

# The $S_3$ symmetry: Flavour and texture zeroes

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**Abstract.** We use the permutational symmetry group  $S_3$  as a symmetry of flavour, which leads to a unified treatment of masses and mixings of the quarks and leptons. In this framework all mass matrices of the fermions in the theory have the same form with four texture zeroes of class I. Also, with the help of six elements of real matrix representation of  $S_3$  as transformation matrices of similarity classes, we make a classification of the sets of mass matrices with texture zeroes in equivalence classes. This classification reduce the number of phenomenologically viable textures for the non-singular mass matrices of  $3 \times 3$ , from thirty three down to only eleven independent sets of matrices. Each of these sets of matrices has exactly the same physical content.

## 1. Introduction

The Standard Model (SM) can be extended by adding of three right-handed neutrino states, which would be singlets of the gauge group of SM, but coupled to matter just through the neutrino masses [1]. But in minimal extensions of the Standard Model, considering a mass term for left-handed neutrinos purely of Dirac is not theoretically favored, because it can not explain easily why neutrinos are much lighter than the charged leptons. Thus, we assume that neutrinos have Majorana masses and acquire their small masses through of the type I seesaw mechanism.

In both lepton and quark sectors of the extended Standard Model, analogous fermions in different generations, say  $u, c$  and  $t$  or  $d, s$  and  $b$ , have completely identical couplings to all gauge bosons of the strong, electroweak interactions. Prior to the introduction of the Higgs boson and mass terms, the Lagrangian is chiral and invariant with respect to any permutation of the left and right quark fields. The introduction of a Higgs boson and the Yukawa couplings give mass to the quarks and leptons when the gauge symmetry is spontaneously broken. The mass term in the Lagrangian, obtained by taking the vacuum expectation value of the Higgs field in the quark and lepton Higgs couplings, gives rise to mass matrices  $\mathbf{M}_d$ ,  $\mathbf{M}_u$ ,  $\mathbf{M}_l$  and  $\mathbf{M}_\nu$ ;

$$\mathcal{L}_Y = \bar{\mathbf{q}}_{d,L} \mathbf{M}_d \mathbf{q}_{d,R} + \bar{\mathbf{q}}_{u,L} \mathbf{M}_u \mathbf{q}_{u,R} + \bar{\mathbf{L}}_L \mathbf{M}_l \mathbf{L}_R + \bar{\nu}_L \mathbf{M}_\nu (\nu_L)^c + h.c. \quad (1)$$

Therefore, we propose, as well as many other authors [2, 3, 4, 5, 6], that the texture zeroes of the mass matrices of the quarks and leptons, are the result of a flavour permutational symmetry  $S_3$  and its spontaneous or explicit breaking.

On the other hand, in the last ten years, important theoretical advances have been made in the understanding of the mechanisms for the mass fermion generation and flavour mixing. Phenomenologically, some striking progress has been made with the help of the texture zeroes and flavour symmetries in specifying the quantitative relationship between flavour mixing angles and quark or lepton mass ratios [3, 4, 7, 5] with a minimum of free parameters. In fact it, can be

noted that different mass matrices with texture zeroes located in different positions may have exactly the same physical content. Therefore the question arises, is there any relation between these matrices?. We find an answer to this question through the similarity classes, recently proposed by Branco [8].

In this paper, we use the permutational symmetry  $S_3$  as a flavour symmetry, in a unified treatment of masses and mixings of quarks and leptons. Also, with the help of a real matrix representation of the group  $S_3$ , as a basis for the transformation matrices of the similarity classes, we make a classification of the set of mass matrices with texture zeroes.

## 2. Flavour permutational symmetry $S_3$

A phenomenologically and theoretically meaningful approach for reducing the number of free parameters in the Standard Model is the imposition of texture zeroes [5, 7] or flavour symmetries. Recent flavour symmetry models are reviewed in [9]. Also, certain texture zeroes may be obtained from a flavour symmetry. In particular, a permutational  $S_3$  flavour symmetry and its sequential explicit breaking justifies taking the same generic form for the mass matrices of all Dirac fermions, conventionally called a four texture zeroes form [3, 4]. Some reasons to propose the validity of a matrix with four texture zeroes as a universal form for the mass matrices of all Dirac fermions in the theory are the following:

- (i) The idea of  $S_3$  flavour symmetry and its explicit breaking has been successfully realized as a mass matrix with four texture zeroes in the quark sector to interpret the strong mass hierarchy of up and down type quarks [2, 3]. Also, the numerical values of mixing matrices of the quarks determined in this framework are in good agreement with the experimental data [3].
- (ii) Since the mass spectrum of the charged leptons exhibits a hierarchy similar to the quark's one, it would be natural to consider the same  $S_3$  symmetry and its explicit breaking to justify the use of the same generic form with four texture zeroes for the charged lepton mass matrix.
- (iii) As for the Dirac neutrinos, we have no direct information about the absolute values or the relative values of the Dirac neutrino masses, but the mass matrix with four texture zeroes can be obtained from an grand unified theory  $SO(10)$  which describes well the data on masses and mixings of Majorana neutrinos [10]. Furthermore, from supersymmetry arguments, it would be sensible to assume that the Dirac neutrinos have a mass hierarchy similar to that of the u-quarks and it would be natural to take for the Dirac neutrino mass matrix also a matrix with four texture zeroes.

### 2.1. Mass matrices from the breaking of $S_{3L} \otimes S_{3R}$

Some authors have pointed out that realistic Dirac fermion mass matrices results from the flavour permutational symmetry  $S_L(3) \otimes S_R(3)$  and its spontaneous or explicit breaking [2, 3, 4, 5, 6]. The group  $S_3$  treats three objects symmetrically, while its  $3 \times 3$  representation structure  $\mathbf{1} \oplus \mathbf{2}$  treats the generations differently and adapts itself readily to the hierarchical nature of the mass spectrum. Under exact  $S_{3L} \otimes S_{3R}$  symmetry, the mass spectrum for either quark sector (up or down quarks) or leptonic sector (charged leptons or Dirac neutrinos) consists of one massive particle in a singlet irreducible representation and a pair of massless particles in a doublet irreducible representation, the corresponding quark mass matrix with the exact  $S_{3L} \otimes S_{3R}$  symmetry will be denoted by  $\mathbf{M}_{i3}$  with  $i = u, d, l, \nu_D$ . Here,  $u$ ,  $d$ ,  $l$  and  $\nu_D$  denote the up quarks, down quarks, charged leptons and Dirac neutrinos, respectively. Assuming that there is only one Higgs boson in the theory, this  $SU(2)_L$  doublet can be accommodated in a singlet representation of  $S_3$ . In order to generate masses for the first and second families, we add

the terms  $\mathbf{M}_{i2}$  and  $\mathbf{M}_{i1}$  to  $\mathbf{M}_{i3}$ . The term  $\mathbf{M}_{i2}$  breaks the permutational symmetry  $S_{3L} \otimes S_{3R}$  down to  $S_{2L} \otimes S_{2R}$  and mixes the singlet and doublet representation of  $S_3$ .  $\mathbf{M}_{i1}$  transforms as the mixed symmetry term in the doublet complex tensorial representation of  $S_3^{diag} \subset S_{3L} \otimes S_{3R}$ . Putting the first family in a complex representation will allow us to have a CP violating phase in the mixing matrix. Then, in a symmetry adapted basis,  $\mathbf{M}_i$  takes the form

$$M_i = m_{i3} \left[ \begin{pmatrix} 0 & |A_i|e^{i\phi_i} & 0 \\ |A_i|e^{-i\phi_i} & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 & 0 \\ 0 & -\Delta_i + \delta_i & C_i \\ 0 & C_i & \Delta_i - \delta_i \end{pmatrix} + \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 - \Delta_i \end{pmatrix} \right]$$

$$M_i = m_{i3} \begin{pmatrix} 0 & A_i & 0 \\ A_i^* & B_i & C_i \\ 0 & C_i & D_i \end{pmatrix}. \quad (2)$$

where  $A_i = |A_i|e^{i\phi_i}$ ,  $B_i = -\Delta_i + \delta_i$  and  $D_i = 1 - \delta_i$ . From the strong hierarchy in the masses of the Dirac fermions,  $m_{i3} \gg m_{i2} > m_{i1}$ , we expect  $1 - \delta_i$  to be very close to unity. The small parameter  $\delta_i$  is a function of the flavour symmetry breaking parameter  $Z_i^{1/2}$  [3, 4]. In other words, each possible symmetry breaking pattern is now characterized by the flavour symmetry breaking parameter  $Z_i^{1/2}$ , which is defined as the ratio  $Z_i^{1/2} = \frac{(M_i)_{23}}{(M_i)_{22}}$ . This ratio measures the mixing of singlet and doublet irreducible representations of  $S_3$ . The mass matrix (2) may be written as  $M_i = P_i^\dagger \bar{M}_i P_i$ , where  $\bar{M}_i$  is a real symmetric matrix and  $P_i \equiv \text{diag}[1, e^{i\phi_i}, e^{i\phi_i}]$ .

### 3. Classification of texture zeroes in equivalence classes

In this section we make a classification of mass matrices with texture zeroes in terms of the similarity classes. The similarity classes are defined as follows: Two matrices  $M$  and  $M'$  are similar if there exists an invertible matrix  $T$  such that

$$M' = TMT^{-1} \quad \text{or} \quad M' = T^{-1}MT. \quad (3)$$

The equivalence classes associated with a similarity transformation are called similarity classes. Another way to see the similarity classes is that the matrices that satisfy the similarity transformation, eq. (3), have the same invariants; Trace, determinant and  $\chi$ :

$$\text{Tr}\{M\} = \text{Tr}\{M'\}, \quad \det\{M\} = \det\{M'\} \quad \text{and} \quad \chi' = \chi \equiv \frac{\text{Tr}\{M^2\} - \text{Tr}\{M\}^2}{2}. \quad (4)$$

In this paper we will count the texture zeroes in a matrix as follows: two texture zeroes off-diagonal counts as one zero, while one on the diagonal counts as one zero [5]. But in the literature we find that a mass matrix has double number of texture zeroes than the number that we obtain with our rule. This is so, because in the literature the total number of texture zeroes counted in a mass matrix is the sum of the texture zeroes coming from the two mass matrices in the sector of quarks (up and down quarks) or leptons (charged leptons and left-handed neutrinos). Hence, to avoid confusion in the nomenclature of the matrices, we count the number of texture zeroes in a matrix with the rule previously enunciated, but when referring to this matrix we will follow the literature's rule and name it with the double of texture zeroes it actually has.

Now, from the most general form of the mass matrices of  $3 \times 3$ , symmetric and Hermitian:

$$M^S = \begin{pmatrix} g & a & e \\ a & b & c \\ e & c & d \end{pmatrix} \quad \text{and} \quad M^h = \begin{pmatrix} g & a & e \\ a^* & b & c \\ e^* & c^* & d \end{pmatrix}, \quad (5)$$

we can see that only six of the nine elements of these matrices are independent of each other. Therefore, in a certain sense the similarity transformation, eq. (3), realized the permutation of the six independent elements in the nine entries of the mass matrices. But if we want to preserve the invariants (4), the elements on the diagonal can only exchanged positions on the diagonal, while the off-diagonal elements can only exchange positions outside of the diagonal. Thus we have that all these operations reduce to the permutations of three objects. So it is natural propose to as transformation matrices  $T$  in the similarity classes, see eq.(3), the six elements of the real representation of the group of permutations  $S_3$ , which are:

$$\begin{aligned} T(A_0) &= \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, T(A_1) = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}, T(A_2) = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}, \\ T(A_3) &= \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}, T(A_4) = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}, T(A_5) = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}. \end{aligned} \quad (6)$$

Then, applying the transformations (3) and taking the matrices (6) as matrices of transformation, we get the classification of mass matrices with texture zeroes, which is shown in the tables 1, 2, 3 and 4. In this tables, the "★" and "×" denote an arbitrary non-vanishing matrix element on the diagonal and off-diagonal entries, respectively.

Class	Textures	Invariants	
		Symmetric	Hermitian
I	$\begin{pmatrix} 0 & \times & 0 \\ \times & 0 & 0 \\ 0 & 0 & \star \end{pmatrix} \begin{pmatrix} \star & 0 & 0 \\ 0 & 0 & \times \\ 0 & \times & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 & \times \\ 0 & \star & 0 \\ \times & 0 & 0 \end{pmatrix}$	$\text{Tr} = d$ $\det = -a^2 d$ $\chi = a^2$	$\text{Tr} = d$ $\det = - a ^2 d$ $\chi =  a ^2$

**Table 1.** Matrix with eight texture zeroes.

#### 4. Seesaw mechanism

The left-handed Majorana neutrinos acquire their small masses through the type I seesaw mechanism  $M_{\nu_L} = M_{\nu_D} M_{\nu_R}^{-1} M_{\nu_D}^T$ , where  $M_{\nu_D}$  and  $M_{\nu_R}$  denote the Dirac and right handed Majorana neutrino mass matrices, respectively. The form of  $M_{\nu_D}$  is given in eq. (2); which is a matrix with four texture zeroes class I, Hermitian, and from our conjecture of a universal  $S_3$  flavour symmetry in a unified treatment of all fermions, it is natural to take for  $M_{\nu_R}$  also a matrix with four texture zeroes of class I, but symmetric. Let us further assume that the phases in the entries of the  $M_{\nu_R}$  may be factorized out as  $M_{\nu_R} = R \bar{M}_{\nu_R} R$ , where  $R \equiv \text{diag} [e^{-i\phi_c}, e^{i\phi_c}, 1]$  with  $\phi_c \equiv \arg \{c_{\nu_R}\}$  and

$$\bar{M}_{\nu_R} = \begin{pmatrix} 0 & a_{\nu_R} & 0 \\ a_{\nu_R} & |b_{\nu_R}| & |c_{\nu_R}| \\ 0 & |c_{\nu_R}| & d_{\nu_R} \end{pmatrix}, \quad (7)$$

Then, the type I seesaw mechanism takes the form  $M_{\nu_L} = P_D^\dagger \bar{M}_{\nu_D} P_D R^\dagger \bar{M}_{\nu_R}^{-1} R^\dagger P_D \bar{M}_{\nu_D} P_D^\dagger$  and the mass matrix of the left-handed Majorana neutrinos has the following form, with four texture zeroes of class I:

$$M_{\nu_L} = \begin{pmatrix} 0 & a_{\nu_L} & 0 \\ a_{\nu_L} & b_{\nu_L} & c_{\nu_L} \\ 0 & c_{\nu_L} & d_{\nu_L} \end{pmatrix}, \quad (8)$$

Class	Textures	Invariants	
		Symmetric	Hermitian
I	$\begin{pmatrix} \star & 0 & 0 \\ 0 & 0 & \times \\ 0 & \times & \star \end{pmatrix} \begin{pmatrix} 0 & 0 & \times \\ 0 & \star & 0 \\ \times & 0 & \star \end{pmatrix} \begin{pmatrix} \star & \times & 0 \\ \times & 0 & 0 \\ 0 & 0 & \star \end{pmatrix}$ $\begin{pmatrix} \star & 0 & 0 \\ 0 & \star & \times \\ 0 & \times & 0 \end{pmatrix} \begin{pmatrix} 0 & \times & 0 \\ \times & \star & 0 \\ 0 & 0 & \star \end{pmatrix} \begin{pmatrix} \star & 0 & \times \\ 0 & \star & 0 \\ \times & 0 & 0 \end{pmatrix}$	$\text{Tr} = d + g$ $\det = -c^2 g$ $\chi = c^2 - gd$	$\text{Tr} = d + g$ $\det = - c ^2 g$ $\chi =  c ^2 - gd$
II	$\begin{pmatrix} 0 & \times & 0 \\ \times & 0 & \times \\ 0 & \times & \star \end{pmatrix} \begin{pmatrix} 0 & \times & \times \\ \times & 0 & 0 \\ \times & 0 & \star \end{pmatrix} \begin{pmatrix} \star & \times & 0 \\ \times & 0 & \times \\ 0 & \times & 0 \end{pmatrix}$ $\begin{pmatrix} 0 & 0 & \times \\ 0 & \star & \times \\ \times & \times & 0 \end{pmatrix} \begin{pmatrix} 0 & \times & \times \\ \times & \star & 0 \\ \times & 0 & 0 \end{pmatrix} \begin{pmatrix} \star & 0 & \times \\ 0 & 0 & \times \\ \times & \times & 0 \end{pmatrix}$	$\text{Tr} = d$ $\det = -a^2 d$ $\chi = a^2 + c^2$	$\text{Tr} = d$ $\det = - a ^2 d$ $\chi =  a ^2 +  c ^2$
III	$\begin{pmatrix} \star & 0 & 0 \\ 0 & \star & 0 \\ 0 & 0 & \star \end{pmatrix}$	$\text{Tr} = g + b + d$ $\det = gbd$ $\chi = -gb - gd - bd$	$\text{Tr} = g + b + d$ $\det = gbd$ $\chi = -gb - gd - bd$
IV	$\begin{pmatrix} 0 & \times & \times \\ \times & 0 & \times \\ \times & \times & 0 \end{pmatrix}$	$\text{Tr} = 0$ $\det = 2ace$ $\chi = a^2 + e^2 + c^2$	$\text{Tr} = 0$ $\det = a^* c^* e + ace^*$ $\chi =  a ^2 +  e ^2 +  c ^2$

**Table 2.** Matrix with six texture zeroes.

Class	Textures	Invariants	
		Symmetric	Hermitian
I	$\begin{pmatrix} 0 & \times & 0 \\ \times & \star & \times \\ 0 & \times & \star \end{pmatrix} \begin{pmatrix} 0 & 0 & \times \\ 0 & \star & \times \\ \times & \times & \star \end{pmatrix} \begin{pmatrix} \star & 0 & \times \\ 0 & 0 & \times \\ \times & \times & \star \end{pmatrix}$ $\begin{pmatrix} \star & \times & \times \\ \times & 0 & 0 \\ \times & 0 & \star \end{pmatrix} \begin{pmatrix} \star & \times & 0 \\ \times & \star & \times \\ 0 & \times & 0 \end{pmatrix} \begin{pmatrix} \star & \times & \times \\ \times & \star & 0 \\ \times & 0 & 0 \end{pmatrix}$	$\text{Tr} = b + d$ $\det = -a^2 d$ $\chi = a^2 + c^2 - bd$	$\text{Tr} = b + d$ $\det = - a ^2 d$ $\chi =  a ^2 +  c ^2 - bd$
II	$\begin{pmatrix} 0 & \times & \times \\ \times & \star & 0 \\ \times & 0 & \star \end{pmatrix} \begin{pmatrix} \star & \times & 0 \\ \times & 0 & \times \\ 0 & \times & \star \end{pmatrix} \begin{pmatrix} \star & 0 & \times \\ 0 & \star & \times \\ \times & \times & 0 \end{pmatrix}$	$\text{Tr} = b + d$ $\det = -a^2 d$ $-e^2 b$ $\chi = a^2 + e^2 - bd$	$\text{Tr} = b + d$ $\det = - a ^2 d$ $- e ^2 b$ $\chi =  a ^2 +  e ^2 - bd$
III	$\begin{pmatrix} 0 & \times & \times \\ \times & 0 & \times \\ \times & \times & \star \end{pmatrix} \begin{pmatrix} 0 & \times & \times \\ \times & \star & \times \\ \times & \times & 0 \end{pmatrix} \begin{pmatrix} \star & \times & \times \\ \times & 0 & \times \\ \times & \times & 0 \end{pmatrix}$	$\text{Tr} = d$ $\det = 2ace - a^2 d$ $\chi = a^2 + c^2 + e^2$	$\text{Tr} = d$ $\det = a^* c^* e + ace^* -  a ^2 d$ $\chi =  a ^2 +  c ^2 +  e ^2$
IV	$\begin{pmatrix} \star & 0 & 0 \\ 0 & \star & \times \\ 0 & \times & \star \end{pmatrix} \begin{pmatrix} \star & 0 & \times \\ 0 & \star & 0 \\ \times & 0 & \star \end{pmatrix} \begin{pmatrix} \star & \times & 0 \\ \times & \star & 0 \\ 0 & 0 & \star \end{pmatrix}$	$\text{Tr} = g + b + d$ $\det = -gc^2 + gbd$ $\chi = c^2 - gb - gd - bd$	$\text{Tr} = g + b + d$ $\det = -g c ^2 - gbd$ $\chi =  c ^2 - gb - gd - bd$

**Table 3.** Matrix with four texture zeroes.

where

$$\begin{aligned}
a_{\nu_L} &= \frac{|a_{\nu_D}|^2}{a_{\nu_R}}, \quad d_{\nu_L} = \frac{d_{\nu_D}^2}{d_{\nu_R}}, \quad c_{\nu_L} = \frac{c_{\nu_D} d_{\nu_D}}{d_{\nu_R}} + \frac{|a_{\nu_D}|}{|a_{\nu_R}|} \left( c_{\nu_D} e^{-i\phi_{\nu_D}} - \frac{|c_{\nu_R}| d_{\nu_D}}{d_{\nu_R}} e^{i(\phi_c - \phi_{\nu_D})} \right) \\
b_{\nu_L} &= \frac{c_{\nu_D}^2}{d_{\nu_R}} + \frac{|c_{\nu_R}|^2 - |b_{\nu_R}| d_{\nu_R}}{d_{\nu_R}} \frac{|a_{\nu_D}|^2}{a_{\nu_R}^2} e^{i2(\phi_c - \phi_{\nu_D})} + 2 \frac{|a_{\nu_D}|}{|a_{\nu_R}|} \left( b_{\nu_D} e^{-i\phi_{\nu_D}} - \frac{c_{\nu_D} |c_{\nu_R}|}{d_{\nu_R}} e^{i(\phi_c - \phi_{\nu_D})} \right).
\end{aligned} \tag{9}$$

Class	Textures	Invariants	
		Symmetric	Hermitian
I	$\begin{pmatrix} 0 & \times & \times \\ \times & \star & \times \\ \times & \times & \star \end{pmatrix} \begin{pmatrix} \star & \times & \times \\ \times & 0 & \times \\ \times & \times & \star \end{pmatrix} \begin{pmatrix} \star & \times & \times \\ \times & \star & \times \\ \times & \times & 0 \end{pmatrix}$	$\text{Tr} = b + d$ $\det = 2ace - a^2d - be^2$ $\chi = a^2 + c^2 + e^2 - bd$	$\text{Tr} = b + d$ $\det = - a ^2d - b e ^2 + ace^* + a^*c^*e$ $\chi =  a ^2 +  c ^2 +  e ^2 - bd$
II	$\begin{pmatrix} \star & 0 & \times \\ 0 & \star & \times \\ \times & \times & \star \end{pmatrix} \begin{pmatrix} \star & \times & \times \\ \times & \star & 0 \\ \times & 0 & \star \end{pmatrix} \begin{pmatrix} \star & \times & 0 \\ \times & \star & \times \\ 0 & \times & \star \end{pmatrix}$	$\text{Tr} = b + d + g$ $\det = gbd$ $-c^2g - e^2b$ $\chi = e^2 + c^2 - gb - gd - bd$	$\text{Tr} = b + d + g$ $\det = gbd$ $- c ^2g -  e ^2b$ $\chi =  e ^2 +  c ^2 - gb - gd - bd$

**Table 4.** Matrix with two texture zeroes.

The elements  $a_{\nu_L}$  and  $d_{\nu_L}$  are real, while  $b_{\nu_L}$  and  $c_{\nu_L}$  are complex. It may also be noticed that  $M_{\nu_L}$  may also have four texture zeroes of class I when  $M_{\nu_R}$  has four texture zeroes of class I, six texture zeroes of class II, six texture zeroes of class I and eight texture zeroes of class I. From eqs. (9) we conclude that the information of the number of texture zeroes in  $M_{\nu_R}$  is found in elements (2,2) and (2,3) of the matrix  $M_{\nu_L}$ . In this case, without loss of generality, we may choose  $\arg\{b_{\nu_L}\} = 2\arg\{c_{\nu_L}\} = 2\varphi$ , the analysis simplifies since the phases in  $M_{\nu_L}$  may be factorized as  $M_{\nu_L} = Q\bar{M}_{\nu_L}Q$ , where  $Q$  is a diagonal matrix of phases  $Q \equiv \text{diag}[e^{-i\varphi}, e^{i\varphi}, 1]$  and  $\bar{M}_{\nu_L}$  is a real symmetric matrix.

### 5. Mass matrix with four texture zeroes as function of the fermion masses

Now, computing the invariants (4) of the real symmetric matrix  $\bar{M}_{\nu_L}$ , we may express the parameters  $A_i$ ,  $B_i$ ,  $C_i$  and  $D_i$  occurring in (2) in terms of the mass eigenvalues. In this way, we get the  $\bar{M}_i$  matrix ( $i = u, d, l, \nu_L$ ), reparametrized in terms of its eigenvalues and the parameter  $\delta_i$  is

$$\bar{M}_i = \begin{pmatrix} 0 & \sqrt{\frac{\tilde{m}_{i1}\tilde{m}_{i2}}{1-\delta_i}} & 0 \\ \sqrt{\frac{\tilde{m}_{i1}\tilde{m}_{i2}}{1-\delta_i}} & \tilde{m}_{i1} - \tilde{m}_{i2} + \delta_i & \sqrt{\frac{\delta_i}{(1-\delta_i)}}f_{i1}f_{i2} \\ 0 & \sqrt{\frac{\delta_i}{(1-\delta_i)}}f_{i1}f_{i2} & 1 - \delta_i \end{pmatrix}, \quad (10)$$

where  $\tilde{m}_{i1} = \frac{m_{i1}}{m_{i3}}$ ,  $\tilde{m}_{i2} = \frac{|m_{i2}|}{m_{i3}}$ ,  $f_{i1} = 1 - \tilde{m}_{i1} - \delta_i$  and  $f_{i2} = 1 + \tilde{m}_{i2} - \delta_i$ . The small parameters  $\delta_i$  are also functions of the mass ratios and the flavour symmetry breaking parameter  $Z_i^{1/2}$ . The small parameter  $\delta_i$  is obtained as the solution of the cubic equation  $(1-\delta_i)(\tilde{m}_{i1} - \tilde{m}_{i2} + \delta_i)^2 Z - \delta_i f_{i1} f_{i2} = 0$ , and may be written as

$$\delta_i = \frac{Z_i}{Z_i + 1} \frac{(\tilde{m}_{i2} - \tilde{m}_{i1})^2}{W_i(Z)} \quad (11)$$

where  $W_i(Z)$  is the product of the two roots that do not vanish when  $Z_i$  vanishes

$$W_i(Z) = \left[ p_i^3 + 2q_i^2 + 2q_i\sqrt{p_i^3 + q_i^2} \right]^{\frac{1}{3}} - |p_i| + \left[ p_i^3 + 2q_i^2 - 2q_i\sqrt{p_i^3 + q_i^2} \right]^{\frac{1}{3}} + \\ + \frac{1}{9} (Z_i (2(\tilde{m}_{i2} - \tilde{m}_{i1}) + 1) + (\tilde{m}_{i2} - \tilde{m}_{i1}) + 2)^2 - \frac{1}{3} \left( \left[ q_i + \sqrt{p_i^3 + q_i^2} \right]^{\frac{1}{3}} + \left[ q_i - \sqrt{p_i^3 + q_i^2} \right]^{\frac{1}{3}} \right) \times \\ \times (Z_i (2(\tilde{m}_{i2} - \tilde{m}_{i1}) + 1) + (\tilde{m}_{i2} - \tilde{m}_{i1}) + 2)$$

$$\text{with } p_i = -\frac{1}{3} \frac{Z_i (Z_i (2(\tilde{m}_{i2} - \tilde{m}_{i1}) + 1) + \tilde{m}_{i2} - \tilde{m}_{i1} + 2)}{Z_i + 1} + \frac{[Z_i (\tilde{m}_{i2} - \tilde{m}_{i1}) (\tilde{m}_{i2} - \tilde{m}_{i1} + 2) (1 + \tilde{m}_{i2}) (1 - \tilde{m}_{i1})]}{Z_i + 1}, \text{ and } q_i = \\ \frac{1}{6} \frac{[Z_i (\tilde{m}_{i2} - \tilde{m}_{i1}) (\tilde{m}_{i2} - \tilde{m}_{i1} + 2) (1 + \tilde{m}_{i2}) (1 - \tilde{m}_{i1})] (Z_i (2(\tilde{m}_{i2} - \tilde{m}_{i1}) + 1) + \tilde{m}_{i2} - \tilde{m}_{i1} + 2)}{(Z_i + 1)^2} - \frac{1}{27} \frac{(Z_i (2(\tilde{m}_{i2} - \tilde{m}_{i1}) + 1) + \tilde{m}_{i2} - \tilde{m}_{i1} + 2)^3}{(Z_i + 1)^3}.$$

Also, the values allowed for the parameters  $\delta_i$  are in the following range  $0 < \delta_i < 1 - \tilde{m}_{i1}$ . Now, the entries in the real orthogonal matrix  $\mathbf{O}$  that diagonalize the matrix  $\bar{M}_i$ , may also be expressed as

$$\mathbf{O}_i = \begin{pmatrix} \left[ \frac{\tilde{m}_{i2} f_{i1}}{\mathcal{D}_{i1}} \right]^{\frac{1}{2}} & - \left[ \frac{\tilde{m}_{i1} f_{i2}}{\mathcal{D}_{i2}} \right]^{\frac{1}{2}} & \left[ \frac{\tilde{m}_{i1} \tilde{m}_{i2} \delta_i}{\mathcal{D}_{i3}} \right]^{\frac{1}{2}} \\ \left[ \frac{\tilde{m}_{i1} (1-\delta_i) f_{i1}}{\mathcal{D}_{i1}} \right]^{\frac{1}{2}} & \left[ \frac{\tilde{m}_{i2} (1-\delta_i) f_{i2}}{\mathcal{D}_{i2}} \right]^{\frac{1}{2}} & \left[ \frac{(1-\delta_i) \delta_i}{\mathcal{D}_{i3}} \right]^{\frac{1}{2}} \\ - \left[ \frac{\tilde{m}_{i1} f_{i2} \delta_i}{\mathcal{D}_{i1}} \right]^{\frac{1}{2}} & - \left[ \frac{\tilde{m}_{i2} f_{i1} \delta_i}{\mathcal{D}_{i2}} \right]^{\frac{1}{2}} & \left[ \frac{f_{i1} f_{i2}}{\mathcal{D}_{i3}} \right]^{\frac{1}{2}} \end{pmatrix}, \quad (12)$$

where,  $\mathcal{D}_{i1} = (1 - \delta_i)(\tilde{m}_{i1} + \tilde{m}_{i2})(1 - \tilde{m}_{i1})$ ,  $\mathcal{D}_{i2} = (1 - \delta_i)(\tilde{m}_{i1} + \tilde{m}_{i2})(1 + \tilde{m}_{i2})$  and  $\mathcal{D}_{i3} = (1 - \delta_i)(1 - \tilde{m}_{i1})(1 + \tilde{m}_{i2})$ .

## 6. Mixing Matrices as Functions of the Fermion Masses

The quark mixing matrix  $V_{CKM} = U_u U_d^\dagger$  takes the form

$$V_{CKM}^{th} = \mathbf{O}_u^T P^{(u-d)} \mathbf{O}_d, \quad (13)$$

where  $P^{(u-d)} = \text{diag} [1, e^{i\phi}, e^{i\phi}]$  with  $\phi = \phi_u - \phi_d$ , and  $\mathbf{O}_{u,d}$  are the real orthogonal matrices (12) that diagonalize the real symmetric mass matrices  $\bar{M}_{u,d}$ . In a similar way, the lepton mixing matrix  $U_{PMNS} = U_l^\dagger U_\nu$  may be written as

$$U_{PMNS}^{th} = \mathbf{O}_l^T P^{(\nu-l)} \mathbf{O}_\nu K, \quad (14)$$

where  $P^{(\nu-l)} = \text{diag} [1, e^{i\Phi_1}, e^{i\Phi_2}]$  is the diagonal matrix of the Dirac phases, with  $\Phi_1 = 2\varphi - \phi_l$  and  $\Phi_2 = \varphi - \phi_l$ . The real orthogonal matrices  $\mathbf{O}_{\nu,l}$  are defined in eq. (12). Exact explicit expressions for the unitary matrices (13) and (14) are given by J. Barranco, F. Gonzalez Canales and A. Mondragon [4].

### 6.1. The $\chi^2$ fit for the Quark Mixing Matrix

We made a  $\chi^2$  fit of the theoretical expressions for the moduli of the entries of the quark mixing matrix  $|(V_{CKM}^{th})_{ij}|$ , eq.(13) to the experimental values  $|(V_{CKM}^{exp})_{ij}|$  [11]. We computed the moduli of the entries of the quark mixing matrix and the inner angles of the unitarity triangle from the theoretical expression (13) with the following numerical values of the quark mass ratios [11]:

$$\tilde{m}_u = 2.5469 \times 10^{-5}, \quad \tilde{m}_c = 3.9918 \times 10^{-3}, \quad \tilde{m}_d = 1.5261 \times 10^{-3}, \quad \tilde{m}_s = 3.2319 \times 10^{-2}. \quad (15)$$

The resulting best values of the parameters  $\delta_u = 3.829 \times 10^{-3}$ ,  $\delta_d = 4.08 \times 10^{-4}$  and the Dirac CP violating phase is  $\phi = 90^\circ$ . The best values for the moduli of the entries of the  $CKM$  mixing matrix are given in the following expression

$$|V_{CKM}^{th}| = \begin{pmatrix} 0.97421 & 0.22560 & 0.003369 \\ 0.22545 & 0.97335 & 0.041736 \\ 0.008754 & 0.04094 & 0.99912 \end{pmatrix} \quad (16)$$

and the inner angles of the unitary triangle  $\alpha^{th} = 91.24^\circ$ ,  $\beta^{th} = 20.41^\circ$  and  $\gamma^{th} = 68.33^\circ$ . The Jarlskog invariant takes the value  $J_q^{th} = 2.9 \times 10^{-5}$ . All these results are in very good agreement with the experimental values. The minimum value of  $\chi^2$  obtained in this fit is 4.6 and the resulting value of  $\chi^2$  for degree of freedom is  $\frac{\chi_{min}^2}{d.o.f.} = 0.77$ . In this way, we obtain the following numerical values for the mixing angles:  $\theta_{12}^{qth} = 13^\circ$ ,  $\theta_{23}^{qth} = 2.38^\circ$  and  $\theta_{13}^{qth} = 0.19^\circ$ . Which are also in very good agreement with the latest analysis of the experimental data [11].

## 6.2. The $\chi^2$ fit for the Lepton Mixing Matrix

In the case of the lepton mixing matrix, we made a  $\chi^2$  fit of the theoretical expressions for the moduli of the entries of the lepton mixing matrix  $|(U_{PMNS}^{th})_{ij}|$  to the values extracted from experiment as given by Gonzalez-Garcia [12]. The computation was made using the following values for the charged lepton masses [11]:

$$m_e = 0.5109 \text{ MeV}, \quad m_\mu = 105.685 \text{ MeV}, \quad m_\tau = 1776.99 \text{ MeV}. \quad (17)$$

We took for the masses of the left-handed Majorana neutrinos a normal hierarchy. From the best values obtained for  $m_{\nu_3}$  and the experimental values of the  $\Delta m_{32}^2$  and  $\Delta m_{21}^2$ , we obtained the following best values for the neutrino masses

$$m_{\nu_1} = 2.7 \times 10^{-3} \text{ eV}, \quad m_{\nu_2} = 9.1 \times 10^{-3} \text{ eV}, \quad m_{\nu_3} = 4.7 \times 10^{-2} \text{ eV}. \quad (18)$$

The resulting best values of the parameters  $\delta_l = 0.06$ ,  $\delta_\nu = 0.522$  and the Dirac CP violating phases are  $\Phi_1 = \pi$  and  $\Phi_2 = 3\pi/2$ . The best values for the moduli of the entries of the  $PMNS$  mixing matrix are given in the following expression

$$|U_{PMNS}^{th}| = \begin{pmatrix} 0.820421 & 0.568408 & 0.061817 \\ 0.385027 & 0.613436 & 0.689529 \\ 0.422689 & 0.548277 & 0.721615 \end{pmatrix}. \quad (19)$$

The value of the rephasing invariant related to the Dirac phase is  $J_l^{th} = 8.8 \times 10^{-3}$ . In this numerical analysis, the minimum value of the  $\chi^2$ , corresponding to the best fit, is  $\chi^2 = 0.288$  and the resulting value of  $\chi^2$  degree of freedom is  $\frac{\chi_{min}^2}{d.o.f.} = 0.075$ . All numerical results of the fit are in very good agreement with the values of the moduli of the entries in the matrix  $U_{PMNS}^{exp}$  as given in Gonzalez-Garcia [12]. We obtain the following numerical values for the mixing angles:  $\theta_{12}^{th} = 34.7^\circ$ ,  $\theta_{23}^{th} = 43.6^\circ$  and  $\theta_{13}^{th} = 3.5^\circ$ . Which are also in very good agreement with the latest experimental data [12].

## 7. Conclusions

In this work we have shown that, starting from the flavour permutational symmetry  $S_3$ , a simple and explicit ansatz about the pattern of symmetry breaking leads to a unified treatment of masses of quarks and leptons, in which the left-handed Majorana neutrinos acquire their masses via the type I seesaw mechanism. The mass matrices of all Dirac fermions have a similar form with four texture zeroes of class I and a normal hierarchy of masses. Then, the mass matrix of the left-handed Majorana neutrinos also has a four texture zeroes class I and a normal hierarchy of masses. In this scheme, we have a parametrization of the  $CKM(PMNS)$  mixing matrix in terms of four quark (lepton) mass ratios and one (two)  $CP$  violating phase in very good agreement with all available experimental data. Also, with help of the symmetry group  $S_3$  we made a classification of mass matrices with texture zeroes in equivalence classes. In this classification we use the similarity transformation  $M' = TMT^{-1}$  or  $M' = T^{-1}MT$ , in which we take the transformation matrices  $T$  as six elements of real representation of  $S_3$ . With this classification we reduce the number of phenomenologically viable textures for non-singular mass matrices of  $3 \times 3$ , from thirty three down to only eleven independent sets of matrices. Each of these sets of matrices has exactly the same physical content.

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