Scalar Fields in

Non-Commutative Spaces

I. **Non-commutative (NC) space and lattice regularization**

II. **The NC $\lambda\phi^4$-model: formulation**
Phase diagram in space-time dimension $d = 3$ and $d = 2$

III. **Can translation symmetry break spontaneously in $d = 2$?**

Héctor Mejía-Díaz, Marco Panero, W.B., JHEP 1410:56

I. Lattice structure of a non-commutative plane

NC plane, $[\hat{x}_i, \hat{x}_j] = \mathrm{i}\,\Theta_{ij} = \mathrm{i}\,\theta\,\epsilon_{ij}$, $\theta = \text{const.}$ $\qquad (i,\, j = 1, 2)$

A (fuzzy) lattice structure is imposed by the operator identity

$$\exp\left(\mathrm{i}\,\frac{2\pi}{a}\hat{x}_i\right) = \hat{1\!\!1}$$

Momentum components are commutative and periodic over Brillouin zone:

$$\begin{aligned} e^{\mathrm{i}\,k_i \hat{x}_i} &= e^{\mathrm{i}\,(k_i + \frac{2\pi}{a})\hat{x}_i} \\ \hat{1\!\!1} &= e^{\mathrm{i}\,(k_i + \frac{2\pi}{a})\hat{x}_i} e^{-\mathrm{i}\,k_j \hat{x}_j} = \cdots = \hat{1\!\!1}\exp\left(\frac{\mathrm{i}\,\pi}{a}\theta(k_2 - k_1)\right) \\ &\Rightarrow\ \frac{\theta}{2a} k_i \in \mathbb{Z} \end{aligned}$$

k_i discrete $\Rightarrow$ lattice is automatically <u>periodic</u>

Assume periodicity over $N \times N$ $\rightarrow$ momenta $k_n = \frac{2\pi}{aN} n \quad (n_i \in \mathbb{Z})$

$$\theta = \frac{1}{\pi} N a^2$$

- continuum limit: $a \rightarrow 0$
- infinite volume limit: $Na \rightarrow \infty$

The Double-Scaling Limit

$$a \rightarrow 0, \; N \rightarrow \infty \quad \text{at} \quad Na^2 = \text{const.}$$

combines both at $\theta = \text{const.}$: continuous NC plane of infinite extent.

Simultaneous limit in the spirit of UV/IR mixing.

II. The NC $\lambda\phi^4$-model

Formulation for NC field theory in terms of ordinary coordinates x_μ, if all fields are multiplied by $\star$-products :

$$\phi(x) \star \psi(x) := \phi(x) \exp\left(\frac{\mathrm{i}}{2} \overleftarrow{\partial}_\mu \Theta_{\mu\nu} \overrightarrow{\partial}_\nu\right) \psi(x)$$

based on plane wave decomposition, $e^{\mathrm{i}\, p_\mu \hat{x}_\mu} e^{\mathrm{i}\, q_\nu \hat{x}_\nu} = e^{\mathrm{i}\,(p+q)_\mu \hat{x}_\mu - \frac{\mathrm{i}}{2} p_\mu \Theta_{\mu\nu} q_\nu}$

Euclidean action:

$$S[\phi] = \int d^d x \left[\frac{1}{2} \partial_\mu \phi \, \partial_\mu \phi + \frac{m^2}{2} \phi^2 + \frac{\lambda}{4} \phi \star \phi \star \phi \star \phi \right]$$

Bilinear terms under $\int$: $\star$-product $\equiv$ standard product (since $\Theta_{\mu\nu} = -\Theta_{\nu\mu}$)

$\Rightarrow$ λ determines extent of NC effects.

Perturbation theory:

1-loop diagrams: $\int d^d k \, \frac{1}{k^2+m^2}$, $\int d^d k \, \frac{\exp(\mathrm{i}\, k_\mu \Theta_{\mu\nu} p_\nu)}{k^2+m^2}$

planar non-planar

leading divergence
in $d = 4$, $|k| < \Lambda$: $\propto \Lambda^2$ $\propto [1/\Lambda^2 + p_\mu (\Theta^2)_{\mu\nu} p_\nu]^{-1}$

(Minwalla/Van Raamsdonk/Seiberg, '00)

$\|\Theta\| > 0$: removes non-planar UV divergence, unless $p \to 0$

Limit $\Theta \to 0$ is *not smooth*, beware of expansion in small $\|\Theta\|$

$\|\Theta\| \to \infty$ is commutative, but different from $\Theta = 0$

First consider $d = 3$

$\phi(\vec{x}, t)$, NC plane + commutative Euclidean time t

Action on a $N^2 \times T$ lattice can be mapped onto a matrix model with twisted boundary conditions (Ambjørn/Makeenko/Nishimura/Szabo, '00)

$$S[\bar{\phi}] = \mathrm{Tr} \sum_{t=1}^{T} \Big[\frac{1}{2} \sum_{i=1}^{2} \Big(\Gamma_i \bar{\phi}(t) \Gamma_i^\dagger - \bar{\phi}(t) \Big)^2 + \frac{1}{2} \Big(\bar{\phi}(t+1) - \bar{\phi}(t) \Big)^2 + \frac{m^2}{2} \bar{\phi}^2(t) + \frac{\lambda}{4} \bar{\phi}^4(t) \Big]$$

$\bar{\phi}(t)$: Hermitian $N \times N$ matrices, at $t = 1 \dots T$

- Time direction: ordinary (discrete) kinetic term

• NC plane: unitary "twist eaters" Γ_i provide shift by one lattice unit, if

$$\Gamma_i \Gamma_j = Z_{ji} \Gamma_j \Gamma_i \qquad (\text{'t Hooft-Weyl algebra}).$$

We use $Z_{21} = Z_{12}^* = e^{2\pi \mathrm{i}\, k/N}$ with $k = (N+1)/2$, $\underline{N \text{ odd}}$

Solution for twist eaters: shift- and clock-operator

$$\Gamma_1 = \begin{pmatrix} 0 & 1 & & & \\ & . & . & & \\ & & . & . & \\ & & & . & 1 \\ 1 & & & & 0 \end{pmatrix}, \quad \Gamma_2 = \begin{pmatrix} 1 & & & & \\ & Z_{21} & & & \\ & & Z_{21}^2 & & \\ & & & . & \\ & & & & . \end{pmatrix}$$

Gubser/Sondhi, '01: 1-loop calculation in Hartree-Fock approximation

$\Rightarrow$ **Conjectured phase diagram** (in $d = 3, 4$) :

- small θ : Ising-type transition: disorder $\leftrightarrow$ uniform order
- larger θ : disorder $\leftrightarrow$ striped order (new!)

(order at $m^2 \ll -||\Theta||^{-1}$ $\sim$ very low temperature)

- Chen/Wu, '02: RG study in $d = 4 - \varepsilon$: striped phase for $\theta > 12/\sqrt{\varepsilon}$

- Castorina/Zappalà, '02: approach with $S_{\rm eff}$
  supports Gubser/Sondhi conjecture

- W.B./F. Hofheinz/J. Nishimura '04: **numerical study**
  Striped phase observed, persists in Double-Scaling Limit

Simulations reveal phase diagram in $m^2 - \lambda$ plane
(large $\lambda \to$ strong NC effects)

$N = T = 15,\ 25,\ 35,\ 45,$ phase transitions stable for $N \geq 25$

Ordered regime splits indeed into

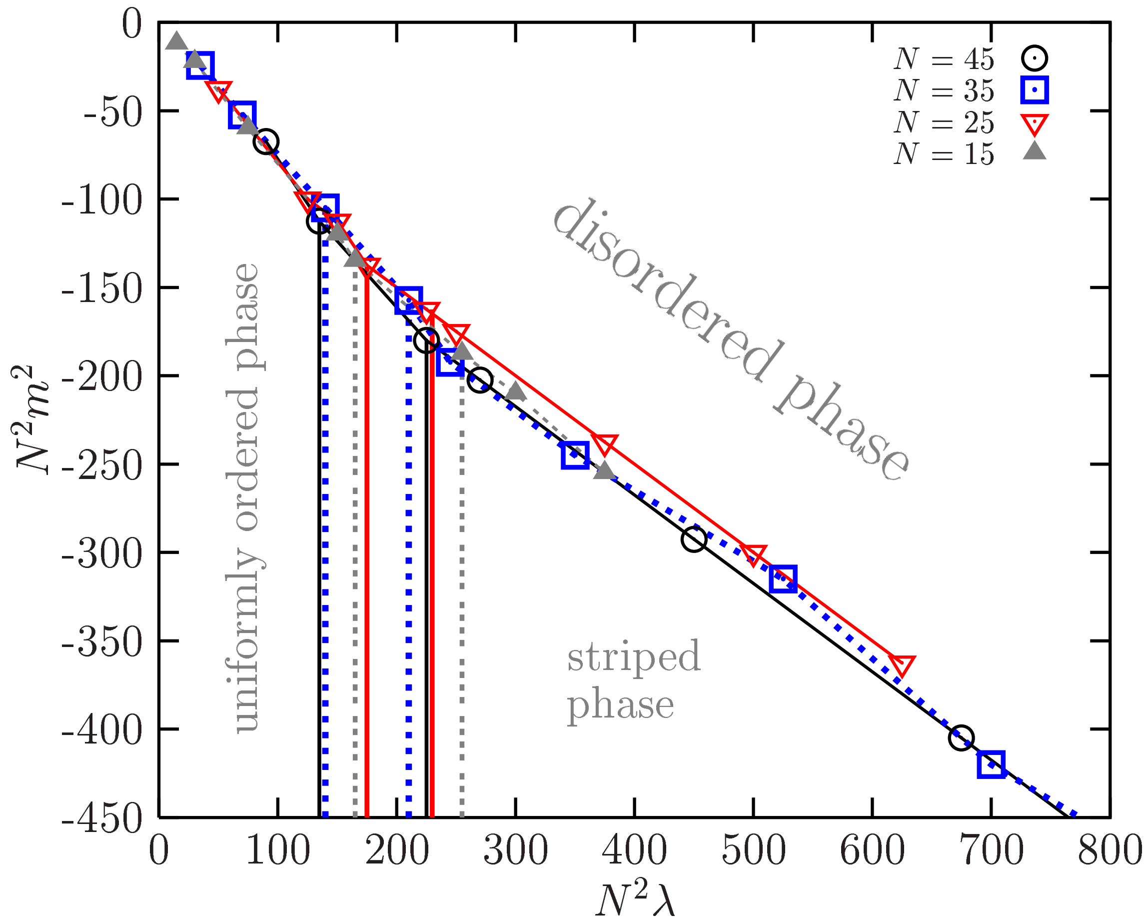
- uniform phase: small λ
- striped phase: larger λ

Evaluation relies on **momentum dependent order parameter**

$$M(k) = \frac{1}{NT} \max_{k=|\vec{p}|N/2\pi} \Big| \sum_t \tilde{\phi}(\vec{p}, t) \Big|$$

(rotation to capture pattern of each configurations)

$M(0)$ uniform order, $M(k > 0)$ detects stripes with width $\propto 1/k$



Thermal cycle: • phase transitions order-disorder of 2^{nd} order (in both cases)

• transition uniform-striped : 1^{st} order (hysteresis cycles don't close)

Striped phase persists in Double-Scaling Limit (W.B./Hofheinz/Nishimura '04)

Corresponding model in $\mathbf{d = 2}$ (skip time coordinate)

Usually a continuous, global sym. cannot break spontaneously in $d \leq 2$

However, **Mermin-Wagner-Coleman Theorem** assumes
locality and **IR regularity**.

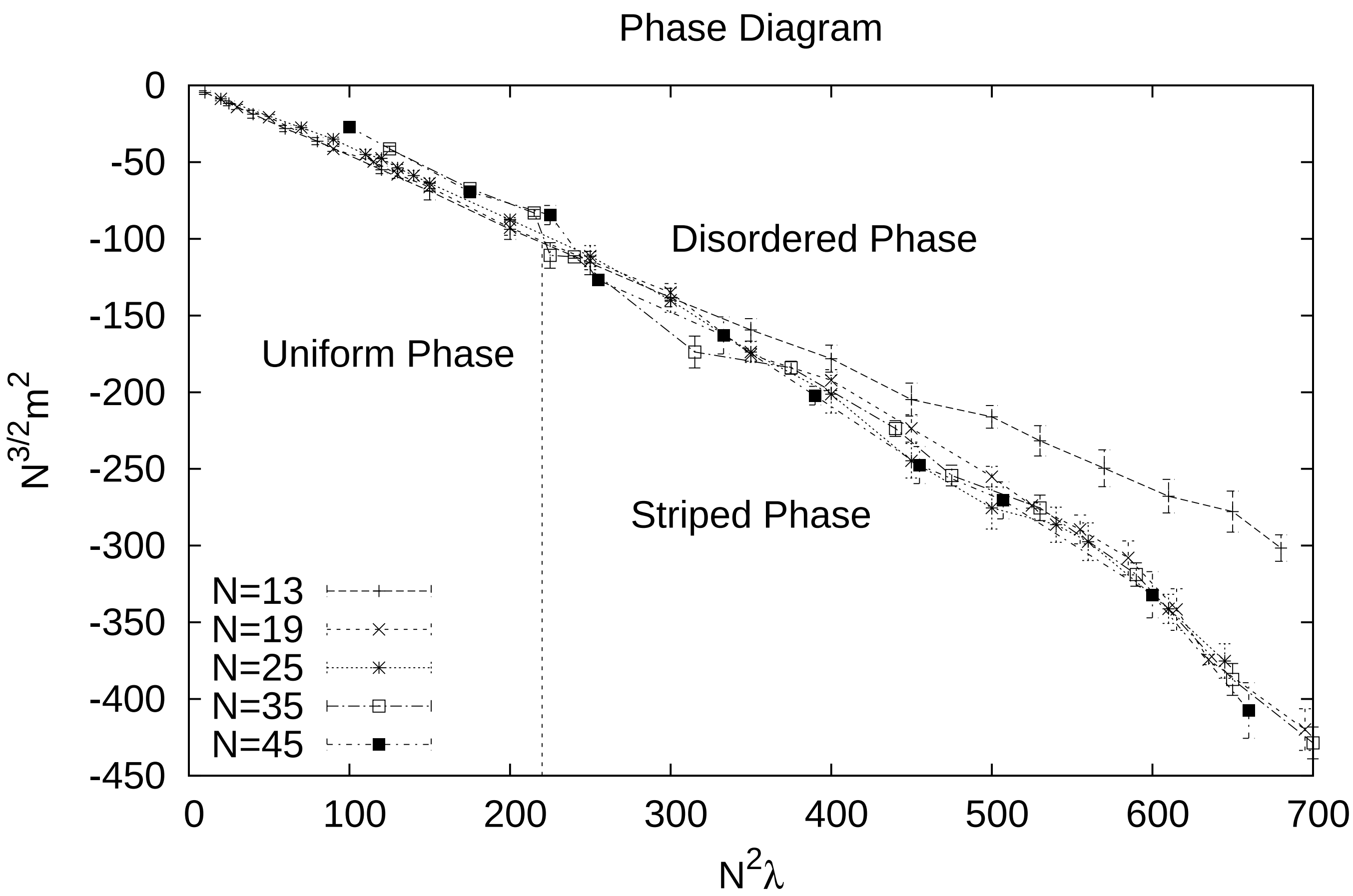
Sill, Gubser/Sondhi '01 do not expect a striped phase (generalised M-W-C)

But: Castorina/Zappalà '07: analysis of S_{eff} seems to allow stripes.

Numerical:
Ambjørn/Catterall '02, W.B./Hofheinz/Nishimura '04 see stripes.

But: does it survive the Double-Scaling Limit?
Or fate like confinement phase of lattice QED?

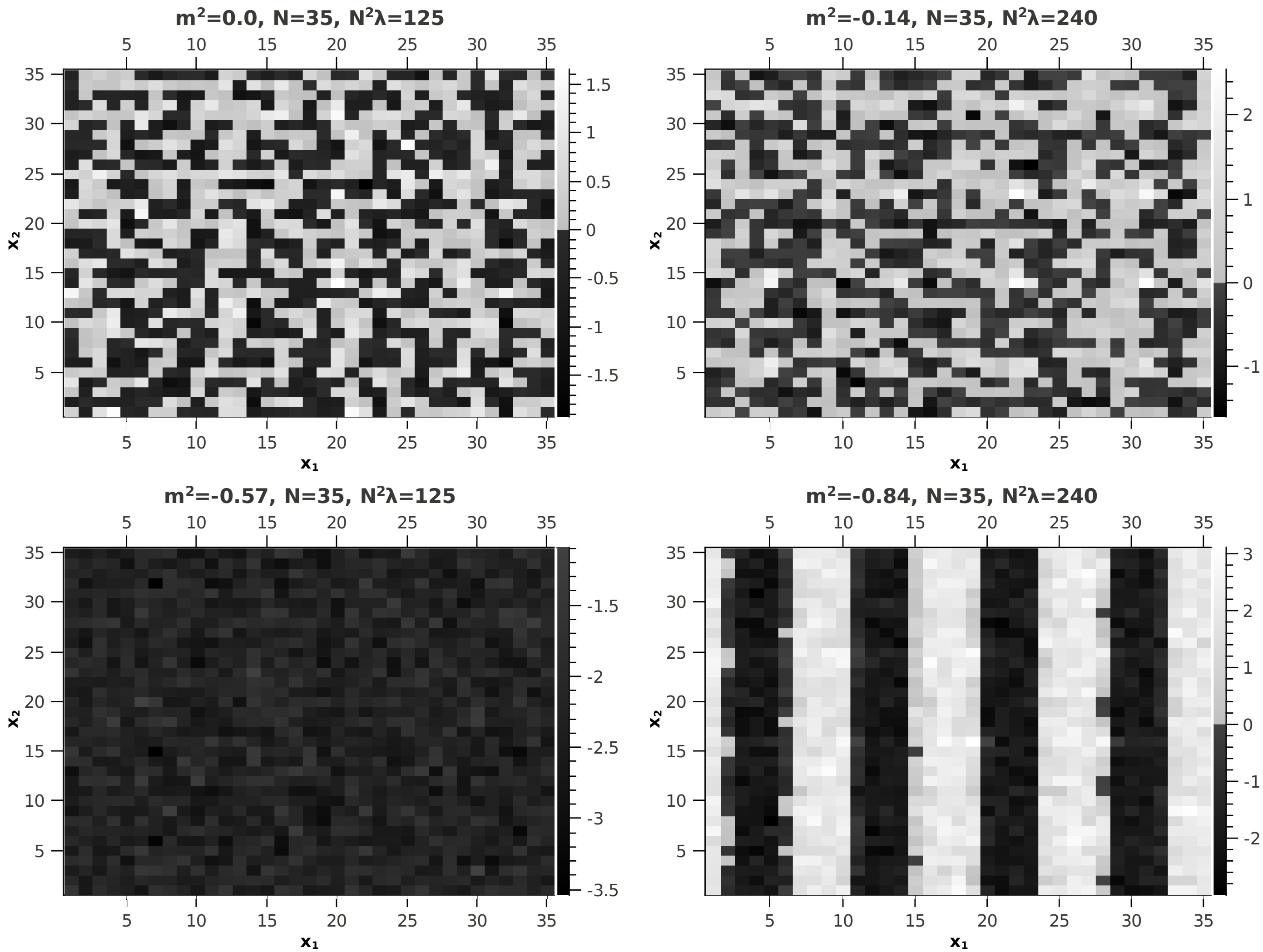
Solved in thesis by Héctor Mejía-Díaz (UNAM)



Phase diagram in $\mathbf{d = 2}$:

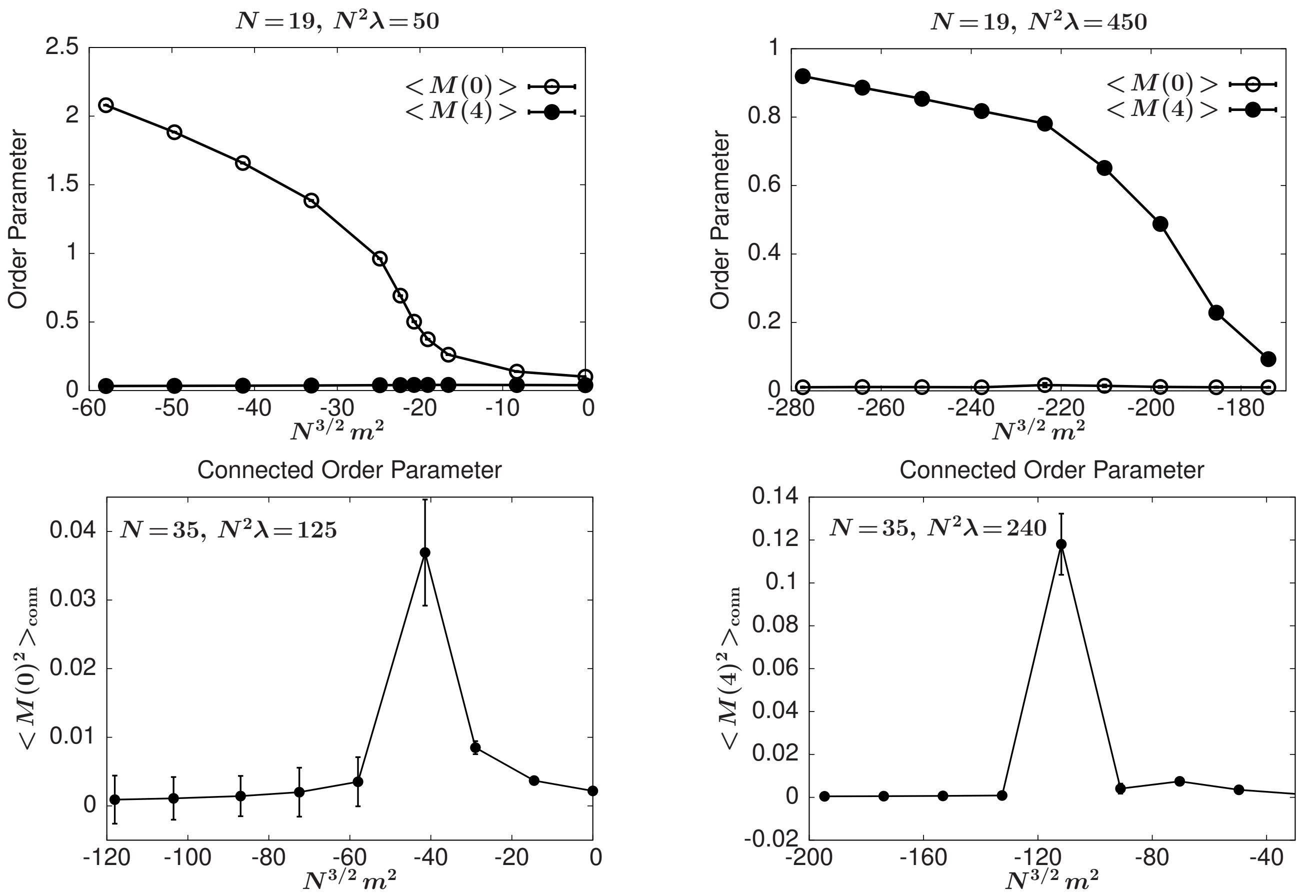
requires scaling of the axes different from $d = 3$: $N^{3/2}m^2$ vs. $N^2\lambda$

Stabilisation for $N \geq 19$



Snapshots: above: disordered, below: uniform/striped order

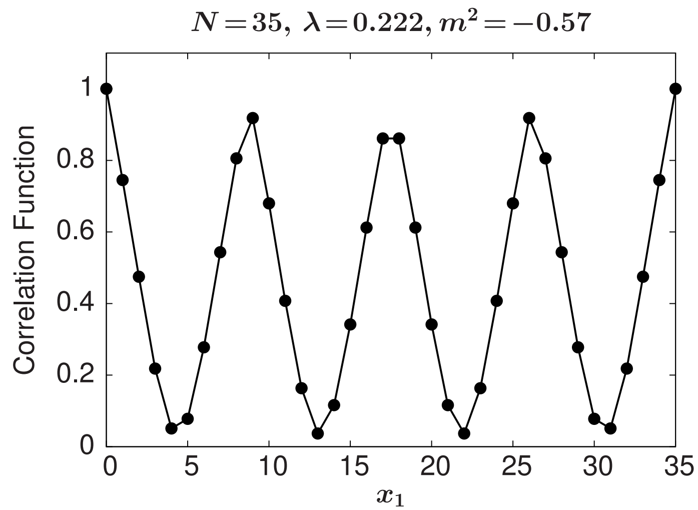
Map matrices back to lattice: $+/- \sim$ dark/bright



Identification of the phase transition order/disorder :

Above: uniform/striped order parameter takes off for decreasing m^2

Below: peak of connected correlator $\langle M^2\rangle_{\rm C} = \langle M^2\rangle - \langle M\rangle^2$ localises critical value $m_{\rm c}^2$



Correlation $\langle\phi_{(0,0)}\phi_{(x_1,0)}\rangle$ near striped phase $(N^{3/2}m^2, N^2\lambda) \simeq (-118, 272)$, pattern not condensed $\longrightarrow$ disordered.

Concept: approach $m^2 \searrow m_c^2$ for increasing N such that the correlator down to the first dip stabilises.

Thus $\Delta m^2 := m^2 - m_c^2$ defines a scale, which translates — with a suitable exponent — into the desired Double-Scaling Limit: $a^2 \propto (\Delta m^2)^\sigma$

Question: does proximity to striped phase persist in this limit?

Ansatz: define $a = 1$ at $N = 35$: $\underline{Na^2 = \text{const.}}$ $\Rightarrow$ $a = \sqrt{\frac{35}{N}}$

Adjust dimension, like reduced temperature $\tau = (T - T_\mathrm{c})/T_\mathrm{c}$,

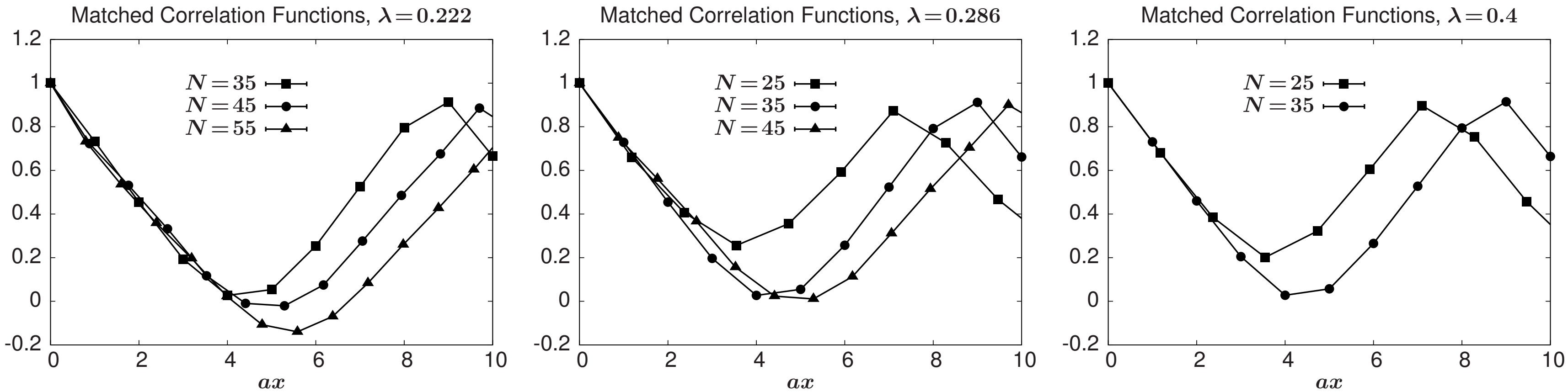
$$a^2 = \frac{(\Delta m^2)^{\sigma}}{(m_\mathrm{c}^2)^{1+\sigma}}$$

Take two sizes N_1, N_2 with Δm_1^2, Δm_2^2, at fixed λ ($\rightarrow$ the dim'less term $\lambda\theta$ remains const.), same correlation decay. Extract exponent

$$\sigma = \frac{\ln(m_{1,\mathrm{c}}^2 / m_{2,\mathrm{c}}^2)}{\ln(\Delta m_{1,\mathrm{c}}^2 / \Delta m_{2,\mathrm{c}}^2) + \ln(m_{1,\mathrm{c}}^2 / m_{2,\mathrm{c}}^2)}$$

σ will stabilise for sufficiently large N_i and small $\Delta m_{i,\mathrm{c}}^2$,

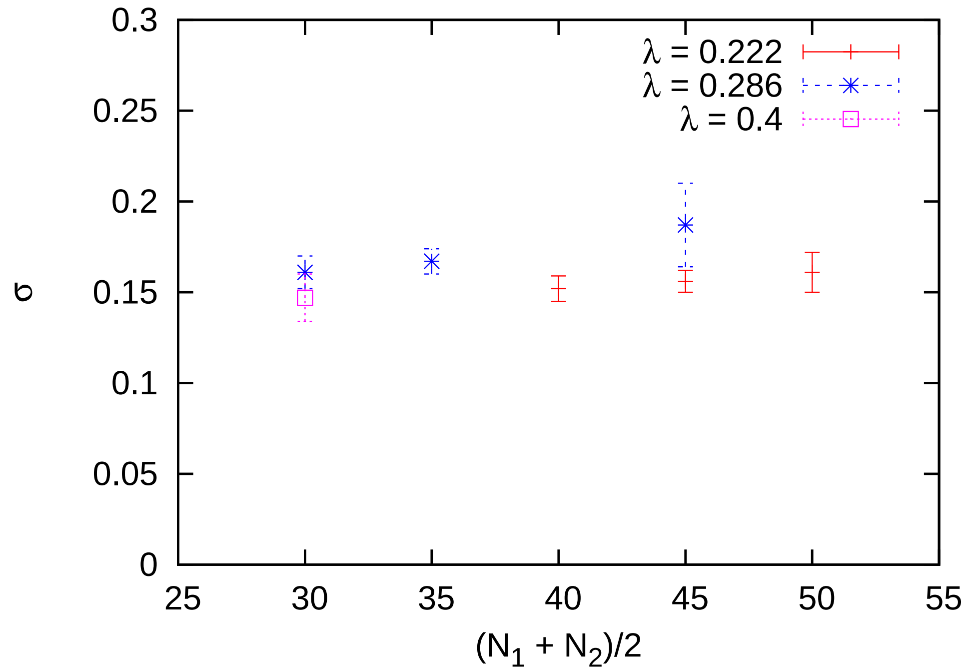
iff we stay near the striped phase.



λ	N_1	N_2	σ
0.222	35	45	0.152 (7)
	35	55	0.156 (6)
	45	55	0.161 (11)
0.286	25	35	0.161 (9)
	25	45	0.167 (7)
	35	45	0.178 (23)
0.4	25	35	0.147 (13)

(Feasibility of the simulation restricts the accessible values of $N^2\lambda$;
too large $\rightarrow$ landscape of deep semi-stable minima)

Stabilisation of σ is manifest:



$$\boxed{\sigma = 0.16(1)}$$

Striped phase persists in the Double-Scaling Limit, translation symmetry does break spontaneously.

IV. Conclusions

We studied the 3d and 2d $\lambda\phi^4$-model with a NC plane.

Lattice version can be mapped on a **Hermitian matrix model.**

This enables MC simulations (standard Metropolis algorithm).

$m^2 \ll 0$ **enforces order** :

λ resp. θ **small**: **uniform** order ; λ resp. θ **large**: **striped** order

Striped phase survives the Double-Scaling Limit
($a \to 0$ *and* $L = Na \to \infty$, at $\theta = \text{const.}$)

SSB of translation invariance **even in $\mathbf{d = 2}$**

Mermin-Wagner-Coleman Theorem evaded
by IR divergence and non-locality