3D O(2) non-linear sigma model on the lattice and cosmic strings

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RADPyC 23

Non-linear sigma model 3D O(2)

- 3D Euclidean space
- Field variables in S^1.
- 3D O(2) model, 3D XY model or 3D plane rotator.
- Relevant to the critical behavior of superfluids and planar magnets.

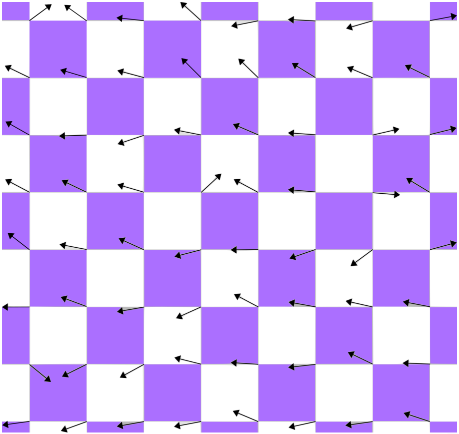
Partition Function

$$Z = \int \mathcal{D}\vec{\sigma} \;\; e^{-\beta \mathcal{H}[\vec{\sigma}]}.$$

- Where $\beta = 1/T$, with $k_B = 1$.

Non-linear sigma model 3D O(2)

- In the lattice regularization, at each site there is a spin $\vec{\sigma}_x = (\cos\theta_x, \sin\theta_x)$, where $\theta_x \in (-\pi, \pi]$ and $x = (x_1, x_2, x_3)$.
- We employ a cubic lattice Λ with periodic boundary conditions.
- Example of a 2D O(2) configuration:



Non-linear sigma model 3D O(2)

- We consider two distinct Hamiltonians.

Standard Ferromagnetic Hamiltonian

$$\mathcal{H}[\vec{\sigma}] = -\sum_{\langle ij \rangle} \vec{\sigma}_i \cdot \vec{\sigma}_j$$

- $\sum_{\langle ij \rangle}$ denotes sum over nearest-neighbors.

Topological Hamiltonian

$$\mathcal{H}[\vec{\sigma}] = \begin{cases} 0 & \text{if} \quad |\Delta\varphi_{x,x+\hat{\mu}}| < \delta, \qquad \forall x, \hat{\mu} \\ +\infty & \text{otherwise} \end{cases}$$

- $\Delta\varphi_{x,x+\hat{\mu}} = \varphi_{x+\hat{\mu}} - \varphi_x \mod 2\pi \in (-\pi, \pi)$. $|\hat{\mu}| = 1$.
- φ_x is the angle of $\vec{\sigma}_x$ with respect to a fixed point in S^1.
- Most configurations are invariant under infinitesimal transformations.

String defects within the Standard Model

- In the Early Universe, the unbroken electroweak symmetry was,

$$\mathrm{SU}(2)_{\mathrm{L}} \times \mathrm{U}(1)_{\mathrm{Y}}.$$

- Electroweak symmetry "breaks spontaneously", at $\approx$ 159 GeV[1], when the Higgs field acquires a vacuum expectation value (VEV),

$$\mathrm{SU}(2)_{\mathrm{L}} \times \mathrm{U}(1)_{\mathrm{Y}} \longrightarrow \mathrm{U}(1)_{\mathrm{EM}}.$$

[1] d'Onofrio, Michela; Rummukainen, Kari (2016). "Standard model cross-over on the lattice". Physical Review D. 93 (2): 025003.

String defects within the Standard Model

- When the temperature of the Universe dropped below 159 GeV, the Higgs field in most regions will acquire a VEV, but for causality reasons the phases in widely separated regions will be uncorrelated (Kibble mechanism[2]).
- String defects are trapped in interfaces where the phase change around a loop is 2π.

[2] Kibble, Tom W K (1976). "Topology of cosmic domains and strings". Journal of Physics A: Mathematical and General. 9 (8): 1387–1398.

Time evolution of string defects

- A random network of strings may arise.
- The characteristic scale of this network is the correlation length ξ.
- At a short time after string formation the Universe is very dense, which damps the string fluctuations.
- The damping force on strings increases ξ, but for causality reasons, ξ can never grow faster than ct. This requires a mechanism for energy dissipation, which involves gravitational radiation and particle emission.

3D O(2) string defects

- The 3D O(2) model undergoes a second-order phase transition, where the O(2) symmetry breaks,
- The vacuum manifold is isomorphic to S^1,

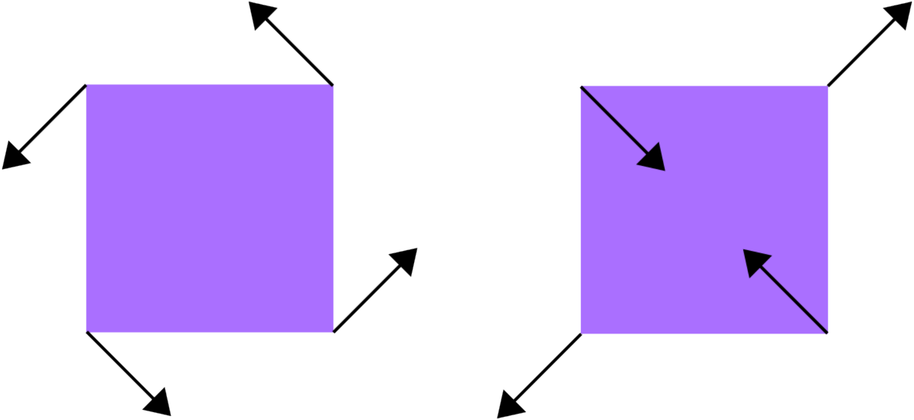
$$\pi_1(S^1) = \mathbb{Z},$$

which suggest the existence of string defects.

- Strings defects in this model are formed from vortex lines.

Vortices

- Field configurations with vortices and anti-vortices.



Vorticity

- Each plaquette has a vorticity,

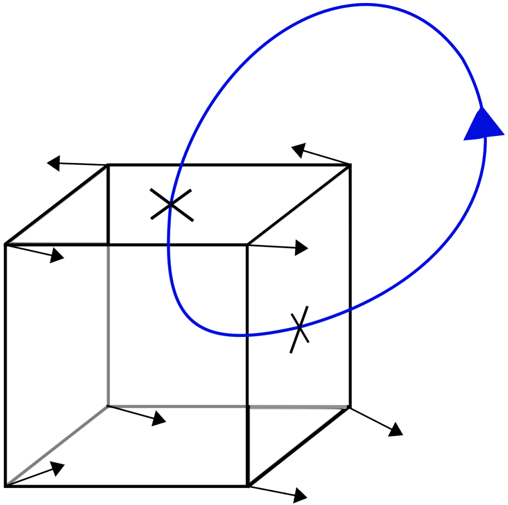
$$v_{\mu\nu}(x) = \frac{1}{2\pi}\left(\Delta\varphi_{x,x+\hat{\mu}} + \Delta\varphi_{x+\hat{\mu},x+\hat{\mu}+\hat{\nu}} + \Delta\varphi_{x+\hat{\mu}+\hat{\nu},x+\hat{\nu}} + \Delta\varphi_{x+\hat{\nu},x}\right),$$

  where $|\hat{\mu}|, |\hat{\nu}| = 1$.
- There are only 3 values for the plaquettes vorticity,

$$v_{\mu\nu}(x) \in \{-1 \text{ (antivortex)}, 0, 1 \text{ (vortex)}\}.$$

Vortex lines

- Vortices tend to form vortex lines.



Observables

- Magnetization density,

$$m = \frac{1}{L^3}\langle|\vec{M}[\vec{\sigma}]|\rangle, \;\; \vec{M}[\vec{\sigma}] = \sum_{x\in\Lambda}\vec{\sigma}_x.$$

- Magnetic susceptibility,

$$\chi_M = \frac{1}{L^3}\left(\langle|\vec{M}[\vec{\sigma}]|^2\rangle - \langle|\vec{M}[\vec{\sigma}]|\rangle^2\right).$$

- Connected correlation function,

$$C(x,y) = \langle\vec{\sigma}_x\cdot\vec{\sigma}_y\rangle - \langle\vec{\sigma}_x\rangle\cdot\langle\vec{\sigma}_y\rangle.$$

- At large distance $|x| \gg 1$,

$$C(0,x) = C(x) \propto \cosh\left(\frac{x - L/2}{\xi}\right).$$

- ξ correlation length. Physical scale.

Critical exponents

In a cubic lattice of volume L^3, at T_c, observables grow as follows,

$$\xi \propto |T - T_c|^{-\nu},$$

$$m \propto L^{-\beta/\nu},$$

$$\chi_M \propto L^{\gamma/\nu},$$

Models in the same universality class share same values of critical exponents.

Critical exponents

- For the standard action $\beta_c = 0.454168(5)$.
- For the topological action $\delta_c = 2.5156(2)$.
- Critical exponents,

Standard action

β/ν	γ/ν	ν
0.512(3)	1.97(1)	0.68(2)

Topological action

β/ν	γ/ν	ν
0.506(4)	1.99(1)	0.63(4)

- Compatible critical exponents.
- Both actions fall into the same universality class.

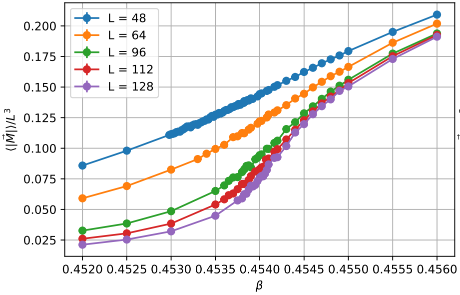
Step scaling function

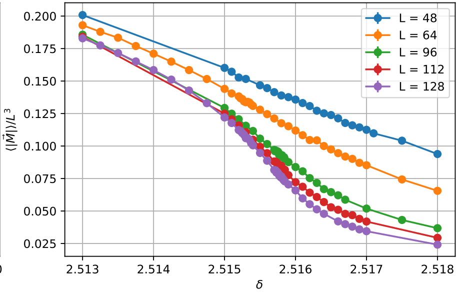
- At some fixed value u_0 of $\xi(L)/L$ we define,

$$\sigma(2, u_0) = \frac{2L}{\xi(2L)}.$$

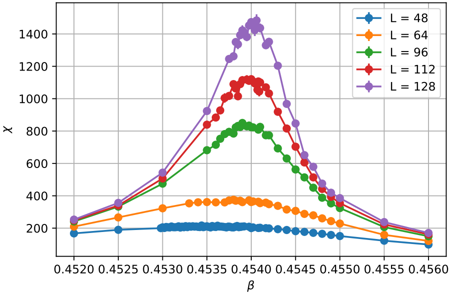
- σ is a universal quantity, based on finite-size effects.".
- σ is sensitive to finite-size effects.
- It allows us to study lattice artifacts.
- $\sigma(2, 1.3) = 0.95(1)$, with the standard action.
- $\sigma(2, 1.3) = 0.94(4)$, with the topological action.

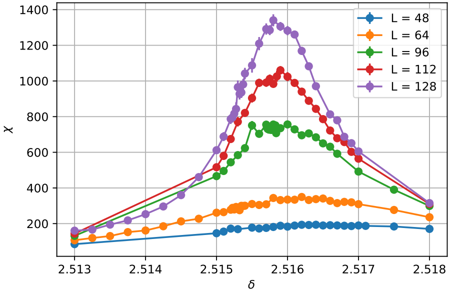
Magnetization





Magnetic susceptibility





Gradient Flow Equation

- We explore the persistence of topological defects after evolution of field configurations through gradient flow equation.
- The general gradient flow equation reads[3],

$$\frac{\delta \sigma_t^a(x)}{\delta t} = -g^{ab}\frac{\delta S(\sigma_t)}{\delta \sigma_t^b(x)},$$

  where S is the action and g^{ab} is non-trivial for non-linear sigma models.
- The evolution of the field through this equation implies the reduction of the action,

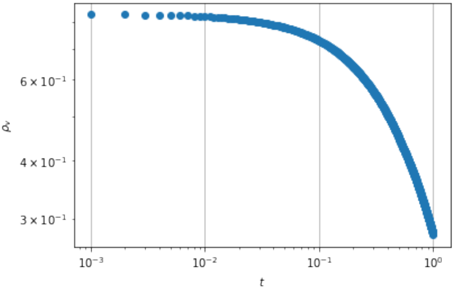
$$\partial_t S(\sigma_t) \leq 0.$$

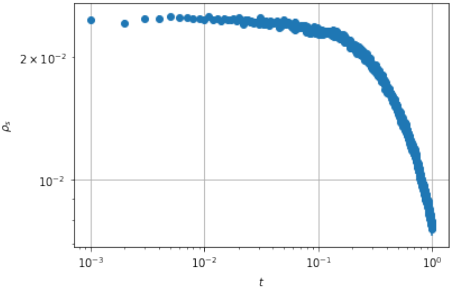
  In the 3D O(2) model, this flow causes vorticity and string density suppression.

[3] Kikuchi, K., Onogi, T. (2014). "Generalized gradient flow equation and its application to super Yang-Mills theory". J. High Energ. Phys. 2014, 94.

Vorticity and String density suppression

- Suppression of densities of topological defects in a volume of 16^3 initially at $T = 2T_c$.





Conclusions

- There is evidence that the proposed topological action is in the same universality class as the standard action.
- The persistence of topological defects can be explored with the evolution through gradient flow equation.
- Vortex lines in the 3D O(2) non-linear sigma model bear strong analogies with global cosmic strings.
- This is a model for the evolution of the Early Universe.