Fock-like representations for algebraically interacting paraparticles

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(1). The **Relative Parabose Set algebra** (in a single parabosonic and a single parafermionic degree of freedom): $P_{BF}^{(1,1)}$ with generators b^+, b^-, f^+, f^- and relations

"interaction (mixed) relations"

$$\big[\{b^+,b^+\},f^-\big]=0,\quad \big[\{b^-,b^-\},f^-\big]=0,\quad \big[\{f^-,b^+\},b^-\big]=-2f^-,\quad \big[\{b^-,f^-\},b^+\big]=2f^-,\quad \big[\{b^-,b^+\},f^+\big]=0,\quad \big[\{f^+,b^-\},b^+\big]=2f^+,$$

$$\big[[f^+,f^-],b^-\big]=0,\quad \big[\{b^+,b^-\},f^-\big]=0,\quad \{\{b^-,f^+\},f^-\}=2b^-,\quad \{\{f^-,b^-\},f^+\}=2b^-,\quad \big[[f^-,f^+],b^+\big]=0,\quad \big[\{b^+,f^+\},b^-\big]=-2f^+,$$

$$\big[\{b^+,b^+\},f^+\big]=0=\big[\{b^-,b^-\},f^+\big],\quad \big[\{f^-,b^-\},b^-\big]=0=\big[\{f^-,b^+\},b^+\big],\quad \big[\{f^+,b^+\},b^+\big]=0=\big[\{f^+,b^-\},b^-\big],$$

$$\{\{b^-,f^-\},f^-\}=0=\{\{b^-,f^+\},f^+\},\quad \{\{b^+,f^+\},f^+\}=0=\{\{b^+,f^-\},f^-\},\quad \{\{b^+,f^-\},f^+\}=2b^+=\{\{f^+,b^+\},f^-\}$$

"free (unmixed) relations"

$$\big[b^-,\{b^+,b^-\}\big]=2b^-,\quad \big[b^-,\{b^-,b^-\}\big]=0,\quad \big[b^+,\{b^+,b^+\}\big]=0,\quad \big[b^-,\{b^+,b^+\}\big]=4b^+,\quad \big[f^-,[f^+,f^-]\big]=2f^-,\quad \big[f^+,[f^-,f^+]\big]=2f^+,$$

$$\big[b^+,\{b^-,b^-\}\big]=-4b^-,\quad \big[b^+,\{b^-,b^+\}\big]=-2b^+,$$

(2). Earlier results:

Conjecture *(Greenberg-Messiah, Phys. Rev., 1965)*:

*If we consider representations of $P_{BF}^{(1,1)}$, satisfying the **adjointness conditions** $(b^-)^\dagger=b^+$ and $(f^-)^\dagger=f^+$, on a **complex, infinite dimensional, Euclidean** [a] **space** possessing **a unique vacuum vector** $|0\rangle$ satisfying $b^-|0\rangle=f^-|0\rangle=0$, then the following conditions (p may be an **arbitrary positive integer**)*

$$b^-b^+|0\rangle=f^-f^+|0\rangle=p|0\rangle \qquad b^-f^+|0\rangle=f^-b^+|0\rangle=0$$

*single out an **irred. repr., unique up to unitary equivalence**.*

[a]Euclidean or pre-Hilbert space (inner product space, but not necessarily complete)

Carrier space structure for the $P_{BF}^{(1,1)}$-module (Yang-Jing Mod. Phys. Lett. A, 2001)

Fock space: $\rightsquigarrow \bigoplus_{n=0}^{p}\bigoplus_{m=0}^{\infty}\mathcal{V}_{m,n}$

$\mathcal{V}_{m,n}\rightsquigarrow$ <u>**2 − dim.**</u>
except : $\mathcal{V}_{0,n},\mathcal{V}_{m,0},\mathcal{V}_{m,p}\rightsquigarrow$ <u>**1 − dim.**</u>

■ If $1\le m$ and $1\le n<p$ $\qquad \mathcal{V}_{m,n}$ is spanned by

$$\left|\begin{matrix} m_1,m_2,\dots,m_l\\ n_0,n_1,n_2,\dots,n_l\end{matrix}\right\rangle \equiv (f^+)^{n_0}(b^+)^{m_1}(f^+)^{n_1}(b^+)^{m_2}(f^+)^{n_2}\dots(b^+)^{m_l}(f^+)^{n_l}|0\rangle$$

where $m_1+m_2+\dots+m_l=m$, $n_0+n_1+n_2+\dots+n_l=n$ and $m_i\ge1$ (for $i=1,2,\dots,l$), $n_i\ge1$ (for $i=1,2,\dots,l-1$), $n_0,n_l\ge0$.

The corresponding subspace $\mathcal{V}_{m,n}$ has a basis consisting of the two vectors

$$|m,n,\alpha\rangle\equiv(f^+)^n(b^+)^m|0\rangle,\qquad |m,n,\beta\rangle\equiv(f^+)^{(n-1)}(b^+)^{(m-1)}R^+|0\rangle$$

where: $R^\eta=\frac{1}{2}\{b^\eta,f^\eta\}$ for $\eta=\pm$.

In other words : $\left|\begin{matrix} m_1,m_2,\dots,m_l\\ n_0,n_1,n_2,\dots,n_l\end{matrix}\right\rangle=c_1|m,n,\alpha\rangle+c_2|m,n,\beta\rangle$

■ If $m=0$ or $n=0,p$

$$|0,n,\beta\rangle=|m,0,\beta\rangle=0 \textit{ and : } |m,p,\beta\rangle=\frac{1}{p}|m,p,\alpha\rangle$$

$\mathcal{V}_{0,n},\mathcal{V}_{m,0},\mathcal{V}_{m,p}\rightsquigarrow$ 1-dim. with **bases the single vectors** $|0,n,\alpha\rangle$, $|m,0,\alpha\rangle$, $|m,p,\alpha\rangle$

■ If $n\ge p+1\rightsquigarrow$ all basis vectors vanish

(3). Our work:

3a. Action of the generators ($0\le m,\ 0\le n\le p$):

$$b^-\cdot|m,n,\alpha\rangle=\begin{cases}(-1)^{n+2}m|m-1,n,\alpha\rangle+2(-1)^{n+1}nm|m-1,n,\beta\rangle, & \underline{m: even}\\ (-1)^{n+1}\big(2n-m-(p-1)\big)|m-1,n,\alpha\rangle+\\ \qquad +2(-1)^{n+1}n(m-1)|m-1,n,\beta\rangle, & \underline{m: odd}\end{cases}$$

$$b^-\cdot|m,n,\beta\rangle=\begin{cases}-(-1)^n|m-1,n,\alpha\rangle+(-1)^n\big(2n-m-p\big)|m-1,n,\beta\rangle, & \underline{m: even}\\ -(-1)^n|m-1,n,\alpha\rangle-(-1)^n(m-1)|m-1,n,\beta\rangle, & \underline{m: odd}\end{cases}$$

$$f^-\cdot|m,n,\alpha\rangle=n(p+1-n)|m,n-1,\alpha\rangle$$
$$f^-\cdot|m,n,\beta\rangle=|m,n-1,\alpha\rangle+(n-1)(p-n)|m,n-1,\beta\rangle$$

$$b^+\cdot|m,n,\alpha\rangle=(-1)^n|m+1,n,\alpha\rangle+(-1)^{n-1}2n|m+1,n,\beta\rangle$$
$$b^+\cdot|m,n,\beta\rangle=(-1)^{n-1}|m+1,n,\beta\rangle$$

$$f^+\cdot|m,n,\alpha\rangle=\begin{cases}|m,n+1,\alpha\rangle, & \underline{\text{if } n\le p-1}\\ 0, & \underline{\text{if } n\ge p}\end{cases}\qquad f^+\cdot|m,n,\beta\rangle=\begin{cases}|m,n+1,\beta\rangle, & \underline{\text{if } n\le p-1}\\ 0, & \underline{\text{if } n\ge p}\end{cases}$$

Fock-like representations of $P_{BF}^{(1,1)}$

In other words **:**

for any positive integer p, there is an irreducible representation of $P_{BF}^{(1,1)}$, uniquely specified (up to unitary equivalence) by $b^-|0\rangle=f^-|0\rangle=0$ together with

$$b^-b^+|0\rangle=f^-f^+|0\rangle=p|0\rangle \qquad b^-f^+|0\rangle=f^-b^+|0\rangle=0$$

We emph. the fact that **each one of these representations is characterized by the positive integer p**, in other words **the value of p is part of the data which uniquely specifies the representation**

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3b. Irreducibility of the Fock-like space

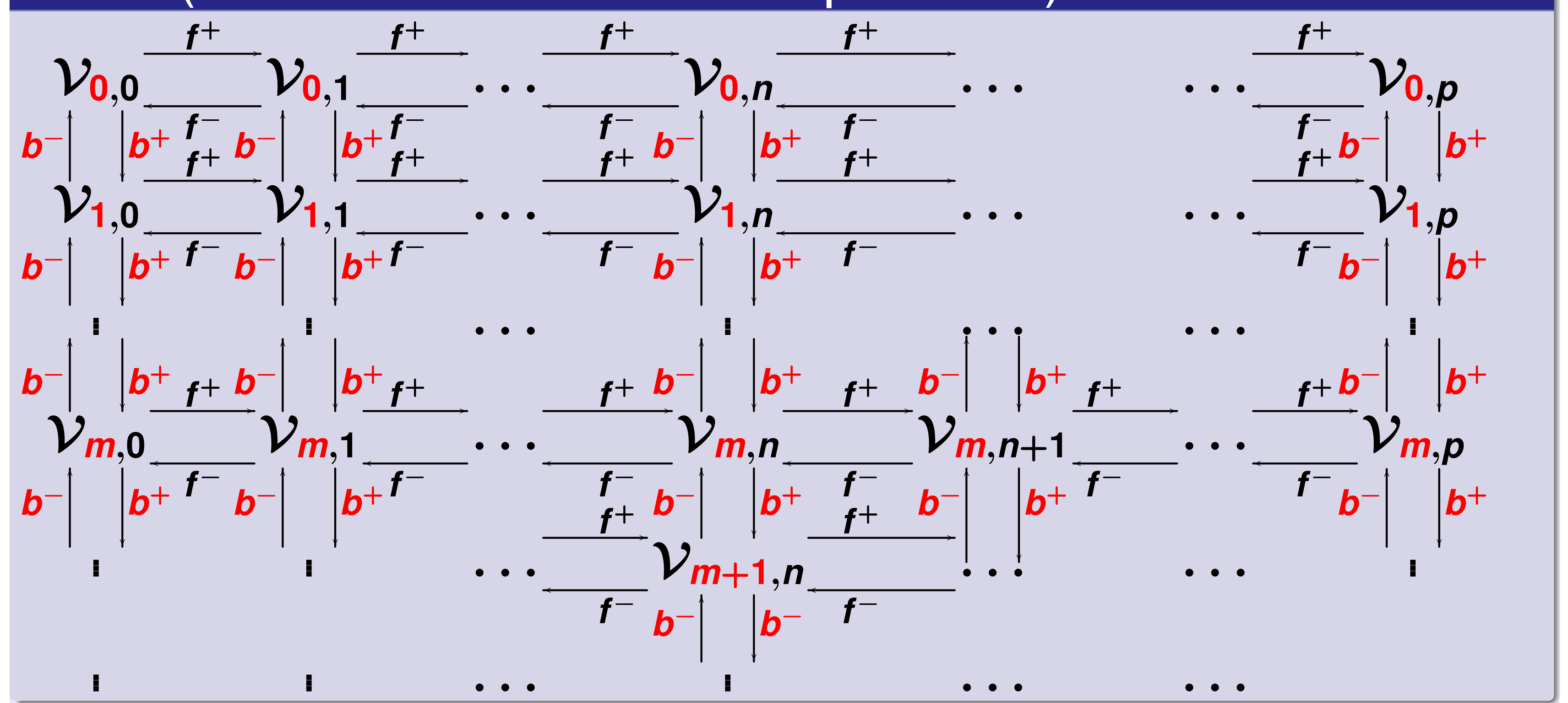
Theorem (Irreducibility)

*The Fock-like representation of $P_{BF}^{(1,1)}$ is uniquely identified given that we have chosen some (arbitrary but fixed) value for the positive integer **p**. The carrier space of this representation is the v.s.*

$$\bigoplus_{n=0}^{p}\bigoplus_{m=0}^{\infty}\mathcal{V}_{m,n}$$

*The above vector space has no irreducible subspaces under the above defined $P_{BF}^{(1,1)}$-action, thus it is **an irreducible representation** or equivalently **a simple $P_{BF}^{(1,1)}$-module**.*

Proof (construction of ladder operators):



3c. the Fock-like repres. as a $(\mathbb{Z}_2\times\mathbb{Z}_2)$-graded module

- Assign a $(\mathbb{Z}_2\times\mathbb{Z}_2)$-grading to the carrier space through:

$$deg|m,n,\alpha\rangle=deg|m,n,\beta\rangle=\big(m \bmod 2,\ n \bmod 2\big)\in\mathbb{Z}_2\times\mathbb{Z}_2$$

In other words $deg\mathcal{V}_{m,n}=(0,0)$ if m,n are both even, $deg\mathcal{V}_{m,n}=(1,1)$ if m,n are both odd and $deg\mathcal{V}_{m,n}=(0,1)$ (or: $(1,0)$) if m is even and n is odd (or: m is odd and n is even)

- Assign a $(\mathbb{Z}_2\times\mathbb{Z}_2)$-grading to the algebra through:

$$degb^\pm=(1,0)\qquad degf^\pm=(0,1)$$

- $\bigoplus_{n=0}^{p}\bigoplus_{m=0}^{\infty}\mathcal{V}_{m,n}$ represents an infinite family (parametrized by the values of the pos. int. p) of infinite dimensional, non-equivalent, irreducible, $(\mathbb{Z}_2\times\mathbb{Z}_2)$-graded modules.