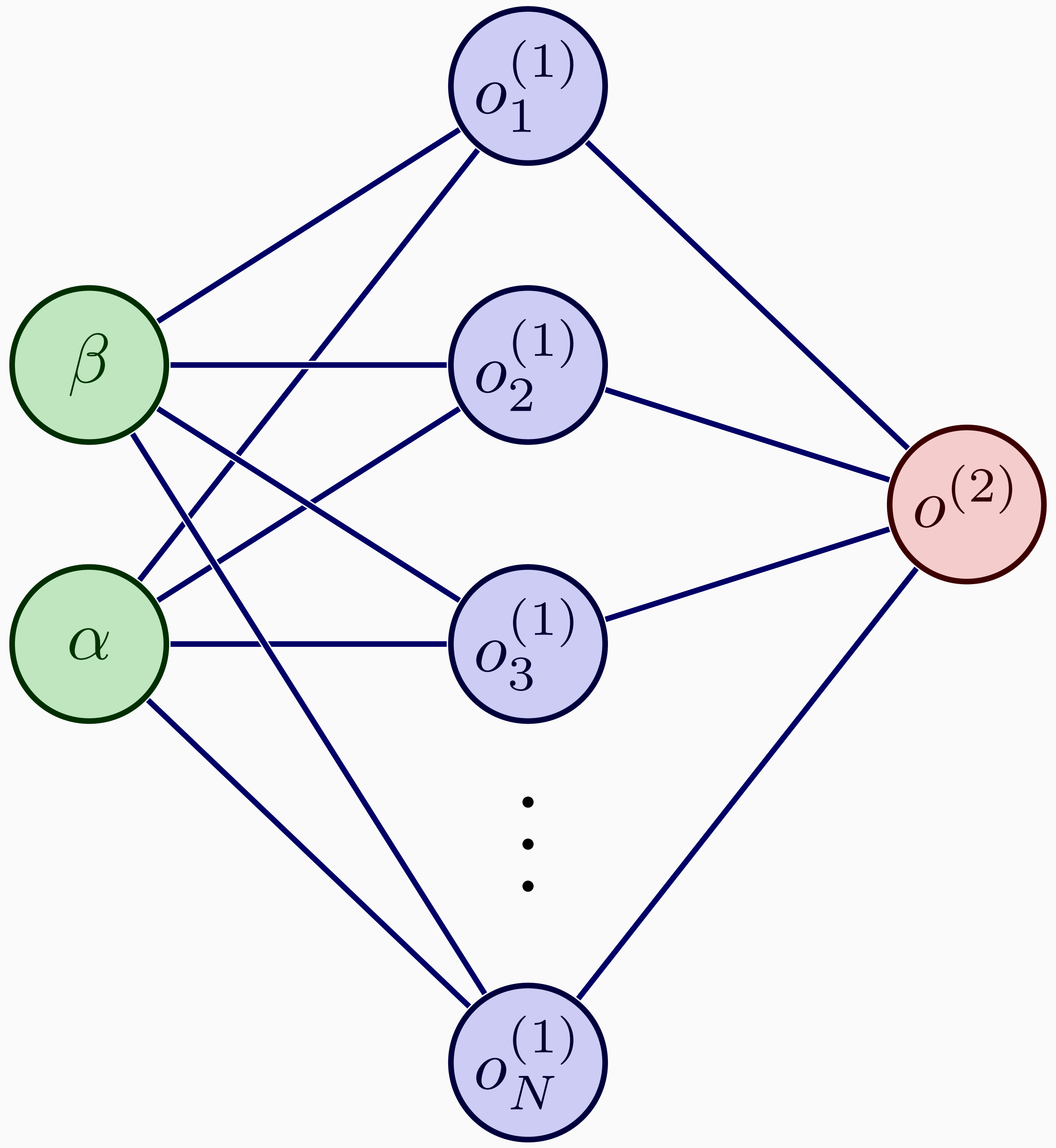

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$$\sigma(x) = \frac{1}{1 + e^{-x}}$$

- Adam optimizer (gradient descent)
- Dropout regularization

Algorithm

- Initialize the ANN parameters (randomly).
- Given a sampling set of GPD values $H_i(x_i, \xi_i)$ in the DGLAP region, numerically evaluate the RT along each line (x_i, ξ_i) using $h_{ANN}(\beta, \alpha)$ as DD. $\mathscr{R}h_{ANN}(x_i, \xi_i) = \hat{H}(x_i, \xi_i)$.
- Update the ANN parameters using Adam optimization algorithm in order to minimize

$$MSE = \frac{1}{N_{sample}} \sum_{i=1}^{N_{sample}} \left(H_i - \hat{H}_i\right)^2$$

- Iterate until convergence.

Testing with analytical models

Nakanishi based model for pion

N.Chouika et al. *Phys.Lett.B*:780(2018)

$$H(x,\xi,t=0) = \begin{cases} 30\,\dfrac{(1-x)^2(x^2-\xi^2)}{(1-\xi^2)^2}, & |x| > \xi \\ 15\,\dfrac{(1-x)(\xi^2-x^2)(x+2x\xi+\xi^2)}{2\xi^3(1+\xi)^2}, & |x| < \xi \end{cases}$$

$$H(x,\xi,t=0) = (1-x)\int_{\Omega^+} d\beta d\alpha \delta(x-\beta-\alpha\xi)\, h(\beta,\alpha)$$

$$h(\beta,\alpha) = \frac{15}{2}\left(1-3(\alpha^2-\beta^2)-2\beta\right)$$

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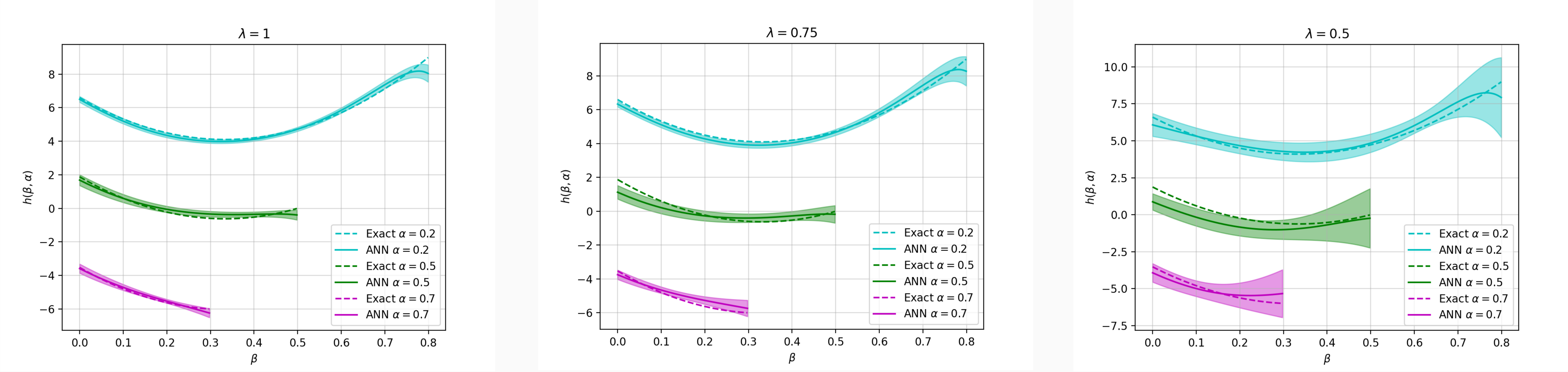
$N_{neurons} = 10^2, \quad N_{sample} = 10^4$

$\xi \in [0,\, \lambda x]$



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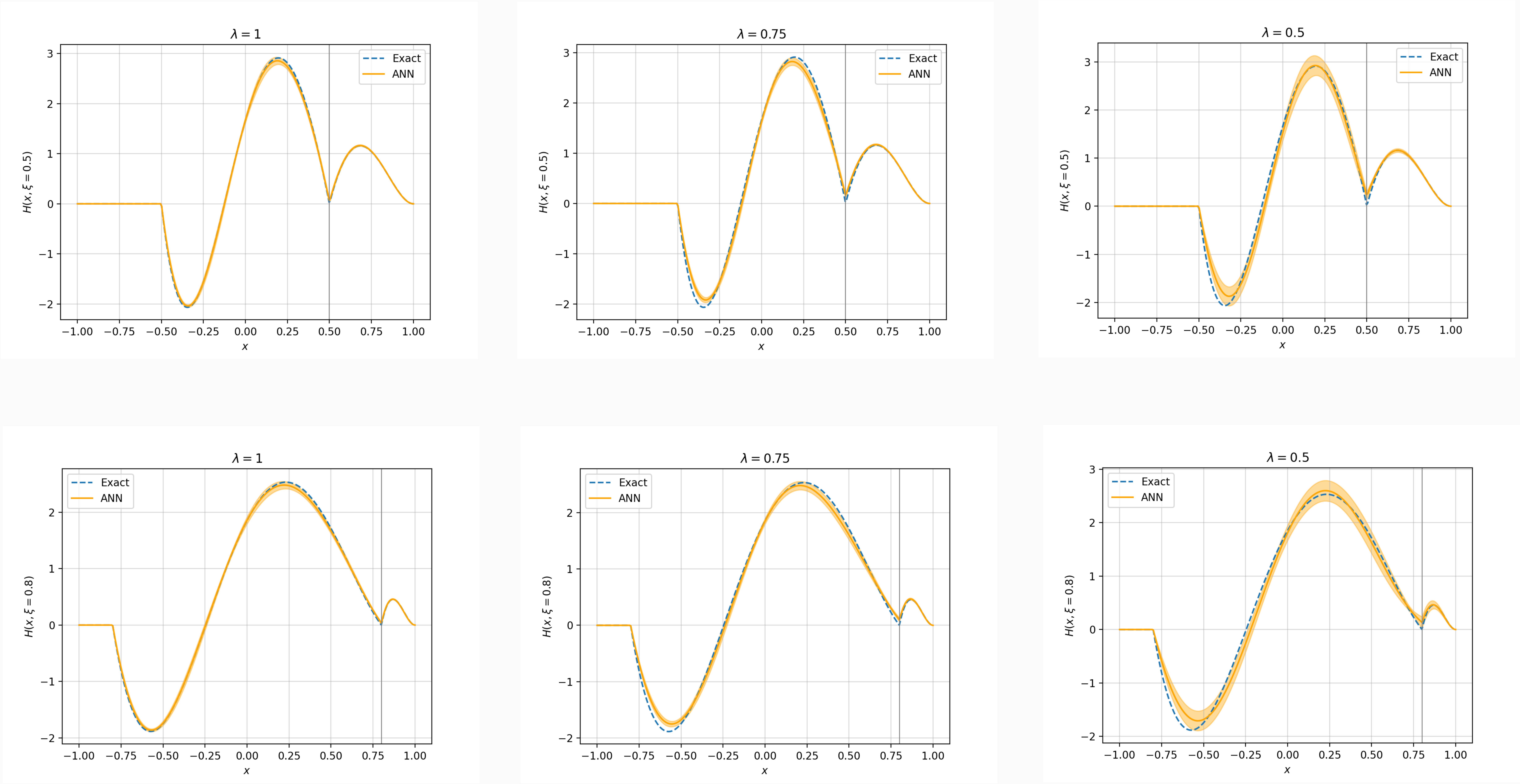

Goloskokov-Kroll model

S.V. Goloskokov, P. Kroll, Eur.Phys.J.C. 50 (2007)

$$H(x,\xi) = \int_\Omega d\beta d\alpha\, \delta\left(x - \beta - \alpha\xi\right) q(\beta)\, h_{GK}(\beta,\alpha)\,,$$

$$h_{GK}(\beta,\alpha) = \frac{\Gamma(n+2)}{2^{n+1}\Gamma^2(n+1)} \frac{\left[(1-\beta)^2 - \alpha^2\right]^n}{(1-\beta)^{2n+1}}$$

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Valence distribution $q^u_{val}(\beta) = \beta^{-\delta}(1-\beta)^{2n+1} \sum_{j=0}^{2} c_j\, \beta^{j/2}\,, \quad n = 1,\;\; \delta = 0.48$

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Sea distribution $$q_{sea}(\beta) = \beta^{-\delta}(1-\beta)^{2n+1} \sum_{j=0}^{3} c_j\, \beta^{j/2}\,, \quad n = 2,\;\; \delta = 1.1$$

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$$H_{sea}(x,\xi) = \int_\Omega d\beta d\alpha\, \delta\left(x-\beta-\alpha\xi\right)\ sign(\beta)\, q_{sea}(|\beta|)\, h_{GK}(|\beta|,\alpha) \longrightarrow H_{sea}(-x,\xi) = -H_{sea}(x,\xi)$$

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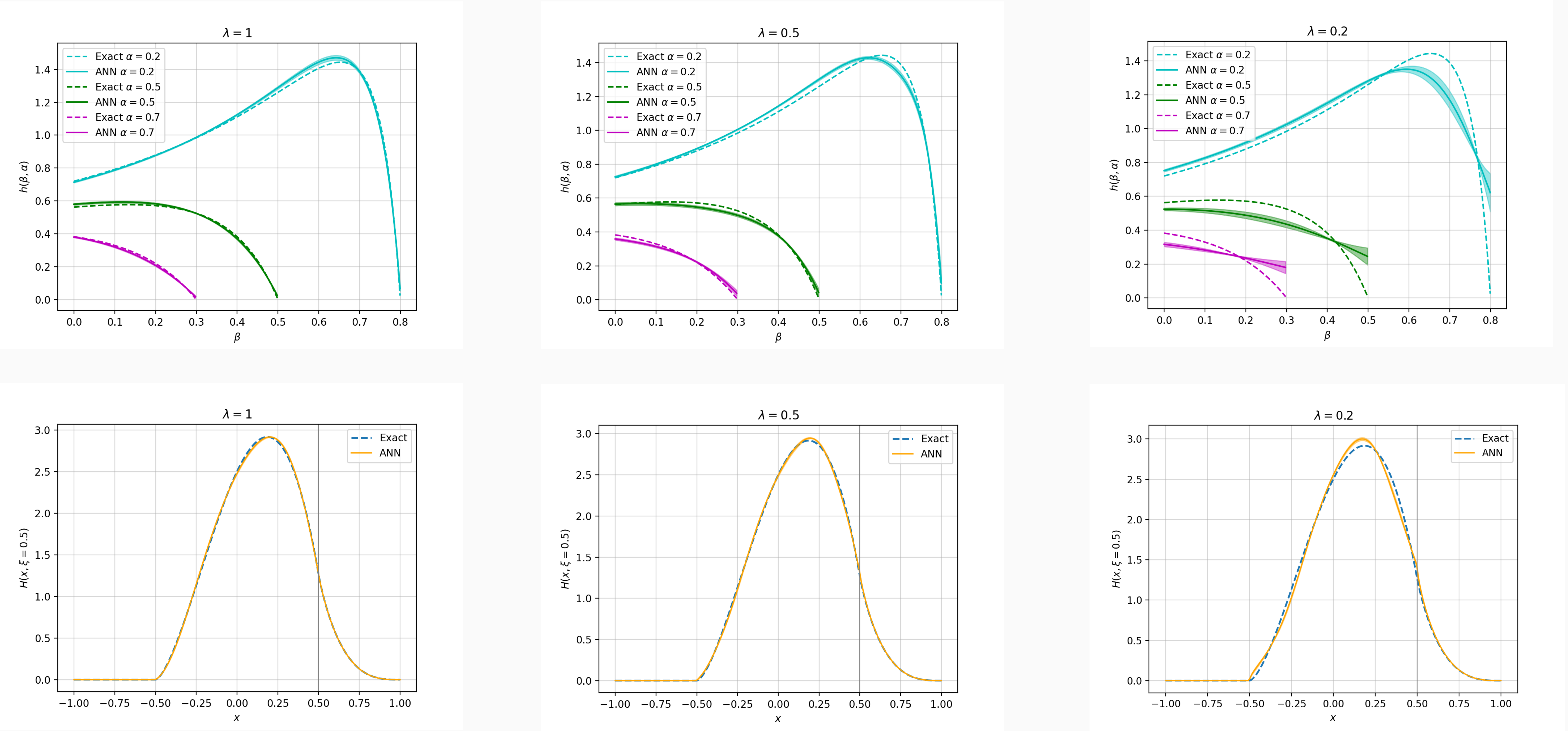
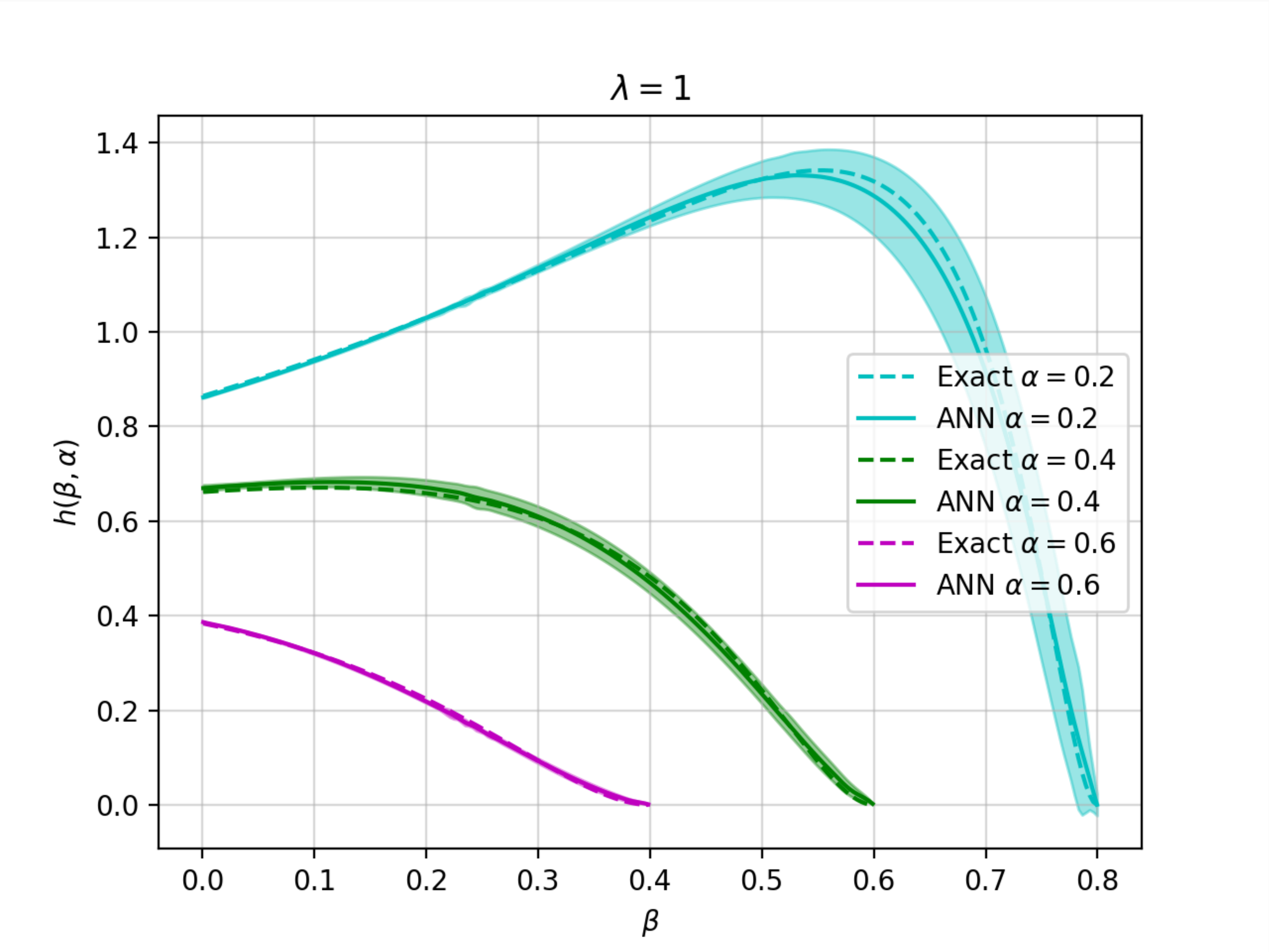
$N_{neurons} = 25\,, \quad N_{sample} = 5\times 10^3$

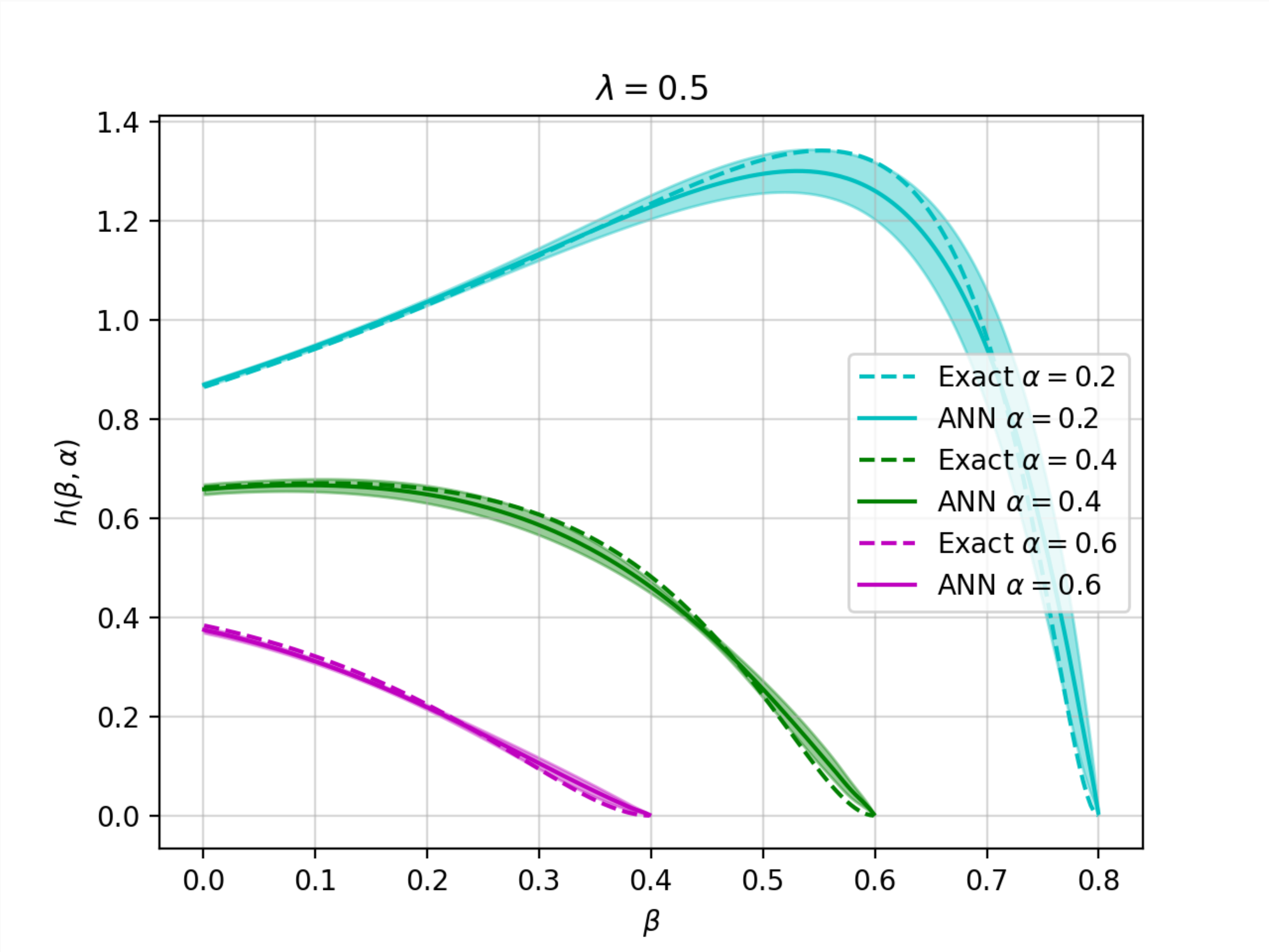
$$h_{GK}(\beta,\alpha) \simeq \frac{h_{ANN}(\beta,\alpha)}{\int_{-1+|\beta|}^{1-|\beta|} d\alpha\, h_{ANN}(\beta,\alpha)}$$

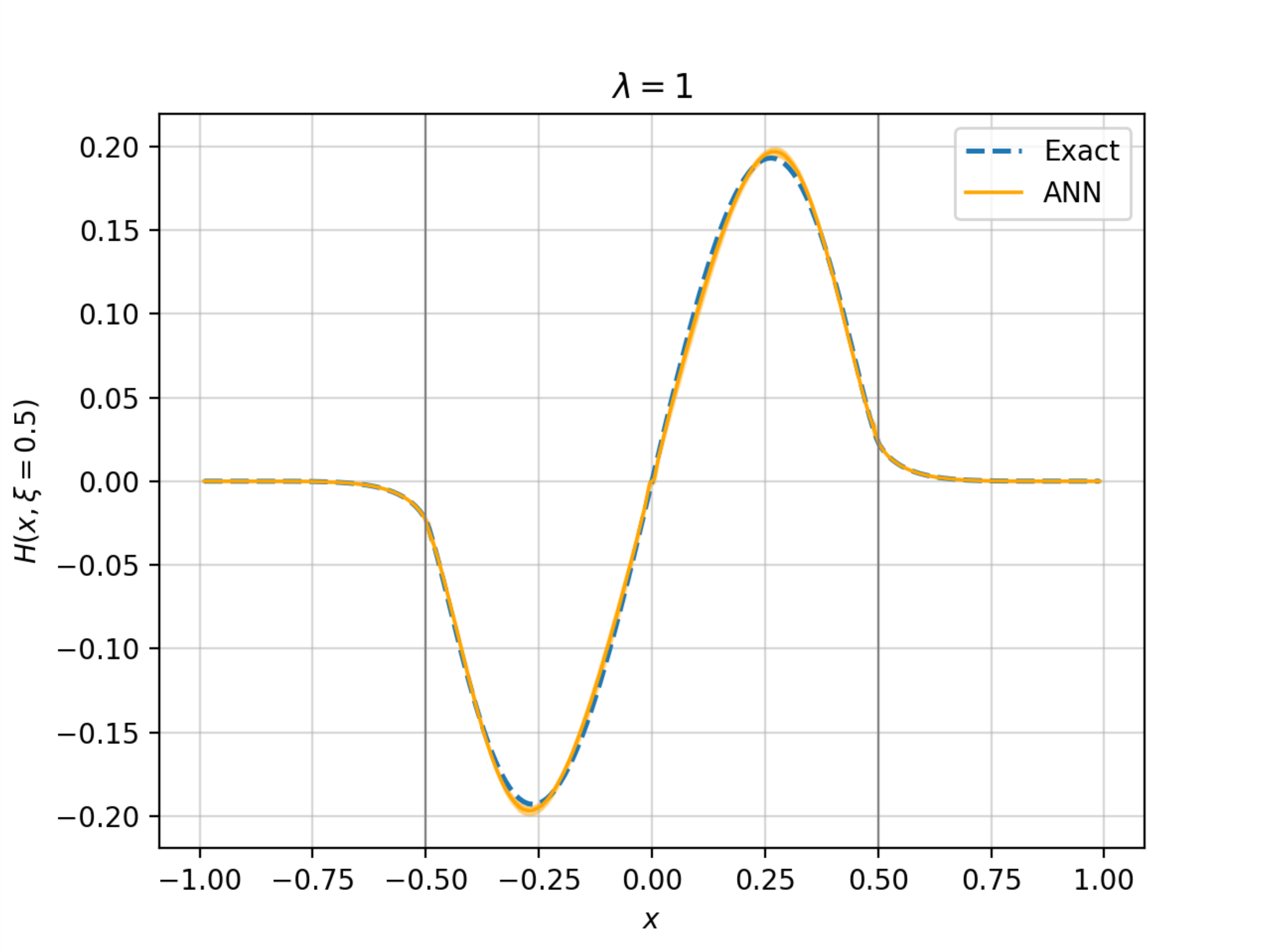
Valence DD/GPD

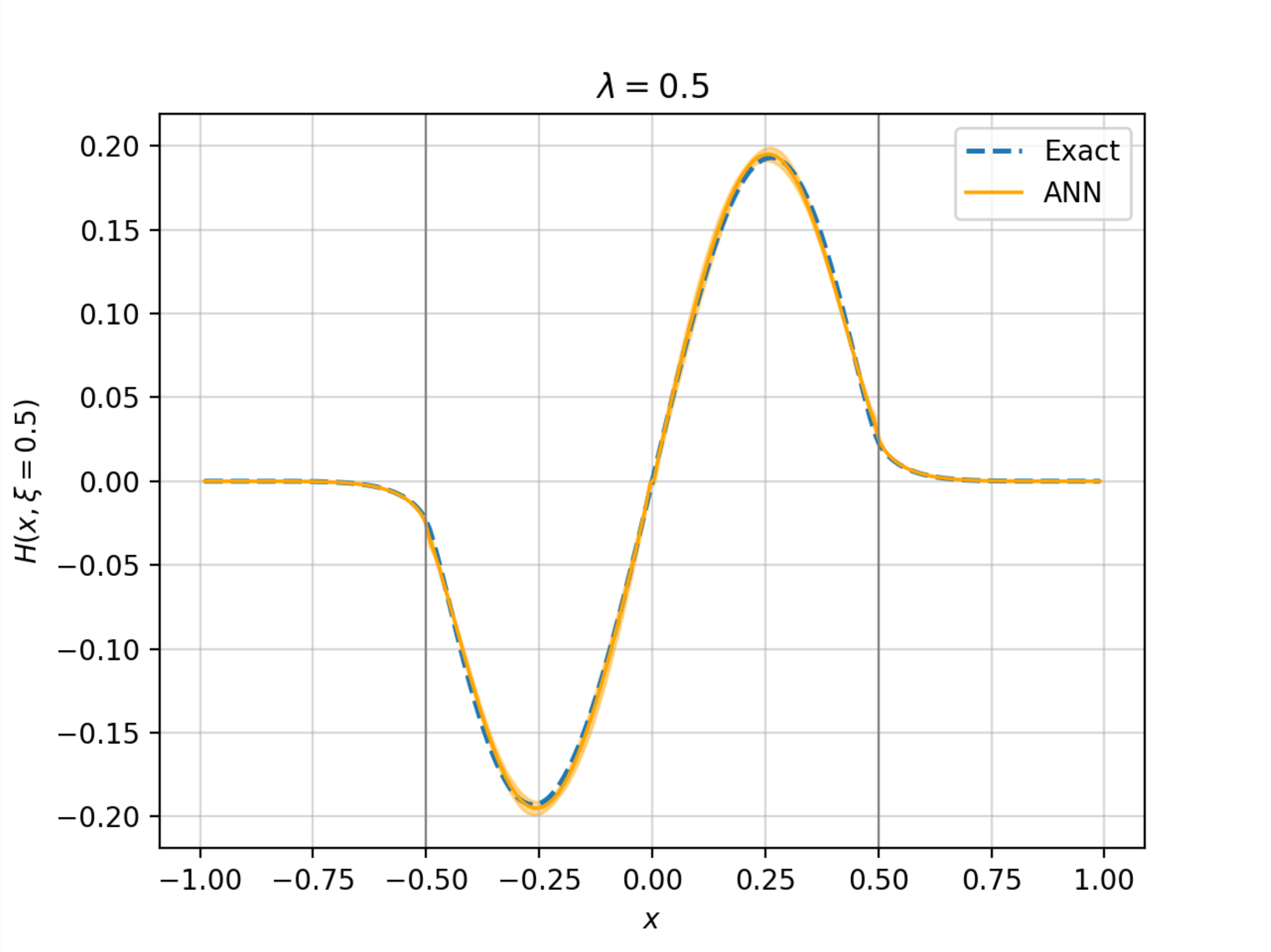


Sea DD/GPD


λ = 1
h(β, α)
β
Exact α = 0.2
ANN α = 0.2
Exact α = 0.4
ANN α = 0.4
Exact α = 0.6
ANN α = 0.6


λ = 0.5
h(β, α)
β
Exact α = 0.2
ANN α = 0.2
Exact α = 0.4
ANN α = 0.4
Exact α = 0.6
ANN α = 0.6


λ = 1
H(x, ξ = 0.5)
x
Exact
ANN


λ = 0.5
H(x, ξ = 0.5)
x
Exact
ANN

Conclusions and Outlook

- GPD can be extended from DGLAP (and proper subsets) to ERBL inverting its RT
- ANNs are a good tool for inverting RT
- Testing the method on experimental results (EIC)
- Applying ANNs to PDF reconstruction by a sequence of Mellin moments (with Khépani Raya and PP)