LKF transformation of interaction vertices in QED

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Non-Perturbative Physics: Tools and Applications
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In collaboration with **Christian Schubert** (UMSNH), **Naser Ahmadiniaz** (HZDR), **José Nicasio** (Missouri), **Adnan Bashir** (Jlab and UMSNH) and **Ulrich Jentschura** (Missouri).

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Outline

Introduction

Classical and quantum fields

Classical field theory from the action principle $\delta S = 0$. For electrodynamics:

$$S[\Psi, A] = \int d^D x \left[\frac{1}{4}\mathrm{tr}F^2 + \bar{\Psi}(i\not{D} - m)\Psi\right] \qquad (D_\mu := \partial_\mu + ieA_\mu)\,,$$

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Quantum fields $\longrightarrow$ operator-valued distributions: $\Psi(x) \to \widehat{\Psi}(x)$ and $A_\mu(x) \to \widehat{A}_\mu(x)$.
Particles $\longrightarrow$ states e^-, γ $\longrightarrow$, excitations about the vacuum $|\Omega\rangle$:

$$e^- \sim |p, \sigma\rangle = \widehat{a}^\dagger_{p,\sigma}|\Omega\rangle \qquad\qquad \gamma \sim |k, h\rangle = \widehat{\alpha}^\dagger_{k,h}|\Omega\rangle$$

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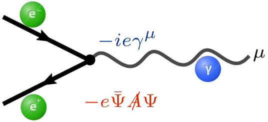
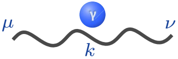
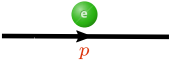
Perturbative QFT – Feynman rules (plus asymptotic states)





$\frac{-i}{k^2}\left(\eta_{\mu\nu} - (1-\xi)\frac{k_\mu k_\nu}{k^2}\right)$



$\frac{i}{\not{p}+m} = \frac{i(\not{p}-m)}{p^2+m^2}$

Gauge symmetry

An $SU(N)$ gauge theory: invariant under *local* "phase rotations"

$$\Psi_a(x) \to U_{ab}(x)\Psi_b(x) = \mathrm{e}^{ie\alpha(x)^R T^R_{ab}}\Psi_b(x)$$

with group generators $\{T^R\}$.

Action contains kinetic terms plus interaction vertices:

$$S[\Psi, A] = \int d^D x \left[\frac{1}{2}A_\mu\left(\eta^{\mu\nu}\partial^2 - \partial^\mu\partial^\nu\right)A_\nu + \bar{\Psi}[i\partial\!\!\!/ - m]\Psi - e\bar{\Psi}A\!\!\!/\Psi \quad (+ \text{ other vertices })\right]$$

Gauge potential transforms to absorb derivatives of α:

$$\begin{aligned} A_\mu(x) &\longrightarrow U(x)A_\mu(x)U^\dagger(x) + \frac{i}{e}(\partial_\mu U(x))U^\dagger(x) \\ &= \mathrm{e}^{ie\alpha^R(x)T^R}A_\mu(x)\mathrm{e}^{-ie\alpha^R(x)T^R} + \frac{i}{e}(\partial_\mu \mathrm{e}^{ie\alpha^R(x)T^R})(\mathrm{e}^{-ie\alpha^R(x)T^R}) \end{aligned}$$

Propagators

Physical observables should be gauge invariant.

- Non-perturbatively – basically clear
- Lattice discretisation – gauge invariance of lattice QCD
- Schwinger-Dyson equations – gauge covariant truncation schemes

Basic gauge invariant objects: (traces of) Wilson loops (holonomy).

Problem:

Perturbation theory propagators. Photon kinetic term has non-trivial kernel,

$$\left(\eta^{\mu\nu}\partial^2 - \partial^\mu\partial^\nu\right)\partial_\nu a(x) = 0 \Longleftrightarrow \left(\eta^{\mu\nu}k^2 - k^\mu k^\nu\right)k_\nu = 0 \tag{1}$$

(Gauge orbit of zero function) $\Longrightarrow$ **operator is not invertible**.

Add a **gauge fixing** term,

$$S_{\rm gf}(\xi) = -\int d^4x\,(\partial\cdot A)^2/(2\xi)$$

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Fixes the (linear) covariant "gauge" with parameter ξ:

$$G_{\mu\nu}(k;\xi) := \langle A_\mu(k)A_\nu(-k)\rangle_\xi = \frac{1}{k^2}\left(\eta_{\mu\nu} - (1-\xi)\frac{k_\mu k_\nu}{k^2}\right)\,.$$

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Position space family of Green functions:

$$G_{\mu\nu}(x-x';\xi) := \langle A_\mu(x)A_\nu(x')\rangle_\xi = G_{\mu\nu}(x-x';\hat{\xi}) + \Delta\xi\partial_\mu\partial_\nu\Delta_D(x-x'),$$

(Reference gauge $\hat{\xi}$ and $\Delta\xi = \xi - \hat{\xi}$).

Δ_D: gauge fixes the longitudinal part of photon propagator

$$\Delta_D(y) = -ie^2(\mu)\mu^{4-D}\int d^D\bar{k}\,\frac{{\rm e}^{-ik\cdot y}}{k^4} = -\frac{ie^2(\mu)}{16\pi^{\frac{D}{2}}}\Gamma\Big[\frac{D}{2}-2\Big](\mu y)^{4-D},$$

Choosing a gauge?

Some examples of “good” gauge parameters:

1. Feynman gauge ($\xi = 1$): minimises number of terms in loop calculations. (Also – simple Ward identities that decouple from ghosts).
2. Landau gauge ($\xi = 0$) in QCD: UV finite ghost-gluon vertex that allows lattice access to IR fixed point.
3. Traceless (Yennie-Fried) gauges ($\xi = 3$ in $D = 4$): removes some IR divergences in propagator and vertex.

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Changing the gauge parameter reorganises physical information between Feynman diagrams:

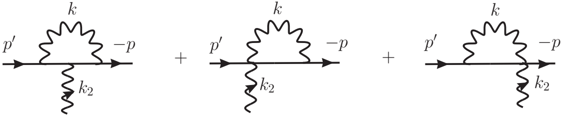

Figure: 1-loop scalar-Photon-Scalar vertex in scalar QED

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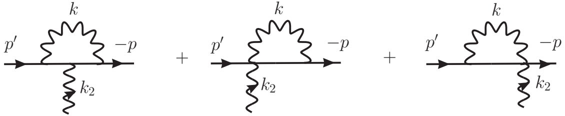

Figure: 1-loop scalar-Photon-Scalar vertex in scalar QED

¿How can we transform results from one gauge to another?

LKFT for propagators

Propagator dependence on ξ

Physics independent of gauge parameter but intermediate objects depend on ξ.

- Photon propagator depends linearly on ξ:

$$\langle A_\mu(k)A_\nu(-k)\rangle_{\xi+\Delta\xi} = \langle A_\mu(k)A_\nu(-k)\rangle_\xi + \Delta\xi\frac{k_\mu k_\nu}{k^4}$$

$$\langle A_\mu(x)A_\nu(x')\rangle_{\xi+\Delta\xi} = \langle A_\mu(x)A_\nu(x')\rangle_\xi + \Delta\xi\partial_\mu\partial_\nu\Delta_D(x-x')$$

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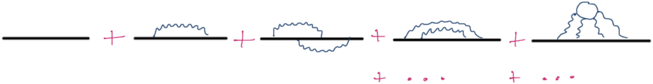
- Matter propagator difficult: loop (virtual) photon propagators depend ξ



Gauge dependence studied[1] by Landau and Khalatnikov and Fradkin. From

$$S(x', x \,|\, \xi) = \langle \hat{\Psi}(x) \hat{\bar{\Psi}}(x')\rangle_{\xi} \,,$$

a variation in ξ induces the non-perturbative transformation

$$S(x', x \,|\, \xi + \Delta\xi) = S(x', x \,|\, \xi)\, \mathrm{e}^{i\Delta\xi[\Delta_D(x-x') - \Delta_D(0)]} \,. \tag{2}$$

[1] Landau, Khalatnikov, Sov. Phys. JETP2, 69 (1956), Fradkin, Zh. Eksp. Teor. Fiz. 29, 258261 (1955).

N-point Green functions

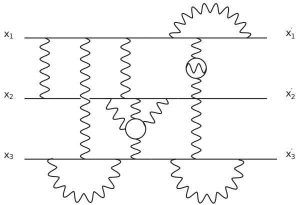
Extended to higher-order Green functions[2] using first quantised

Worldline Formalism.

Consider the $(N = 2n)$-point Green functions

$$S_N = \langle \hat{\bar{\Psi}}(x_1) \cdots \hat{\bar{\Psi}}(x_n) \hat{\Psi}(x_1') \cdots \hat{\Psi}(x_n') \rangle$$

(All contractions between Ψ and the $\bar{\Psi}$). Representative Feynman diagram:



[2] Ahmadiniaz, Bashir, Schubert, Phys. Rev. **D93**, 045023 (2016) and Russ. Phys. J.**59**, 1752 (2017). Nicasio, Ahmadiniaz, **JPE**, Schubert, Phys. Rev. **D 104** (2021) 2, 025014 [arXiv:2010.04160 [hep-th]]

Correlation functions

Demonstrable claim:

Gauge variation induced in propagator depends only on **internal** photons *not* attached to matter loops

Easier to work with **quenched, partial amplitudes** $\left(S_{2n} = \sum_{\pi \in S_n} \mathcal{S}_{n\pi}\right)$:

$$\mathcal{S}(x_1, \ldots, x_n; x'_{\pi(1)}, \ldots, x'_{\pi(n)} | \xi + \Delta\xi) = \prod_{k,l=1}^{N} \mathrm{e}^{-\Delta_\xi S_{i\pi}^{(k,l)}} \mathcal{S}(x_1, \ldots, x_n; x'_{\pi(1)}, \ldots, x'_{\pi(n)} | \xi)$$

with one LKF exponent for each way of pairing up endpoints:

$$\Delta_\xi S_{i\pi}^{(k,l)} = \frac{\Delta\xi e^2}{32\pi^{\frac{D}{2}}} \Gamma[\nu] \Big\{ \big[(x_k - x_l)^2 \big]^{-\nu} - \Big[\Big(x_k - x'_{\pi(l)} \Big)^2 \Big]^{-\nu} - \Big[\Big(x'_{\pi(k)} - x_l \Big)^2 \Big]^{-\nu} + \Big[\Big(x'_{\pi(k)} - x'_{\pi(l)} \Big)^2 \Big]^{-\nu} \Big\},$$

where $\nu = \frac{D}{2} - 2$.

The LKF exponent

Lift from partial amplitudes to full propagator: sum over worldline endpoints,

$$\sum_{k,l=1}^{n} \Delta_\xi S_{i\pi}^{(k,l)} = \sum_{k,l=1}^{n} \Delta_\xi S_{i\mathbb{1}}^{(k,l)} .$$

Sum is actually independent of the permutation π,
$\Longrightarrow$ Full correlation function transforms as

$$S(x_1 \ldots x_n; x_1' \ldots x_n' | \xi + \Delta\xi) = T_n \, S(x_1 \ldots x_n; x_1' \ldots x_n' | \xi) ,$$

with global LKF factor

$$T_n = \mathrm{e}^{-\sum_{k,l=1}^{n} \Delta_\xi S_i^{(k,l)}} \tag{3}$$

(Calculate exponent using *any* permutation).

Remember: $\Delta_\xi S_i^{(k,l)} \sim \mathcal{O}(\alpha) \Longrightarrow T_n$ contains **arbitrarily high powers** of α!

Applications

Low order diagrams contain **gauge-dependent** info on higher order ones.

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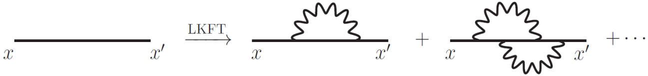

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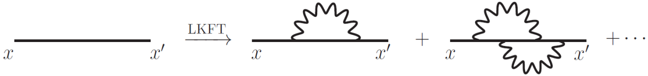

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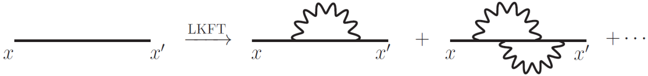

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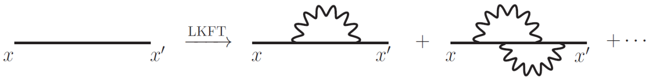

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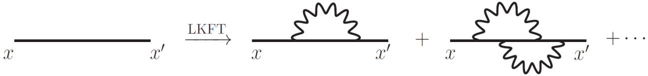

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LKFTs less well studied than Ward-Takahashi / Slavnov-Taylor identities.

LKFT in momentum space

Position space transformation induces a **momentum space** transformation[3].

Starting with momentum space propagator $\mathcal{S}_{\mathcal{L}}(p\,|\,\hat{\xi})$ in a fixed reference gauge, $\hat{\xi}$:

1. Transformed to position space for $\mathcal{S}_{\mathcal{L}}(x\,|\,\hat{\xi})$
2. Apply LKF transformation to transform $\mathcal{S}_{\mathcal{L}}(x\,|\,\hat{\xi}) \to \mathcal{S}_{\mathcal{L}+}(x\,|\,\hat{\xi}+\Delta\xi)$
3. Transform back to momentum space for $\mathcal{S}_{\mathcal{L}+}(p\,|\,\xi+\Delta\xi)$

Recently worked out for a general input $S(p',p) := (2\pi)^D\delta^D(p'+p)\mathcal{S}(p^2,\not{p})$:

$$S(p',p\,|\xi+\Delta\xi) = \int \frac{d^Dq}{(2\pi)^D}\,\Pi_D(q,\Delta\xi)S(p'-q,p+q\,|\,\xi)$$

where

$$\begin{aligned}\Pi_D(q,\Delta\xi) &:= \int d^Dx\,\mathrm{e}^{i\Delta\xi\left[\Delta_D(x)-\Delta_D(0)\right]+iq\cdot x}\\ &= 2\pi^{\frac{D}{2}}\left(\frac{q}{2}\right)^{1-\frac{D}{2}}\mathrm{e}^{-i\Delta\xi\Delta_D(0)}\int_0^\infty dx\,x^{\frac{D}{2}}J_{\frac{D-2}{2}}(qx)\mathrm{e}^{i\Delta\xi\Delta_D(x)}\,.\end{aligned}$$

[3] Bashir et al., Phys. Rev. **D66**, (2002), 105005. Kotikov, Teber, Phys. Rev. **D100**, (2019), 105017. Villanueva-Sandoval et al., Jnl. Phys: Conference Series 1208, (2019), 012001

Explicit formulae

For $D = 3$:

$$S_3(p', p \,|\, \xi) = 8\pi \frac{\alpha\Delta\xi}{2} \int \frac{d^3 q}{(2\pi)^3} \frac{S_3(p' - q, p + q \,|\, \xi)}{\left[q^2 + \left(\frac{\alpha\Delta\xi}{2}\right)^2\right]^2} \,.$$

Decompose into Dirac structure

$$\mathcal{S}(p^2, \not{p} \,|\, \xi) = \mathcal{A}(p^2 \,|\, \xi) + \not{p}\mathcal{B}(p^2 \,|\, \xi) \,,$$

The coefficients of these structures transform **without mixing**:

$$\mathcal{A}_3(p^2 \,|\, \xi + \Delta\xi) = \frac{\alpha\Delta\xi}{2\pi p} \int_{-\infty}^{\infty} dq \, \frac{q\mathcal{A}_3(q^2 | \, \xi)}{(p-q)^2 + \left(\frac{\alpha\Delta\xi}{2}\right)^2} \,,$$

$$\mathcal{B}_3(p^2 \,|\, \xi + \Delta\xi) = \frac{\alpha\Delta\xi}{2\pi p^2} \int_{-\infty}^{\infty} dq \, \mathcal{B}_3(q^2 \,|\, \xi) \left\{ \frac{q^2}{(p-q)^2 + \left(\frac{\alpha\Delta\xi}{2}\right)^2} + \frac{q}{2p} \log\left((p-q)^2 + \left(\frac{\alpha\Delta\xi}{2}\right)^2 \right) \right\}$$

Explicit formulae

D=4:

$$S_4(p', p\,|\,\xi\,) = (4\pi)^2 \left(\frac{x_0^2}{4}\right)^{\frac{\alpha\Delta\xi}{4\pi}} \frac{\Gamma\left[2 - \frac{\alpha\Delta\xi}{4\pi}\right]}{\Gamma\left[\frac{\alpha\Delta\xi}{4\pi}\right]} \int \frac{d^4q}{(2\pi)^4} \frac{S(p'-q, p+q\,|\,\xi)}{\left[q^2\right]^{2-\frac{\alpha\Delta\xi}{4\pi}}}\,.$$

Form factors ($\mathcal{C}_4 = \frac{2\Gamma\left[2-\frac{\alpha\Delta\xi}{4\pi}\right]\Gamma\left[1-\frac{\alpha\Delta\xi}{4\pi}\right]}{\pi p^{4-\frac{\alpha\Delta\xi}{2\pi}}}\left(\frac{x_0^2}{4}\right)^{\frac{\alpha\Delta\xi}{4\pi}}$):

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$$\mathcal{A}_4(p^2\,|\,\xi + \Delta\xi) = -\mathcal{C}_4 \mathrm{Im}\left\{\mathrm{e}^{i\frac{\alpha\Delta\xi}{4\pi}\pi} \int_0^\infty dq\, q^3 \mathcal{A}_4(q^2\,|\,\xi)\, {}_2F_1\left(1 - \frac{\alpha\Delta\xi}{4\pi}, 2 - \frac{\alpha\Delta\xi}{4\pi}; 2; \left(\frac{q}{p}\right)^2\right)\right\},$$

and ($\mathcal{D}_4 = \frac{\Gamma\left[3-\frac{\alpha\Delta\xi}{4\pi}\right]\Gamma\left[1-\frac{\alpha\Delta\xi}{4\pi}\right]}{\pi p^{6-\frac{\alpha\Delta\xi}{2\pi}}}\left(\frac{x_0^2}{4}\right)^{\frac{\alpha\Delta\xi}{4\pi}}$)

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Form factors ($\mathcal{C}_4 = \frac{2\Gamma\left[2-\frac{\alpha\Delta\xi}{4\pi}\right]\Gamma\left[1-\frac{\alpha\Delta\xi}{4\pi}\right]}{\pi p^{4-\frac{\alpha\Delta\xi}{2\pi}}}\left(\frac{x_0^2}{4}\right)^{\frac{\alpha\Delta\xi}{4\pi}}$):

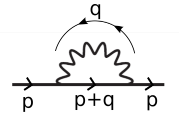
$$\mathcal{A}_4(p^2\,|\,\xi+\Delta\xi) = -\mathcal{C}_4\,\mathrm{Im}\left\{e^{i\frac{\alpha\Delta\xi}{4\pi}\pi}\int_0^\infty dq\,q^3\mathcal{A}_4(q^2\,|\,\xi)\,{}_2F_1\left(1-\frac{\alpha\Delta\xi}{4\pi}, 2-\frac{\alpha\Delta\xi}{4\pi}; 2; \left(\frac{q}{p}\right)^2\right)\right\},$$

and ($\mathcal{D}_4 = \frac{\Gamma\left[3-\frac{\alpha\Delta\xi}{4\pi}\right]\Gamma\left[1-\frac{\alpha\Delta\xi}{4\pi}\right]}{\pi p^{6-\frac{\alpha\Delta\xi}{2\pi}}}\left(\frac{x_0^2}{4}\right)^{\frac{\alpha\Delta\xi}{4\pi}}$)

$$\mathcal{B}_4(p^2\,|\,\xi+\Delta\xi) = -\mathcal{D}_4\,\mathrm{Im}\left\{e^{i\frac{\alpha\Delta\xi}{4\pi}\pi}\int_0^\infty dq\,q^5\mathcal{B}_4(q^2\,|\,\xi)\,{}_2F_1\left(1-\frac{\alpha\Delta\xi}{4\pi}, 3-\frac{\alpha\Delta\xi}{4\pi}; 3; \left(\frac{q}{p}\right)^2\right)\right\}$$

Verification

- Results obtained checked against gauge dependent parts of the propagator to one-loop order (scalar and spinor QED).



$$\Big|_{\xi} = \xi \int \frac{d^D q}{(2\pi)^D} \not{q} \frac{i}{\not{p} + m} \not{q} \frac{-i}{(q^2)^2}$$

Example: Spinor QED in $D = 4$

- $\mathcal{A}_4(p^2 \,|\, \xi + \Delta\xi) = \frac{1}{m}\left(\frac{m^2 x_0^2}{4}\right)^{\frac{\alpha\Delta\xi}{4\pi}} \Gamma\left[2 - \frac{\alpha\Delta\xi}{4\pi}\right]\Gamma\left[1 - \frac{\alpha\Delta\xi}{4\pi}\right] {}_2F_1\left(1 - \frac{\alpha\Delta\xi}{4\pi}, 2 - \frac{\alpha\Delta\xi}{4\pi}; 2; -\frac{p^2}{m^2}\right)$
- $\mathcal{B}_4(p^2 \,|\, \xi + \Delta\xi) =$
  $-\frac{1}{2m^2}\left(\frac{m^2 x_0^2}{4}\right)^{\frac{\alpha\Delta\xi}{4\pi}} \Gamma\left[1 - \frac{\alpha\Delta\xi}{4\pi}\right]\Gamma\left[3 - \frac{\alpha\Delta\xi}{4\pi}\right] {}_2F_1\left(1 - \frac{\alpha\Delta\xi}{4\pi}, 3 - \frac{\alpha\Delta\xi}{4\pi}; 3; -\frac{p^2}{m^2}\right)$

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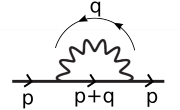

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- To one loop order ($\mathcal{O}(\alpha)$) using *dim. reg.*:

$$\mathcal{S}^{(1)}(p^2, \not{p}) = \frac{\alpha\Delta\xi}{4\pi}\Bigg\{ -\frac{m}{p^2+m^2}\left[\frac{1}{\epsilon} - \left(1 - \frac{m^2}{p^2}\right)\log\left[1 + \frac{p^2}{m^2}\right] - \log\left[\frac{m^2}{4\pi}\right] - \gamma_E\right)\right]$$
$$+ \frac{\not{p}}{p^2+m^2}\left[\frac{1}{\epsilon} - \left(1 + \frac{m^4}{p^4}\right)\log\left[1 + \frac{p^2}{m^2}\right] - \log\left[\frac{m^2}{4\pi}\right] + \left(1 + \frac{m^2}{p^2}\right) - \gamma_E\right]\Bigg\}.$$

LKFT for vertices

Vertices' dependence on ξ

Interaction vertex strongly dependent on gauge parameter:

- On-shell interaction vertex

$$ie\gamma^{\mu} \to ie\Gamma^{\mu}(p, p', q) = \gamma^{\mu} F_1(q^2) + \frac{\sigma^{\mu\nu} q_{\nu}}{2m} F_2(q^2)$$

- General (Ball-Chiu[4]) decomposition:

$$\Gamma^{\mu}(p, p', q) = A(\not{p}' + \not{p})\gamma^{\mu} + B\gamma^{\mu} + C(p' + p)^{\mu} + \sum_{i=1}^{8} t_i T_i^{\mu}$$

  into *longitudinal* and *transverse* $(q_{\mu} T_i^{\mu} = 0)$ parts.
  (Adkins, Lymberopoulos and Velkov, Kizilersu, Reenders, and Pennington).

- Quark-gluon vertex tensor decomposition studied at two-loop order. (Lots of work in scalar QED, QED_3 and reduced QED).
- Three and four gluon vertices of QCD: calculated off-shell in different covariant gauges for special kinematics up to two-loop order.

[4] Ball and Chiu, Phys. Rev. **D22**, 10 (1980), 2542

LKFT for vertex

LKF transformations associated to external legs: in momentum space define

$$\Lambda^{\mu}(p', p, k) = S(p)\Gamma^{\mu}(p', p, k)S(p')$$

($\Lambda^{\mu}(p', p, k)$ is the partially amputated vertex).

Two external legs $\Longrightarrow$ same LKFT as propagator. In position space:

$$\Lambda^{\mu}(x', x, z \,|\, \xi + \Delta\xi) = \Lambda^{\mu}(x', x, z \,|\, \xi)\mathrm{e}^{i\Delta\xi\left[\Delta_D(|x'-x|)-\Delta_D(0)\right]} .$$

Result previously obtained by Landau and Khalatnikov, Zumino etc.

Can we find the **momentum space** version?

Momentum space LKFT

As before: $\Lambda^{\mu}(p|\xi) \longrightarrow \Lambda^{\mu}(x|\xi) \longrightarrow$ LKFT $\longrightarrow \Lambda^{\mu}(x|\xi+\Delta\xi) \longrightarrow \Lambda^{\mu}(p|\xi+\Delta\xi)$.

- Define $\Lambda^{\mu}(p', p, k \,|\, \xi) = (2\pi)^{D}\delta^{D}(p'+p+k)\Lambda^{\mu}(p', p \,|\, \xi)$:

$$\Lambda^{\mu}(p', p \,|\, \xi+\Delta\xi) = \int \frac{d^{D}q}{(2\pi)^{D}} \Pi_{D}(q, \Delta\xi)\, \Lambda^{\mu}(p'-q, p+q \,|\, \xi)\,,$$

- Note that these LKFTs are compatible with the Ward identity, $-k_{\mu}\Lambda^{\mu}(p', p, k) = S(p') - S(p)$:

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- Note that these LKFTs are compatible with the Ward identity, $-k_\mu \Lambda^\mu(p', p, k) = S(p') - S(p)$:

$$\begin{aligned} -k_\mu \Lambda^\mu(p', p\,|\,\xi+\Delta\xi) &= -\int \frac{d^D q}{(2\pi)^D} \Pi_D(q, \Delta\xi)\, k_\mu \Lambda^\mu(p'-q, p+q\,|\,\xi) \\ &= \int \frac{d^D q}{(2\pi)^D} \Pi_D(q, \Delta\xi)\, \left[\mathcal{S}(p'-q\,|\,\xi) - \mathcal{S}(p+q\,|\,\xi)\right] \\ &= \mathcal{S}(p'\,|\,\xi+\Delta\xi) - \mathcal{S}(p\,|\,\xi+\Delta\xi)\,. \end{aligned}$$

(Used $k_\mu = (p'-q)_\mu + (p+q)_\mu$ and symmetry under $q \to -q$).

Infinitesimal transformations

The momentum space transformations allow us to explore the ξ-dependence of the vertex and propagator:

$$\frac{\partial}{\partial \xi}\Lambda(p', p\,|\,\xi) = i\int \frac{d^D q}{(2\pi)^D}\Big[\widetilde{\Delta}_D(q)\Lambda(p' - q, p + q\,|\,\xi) - \widetilde{\Delta}_D(q = 0)\Lambda(p', p\,|\,\xi)\Big]$$

$$= 4\pi\alpha\int \frac{d^D q}{(2\pi)^D}\frac{1}{q^4}\Big[\Lambda(p' - q, p + q\,|\,\xi) - \Lambda(p', p\,|\,\xi)\Big]\,.$$

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Transfer to the fully amputated vertex, $\Gamma^\mu(p',p,k\,|\,\xi)$:

$$\frac{\partial}{\partial\xi}\Gamma(p',p\,|\,\xi) =$$
$$4\pi\alpha\int\frac{d^Dq}{(2\pi)^D}\frac{1}{q^4}\Big[\mathcal{S}^{-1}(p'\,|\,\xi)\mathcal{S}(p'-q\,|\,\xi)\Gamma(p'-q,p+q\,|\,\xi)\mathcal{S}(p+q\,|\,\xi)\mathcal{S}^{-1}(p\,|\,\xi) - \Gamma(p',p\,|\,\xi)\Big]$$
$$+\Big(\frac{\partial}{\partial\xi}\mathcal{S}^{-1}(p'\,|\,\xi)\Big)\mathcal{S}(p'\,|\,\xi)\Gamma(p',p\,|\,\xi) + \Gamma(p',p\,|\,\xi)\mathcal{S}(p\,|\,\xi)\Big(\frac{\partial}{\partial\xi}\mathcal{S}^{-1}(p\,|\,\xi)\Big)\,.$$

Direct derivation of known result for the propagator

Bashir, Dall'Olio J. Phys. Conf. Ser. 1208 (2019) 012002:

$$\frac{\partial}{\partial\xi}\mathcal{S}(p\,|\,\xi) = 4\pi\alpha\int\frac{d^Dq}{(2\pi)^D}\frac{1}{q^4}\Big[\mathcal{S}(p-q\,|\,\xi) - \mathcal{S}(p\,|\,\xi)\Big]\,.$$

Verification – one-loop order

Suppose the tree-level (partially-amputated) vertex is given as input:

$$\Lambda^{\mu\,(0)}(p' - q, p + q \,|\, \xi) = \frac{-ie\gamma^{\mu}}{(\not{p} + m)(\not{p}' + m)}\,.$$

Truncating the LKF transformation to $\mathcal{O}(\alpha)$ gets the gauge-dependent parts of the one-loop vertex:

$$\Lambda^{\mu\,(1)}(p', p \,|\, \xi + \Delta\xi) = 4\pi\Delta\xi \int \frac{d^D q}{(2\pi)^D}\, \frac{1}{q^4} \Big(\Lambda^{\mu\,(0)}(p' - q, p + q \,|\, \xi) - \Lambda^{\mu\,(0)}(p', p \,|\, \xi) \Big)$$

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This should agree with the known results for the fully amputated vertex, **but...** now with the shifted gauge parameter

$$\begin{aligned}\Lambda^{\mu\,(1)}(p',p\,|\,\xi+\Delta\xi) = {}& S^{(0)}(p'\,|\,\xi+\Delta\xi)\Gamma^{\mu\,(1)}(p',p\,|\,\xi+\Delta\xi)S^{(0)}(p\,|\,\xi+\Delta\xi)\\ &+ S^{(1)}(p'\,|\,\xi+\Delta\xi)\Gamma^{\mu\,(0)}(p',p\,|\,\xi+\Delta\xi)S^{(0)}(p\,|\,\xi+\Delta\xi)\\ &+ S^{(0)}(p'\,|\,\xi+\Delta\xi)\Gamma^{\mu\,(0)}(p',p\,|\,\xi+\Delta\xi)S^{(1)}(p\,|\,\xi+\Delta\xi)\,.\end{aligned}$$

Verified by direct calculation in perturbation theory $\surd$!

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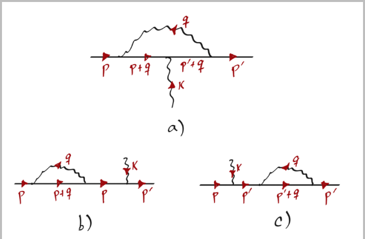

This … ut...
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Longitudinal / Transverse parts

Ball-Chiu decomposition into longitudinal, L^μ, and transverse, T^μ parts:

$$k_\mu T^\mu(p', p, k) = 0 \Longrightarrow (p + p') \cdot T(p', p, k) = 0\,.$$

This time LKFTs **do mix** the structures, but only in the *transverse part*:

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$$\begin{aligned} k_{\mu}\Lambda^{\mu}_{T}(p', p \,|\, \xi + \Delta\xi) &= -\int \frac{d^D q}{(2\pi)^D}\Pi_D(q, \Delta\xi)\,(p'_{\mu} + p_{\mu})\Lambda^{\mu}_{T}(p' - q, p + q \,|\, \xi) \\ &= -\int \frac{d^D q}{(2\pi)^D}\Pi_D(q, \Delta\xi)\,[(p' - q) + (p + q)]_{\mu}\Lambda^{\mu}_{T}(p' - q, p + q \,|\, \xi) = 0 \,. \end{aligned}$$

But the longitudinal vertex generates both longitudinal and transverse parts:

- The transverse part of the LKF-transformed vertex comes from $\Lambda^{\mu}_{T}(p', p, k \,|\, \xi)$ AND $\Lambda^{\mu}_{L}(p', p, k \,|\, \xi)$
- The longitudinal part of the LKF-transformed vertex comes only from $\Lambda^{\mu}_{L}(p', p, k \,|\, \xi)$

¡See article for explicit formulae!

Conclusion

LKF transformations – interesting non-perturbative window onto QFT.

Compatible with WT identity, but generate parts of higher order diagrams.

1. First integral representations of general momentum space LKFT
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De Meerleer, Dudal, Sorella, Dall'Olio, Bashir, Acta Phys. Polon. Supp. 11 (2018) 589-594

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¡Thank you for your attention!