Schwinger pair creation of particles and strings

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Abstract. I shortly review the worldline instanton method for calculating Schwinger pair production rates in (i) one-loop QED (ii) multiloop QED and (iii) one-loop open string theory.

1. History

Already in 1931 F. Sauter [1] realized that Dirac's theory of the electron predicts that an electric field of sufficient strength and extent can induce spontaneous creation of electron – positron pairs from the vacuum. Naively, a virtual pair turns real by separating out along the field and drawing a sufficient amount of energy from it to make up for both rest mass energies (fig. 1).

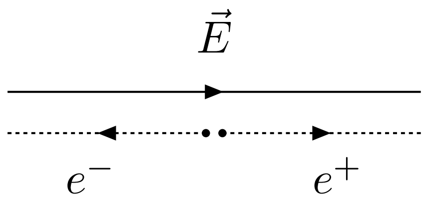


Figure 1. Pair creation by an external field.

Twenty years later, J. Schwinger [2] used the more advanced methods then available for QED to obtain his famous formulas for the imaginary parts of the one-loop effective actions in a constant homogeneous field, for both spinor and scalar QED:

$$\mathrm{Im}\mathcal{L}^{(1)}_{\mathrm{spin}}(E) = \frac{m^4}{8\pi^3}\beta^2 \sum_{n=1}^{\infty} \frac{1}{n^2} \exp\left[-\frac{\pi n}{\beta}\right] \tag{1.1}$$

$$\mathrm{Im}\mathcal{L}^{(1)}_{\mathrm{scal}}(E) = -\frac{m^4}{16\pi^3}\beta^2 \sum_{n=1}^{\infty} \frac{(-1)^n}{n^2} \exp\left[-\frac{\pi n}{\beta}\right] \tag{1.2}$$

($\beta = \frac{eE}{m^2}$; the upper index on $\mathcal{L}$ refers to the loop order). For low pair creation rates, these imaginary parts relate directly to the pair creation rate per volume w, namely one has

$w \approx 2\mathrm{Im}\mathcal{L}$. Schwinger's formulas (1.1),(1.2) are written in terms of an infinite sum of "Schwinger exponentials", where the nth term relates to the coherent production of n pairs by the field. Their dependence on the field strength is nonperturbative, which confirms Sauter's picture of the pair production as a tunneling process. Unfortunately, this also implies that the pair creation rate is exponentially small for $E \ll E_{\mathrm{cr}} := 10^{16} V/\mathrm{cm}$. Until recently, macroscopic fields of this strength were considered unattainable. However, this has changed due to the rapid evolution of laser technology. Presently there are various lasers in planning or operation, e.g. POLARIS [3], ELI [4] and XFEL [5], whose maximum field strength falls only a few orders of magnitude short of E_{cr} (only about one order of magnitude in the case of XFEL). This has led to a number of attempts to bring down the pair creation threshold by a superposition of various optical or X-ray beams; to mention just two recent proposals, counterpropagating linearly polarized lasers were considered in [6] and the superposition of a plane-wave X-ray beam with a strongly focused optical laser pulse in [7]. Obviously, such configurations have field strengths very far from the constant homogeneous case considered by Schwinger, and there is little hope for an exact calculation of the associated pair creation rates; approximation methods will have to be used. The tunnel effect picture suggest the use of WKB, and this is indeed the method which in the past has been almost universally used for the calculation of pair creation rates in nonconstant fields; see, e.g., [8, 9, 10, 11, 12].

The subject of this talk, the *worldline instanton method*, is related to the WKB approximation, but closer in spirit to modern relativistic quantum field theory, which leads to certain advantages. This approach was invented in 1982 by I. K. Affleck, O. Alvarez and N. S. Manton [13] ("AAM" in the following) for the case of a constant field in scalar QED, using Feynman's worldline representation of the scalar QED effective action [14]. In the quenched (single scalar loop) approximation, this representation is

$$\Gamma_{\mathrm{scalar}}[A] \;=\; \int d^4x\, \mathcal{L}(A) = \int_0^\infty \frac{dT}{T}\, \mathrm{e}^{-m^2 T} \int_{x(T)=x(0)} \mathcal{D}x(\tau)\, e^{-S[x(\tau)]} \tag{1.3}$$

Here m and T are the mass and proper time of the loop scalar, and the path integral $\int \mathcal{D}x(\tau)$ is over closed trajectories in (Euclidean) spacetime with a worldline action $S[x(\tau)]$ that has three parts:

$$S \;=\; S_0 + S_{\mathrm{ext}} + S_{\mathrm{int}} \tag{1.4}$$

where

$$\begin{aligned} S_0 &= \int_0^T d\tau \frac{\dot{x}^2}{4} \\ S_{\mathrm{ext}} &= ie \int_0^T \dot{x}^\mu A_\mu(x(\tau)) \\ S_{\mathrm{int}} &= -\frac{e^2}{8\pi^2} \int_0^T d\tau_1 \int_0^T d\tau_2 \frac{\dot{x}(\tau_1)\cdot\dot{x}(\tau_2)}{(x(\tau_1)-x(\tau_2))^2} \end{aligned} \tag{1.5}$$

S_0 describes the free propagation of the scalar particle, S_{ext} its interaction with the background field, and S_{int} internal photon exchanges in the loop.

The basic idea is to calculate the path integral by a stationary phase approximation. We will carry it through first at the one-loop level.

2. One-loop QED

The one-loop effective action is obtained from (1.3) by omitting S_{int},

$$\Gamma^{(1)}_{\text{scalar}}[A] \;=\; \int_0^\infty \frac{dT}{T}\,\mathrm{e}^{-m^2T}\int \mathcal{D}x\,\mathrm{e}^{-\int_0^T d\tau\left(\frac{\dot{x}^2}{4}+ieA\cdot\dot{x}\right)} \tag{2.1}$$

Rescaling $\tau = Tu$, this becomes

$$\Gamma^{(1)}_{\text{scalar}}[A] \;=\; \int_0^\infty \frac{dT}{T}\,\mathrm{e}^{-m^2T}\int \mathcal{D}x\,\mathrm{e}^{-\left(\frac{1}{T}\int_0^1 du\,\dot{x}^2+ie\int_0^1 duA\cdot\dot{x}\right)} \tag{2.2}$$

This makes it apparent that the T integral has a stationary point at $T_c = \sqrt{\int_0^1 du\,\dot{x}^2}/m$. If we are only interested in the imaginary part of the effective action at large mass, we can use this stationary point to eliminate the T integral, yielding

$$\mathrm{Im}\Gamma^{(1)}[A] \;=\; \frac{1}{m}\sqrt{\frac{2\pi}{T_c}}\,\mathrm{Im}\int \mathcal{D}x\,\mathrm{e}^{-\left(m\sqrt{\int du\,\dot{x}^2}+ie\int_0^1 duA\cdot\dot{x}\right)} \tag{2.3}$$

The new worldline action,

$$S \;=\; m\sqrt{\int_0^1 du\,\dot{x}^2}+ie\int_0^1 duA\cdot\dot{x} \tag{2.4}$$

is stationary if

$$m\frac{\ddot{x}_\mu}{\sqrt{\int_0^1 du\,\dot{x}^2}} \;=\; ieF_{\mu\nu}\dot{x}_\nu \tag{2.5}$$

Contracting this equation with $\dot{x}^\mu$ shows that $\dot{x}^2 = \text{const.} \equiv a^2$, so that $m\ddot{x}_\mu = ieaF_{\mu\nu}\dot{x}_\nu$. Thus the extremal action trajectory $x^{\text{cl}}(u)$, to be called *worldline instanton*, is simply a periodic solution of the Lorentz force equation. The semiclassical (large mass) approximation becomes

$$\mathrm{Im}\mathcal{L}(E) \;\sim\; \mathrm{e}^{-S[x^{\text{cl}}]} \tag{2.6}$$

In the case of a constant field $\mathbf{E} = (0,0,E)$ considered by AAM, the worldline instanton turns out to be a circle in the $x_3 - x_4$ plane, of radius m/eE and winding number n:

$$x^{\text{cl}}(u) \;=\; \frac{m}{eE}\Big(x_1, x_2, \cos(2n\pi u), \sin(2n\pi u)\Big) \tag{2.7}$$

$$S[x^{\text{cl}}] \;=\; n\pi\,\frac{m^2}{eE} \tag{2.8}$$

Thus the instanton action for winding number n reproduces the nth exponent in Schwinger's representation (1.2) of $\mathrm{Im}\mathcal{L}^{(1)}_{\text{scal}}(E)$.

Worldline instantons for more general electric fields where obtained in [15]. As an example, let us show the case of the "single-bump" field $E(x_3) = E\,\mathrm{sech}^2(kx_3)$. Here the nth instanton solution is

$$\begin{aligned} x_3(u) &= \frac{m}{eE}\frac{1}{\tilde{\gamma}}\,\mathrm{arcsinh}\left(\frac{\tilde{\gamma}}{\sqrt{1-\tilde{\gamma}^2}}\,\sin(2n\pi u)\right)\\ x_4(u) &= \frac{m}{eE}\frac{1}{\tilde{\gamma}\sqrt{1-\tilde{\gamma}^2}}\,\arcsin(\tilde{\gamma}\cos(2n\pi u)) \end{aligned} \tag{2.9}$$

where the parameter $\tilde{\gamma} \equiv \frac{mk}{eE}$ is a measure for the inhomogeneity of the field. The instanton action is

$$S[x^{\rm cl}] = n \frac{m^2\pi}{eE}\left(\frac{2}{1+\sqrt{1-\tilde{\gamma}^2}}\right) \tag{2.10}$$

For $\tilde{\gamma} \to 0$ one recovers the constant field instanton. With increasing $\tilde{\gamma}$ the instanton action increases, which means that the associated pair creation rate goes down. For $\tilde{\gamma} > 1$ the instanton ceases to exist, in accordance with the fact that the maximal energy which can be extracted from the field by a virtual pair drops below $2m$. In fig. 2 the instanton trajectories are plotted for various values of $\tilde{\gamma}$.

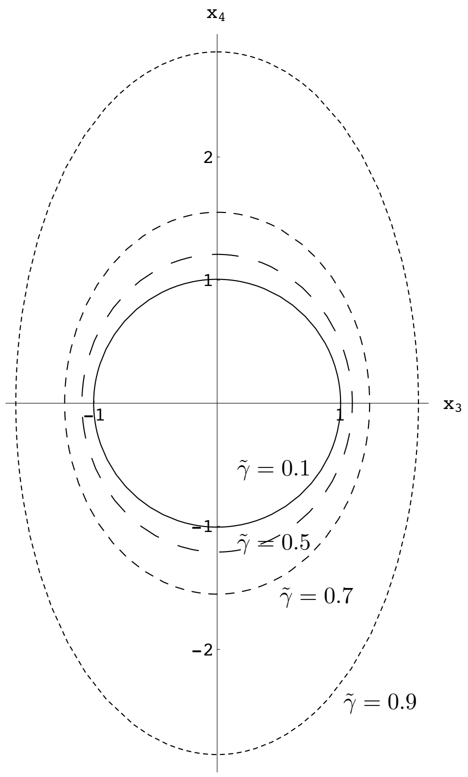


Figure 2. Plot of the instanton paths for the case $E(x) = E\mathrm{sech}^2(kx)$.

In [16] a general method was developed to also calculate the prefactor of the instanton-induced Schwinger exponential $\mathrm{e}^{-S[x^{\rm cl}]}$, that is, the fluctuation determinant of the expansion of the worldline path integral around the stationary trajectory. Including this prefactor for the single-bump case one finds excellent numerical agreement [16] with an exact result by A.I. Nikishov [17] and as well with a direct Monte Carlo integration of the worldline path integral [18].

See [16] for some other cases where the instantons can be found in closed form, [19] for more general classes of fields, and [15] for the extension to spinor QED.

3. Multiloop QED

As argued already by [13], the constant field worldline instantons (2.7) remain stationary trajectories even in the presence of photon insertions. Evaluating the photon insertion term S_{int} on the principal ($n = 1$) trajectory gives the following all-loop formula for the imaginary part of the effective Lagrangian in the large mass (= weak field) approximation:

$$\mathrm{Im}\mathcal{L}_{\mathrm{scal}}(E) = \sum_{l=1}^{\infty} \mathrm{Im}\mathcal{L}^{(l)}_{\mathrm{scal}}(E) \quad \overset{\beta\to 0}{\sim} \quad \frac{m^4\beta^2}{8\pi^3} \exp\Big[-\frac{\pi}{\beta} + \alpha\pi\Big] = \mathcal{L}^{(1)}_{\mathrm{scal}}\, \mathrm{e}^{\alpha\pi} \tag{3.1}$$

This formula is rather remarkable, since in terms of standard field theory it corresponds to a summation of the infinite set of Feynman diagrams shown in fig. 3:

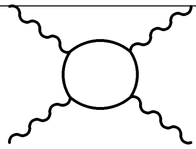
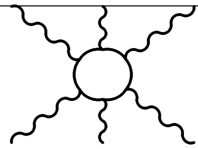
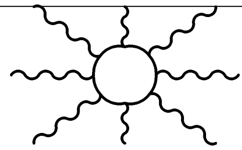
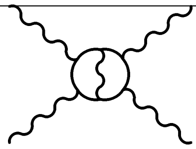
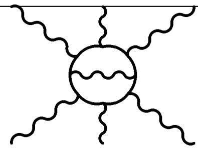
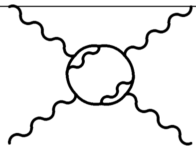
	Number of external legs			
Number of loops	4	6	8	$\cdots$
1				$\cdots$
2			$\cdots$	$\cdots$
3		$\cdots$	$\ddots$	$\vdots$
$\vdots$	$\vdots$	$\ddots$	$\ddots$	$\vdots$

Figure 3. Feynman diagrams contributing to the AAM formula.

Moreover, the mass in (3.1) is the physical mass [13], so that fig. 3 effectively includes also an infinite set of mass renormalization counterdiagrams.

In 1984, S. L. Lebedev and V. I. Ritus [20, 21] obtained the same exponentiation by an analysis of the corrections due to photon exchanges in Sauter's tunnelling picture, for both scalar and spinor QED. They also confirmed this $\mathrm{e}^{\alpha\pi}$ factor to linear order in α by a direct calculation of $\mathcal{L}^{(2)}_{\mathrm{spin}}(E)$.

Apart from the Schwinger pair creation effect, the AAM formula (3.1) and its spinor QED equivalent hold also interesting information on the multiloop QED photon amplitudes, in the limit of low photon energy and large photon number. This information can be extracted using Borel dispersion relations [22, 23], which allows one to show, for example, that the large N behaviour of these amplitudes for $l \geq 2$ is dominated by the one-fermion loop contributions, and qualitatively different for physical and generic mass renormalization. More interestingly, it suggests that the QED perturbation series may converge for the one-fermion loop contributions to the N − photon amplitudes [23, 24, 25, 26]. If correct, this would extend an old conjecture by P. Cvitanovic [27] for the $g - 2$ factor.

4. One-loop open string theory

At the one-loop level, pair creation by a constant homogeneous electric field has also been studied for string theory, both for the open [28, 29] and the heterotic string [29]. For the open string case, C. Bachas and M. Porrati [29] obtained the following formula for the imaginary part of the effective action:

$$
\mathrm{Im}\mathcal{L}^{(1)}_{\mathrm{string}}(E) = \frac{1}{4(2\pi)^{D-1}} \sum_{\mathrm{states\,S}} \frac{\beta_1+\beta_2}{\pi\epsilon} \sum_{n=1}^{\infty} (-)^{n+1} \Bigl(\frac{|\epsilon|}{n}\Bigr)^{D/2} \exp\Bigl(-\frac{\pi n}{|\epsilon|}(M_S^2+\epsilon^2)\Bigr) \tag{4.1}
$$

Here the first sum is over the physical states of the bosonic string, with M_S the mass of the state. The second sum is a Schwinger-type sum. $D = 26$ is the critical dimension for the open string. We have defined

$$
\beta_{1,2} \;=\; \pi q_{1,2} E \tag{4.2}
$$

where $q_{1,2}$ are the $U(1)$ charges at the string endpoints, and

$$
\epsilon \;=\; \frac{1}{\pi}\Bigl(\mathrm{arctanh}\beta_1 + \mathrm{arctanh}\beta_2\Bigr) \tag{4.3}
$$

In the weak field limit, (4.1) reproduces the scalar Schwinger formula (1.2), as well as its generalization to arbitrary integer spin J. For stronger fields the right hand grows much faster than for the particle case, and even diverges at a critical field strength

$$
E_{\mathrm{cr}} \;=\; \frac{1}{\pi\,|\max q_i|} \tag{4.4}
$$

Presumably this means that a field of this strength would break the string apart.

We will now show how to reproduce, in the large mass limit, the Schwinger exponents in the open-string result of (4.1) by a worldsheet instanton which is a natural generalization of the AAM worldline instanton (2.7) [30].

To write down a path integral for the open string at one-loop in an external Maxwell field, one has to replace the particle loop by a string annulus and insert the interaction term S_{ext} of (1.5) on both boundaries of the loop, with the charge e replaced by $q_{1,2}$ (we assume $q_1 \neq q_2$ to exclude the Möbius strip contribution to the amplitude). The effective action becomes, in conformal gauge,

$$
\Gamma^{(1)}[F] \;=\; \frac{1}{2}\int_0^\infty \frac{dT}{T}(4\pi^2 T)^{-\frac{D}{2}} Z(T) \int \mathcal{D}x\, \mathrm{e}^{-S_E[x,F]} \tag{4.5}
$$

where the path integral is over the embeddings of the annulus at fixed T into D - dimensional flat space. The worldsheet action is

$$
S_E \;=\; \frac{1}{4\pi\alpha'}\int d\sigma d\tau\, \partial_a x^\mu \partial^a x_\mu - i\frac{q_1}{2}\int d\tau\, x^\mu \partial_\tau x^\nu F_{\mu\nu}\Big|_{\sigma=0} - i\frac{q_2}{2}\int d\tau\, x^\mu \partial_\tau x^\nu F_{\mu\nu}\Big|_{\sigma=\frac{1}{2}} \tag{4.6}
$$

Here α' is the Regge slope, which will be set equal to $\frac{1}{2}$ in the following. The worldsheet is parametrized as a rectangle $\sigma \in [0, \frac{1}{2}]$ and $\tau \in [0, T]$ where $\tau = T$ is identified with $\tau = 0$. We use euclidean conventions where $\sigma^0 = -i\sigma^2 = -i\tau$, $x^0 = -ix^D$, and $A_D = -iA_0$. $Z(T)$ is the partition function,

$$Z(T) = \sum_{\text{oriented states}} \mathrm{e}^{-\pi T M_S^2} \tag{4.7}$$

Let us now consider the constant electric field case, $F_{D,D-1} = -F_{D-1,D} = iE$. We rescale $\tau = Tu$ and do the T - integral by the method of steepest descent. The stationary point is

$$T_0 = \sqrt{\frac{I_u}{I_\sigma + 2\pi^2 M_S^2}} \tag{4.8}$$

where

$$\begin{aligned} I_\sigma &:= \int_0^1 du \int_0^{\frac{1}{2}} d\sigma\, \partial_\sigma x^\mu \partial_\sigma x_\mu \\ I_u &:= \int_0^1 du \int_0^{\frac{1}{2}} d\sigma\, \partial_u x^\mu \partial_u x_\mu \end{aligned} \tag{4.9}$$

The new worldsheet action is

$$S_{\text{eff}} = \frac{1}{\pi}\sqrt{I_u}\sqrt{I_\sigma + 2\pi^2 M_S^2} - i\frac{q_1}{2}\int d\tau\, x^\mu \partial_\tau x^\nu F_{\mu\nu}\Big|_{\sigma=0} - i\frac{q_2}{2}\int d\tau\, x^\mu \partial_\tau x^\nu F_{\mu\nu}\Big|_{\sigma=\frac{1}{2}} \tag{4.10}$$

It leads to the equations of motion

$$\begin{aligned} \Big[I_u \partial_\sigma^2 + (I_\sigma + 2\pi^2 M_S^2)\partial_u^2\Big] x^\mu &= 0 \\ T_0 \partial_\sigma x^\mu &= i\pi q_2 F_{\mu\nu} \partial_u x^\nu \qquad (\sigma = \frac{1}{2}) \\ T_0 \partial_\sigma x^\mu &= -i\pi q_1 F_{\mu\nu} \partial_u x^\nu \qquad (\sigma = 0) \end{aligned} \tag{4.11}$$

The nth worldsheet instanton solving these equations is found by the ansatz

$$\begin{aligned} x_n^{D-1} &= \frac{2\pi M_S}{|a|}\cos(2\pi n u)\cosh(b - a\sigma) \\ x_n^D &= \frac{2\pi M_S}{|a|}\sin(2\pi n u)\cosh(b - a\sigma) \end{aligned} \tag{4.12}$$

(with the remaining coordinates constants). We take equal signs for n and a. Plugging this ansatz into (4.11) one finds that all the parameters can be determined, namely

$$
\begin{aligned}
T_0 &= \frac{2\pi n}{a} \\
b &= \mathrm{arctanh}\beta_1 \\
a &= 2\Big(\mathrm{arctanh}\beta_1 + \mathrm{arctanh}\beta_2\Big) \\
N &= \frac{2\pi M_S}{|a|}
\end{aligned}
\tag{4.13}
$$

This yields the worldsheet action

$$
S[x^\mu] = 2\pi^2 M_S^2 \frac{n}{a} \tag{4.14}
$$

Noting that $a = 2\pi\epsilon$ this correctly reproduces the exponents of the Bachas-Porrati formula (4.1) in the large M_S limit. The method of calculation is, however, substantially simpler than the ones applied in [28] and [29]. Thus we hope that this worldsheet instanton technique will make it feasible to obtain string pair creation rates also for certain classes of nonconstant electric fields. An interesting aspect of the constant field instanton (4.12) is, that its restriction to each boundary coincides with the worldline instanton (2.7). It is not clear whether this property may extend to more general fields, but if so it would greatly facilitate finding the corresponding worldsheet instantons.

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