String Theory: a mini-course

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Abstract. We provide a brief introduction to string theory. We review some basic aspects of the theory as the quantization of the bosonic and the supersymmetric string. D-branes are studied in some detail as well as some aspects of T-duality. The course is intended for students with some previous knowledge in Quantum Field Theory and General Relativity.

1. Introduction

String theory is our best candidate for a model beyond the Standard Model (SM) in the sense that it provides a quantum mechanically well-defined theory underlying gauge and gravitational interactions. As a theory of quantum gravity, it has the potential to answer questions so far unknown as the nature of space-time and the black hole information paradox. As a gauge theory, it has the potential to unify all forces. More over, it offers a way to relate aspects of gauge theories with gravity and vice versa.

The basic feature of string theory proposes that elementary particles are not point-like, but rather they are small 1-dimensional extended objects. By this simple fact, we shall see that, every symmetry we know is relevant in the real world, can be deduced from the supersymmetric quantum string. More over, we shall review how important aspects of the non-perturbative physics of strings are easily achieved.

From the phenomenological point of view, we are interested in constructing a theory which goes far beyond the Standard Model of Particles (SM) since there are basic questions that can not be answered in the context of quantum field theories. For instance, how is gravity coupled to other forces at the quantum level? Is there an unification of all forces? Why are there three families in the SM? How is supersymmetry broken? Why are there so many free parameters in the SM and even more in the Supersymmetric Standard Model? Why is the cosmological constant so small? Why and what is the dark energy? The presence of singularities, as the Big Bang and the black hole singularity, tells us that the theory breaks down in such limits. What is the extension to General Relativity at the quantum level? and so on. All these questions might be answered in the context of string theory and for this reason, it is worth to study it.

String theory is however, a theory under construction, meaning that we do not know the basic physical principles underlying it. Although a lot of progress has been done in the past years,

[1] Speaker.

still there is not a concrete prediction so far. This is a crucial point we must face in the years to come. Meanwhile, let us begin studying the basics of this beautiful model called *string theory.*

Our review is organized as follows. In section 1 we study the bosonic string and its quantization. Section 2 is devoted to the superstring. Special emphasis is given to the construction of the spectrum of Type IIA and Type IIB theories. In section 4 we concentrate on the description of T-duality and the physics of supersymmetric D-branes. At the end, we briefly comment on some recent studies in string theory. References are restricted to some text books on string theory [1]-[9] and to some review articles written by one of the authors [11]-[12] . We apologize for not given an extensive reference list about the hundreds of excellent reviews on string theory. However, we encourage the interested reader to consult the bibliography reported in the text books we refer to.

2. The Bosonic String

It is well known that the Lagrangian of a relativistic point particle is proportional to the corresponding invariant length, the world-line described by the particle while moving in the space-time. Our first attempt to describe the dynamics of a string is to construct its Lagrangian, which based on the relativistic particle case, we expect it to be proportional to the invariant area described by the string as it moves in a D-dimensional space-time[2]. This is called the string world-sheet and it is a 2-dimensional object with an induced metric denoted by $G_{\alpha\beta}$. The action is given by

$$S = -T \int d^2\sigma \sqrt{-det(G_{\alpha\beta})}, \tag{1}$$

where the world-sheet is parametrized by $\sigma_1 = \sigma$ (a space-like coordinate), and $\sigma_0 = \tau$ (a time-like coordinate). After some direct calculations, one can see that the above action can be written in terms of the D-dimensional coordinates X^M which characterize the position of the world-sheet in the space-time[3] This is the *Nambu-Goto* action and it is given by

$$S_{NG} = -T \int d\sigma d\tau \sqrt{\left(\dot{X} \cdot X'\right)^2 + \dot{X}^2 X'^2}, \tag{2}$$

where

$$\dot{X}^M = \frac{\partial X^M}{\partial \tau} \quad X'^M = \frac{\partial X^M}{\partial \sigma}. \tag{3}$$

However this action is difficult to quantize. An easiest way to do it, requires an action without the square root of the field derivatives[4]. The desired action is known as the *Polyakov action* and it is equivalent to the Nambu-Goto action at the classical level in the sense that it give rise to the same equations of motion for the fields X^M. The Polyakov action is

$$S_P \quad = \quad -\frac{1}{2} \int d^2\sigma \sqrt{-h} h^{\alpha\beta} \partial_\alpha X \cdot \partial_\beta X, \tag{4}$$

where $h = det(h_{\alpha\beta})$, $h_{\alpha\beta}$ is the induced metric on the world-sheet, and α and β runs on 0 and 1. This action posses the following symmetries.

[2] Notice that we start from a space-time with an arbitrary dimensionality $D = d+1$. One of the most interesting things of string theory, is that it is possible to fix such number. This constitutes an exceptional case in theoretical models as Quantum Field Theory, where the number of dimensions are fixed by hand. In String Theory, such number is fixed by the theory itself under certain circumstance we shall comment later on.

[3] This is formally known as a sigma model.

[4] Here we refer to the path integral method of quantization. In this mini-course we shall take the canonical quantization procedure. However, using the Polyakov action represents as well an easy way to realize the full set of symmetries involved in the bosonic string.

(i) Space-time Poincaré symmetry. The action is invariant under Poincaré transformations. It is a global symmetry.
(ii) Reparametrizations. This is a non-trivial gauge symmetry and represent a large extra symmetry on the world sheet. Essentially tells us that we can select different coordinate frames on the world-sheet.
(iii) Weyl transformation. This symmetry represents the invariance of the action under rescaling of the metric. By transforming the induced metric $h_{\alpha\beta}$ to $\Omega^2 h_{\alpha\beta}$ the equations of motion are the same.

Since we have three constraints given by these symmetries, it is possible to fix the corresponding three parameters in $h_{\alpha\beta}$. By reparametrization invariance we can choose two components of $h_{\alpha\beta}$, such that only one component remains free. But Weyl invariance fixes it. Therefore, the induced metric $h_{\alpha\beta}$ is fixed as $h_{\alpha\beta} = \eta_{\alpha\beta}$, where $\eta_{\alpha\beta}$ is the flat 2-dimensional Minkowskian metric. With this gauge, the Polyakov action reduces to

$$S = \frac{T}{2}\int d^2\sigma(\dot{X}^2 - X'^2). \tag{5}$$

The corresponding equation of motion with respect to the fields X^M, up to some boundary conditions, are given by

$$\partial_\alpha \partial^\alpha X = 0, \tag{6}$$

which is a differential wave equation. Their solutions are then a linear combination of linear terms in τ and σ and periodic functions. Notice that such solutions represents undulating modes of the string. Later on we shall see that such modes can be quantized, and they will represent different massive and massless fields. If some of the massless fields have the properties of already known fields in the context of quantum field theory, we shall conclude that quantum oscillations of a string produces the particles we are familiar with. For instance, let us say that we find that one of the modes is massless, transforms under the Lorentz group as a vector boson, has a $U(1)$ internal symmetry and has a spin 1. We would assure that such mode is indeed a photon. Things are nevertheless, a little bit more complicated, but we shall see actually that some modes have all the desired properties with respect to the particles we already know.

Fixing the gauge as before, implies that each component of the energy-momentum tensor vanishes,

$$T_{\alpha\beta} = \partial_\alpha X \partial_\beta X - \frac{1}{2} h_{\alpha\beta} h^{\gamma\delta} \partial_\gamma X\, \partial_\delta X = 0, \tag{7}$$

and by using the flat induced metric, the components read

$$T_{\alpha\beta} \doteq \begin{bmatrix} \frac{1}{2}\left(\dot{X}^2 + X'^2\right) & \dot{X}X' \\ \dot{X}X' & \frac{1}{2}\left(\dot{X}^2 + X'^2\right). \end{bmatrix} = 0 \tag{8}$$

This expression has a geometrical interpretation. The gauge selection we have performed implies that the coordinate framework on the world-sheet is orthogonal. This allows us to choose a particular useful system of coordinates: the light-cone coordinates. We shall return to this selection later on.

2.1. Boundary Conditions and D-branes

The stationary points are chosen by demanding the invariance of the action under the shifts $X^M \to X^M + \delta X^M$. The variation of the action reads

$$\delta S = \frac{T}{2}\int d^2\sigma(\partial_\alpha \partial^\alpha X_M)\delta X^M - T\int d\tau \left[X'_M \delta X^M \mid_{\sigma=\pi} - X'_M \delta X^M \mid_{\sigma=0}\right], \tag{9}$$

Table 1. Boundary conditions on the bosonic string.

Type	Boundary Condition
Closed	$X^M(\sigma,\tau) = X^M(\sigma+\pi,\tau)$
Open/Neumann	$X'_M = 0, \quad \sigma = 0, \pi$
Open/Dirichlet	$\delta X^M = 0, \quad \sigma = 0, \pi$

from which, the vanishing of the boundary terms leads to different conditions on the string (see Table 1). Strings which endpoints are identified, are called *closed strings*, while *open strings* presents two variations. For Neumann boundary conditions we can see that there is not momentum flowing away from the string endpoints. These strings are free to move over the whole D-dimensional space-time, meaning that we can impose Nuemann conditions on all field coordinates X^M. We can also consider an open string with all its field coordinates fulfilling Dirichlet boundary conditions. This means that each of the endpoints of the strings, are fixed on a point in a D-dimensional space-time. Notice as well that such point is localized in time. Further analysis shows that this is indeed an instanton of the theory. However, let us consider the richest case, in which we impose Dirichlet boundary conditions on some coordinates and Neumann on others. Let us say that for $M = 0, 1, \ldots, p$, X^M satisfies Neumann boundary conditions, while for $M = p+1, \ldots, d$, it satisfies Dirichlet conditions. Henceforth, the endpoints of the string will be attached to move freely on a $p+1$-dimensional hyperplane. Notice that they can not leave this object unless their endpoints coincide. If this happens, the open string becomes closed and it is free to escape away from the D-brane. In this sense we say that D-branes emit closed strings[5].

It turns out that such objects are very important in string theory. They are named *Dp*-branes: p- dimensional branes (generalization of membranes) on which the endpoints of an open string are attached to. Orthogonal coordinates of the endpoints satisfy Dirichlet boundary conditions (the "D" of D-brane refers to Dirtichlet boundary conditions). For the bosonic case, it is possible to have different *Dp*-branes of all dimensionalities, from $p = -1$ to $p = d$.

2.2. Solution to the Equations of Motion

We are now ready to explicitly show the solutions of the equations of motion. However, we shall take a short-way. We are going to solve the equations of motion in a particular coordinate-frame on the string world-sheet, known as the light-cone coordinates. The advantages of this selection are that we shall arrive to the solutions in a faster and easier way. The price to pay is however, that we have lost covariance in our description of the string dynamics. Also, by selecting this frame, one can check that there are two unphysical degrees of freedom, corresponding to stretching and rotating a string. In a covariant formalism, one is able to arrive at the same conclusion, but in a richer way, i.e., by analyzing the anomalies, ghosts fields and by using the powerful conformal field theory in the string world-sheet.

The light-cone coordinates are given by $\sigma^{\pm} = \tau \pm \sigma$, from which the equations of motion read

$$\partial_+ \partial_- X^M = 0. \tag{10}$$

[5] We shall see that gravitons are associated to closed strings. Therefore, a D-brane emits and absorbs gravitons, implying that these objects have mass and tension and mainly, that they are dynamical.

For the closed string, the corresponding solution can be written in terms of two waves by

$$X^M(\sigma,\tau) = X_R^M(\sigma^-) + X_L^M(\sigma^+), \tag{11}$$

which represent left and right movers. The most general solution for the closed string given by

$$X_R^M = \frac{1}{2}x^M + \alpha' p^M \sigma^- + \frac{i}{2} l_s \sum \frac{1}{n} \alpha_n^M e^{-2in\sigma^-}, \tag{12}$$

$$X_L^M = \frac{1}{2}x^M + \alpha' p^M \sigma^+ + \frac{i}{2} l_s \sum \frac{1}{n} \tilde{\alpha}_n^M e^{-2in\sigma^+}. \tag{13}$$

Since $X^M(\sigma,\tau)$ is required to be real, the left and right movers X_L^M and X_R^M must be real function as well. This implies that x^M (center of mass) and p^M (momentum of the center of mass) are also real and in consequence, the modes of the string must obey that $\alpha_{-n}^M = (\alpha_n^M)^*$, $\quad \tilde{\alpha}_{-n}^M = (\tilde{\alpha}_n^M)^*$ for all n and where we have used $\alpha_0^M = \sqrt{\alpha'/2}\, p^M$. Introducing X_L and X_R into the energy-momentum tensor we obtain that

$$T_{--} = \alpha' \sum L_m e^{-2im\sigma^-}, \quad T_{++} = \alpha' \sum \tilde{L}_m e^{-2im\sigma^+}, \tag{14}$$

where the Fourier coefficients are actually the Virasoro generators given by

$$L_m = \frac{1}{2}\sum \alpha_{m-n}\alpha_n \quad \tilde{L}_m = \frac{1}{2}\sum \tilde{\alpha}_{m-n}\tilde{\alpha}_n. \tag{15}$$

So the requirement of $T_{++} = T_{--} = 0$ is reflected in the fact that all the modes must vanish. This constitutes a classical constraint for all modes, $L_m = \tilde{L}_m = 0$. An interesting and special case concerns the zero mode, since it offers a way to compute the mass associated to different Fourier modes. Hence, for $m = 0$,

$$\begin{aligned} L_0 + \tilde{L}_0 &= \frac{1}{2}(\alpha_0^2 + \alpha_0^2) + \sum_{n\neq 0}(\alpha_{-n}\cdot\alpha_n, +\tilde{\alpha}_{-n}\cdot\tilde{\alpha}_n) \\ &= \alpha' p^M p_M + \sum_{n\neq 0}(\alpha_{-n}\cdot\alpha_n + \tilde{\alpha}_{-n}\cdot\tilde{\alpha}_n). \end{aligned} \tag{16}$$

Since in a relativistic theory, $p^2 = -m^2$, we arrive at the mass expression for the closed string modes,

$$M^2 = \frac{2}{\alpha'}\sum_{n=1}^{\infty}(\alpha_{-n}\cdot\alpha_n + \tilde{\alpha}_n\cdot\tilde{\alpha}_n). \tag{17}$$

By thinking that a closed string is made of two open strings glued together, then it is easy to see that the solution of the equations of motion and the expansion in Fourier modes, corresponds to just one of the movers. Therefore, for the open string theory, the mass of the modes is given by

$$M^2 = \frac{1}{\alpha}\sum_{n=1}^{\infty}\alpha_{-n}\cdot\alpha_n, \tag{18}$$

where the most general solution of the equations of motion is given by

$$X^M(\tau,\sigma) = x^M + \frac{\alpha'}{2}p^M\tau + \frac{i\alpha'}{2}\sum\frac{1}{m}\alpha_m^M e^{-im\tau}cos(m\sigma). \tag{19}$$

2.3. Canonical Quantization

Our next step consists in quantizing the vibrations modes of the string. We would like to explore the possibility that some modes be representations of quantum fields. There are many ways to quantize the string, being the most popular the covariant quantization. However, since we want to study the quantum string features in a short way, we shall take the canonical quantization procedure. Essentially, this consists in promoting the Fourier modes α_n to operators which satisfy the algebra[6]

$$\left[\alpha_m^M, \alpha_n^N\right] = \left[\tilde{\alpha}_m^M, \tilde{\alpha}_n^N\right] = \eta^{MN}\delta_{m+n,0}, \quad \left[\alpha_m^M, \tilde{\alpha}_n^N\right] = 0. \tag{20}$$

Notice that this is the algebra that satisfies every rising and annihilator operator in the usual quantum harmonic oscillator, meaning that each quantum mode, now promoted to be an operator, creates or destroys quantum states. However, since now the modes are non-commuting operators for $m = 0$, we must order the product appearing on the Virasoro generator operators.

Therefore, it is possible to order the products $\alpha_{-n} \cdot \alpha_n$ (and similarly for left movers) up to the addition of a constant a. In this context, the mass term for the open string is written as

$$M^2 = \frac{1}{\alpha'}\left(\sum \alpha_{-n} \cdot \alpha_n - a\right), \tag{21}$$

while for the closed string, it reads

$$M^2 = \frac{2}{\alpha'}\left(\sum_{n=1}^{\infty} \left(\alpha_{-n} \cdot \alpha_n + \tilde{\alpha}_{-n} \cdot \tilde{\alpha}_n\right) - 2a\right), \tag{22}$$

where a is the zero point energy. Similarly as the quantum harmonic oscillator, the product of a lowering and a rising operators denotes another operator called the number operator N given by $N = \sum \alpha_{-n}\alpha_n$ for the left movers and $\tilde{N} = \sum \tilde{\alpha}_{-n}\tilde{\alpha}_n$ for the right ones. In this way, the normal ordening constant a cancels out for the difference $\left(L_0 - \tilde{L}_0\right) \mid \phi\rangle = 0$, which implies $N = \tilde{N}$. This is called the level-matching condition of the bosonic string.

The corresponding Hilbert space is constructed by acting with the modes α_n^i on a vacuum $\mid \Omega\rangle$ annihilated by the lowering operators α_{-n} and $\tilde{\alpha}_{-n}$. It is important to notice that, although not explicitly shown in this notes, the use of the light-cone coordinates forces some coordinates to be non-dynamical. For that reason, the index i in the operators modes runs over $D-2$ values.

Before constructing the quantum states, let us comment on the value of the ordering constant a, appearing in the mass expressions for the string. The value of such constant can be computed as follows. At first sight, it seems we are dwelling with a divergent quantity, since

$$a = \frac{1}{2}(D-2)\sum_{n=1}^{\infty} n. \tag{23}$$

However this sum has an analytic continuation and it is computed by using the ζ-function, which is defined as

$$\zeta(s) = \sum_{n=1}^{\infty} n^{-s}. \tag{24}$$

[6] In the classical picture, the Fourier modes α_n satisfy a Poincaré algebra.

Table 2. Closed string spectrum. For $D = 26$, the second state is masless.

State	Level	M^2/α
$\mid \Omega\rangle_L \otimes \mid \Omega\rangle_R$	$N = \tilde{N} = 0$	$-\frac{D-2}{6}$
$\alpha^i_{-1}\tilde{\alpha}^i_{-1} \mid \Omega\rangle_L \otimes \mid \Omega\rangle_R$	$N = \tilde{N} = 1$	$\frac{26-D}{12}$

This complex function has as unique analytic continuation at $s = -1$, where it takes the value $\zeta(-1) = -1/12$. Therefore by comparison we obtain that,

$$a = -\frac{D-2}{24}. \tag{25}$$

Quantum states are constructed by acting the rising operator modes and by restricting to the level match conditions. This means that for every left mover operator, there must be the corresponding right one. Therefore, there are an infinite tower of string states, generically given by the application of rising operators on the closed vacuum $\mid \Omega\rangle_L \otimes \mid \Omega\rangle_R$ or on the open vacuum $\mid \Omega\rangle$. As the reader might expect, one way to construct gauge bosons, is by considering open strings.

2.4. The Bosonic String Spectrum

Let us firstly, consider the closed string spectrum shown in Table 2. The first state corresponds to vanishing number operators, implying that such state is a tachyon, i.e., a quantum state with negative squared mass. Observe that this not depend on the value of D. The presence of tachyons, as it is now understood, reflects the fact that we are constructing a perturbative theory on a false vacuum. This means that there are some variable which must run down to a stable point on which the corresponding perturbative theory would not have tachyons in its spectrum. Closed string tachyons have been considered in literature, mainly in cosmological models.

The next state is given by a left and right number operators $N = \tilde{N} = 1$. Its mass is then $(26-D)/12$. It is a state with $(D-2)^2$ degrees of freedom. However, since the theory is Lorentz invariant, we expect that physical states belong to representations of the little group $SO(D-2)$. Recall that the little group is the subgroup of the Lorentz group which leaves invariant the D-dimensional momentum of a particle. Therefore, only a massless state would have the necessary degrees of freedom of a little group representation (a massive one would transform with respect to the little group $SO(D-1)$). This fixes completely the dimension number D to 26. Some comments are in order: first, the above argument, although not formal, yields the same result as the obtained by a covariant quantization[7]. Second, by requiring the theory to be Lorentz invariant at the perturbative level, we have fixed the number of dimensions. These theories are called *critical*[8].

The critical dimension 26 can be as well obtained from the open string spectrum, shown in Table 3. The first excited state has $(D-2)$ degrees of freedom and must transform as a vector

[7] There, the dimension is fixed by canceling the Weyl anomaly. If this anomaly is present, there would be inconsistencies as the presence of non-unitary states and unphysical degrees of freedom.

[8] There are models in which the Lorentz invariance is preserved and Weyl anomalies are cancelled non-perturbatively with $D \neq 26$. They are called non-critical string theories.

Table 3. Open string spectrum for $D = 26$

State	Level	M^2/α
$\mid \Omega\rangle$	$N = 0$	-1
$\alpha^i_{-1} \mid \Omega\rangle$	$N = 1$	0

under the little group $SO(D-2)$. The number of degrees of freedom match only for a masless state, i.e., for $D = 26$.

Therefore, in the critical bosonic theory, the massless first excited state in the closed string, transforms as a two-index object under $SO(24)$. It contains 24^2 degrees of freedom, which can be decomposed on irreducible representations of $SO(24)$. Actually

$$\mathbf{24^2} = \mathbf{1} \oplus \mathbf{276} \oplus \mathbf{299}, \tag{26}$$

corresponding to (all of them being massless) a scalar field ϕ called *the dilaton*, an antisymmetric two-index field B_{MN} and a symmetric two-index field G_{MN}. This last field is the cause that so many people around the world are interested in string theory and that they have been working on it for the last 30 years. G_{MN} has all the properties we expect from a quantum particle of gravity, *the graviton*!

3. The Superstring

Bosonic string theory opened up a new and huge area of research. Quantum gravity, an old dream, was now a tangible subject. However, it failed by the first and the simplest feature we must have in any physical theory: the presence of fermions. Bosonic string theory has only bosonic fields as quantum vibrations, and that is the main reason that renders it to be unphysical.

The simplest way to incorporate fermions into the theory considers the presence of an extra symmetry at the level of the Polyakov action. A symmetry that relates a fermionic field for each bosonic field present in the two-dimensional string world-sheet. This is called supersymmetry. According with the Ramond-Neveu-Schwarz (R-NS) formalism[9], the bosonic fields $X^M(\sigma,\tau)$ are paired with fermionic partners $\psi^M(\sigma,\tau)$. In this case, the action of the bosonic string is modified by including the standard Dirac action. Therefore, the supersymmetric string action is

$$S_{R-NS} = -\frac{1}{2\pi}\int d^2\sigma\ (\partial_\alpha X_M \partial^\alpha X^M + \overline{\psi}^M \rho^\alpha \partial_\alpha \psi_M) \tag{27}$$

where α is a world-sheet index and ρ^α are 2×2 matrices which obey the Dirac algebra. We shall use the representation given by

$$\rho^0 \doteq \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \quad \rho^1 \doteq \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \tag{28}$$

The two-dimensional spinor ψ^M has two Weyl components according to

$$\psi^M = \begin{pmatrix} \psi^M_- \\ \psi^M_+ \end{pmatrix}. \tag{29}$$

[9] There exists other formalisms as the Green-Schwarz which consider supersymmetry on the space-time, and recently and developed by Berkovits, the pure spinor formalism.

In the same way as the bosonic action, using the ligth-cone coordinates the fermionic part of the string action is rewriten as

$$S_f = \frac{i}{\pi} \int d^2\sigma \ (\psi_-^M \partial_+ \psi_{M-} + \psi_+^M \partial_- \psi_{M-}). \tag{30}$$

The whole action is invariant under supersymmetric transformations of the form

$$\begin{aligned} \delta X^M &= i\left(\epsilon_+ \psi_-^M - \epsilon_- \psi^M\right), \\ \delta \psi_-^M &= -2\partial_- X^M \epsilon_+ \\ \delta \psi_+^M &= 2\partial_+ X^M \epsilon_- \end{aligned} \tag{31}$$

where $\epsilon_\pm$ are infinitesimal supersymmetric parameters corresponding to constant two-dimensional Majorana spinors.

3.1. Equations of Motion and Boundary Conditions

The equations of motion with respect to X^i and ψ^i (notice that there is again just $(D-2)$ degrees of freedom) are given respectively by the wave equation in (10) and the Dirac equations

$$\partial_\pm \psi_\mp^M = 0. \tag{32}$$

The corresponding boundary terms for the fermionic part is given by[10]

$$\delta S = -T \int d\tau \left(\psi_+^M \delta\psi_{M+} - \psi_-^M \delta\psi_{M-}\right)|_{\sigma=\pi} - \left(\psi_+^M \delta\psi_{M+} - \psi_-^M \delta\psi_{M-}\right)|_{\sigma=0}, \tag{33}$$

which vanishes in several different ways. Each way establishes the presence of different uncorrelated sectors of the theory as we shall see shortly.

3.1.1. Boundary Conditions for the Open String Let us consider an open string, i.e., a string which endpoints are not coincident, and let us analyze the boundary conditions on the fermionic component of the action. Being an open string, both boundary terms in the variation of the action, must vanish separately. This is achieved if

$$\psi_+^M = \pm \psi_-^M, \tag{34}$$

wich just reflects the very well known fact that fermions need a double rotation to recover their initial sign[11]. Let us say that for two of the endpoints at $\sigma = 0$, both Weyl spinors are coincident., i.e., that $\psi_+^M \mid_{\sigma=0} = \psi_-^M \mid_{\sigma=0}$. For the other endpoint there are two possible cases.

Periodic boundary conditions, $\psi_+^M \mid_{\sigma=\pi} = \psi_-^M \mid_{\sigma=\pi}$, are commonly called *Ramond (R) boundary conditions.* A solution satisfying this condition is said to belong to the R-sector, and it reads

$$\begin{aligned} \psi_-^M(\sigma,\tau) &= \frac{1}{\sqrt{2}} \sum_{n\in\mathbf{Z}} d_n^M e^{-in\sigma^-}, \\ \psi_+^M(\sigma,\tau) &= \frac{1}{\sqrt{2}} \sum_{n\in\mathbf{Z}} d_n^M e^{-in\sigma^+}. \end{aligned} \tag{35}$$

[10] Boundary conditions for the bosonic part are the same as in the bosonic string. This means that Dp-branes are also present in the superstring theory.

[11] Formally speaking, two-dimensional Weyl fermions are representations of the Lie group $SU(2)$, a double covering of the rotational group $SO(3)$.

The Majorana conditions requires these expansions to be real and hence $d^M_{-n} = d^{M\dagger}_n$. Notice that the index runs over integer numbers.

Anti-periodic boundary conditions, $\psi^M_+ \mid_{\sigma=\pi} = -\psi^M_- \mid_{\sigma=\pi}$, are referred as *Neveu-Schwarz (NS)* boundary conditions. The corresponding solutions belong to the NS-sector, and are of the form

$$\begin{aligned} \psi^M_-(\sigma,\tau) &= \frac{1}{2}\sum_{r\in\mathbf{Z}+\frac{1}{2}} d^M_r e^{-ir\sigma^-}, \\ \psi^M_+(\sigma,\tau) &= \frac{1}{2}\sum_{r\in\mathbf{Z}+\frac{1}{2}} d^M_r e^{-ir\sigma^+}. \end{aligned} \tag{36}$$

Therefore, the supersymmetric open string solutions are separated into two different sectors. We shall see that space-time bosons are constructed in the NS-sector, while space-time fermions are constructed in the R-sector.

3.1.2. Boundary Conditions for the Closed String. As in the bosonic string, the closed superstring solution can be represented in terms of left and right movers. On each of them is possible to impose periodicity or anti-periodicity boundary conditions, $\psi_\pm(\sigma) = \pm\, \psi_\pm(\sigma+\pi)$. This establishes four ways to determine a solution of the equations of motion. For instance, if both left and right movers satisfy NS boundary conditions, we say that the corresponding solution belongs to the Neveu-Schwartz-Neveu-Schwartz (NS-NS) sector. Hence, the four sectors are NS-NS, NS-R, R-NS and R-R. We shall see that bosonic fields arise from the NS-NS and R-R sector, while fermions belong to the R-NS and NS-R sectors. The solutions are denoted as

$$\psi^M_-(\sigma,\tau) = \frac{1}{2}\sum_{n\in\mathbf{Z}} d^M_n e^{-2in\sigma^-}, \quad \psi^M_-(\sigma,\tau) = \frac{1}{2}\sum_{r\in\mathbf{Z}+\frac{1}{2}} b^M_r e^{-2ir\sigma^-}, \tag{37}$$

for the right movers with R and NS boundary conditions respectively, and

$$\psi^M_+(\sigma,\tau) = \frac{1}{2}\sum_{n\in\mathbf{Z}} \tilde{d}^M_n e^{-2in\sigma^-}, \quad \psi^M_+(\sigma,\tau) = \frac{1}{2}\sum_{r\in\mathbf{Z}+\frac{1}{2}} \tilde{b}^M_r e^{-2ir\sigma^-}, \tag{38}$$

for the left movers. The specific fields arising as quantum modes of these solutions, will depend on which type of spinor we consider in the the supersymmetric action. There are two possibilities: chiral or non-chiral. So, there are at least two different superstring theories consisting on closed strings.

3.2. Canonical Quantization of the Supersymmetric String

As in the bosonic string case, we shall take a short-way and we will proceed to quantize the superstring by using the canonical quantization procedure, which means that Fourier modes are promoted to be operators satisfying some algebra. For the bosonic modes, the commutation relations are the same as in the bosonic string. For the fermionic modes we have that

$$\{b^M_r, b^N_s\} = \eta^{MN}\delta_{r+s,0} \quad \{d^M_m, d^N_n\} = \eta^{MN}\delta_{m+n,0}, \tag{39}$$

and similar expressions for the left movers. The reader should observe that such relations are precisely the algebra of raising and lowering operators in a harmonic oscillator satisfying the fermi statistics. Therefore, we can assure that such modes act as creator and annihilator operators. By acting them on a suitable defined background state, we can describe the quantized string vibrations. Let us then define the background state $\mid \Omega\rangle_R$ for the right movers. For $n \geq 0$, $\mid \Omega\rangle_R$ is annihilated by the operators α_n^M and d_n^M in the R sector, while for the NS-sector, the annihilation comes from operator b_r^M, for $r > 0$. Similarly, for the left movers, we have that for the R sector, $\tilde{\alpha}_n^M \mid 0\rangle_L = \tilde{d}_n^M \mid 0\rangle_L = 0$, while for the NS-sector, $\tilde{b}_r^M \mid \Omega\rangle_L = 0$, for positive indices.

Therefore, excited states of the closed string are constructed by acting negative modes operators on the ground state $\mid \Omega\rangle_R \otimes \mid \Omega\rangle_L$. It turns out that the ground state for left or right movers is degenerated in the R-sector and unique in the NS-sector. In order to construct the respective excited states for the closed string, let us start by analyzing the construction on some left or right movers, which is equivalent to study the spectrum of the open string.

In the NS sector, the ground state is unique, since all rising operators increment its mass. This follows, as in the bosonic case, from the energy momentum tensor. Classically, the energy-momentum tensor nonzero components read

$$
\begin{aligned}
T_{++} &= \partial_+ X_M \partial_+ X^M + \frac{1}{2}\psi_+^M \partial_+ \psi_{+M}, \\
T_{--} &= \partial_- X_M \partial_- X^M + \frac{1}{2}\psi_-^M \partial_- \psi_{-M},
\end{aligned} \tag{40}
$$

which vanishes by the super Weyl symmetry. Hence, by substituting the solutions of the equations of motion, these energy-momentum tensor components can be written in terms of super Virasoro generators L_m and G_r satisfying some algebra. The generators L_m, with m an integer, are given by[12]

$$
L_m = \frac{1}{2}\sum_m \alpha_{m-n}\cdot\alpha_m + \frac{1}{4}\sum_r (2r-m)\psi_{m-r}\cdot\psi_r + a\delta_{m,0}, \tag{41}
$$

where the product of modes operators are already ordered. The constant a is, as in the bosonic case, a measure of the zero point energy. For the R-sector, fermionic fields contribute with a same quantity than bosons, but with an opposite sign, i.e., $a = a_b + a_f = -\frac{1}{12}(D-2)(1-1) = 0$.

For the NS-sector, the situation is different. Being the modes labeled by fractional indices, the zero point energy contribution reads $a_f = -\frac{1}{48}(D-2)$, from which the total contribution to the zero point energy is $-1/2$.

Therefore, the mass is computed from the zero mode generator L_0, which in the NS-sector, is given by

$$
\alpha' M^2 = \sum_{n\in\mathbf{Z}} \alpha_{-n}\cdot\alpha_n + \sum_{r\in\mathbf{Z}+1/2} r b_{-r}\cdot b_r - \frac{1}{2}, \tag{42}
$$

Notice that a generic excited state is constructed by arbitrary values of N_B and N_F and that since all the oscillators transform as space-time vectors then all excited sates are bosons. In the R sector the ground state is degenerated since the operators d_0^M can act without changing the mass, which is given by

$$
\alpha' M^2 = \sum_{n\in\mathbf{Z}} \alpha_{-n}\cdot\alpha_n + \sum_{n\in\mathbf{Z}} n d_{-n}\cdot d_n. \tag{43}
$$

[12] We encourage the interested reader to work out the details behind these calculations following standard text books as the ones shown in the reference list.

This follows from the fact that zero modes d_0 satisfy the algebra $\{d_0^M, d_0^N\} = \eta^{MN}$ yielding the non-existence of a unique solution for $d_0^M \mid \Omega\rangle = 0$. Therefore, the set of ground states in the R sector must transform as a representation of Dirac algebra,

$$d_0^M \mid a\rangle = \frac{1}{\sqrt{2}}\Gamma_{ab}^M \mid b\rangle, \tag{44}$$

where Γ is a Dirac matrix. This implies that the R-sector ground state is actually a space-time fermion.

3.3. The Superstring Spectrum

For the NS sector we have seen that there is a negative-quadratic-mass state which is identified as the tachyon. Consistency requires the tachyon to be projected out. This can be achieved if we can project some states out, and the remaining states are self-contained, i.e., that by the action of any operator, the resulting state is contained among that set of states. It turns out that such projection exists and it is called the *the GSO projection* (Gliozzi, Scherk and Olive). Basically the action of the GSO projector on the NS and R sector is to take out a half of the states arising at different energy values of the string.

Being more explicit, the action of the GSO projector on NS states is given by $(-1)^F$, with F being the world-sheet fermion number. Negative states under the GSO projector are removed. Then, the tachyonic state is projected out (the background state has fermion number $F = 1$) and the first state which survives GSO projection is $\psi_{1/2}^M|\Omega\rangle$, with $M = 1, \cdots, 8$. The coordinates $0, 9$ represent non-physical degrees of freedom[13]. It is possible to show that this state is a massless gauge boson transforming in the adjoint of $U(1)$ gauge group, and it is expressed as the vectorial representation $\mathbf{8_V}$ of the little Lorentz group $SO(8)$.

For the R sector, the energy of ground state is zero, implying there is a degeneration in the energy levels on states given by the zero mode ψ_0^M. In the light-cone gauge (or after considering the effect of ghosts) we have eight zero modes which can be used to construct 4 raising and 4 lowering operators by a linear combination of two zero modes. Applying the raising operator to a bosonic vacuum $|\Omega\rangle$ we generate $2^4 = 16$ states.

Now, on the one hand, there are at least 16 supersymmetry generators, corresponding to the Weyl components of the full spinor, which has 32 degrees of freedom. This spinor is a Majorana-Weyl spinor, which splits it into two real spinors with opposite chirality, i.e., $\mathbf{32} = \mathbf{16} \oplus \mathbf{16'}$. Then, under spinorial representations of the Lorentz group $SO(1,9)$, we have the following decomposition

$$\begin{aligned} SO(1,9) \quad &\longrightarrow \quad SO(1,1) \times SO(8) \\ \mathbf{16} \quad &\longrightarrow \quad (\frac{\mathbf{1}}{\mathbf{2}}, \mathbf{8_s}) \oplus (-\frac{\mathbf{1}}{\mathbf{2}}, \mathbf{8_c})\,. \end{aligned} \tag{45}$$

On the other hand, there is one constraint (Dirac equation) that threw out representations with quantum number $-\frac{1}{2}$ of $SO(1,1)$. That is why we kept just 16 states that correspond to those constructed by raising operators described above.

[13] Remember we are using the light-cone gauge. In the context of covariant quantization, this is an effect of ghosts on the string dynamics.

Table 4. Open superstring spectrum

Sector	State	Level	M^2/α	$SO(8)$ rep.	Field	GSO
NS	$\mid\Omega\rangle_{NS}$	$N_b = N_f = 0$	$-1/2$	-	Tachyon	$-$
NS	$\psi^M_{-1}\mid\Omega\rangle_{NS}$	$N_b = 0,\, N_f = 1$	0	$\mathbf{8_v}$	Gauge boson	$+$
R	$\mid\Omega\rangle_R$	$N_b = N_f = 0$	0	$\mathbf{8_c}$	Gauginos	$-$
R	$\mid\Omega\rangle_R$	$N_b = N_f = 0$	0	$\mathbf{8_s}$	Gauginos	$+$

Table 5. Type IIA superstring spectrum

Sector	State	$SO(8)$ rep.	Fields
NS-NS	$\psi^M_{-1}\tilde{\psi}^N_{-1}\mid\Omega\rangle_L\otimes\mid\Omega\rangle_R$	$\mathbf{8_v}\otimes\mathbf{8_v} = \mathbf{1}\oplus\mathbf{26}\oplus\mathbf{35}$	G_{MN}, B_{MN}, ϕ
R-R	$\mid+\rangle_L\otimes\mid-\rangle_R$	$\mathbf{8_s}\otimes\mathbf{8_c} = \mathbf{8_v}\oplus\mathbf{56}_t$	$C_{(1)}$,$C_{(3)}$
NS-R	$\psi^M_{-1}\mid\Omega\rangle_L\otimes\mid-\rangle$	$\mathbf{8_v}\otimes\mathbf{8_c} = \mathbf{8_s}\oplus\mathbf{56_c}$	λ^α, ψ^α_M
R-NS	$\mid+\rangle_L\ \otimes\tilde{\psi}^M_{-1}\mid\Omega\rangle_R$	$\mathbf{8_s}\otimes\mathbf{8_v} = \mathbf{8_c}\oplus\mathbf{56_s}$	$\lambda^{\dot\alpha}$, $\psi^{\dot\alpha}_M$

Table 6. Type IIB superstring spectrum

Sector	State	$SO(8)$ rep.	Fields
NS-NS	$\psi^M_{-1}\tilde{\psi}^N_{-1}\mid\Omega\rangle_L\otimes\mid\Omega\rangle_R$	$\mathbf{8_v}\otimes\mathbf{8_v} = \mathbf{1}\oplus\mathbf{26}\oplus\mathbf{35}$	G_{MN}, B_{MN}, ϕ
R-R	$\mid+\rangle_L\otimes\mid+\rangle_R$	$\mathbf{8_s}\otimes\mathbf{8_s} = \mathbf{1}\oplus\mathbf{26}\oplus\mathbf{35}$	$C_{(0)}$,$C_{(2)}$
NS-R	$\psi^M_{-1}\mid\Omega\rangle_L\otimes\mid+\rangle$	$\mathbf{8_v}\otimes\mathbf{8_s} = \mathbf{8_s}\oplus\mathbf{56_s}$	λ^α, ψ^α_M
R-NS	$\mid+\rangle_L\ \otimes\tilde{\psi}^M_{-1}\mid\Omega\rangle_R$	same as NS-R	

Therefore, GSO projection in RR sector is basically the application of the chiral matrix $\overline{\Gamma} = i\Gamma^0\Gamma^1\cdots\Gamma^9$ to the spinor states. Because the states in the **16** of $SO(8)$ are spinors, the action of $\overline{\Gamma}$ split up them into two Weyl fermions, which are representations of $SO(8)$ given by $\mathbf{8_s}$ and $\mathbf{8_c}$ with different chirality. After GSO projection, the R-sector contains 16 spinorial states in the $\mathbf{8_s}$ of $SO(8)$ that represents the super-partners of $\mathbf{8_v}$ in the NS sector (gauginos). The spectrum of the open string is shown in Table 4.

3.3.1. Closed superstring spectrum Once the spectrum of the open string has been computed, it is straightforward to compute the closed string spectrum. The resulting fields are constructed by the tensorial product of the corresponding states for the left and right movers. However, there are two different ways to do it.

The theory is called *type IIA* if left and right-moving R-sector ground states are chosen to have the opposite chirality (which reflects in the fact that the two R-sectors have opposite parity. In the *type IIB* left and right-moving sector ground states have the same chirality. These two consistent ways to construct a superstring theory spectrum, are summarized in Tables 5 and 6.

3.3.2. The Type IIA Superstring spectrum As mentioned, the representation of the complete states is obtained by tensoring the representation of the left and right movers. The NS-NS sector contains a symmetric tensor of two index corresponding to a graviton (G_{MN}), an antisymmetric tensor or 2-form (B_{MN}) and a dilaton (Φ). The R-R sector contains a set of completely antisymmetric tensors (p-forms). In particular a 1-form(C_M) and a 3-form (C_3). It is sometimes convenient to introduce the Hodge duals of these, which are a 5-form (C_5) and a 7-form (C_7). Finally, it is also useful to introduce a 9-form C_9, with no dynamics. The space-time fermions arise from the R-NS and NS-R sectors which contain the gravitino (56 states) and a dilatino (8 states) with opposite chirality. The R-NS and NS-R sectors contain fermions, which are a spin 3/2 gravitino and a spin 1/2 dilatino, each of these fermions with opposite chirality.

3.3.3. The Type IIB Superstring spectrum For this case, we choose left and right movers in R-sector to have the same chirality (in the $\mathbf{8_s}$ and $\mathbf{8_s}$ of $SO(8)$). The NS-NS sector contains the same fields as in the Type IIA case. The R-NS and NS-R sectors contain fermions with the same chirality. Bassically they are two copies of fermions in the NS-R sector in type IIA. The RR sector contains a 0-form C_0 (scalar), a 2-form C_2 and a 4-form $C_4^\dagger$ with a self-dual field strength. It is sometimes convenient to introduce the Hodge duals of these, which are a 6-form B_6, and 8-form C_8. Finally, it is also useful to introduce a 10-form C_{10}, which does not have any propagating degrees of freedom, since it has no space-time kinetic term.

The Type IIA and IIB superstring theories have as low energy description (when just massless modes are considered) the Types IIA and IIB supergravity (SUGRA) versions. In other words, II refers to supersymmetry $\mathcal{N} = 2$ and we have two gravitinos in each supergravity multiplet with $D = 10$ [14].

4. Non-perturbative aspects of Superstring Theory

Until now we have described the perturbative sector of type II superstring theories[15] in a flat space-time. However, it is interesting and important to compactify some of the spatial extra dimensions that the theory predicts. The first step is wrapping one such dimension on a circle. Here we find a big difference by comparing strings to quantum fields. While for fields on a compact dimension we get only Kaluza-Klein (KK) modes arising by periodicity, strings can also be wrapped on compact coordinates. In this form, for the string there are two values the energy depends on. One is the energy required to stretch the string in a space with less volume because of the compactification. Essentially these are the KK modes; smaller the radius R of the compact dimension, bigger the energy required to stretch the string. KK modes goes as $\frac{n}{R}$ with $n \in \mathbf{Z}$. When the string is wrapped on the compact dimension, then we need energy to wrap it, i.e, the string has a strength tension in order to be wrapped around the circle. Then, smaller the radius of the compact dimension, smaller the energy needed to wrap it. The number of times a string winds a compact dimension is called the winding number, and the energy is

[14] It is surprising that at SUGRA level, space-time anomalies due to gauge and gravitational fields, coupled to chiral fermions, and cancel each other exactly. This is reflected in string theory (or viceversa) as the absence of tadpole diagramas at the fermionic sector. However, if we couple IIB theory (strings or SUGRA) to super Yang-Mills (SYM) gauge bosons, there is an extra term in the anomaly, which makes the theory inconsistent. On the string theory side, this means we are adding a D9-brane to the background which introduces an open string sector and breaks supersymmetry. This theory is not consistent and one can see that by including open strings and SYM gauge theories, more details must be considered. In Type IIA theory such anomalies are not present because the theory is not chiral.

[15] There are other three types, we shall briefly mention at the end of this note.

proportional to mR with $m \in \mathbf{Z}$ being the winding number.

The mass of a string with KK modes and winding number is given by

$$M^2 = \frac{n^2}{R^2} + \frac{m^2 R^2}{\alpha'^2} + \frac{2}{\alpha'}(N_B + \tilde{N_B} r N_F + r\tilde{N_F} - E_0), \tag{46}$$

with $N_B, \tilde{N_B}$ being the bosonic energy number level for the right and left movers respectively, $N_F, \tilde{N_F}$ the fermionic ones and E_0 the ground energy level for each sector.

Now, consider the symmetric two-order tensor G_{MN} from the NS-NS sector. Upon compactification on a circle, one obtains a nine-dimensional two-tensor G_{ij}, a gauge field $A_i^L = G_{i9}$ and a scalar $\phi = G_{99}$. In the same way, the two-form B_{MN} gives rise to a nine-dimensional two-form B_{ij} and a gauge field $A_i^R = B_{i9}$. Stringy states carrying KK modes or winding numbers are charged under such gauge fields. So, by compactification there is an extra gauge symmetry $U(1) \times U(1)$ under which massive states are charged. In other words, there are states that are charged under NS-NS fields. However, at perturbative level, there is not evidence that there exist massive objects charged under RR fields. This is in some sense paradoxical because compactification also affects RR fields. In fact, the p-forms are reduced to $(p-1)$- and p-forms on a nine-dimensional space-time, so we expect that $(p-1)$-forms must be the fields under which some objects are charged. For the NS-R and R-NS sector, under compactification into a circle, we obtain a nine-dimensional theory in which fermionic fields have no chirality (there is no such chiral Dirac matrix $\overline{\Gamma}$). Gravitinos and gauginos in Type IIA theory are mapped according to

$$\begin{array}{llcl}
\text{R-NS:} & \psi^{\alpha}_{\mu}, \lambda^{\alpha} & \longrightarrow & \lambda^{\alpha'}, \lambda^{\beta}, \psi^{\alpha'}_{i} \\
 & \mathbf{8_s} \otimes \mathbf{8_v} = \mathbf{8_c} \oplus \mathbf{56_s} & \longrightarrow & \mathbf{8} + \mathbf{8} + \mathbf{48} \\
\text{NS-R:} & \psi^{\dot\alpha}_{\mu}, \lambda^{\dot\alpha} & \longrightarrow & \lambda^{\alpha'}, \lambda^{\beta}, \psi^{\alpha'}_{i} \\
 & \mathbf{8_c} \otimes \mathbf{8_v} = \mathbf{8_s} \oplus \mathbf{56_s} & \longrightarrow & \mathbf{8} + \mathbf{8} + \mathbf{48}
\end{array} \tag{47}$$

where $i = 0, \cdots, 8$. Notice the two extra gravitinos $\mathbf{8}$ are the superpartners of the gauge fields in the NS-NS sector, A_i^R and A_i^L and of the $(p-1)$-forms in the RR-sector. For Type IIB we have the same fields in the nine-dimensional spacetime.

4.1. T-duality and D-branes

Consider the mass expression given by Eq. (46). We see that mass of states, i.e. their energy, is unchanged under the identification

$$\begin{array}{ccc}
m & \longleftrightarrow & n \\
\frac{\alpha'}{R} & \longleftrightarrow & R\,.
\end{array} \tag{48}$$

Then it follows that under such change of parameters, the theory has the same spectrum and it is indistinguishable of the original one. We said that two such theories are dual one to each other and the transformation is called *T-duality.*

This identification also has an effect on the bosonic and fermionic modes. T-duality maps X_L^μ into $-X_L^\mu$ and by superconformal symmetry, the fermionic modes are changed in a similar way, $\psi_L^\mu \to -\psi_L^\mu$. The latter map has a profound and very important consequence in the theory. Mapping the left fermionic mode into its negative one, implies a change in the chirality of the spinors constructed with them. Let us explain it with more detail. A basis of states in the RR sector is given by $|s_1 s_2 s_3 s_4\rangle$ with $s_i = \pm\frac{1}{2}$ for all i values. They give us the 16 states we studied previously. We set the ground state as the element given by $|\frac{1}{2}\frac{1}{2}\frac{1}{2}\frac{1}{2}\rangle$ and then the rest of the basis is generated by the raising operator $S_a^\dagger = \frac{1}{\sqrt{2}}(\psi_0^a + i\psi^{a+1})$ which raises the value of s_i by a half for each i.

When ψ^a is mapped to its negative mode by performing T-duality on the $a+1$-coordinate, $S_a^\dagger$ transforms into S_a, which corresponds to a lowering operator. For instance, take $a = 9$; this means that the state $|s_1 s_2 s_3 - \frac{1}{2}\rangle$ will be projected out by the lowering operator. On the other hand, if initially we have the state $|s_2 s_2 s_3 \frac{1}{2}\rangle$, then under T-duality this state is replaced by $|s_1 s_2 s_3 - \frac{1}{2}\rangle$, i.e, under T-duality, the number of negative or positive one-half factors are interchanged. But this is the definition of the action of GSO projection on RR states. Then T-duality changes the chirality. The most important consequence of this is that Type IIA and Type IIB theories are T-dual.

Under toroidal compactifications (taking many wrapped dimensions on circles) it is possible to elucidate the existence of certain planes in which the endpoints of open strings are fixed. Although open strings has not been considered, we shall see it is possible to include them under some constraints. Take for instance the ninth-coordinate wrapped into a circle, and consider a closed and an open string in the bulk. It is important to point out an open string is not able to wind a circle because always can be unwinded; this is a big difference between closed and open strings. The latter has no winding number.

Now let the radius R tends to zero. What does happen to the strings? At the limit, the open string looks like being fixed by its endpoint, in a eight-dimensional plane. This is the first evidence of such hyperplanes. It turns out that the y are actually the D-branes we have studied previously. For the closed string the story is more surprising: at the limit $R \to 0$, the closed string is flatten, identifying the points under $\sigma \to 2\pi - \sigma$. This gives two fixed points which are identified to the endpoints of an unoriented open string. The next step is asking, what are the T-dual version for these cases? The last one gives rise to the construction of another consistent string theory called *Type I*, which contains closed an open unortiented strings.

Let us perform T-duality for an open string on a compact dimension. Under T-duality we know that $X_L^\mu \to -X_L^\mu$ and then we can define a T-dual bosonic field $\overline{X}^\mu = X_R^\mu - X_L^\mu$. Working on the dual coordinates the Neumann boundary condition, which an open string satisfies, is transformed into a Dirichlet condition $\partial_\tau \overline{X}^\mu = 0$. In consequence the endpoints are fixed on a D-brane. But, is there any constraint to include open strings to Type II theories? To include oriented open strings we require that D9-branes do not exist because their existence implies also the existence of SYM degrees of freedom in the ten-dimensional space-time, and as we know, the coupling between Type II SUGRA and SYM is anomalous, meaning that R tadpoles are not cancelled.

However, D-branes are more than just hyperplanes. Actually, they are also BPS states (Bogomol'nyi-Prasad-Sommerfeld), i.e., states that break down a half of supersymmetry on the bulk, and their mass were determined by their charge under some fields. Because of that, supersymmetry protects them from radiative corrections. Their mass and charge are exact. The relation between mass and charge is given by the supersymmetry algebra in ten dimensions with $\mathcal{N} = 2$. In the same way, it was found that some solitonic objects described in the context of supergravity (generalizations of black-holes) known as p-branes, are the low energy limit of Dp-branes. In this form, Dp-branes are the solitonic objects (with tension $T \sim \frac{1}{g_s}$) that allow us to study the non-perturbative sector in string theory; we are able to study their excited states by the perturbative description in terms of open strings. This is a crucial point in our knowledge of the theory.

As we said, D-branes must be charged by some fields because of the BPS condition $M = |Q|$. These fields are precisely the RR fields. Then the objects we require to be charged under RR fields are Dp-branes. In general, a $(p+1)$-form C_{p+1} couples to a Dp-brane by

$$\mu_p \int_{W_{p+1}} C_{p+1} \ , \tag{49}$$

where μ_p is the charge of the Dp-brane and W_{p+1} is its worldvolume. Then, by the RR spectrum for Type IIA and IIB theories, we can deduce which values of p are allowed in each theory. For Type IIA we have Dp-branes for p even, and for Type IIB, p must be odd.

The picture we should have in mind is that open strings has their endpoints attached to Dp-branes and supersymmetry is broken to one half of the bulk. This means that for Type II theories we have supersymmetry $\mathcal{N} = 1$ on the brane and $\mathcal{N} = 2$ in the bulk. Far away from the brane, we have locally a closed string theory. This can be regarded also as follows: take an open string on the Dp-brane, and consider the process in which the two endpoints are coming forward to the same point on the D-brane worldvolume. When two endpoints are glued into a single point, we have essentially a closed string (the charge at the endpoints cancel each other) and it is said that a Dp-brane emits a closed string into the bulk.

Emission of closed string has a relevant role in D-brane physics. Consider two Dp-branes parallel to each other and interchanging a closed string and take into account the tree level amplitude of such a process. There is a well studied duality between the 1-loop open string amplitude and the tree level closed one. We can consider an open string connecting the two Dp-branes and forming a cylinder (1-loop) or a closed string interchanged by the branes at tree-level amplitude. At the open-sector we see that the amplitude is zero because the theory is supersymmetric, then there is not net force between the branes. But if one return to the closed-sector is easy to see there must be a force due to NS-NS and RR fields. It is concluded that NS-NS contribution cancels the RR one. In other words, a Dp-brane has tension (mass) and then could emit gravitons. Also it is able to emit B-fields and dilatons, and C_{p+1}-forms. All of them contribute to a non-zero amplitude but all at once give a zero net force.

The bosonic part of the action which describes a Dp-brane classically is,

$$S = \int d^p x\, e^{-\phi} \sqrt{det(G_{\mu\nu} + \frac{1}{2\pi\alpha'}(B_{\mu\nu} + F_{\mu\nu}))} \ + \ \int G_{p+2} \wedge *G_{p+2} \ + \ \mu_p \int C_{p+1} \ ,$$

where G_{p+2} is the strength field given by $G_{p+2} = dC_{p+1}$, $G_{\mu\nu}$ and $B_{\mu\nu}$ are the induced metric and B-field on the D-brane and $F_{\mu\nu}$ is the magnetic flux on the Dp-brane (μ and ν run over the Dp-brane coordinates excepting the light-gauge cone ones). The magnetic flux arises as a background field on the worldvolume of the brane. It is the classical version of the strength two-form $F_2 = dA$ whit A being the gauge field obtained in the NS sector of the open string $\psi^{\mu}_{\frac{1}{2}}|0;k\rangle$.

The presence of open strings in Type II theories are due to the existence of D-branes. Under this context there is an enhancement of gauge symmetry given by the NS gauge bosons. Consider N Dp-branes at the same position. For each open string there are N possible states each endpoint has. It can be attached to the 'first' brane, or to the jth-brane with $j = 1, \cdots, N$. The same holds for the other endpoint. Then for stringy states we have to label the states by ij.

These extra non-dynamical degrees of freedom are called *Chan-Paton factors*. It can be seen that they introduce a $U(N)$ symmetry and NS gauge bosons turn out into NS non-abelian gauge bosons which transform in the adjoint representation of $U(N)$. The endpoints are charged by these 1-form gauge fields because there is a natural topological coupling between point-particles and 1-forms. By this, there is an enhancement of gauge symmetry $U(1) \to U(N)$. We say that there is a gauge field in the worldvolume of a Dp-brane and if this field is non-trivial, there is also a field strength given by the two-form F which we refer previously as a magnetic flux.

T-duality also plays an important role in the non-perturbative regime. Under it a Dp-brane turns into a D-brane with a higher or lower dimension depending in which coordinates T-duality is taken. For instance, if we take T-duality on transverse coordinates to the brane, its dimension will grow up according the number of coordinates we have considered. When T-duality is performed on longitudinal coordinates the brane dimension decreases. For example, if we have a Dp-brane on coordinates $0, 1, \cdots, p-1, p$ and we take T-duality transformation on one of these coordinates we obtain a D$(p-1)$-brane. This because the Neumann boundary conditions are transformed into Dirichlet ones.

Returning to our original picture (before taking T-dual transformation) the existence of a magnetic flux establishes an extra coupling on the classical action. Every $(p+1)$-form will we couple to the worldvolume W_p of a Dp-brane. Then if we have a two-form given by the strength gauge field $\mathcal{F}$, this together with a $(p-1)$-form establish a $(p+1)$-form that couples to W_p. This kind of coupling is known as Chern-Simons term and is given generally by

$$\int_{\Sigma} \Sigma_p C_{p+1} \wedge Tr\, e^{\mathcal{F}} \ , \tag{50}$$

Now, if this kind of couplings are allowed then the presence of RR charges related to low-dimensional D-branes is implied when a magnetic flux is turned on over the Dp-brane. Considering just the first non-trivial term in the expansion of $e^{\mathcal{F}}$ we induce a RR charge of a D$(p-2)$-brane. Then we hope to obtain a similar spectrum of open strings attached to D-branes as the obtained by analyzing D-branes with magnetic fluxes.

The spectrum of open strings attached to N coincident Dp-branes is as follows. For the NS-sector the state is $\psi^i_{1/2}\,|0;k\rangle$ (i labels longitudinal D-brane coordinates) which gives us N^2 $(p-1)$-dimensional gauge vectors of $SO(p-1)$ in the adjoint of $U(N)$. Also we have $(8-p)$ scalars $\psi^a_{1/2}\,|0;k\rangle$ of the Lorentz group $SO(p-1)$. In the R-sector, $\mathbf{8_s}$ flips out under $SO(8) \to SO(p-1) \times SO(8-p)$ into suitable representations of the Lorentz group $SO(p-1)$ in the adjoint of $U(N)$. The values of p for which Dp-branes generates a supersymmetric Yang-Mills theory are $p = 9, 5, 4, 2$. For instance, a D9-brane gives a SYM theory $D = 10$ and $\mathcal{N} = 1$. Also, for a D5-brane, we have a SYM theory $D = 6$ and $\mathcal{N} = 2$.

The RR fields of Type IIA theory are differential forms C_{p+1} with p even and for Type IIB p is odd. Denote by G_{p+2} the field strength given by $G_{p+2} = dC_{p+1}$. The field strength is invariant under gauge transformations $C_{p+1} \to C_{p+1} + d\Lambda_p$, that implies a gauge invariant action given by

$$S \sim \int_{\mathcal{M}_{10}} dC_{p+1} \wedge *dC_{p+1} \ + \ \int_{W_p} C_{p+1} \ . \tag{51}$$

The RR charge is given by

$$Q_E \ = \ \int_{S^{8-p}} *dC_{p+1} \ . \tag{52}$$

Also we can built magnetic dual objects to the Dp-brane, which corresponds to an object coupling to a $(7-p)$-form. This object is a D$(6-p)$-brane with magnetic charge given by

$$Q_M = \int_{S^p+2} dC_{p+1} \ . \tag{53}$$

By this, for instance a D5-brane in Type IIB is the magnetic dual to a D1-brane and so on. In the same way the magnetic dual to a string under NS-NS charge is a five-brane namely NS5-brane.

4.1.1. M-theory Together with type II superstring theories, there are three more supersymmetric string theories. The first one, we already have sai a little about it. It is called type I theory, and consists of a ten-dimensional background in which un-oriented open and closed strings are allowed to exist. Since it contains open string, it also have ten-dimensional gauge bosons transforming in the adjoint of $SO(32)$. Essentially this follows from the fact that there is a gravitational and gauge anomalies which cancel each other only for gauge groups with 496 generators. For type I theory, it turns out that we require the presence of the gauge group $SO(32)$. An alternative way to construct type I theory is by means of type IIB. Type I is constructed by adding 32 D9-branes to type IIB and by projecting out all oriented strings. This is the action of another non-perturbative object called *Orientifold*.

Another way to construct consistent supersymmetric theories involves the mixture between bosonic and fermionic strings. Take a closed string and let us say that left-movers are bosonic, and impose supersymmetry on the right movers. "Half of the string" propagates on a ten-dimensional space-time, while the other, being bosonic, propagates on a space-time with 26 dimensions. The apparent contradiction is solved when one realizes that the extra 16 coordinates on the bosonic side behave as gauge degrees of freedom. Once again, the fact that we have chiral gauged fermions, implies the presence of gauge anomalies, which cancel for gauge groups $SO(32)$ and $E_8 \times E_8$. These are precisely the extra pair of superstring theories we have not mentioned. They are called Heterotic $SO(32)$ (Het($SO(32)$)) and Heterotic $E_8 \times E_8$ (Het($E_8 \times E_8$)).

Five different and consistent string theories seem to much for an unified description of nature. This issue was softened once people realized that these five string theories and actually related by a set of mappings, called dualities. We have already discussed one of the, T-duality. The other one is called S-duality, and essentially relates a weak coupled string theory with a strong coupled one. Nowadays, we believed that the five different theories are just different limits on the moduli space of a bigger 11-dimensional theory, so far called *M-theory*. No much is known about it, but since the theory is still under progress, we expect to develop more powerful techniques which allow us to explore the fundamentals of M-theory.

5. Recent developments

String theory has become a huge framework on which different topics can be addressed. Enumerating all of them is a task far beyond the scope of this course. However, we shall mention some of them according to our interests. First of all, big progress has been performed in the so called string phenomenology branch. There is a lot of people working on effective four-dimensional theories constructed from string theory. See [10] (and references therein) for a more extensive treatment. One of the main topics covered includes the construction of Standard-Model-like scenarios and their corresponding supersymmetric extensions from Type II superstrings and from

Heterotic strings as well.

Historically, people used Type I and Heterotic theories to construct phenomenologically viable models. The main reason for that, was that such theories already had gauge groups, necessary to incorporate the Standard Model gauge groups. This vision changed once D-branes were introduced in Type II theories.

The construction of vacua from string scenarios, established an starting point for some very nice researches. First of all, we wanted to know the conditions, physical or mathematical, that the extra six dimensions must satisfy in order to obtain a four-dimensional effective physics, close enough to what we daily observe. Soon, people realized that the extra six dimensions must be enrolled in a six-dimensional mathematical object known as *Calabi-Yau manifold*. Some properties of the field content in the effective theory, as the number of families, was related to topological properties of such manifold. In the last decade, there was a huge progress towards the construction of realistic vacua, as the flux compactification and the generalization of Calabi-Yau manifolds, a topic that has enriched the develop of new mathematics [13]. People have also explored the possibility that our universe is just one choice among thousands of options in the so called String Landscape.

Other applications of String theory involve the construction of cosmological models, the AdS/CFT correspondence (a correspondence between gauge theories and gravity which has become a huge area of research) and its implications on physical predictions on the quark-gluon plasma. Also, new methods of calculations on amplitude scatterings by the use of twistors have been developed and the always fascinating study of black holes.

String theory is doubtless a very rich scenario in which many open questions of theoretical physics might be answered. The theory is still under construction and we hope that in the near future we realize whether or not, string theory plays a role in our world.

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