A FIRST APPROACH TO LOOP-TREE DUALITY

XVIII Mexican Workshop on Particles and Fields, 2022

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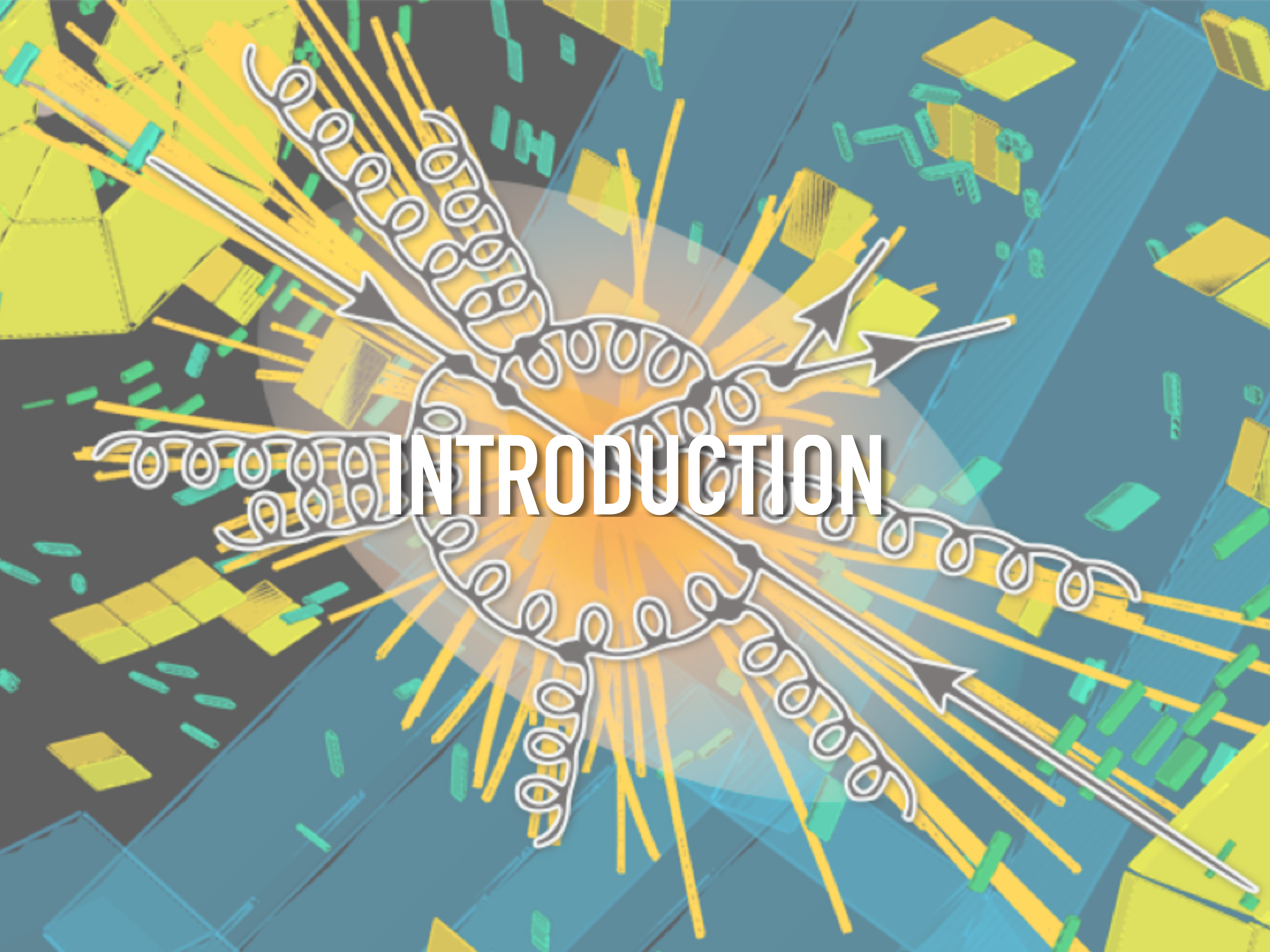
INTRODUCTION



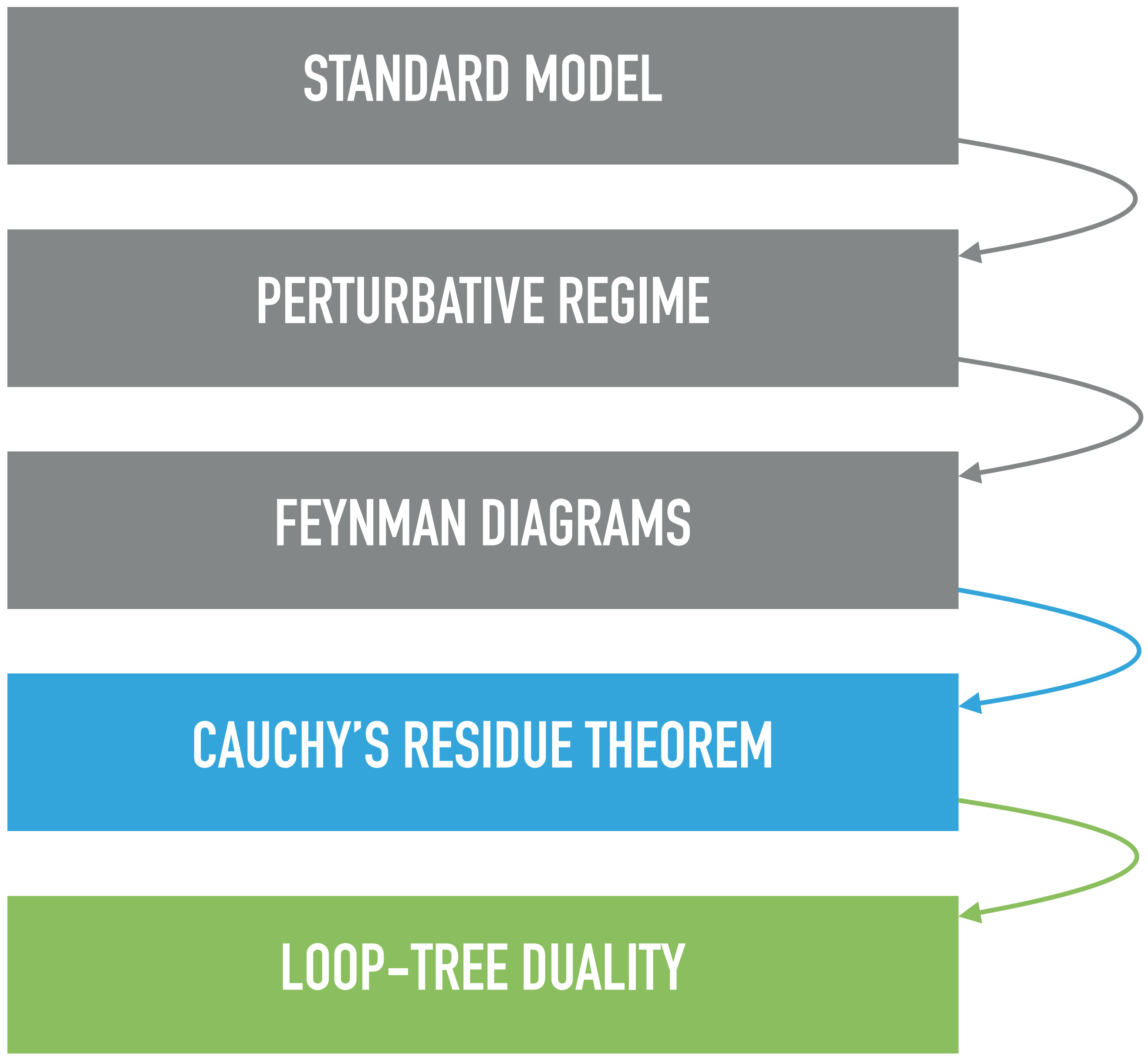
THE LOOP-TREE DUALITY

STANDARD MODEL

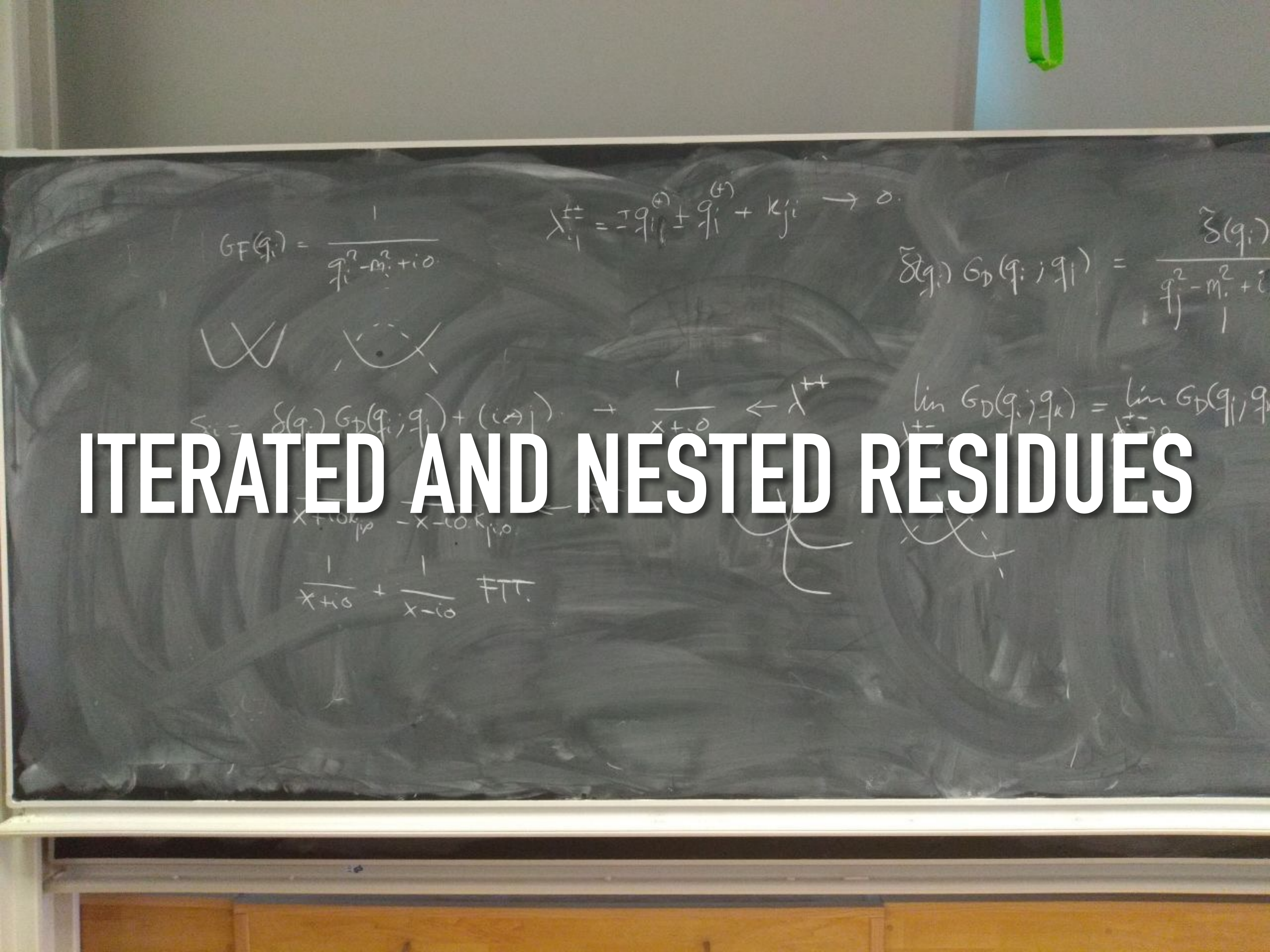
PERTURBATIVE REGIME

FEYNMAN DIAGRAMS

THE LOOP-TREE DUALITY



ITERATED AND NESTED RESIDUES

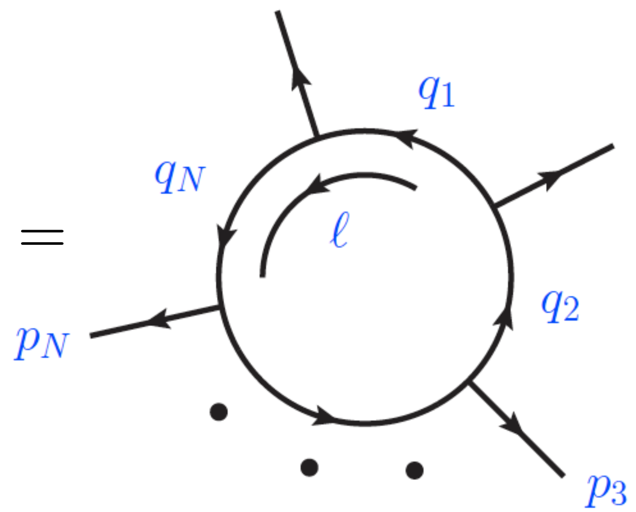


Catani, Gleisberg, Krauss, Rodrigo, Winter, JHEP 09 (2008) 065

ONE-LOOP DIAGRAMS

N-point loop integral

$$\mathcal{A}_N^{(1)} =$$



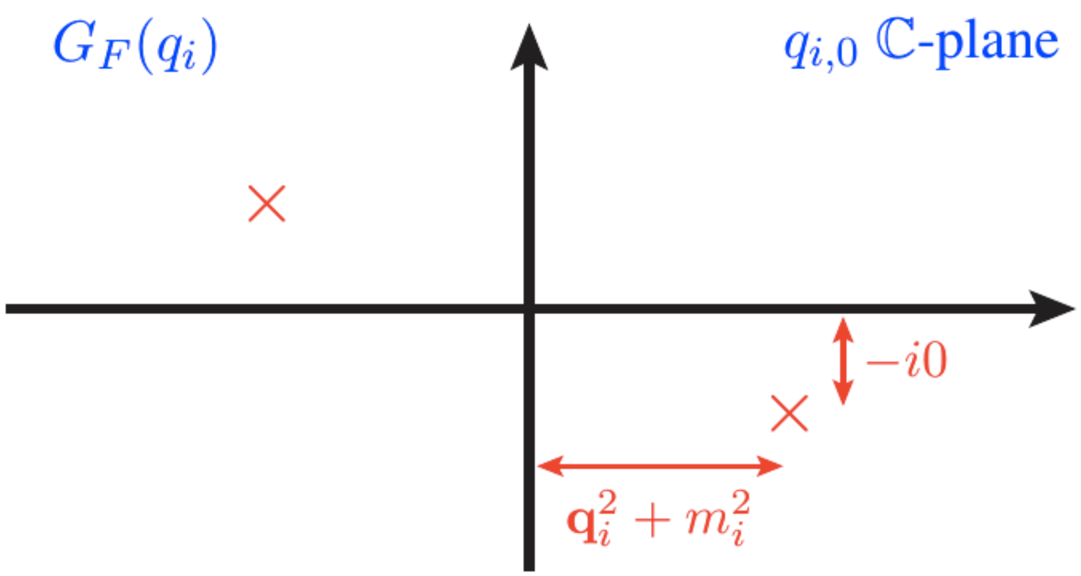
$$= \int_\ell \mathcal{N}(\ell, \{p_i\}_N) \prod_{i=1}^{N} G_F(q_i)$$

$$G_F(q_i) = \frac{1}{q_i^2 - m_i^2 + i0}$$

$$q_i = \ell + \sum_{k=1}^{i} p_k$$

The inverse Feynman propagator $\left(G_F(q_i)\right)^{-1}$ vanishes for

$$q_{i,0} = q_{i,0}^{(\pm)} = \pm\sqrt{\mathbf{q}_i^2 + m_i^2 - i0}$$



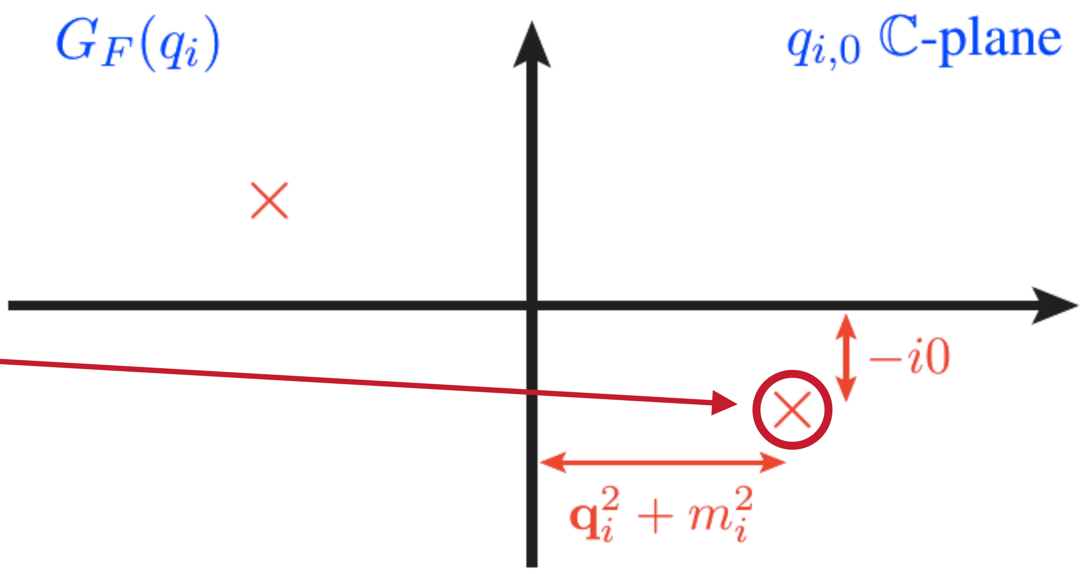
Catani, Gleisberg, Krauss, Rodrigo, Winter, JHEP 09 (2008) 065

ONE-LOOP DIAGRAMS

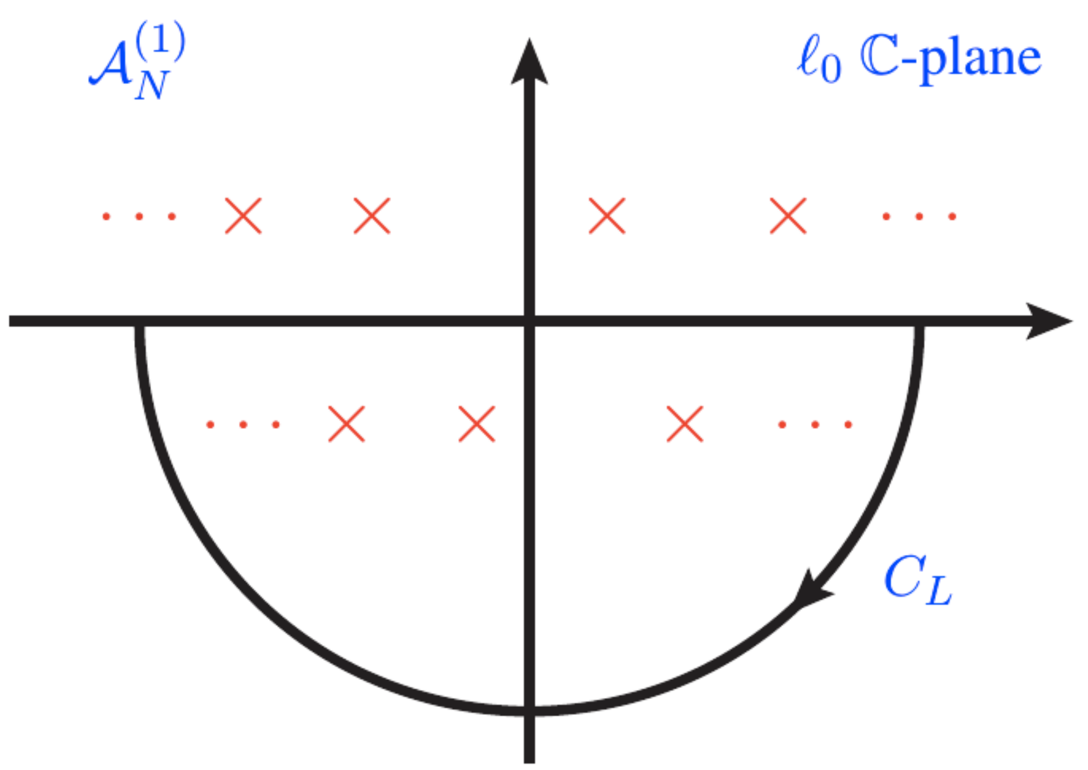
Positive energy mode:

$$\boxed{q_{i,0}^{(+)} = +\sqrt{\mathbf{q}_i^2 + m_i^2 - i0}}$$

(and negative imaginary part)



To obtain the **duality relation**, we apply **Cauchy's residue theorem** (CRT) in the loop energy complex plane and selecting residues with **definite positive energy** and **negative imaginary part** (which is done with the contour C_L)



ONE-LOOP DIAGRAMS

One-loop scalar integrals (or scattering amplitudes in any relativistic, local and unitary QFT) represented as a linear combination of *N* **single-cut** integrals

$$L^{(1)}(\{p_i\}_N) = \sum_{i\in\alpha} \int_{\vec{\ell}} \frac{1}{2q_{i,0}^{(+)}} \prod_{\substack{j\in\alpha \\ j\neq i}} G_F\left(q_{i,0}^{(+)} - p_{j,i,0}, \vec{q}_j\right)$$

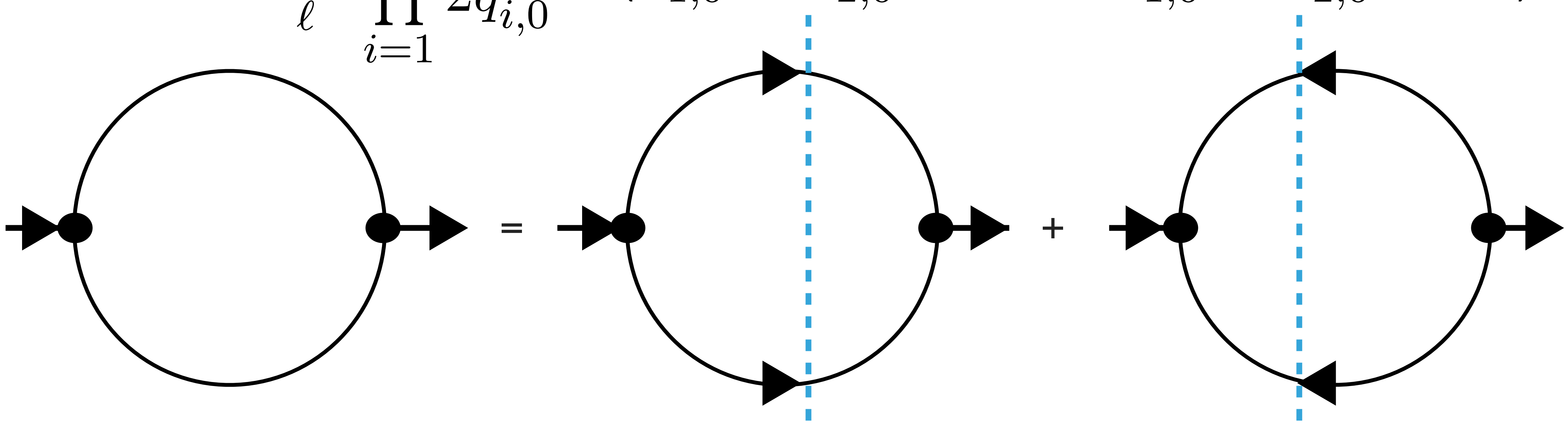
- α the set of indices of the internal lines.
- $q_{i,0}^{(+)}$ is the negative-imaginary-part on-shell energy of the i-th internal particle.
- $G_F\left(q_{i,0}^{(+)} - p_{i,j,0}, \vec{q}_j\right)$ is the propagator of the j-th internal particle when the i-th internal particle set on-shell and $p_{i,j,0}$ is the energy of the momentum $p_i + p_{i-1} + \cdots + p_{j+1}$ whenever $i > j$.
- CRT reduces the dimension of the integration space in 1. If it is applied to the energy of the loop momentum, the final integral is performed over the **phase space**.

Aguilera-Verdugo, Hernandez-Pinto, Rodrigo, Sborlini, Torres Bobadilla, JHEP02 (2021) 112

ONE-LOOP DIAGRAMS

- In the case of the scalar two-point function (Bubble function), the algebraic simplification of the residues add **physical divergencies only**.

$$L^{(1)}(p) = -\int_{\vec{\ell}} \frac{1}{\prod_{i=1}^{2} 2q_{i,0}^{(+)}} \left(\frac{1}{q_{1,0}^{(+)} + q_{2,0}^{(+)} + p_0} + \frac{1}{q_{1,0}^{(+)} + q_{2,0}^{(+)} - p_0} \right) .$$



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MULTI-LOOP DIAGRAMS: ITERATED RESIDUES

- CRT
$$\int_C f(z)dz = 2\pi\imath \sum_{j\in\beta} \Gamma_j \mathrm{Res}(f,\{z,j\})$$

- The application of CRT to a multi-loop diagram demands the promotion of the integration variable to $\mathbb{C}$. All other variables remain real.

- For the same contour of integration, the final result of CRT let us use the equation,

$$\boxed{\int_C f(z)\, dz = -\,2\pi\imath \sum_{z_0\in P} \mathrm{Res}\left(f(z), \{z, z_0\}\right)\theta\left(-\mathrm{Im}\,(z_0)\right)}\ ,$$

ITERATED RESIDUES

where P is the set of all poles of the integrand $f(z)$.

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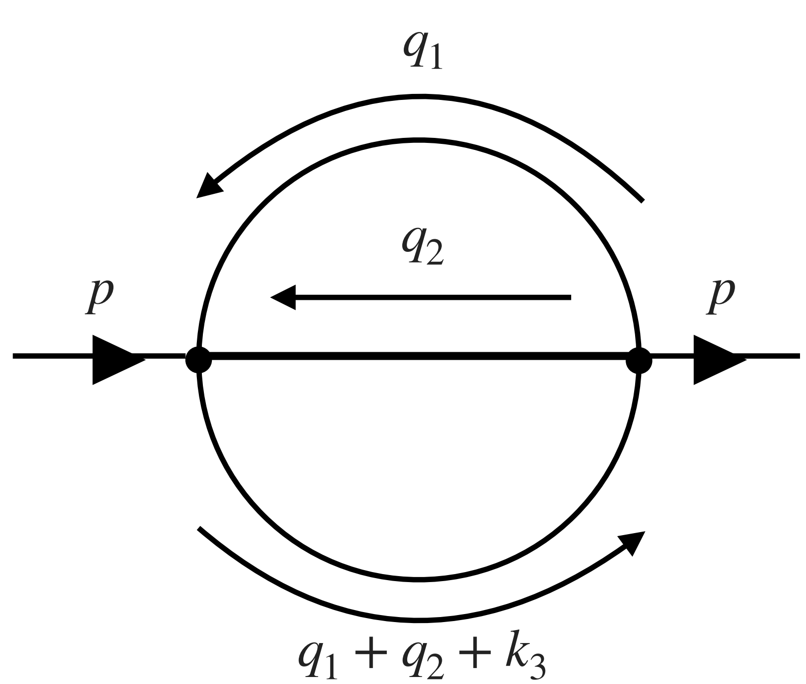
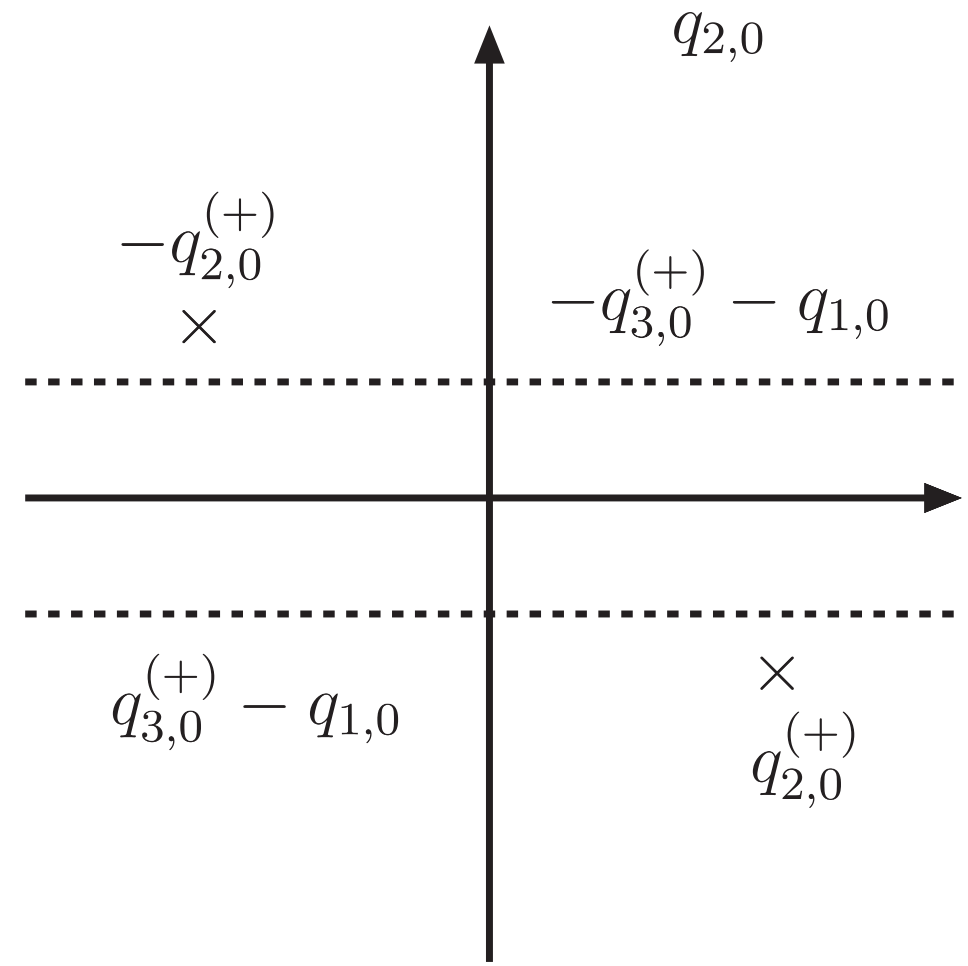
MULTI-LOOP DIAGRAMS

- The iterated residues make evident the presence of poles with non-definite imaginary-part sign poles, the **displaced poles**.



$$\int_{q_1,q_2} \frac{1}{\left(q_{1,0}^2 - \left(q_{1,0}^{(+)}\right)^2\right)\left(q_{2,0}^2 - \left(q_{2,0}^{(+)}\right)^2\right)\left(\left(q_{1,0}+q_{2,0}+p_0\right)^2 - \left(q_{3,0}^{(+)}\right)^2\right)}$$

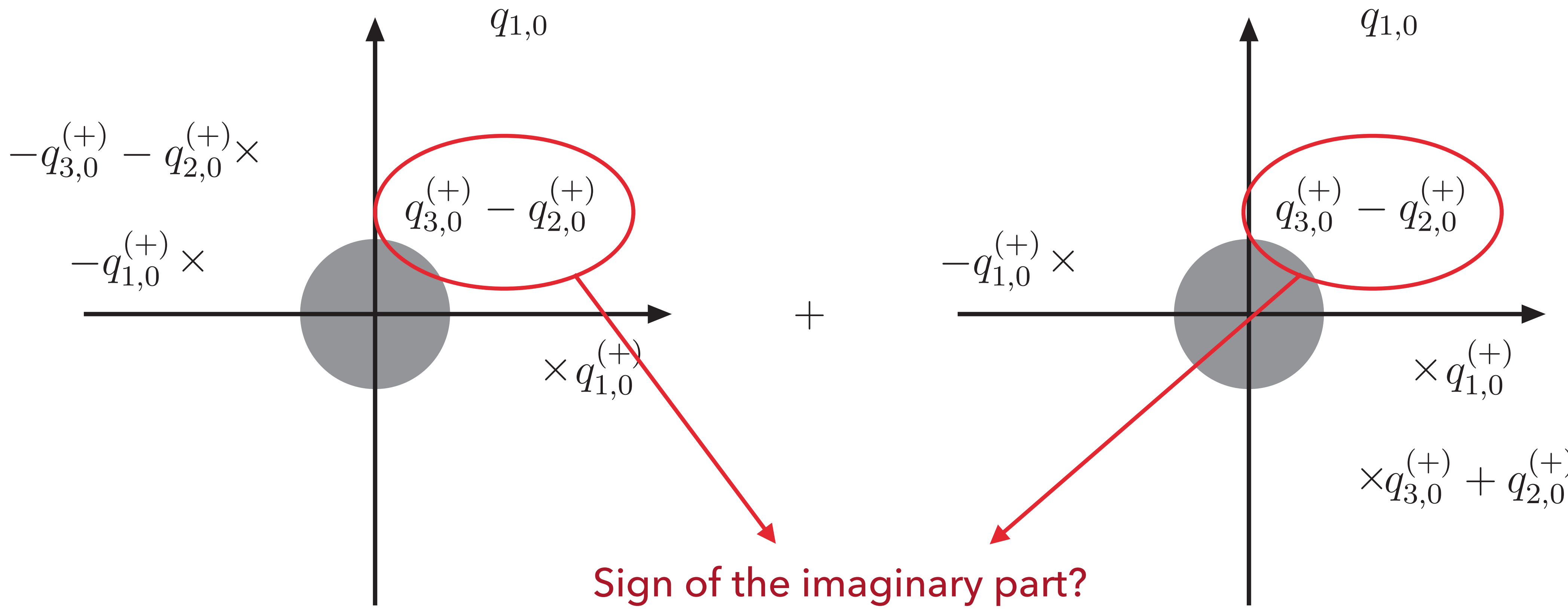


Pole structure of the scalar sunrise diagram.

Aguilera-Verdugo, Hernandez-Pinto, Rodrigo, Sborlini, Torres Bobadilla, JHEP02 (2021) 112

MULTI-LOOP DIAGRAMS

- The computation of the second integral with CRT demands the knowledge of the sign of the imaginary part of the poles.



Diagrammatic representation of the pole structure of both term in the integrand for the second integral.

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NESTED RESIDUES

- The contributions of the displaced poles cancel, leaving the residues of the negative-imaginary part poles **only.**

$$q_{i,0}^{(+)} - q_{j,0}^{(+)}$$

$$\sum_{i\in\Omega} q_{i,0}^{(+)}$$

- Defining the set $P^{(+)}$ as the set of definite negative-imaginary part poles of the integrand, then we can define the nested residues.

$$\boxed{\int_C f(z)\ dz = -2\pi\imath \sum_{z_0\in P^{(+)}} \mathrm{Res}\left(f(z), \{z, z_0\}\right)},$$

NESTED RESIDUES

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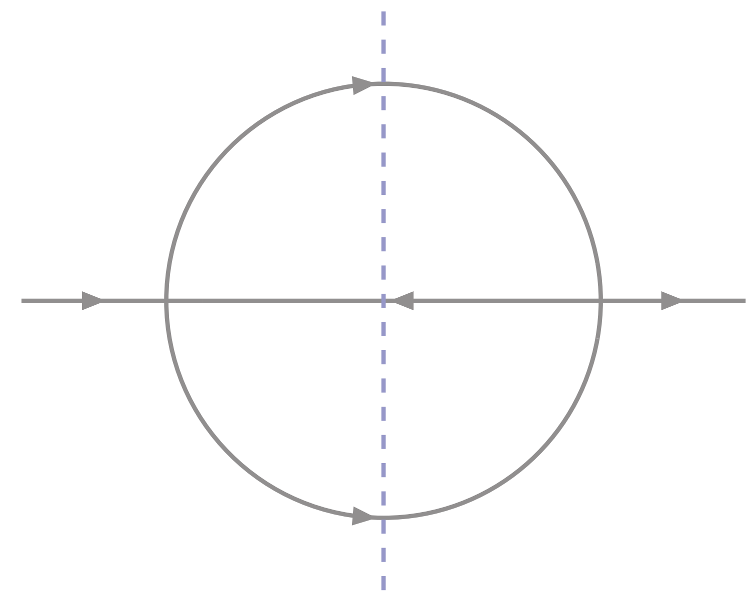
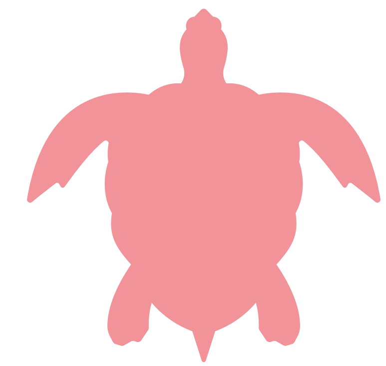
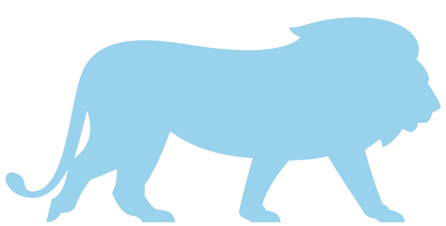
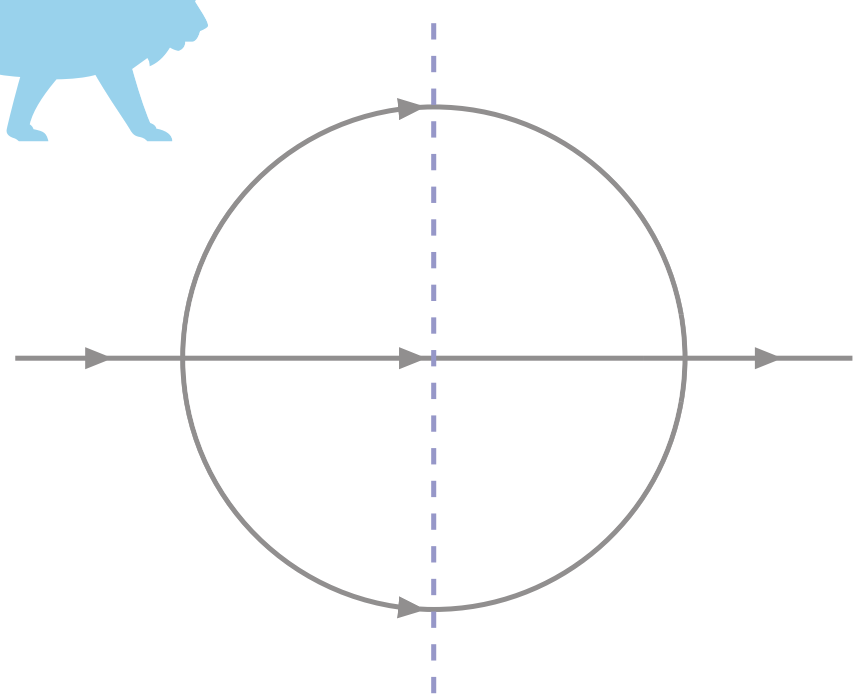
CAUSAL AND NON-CAUSAL DIVERGENCES

- We say a non-causal divergence is the one that appears within the integration domain of a multi-loop phase space.

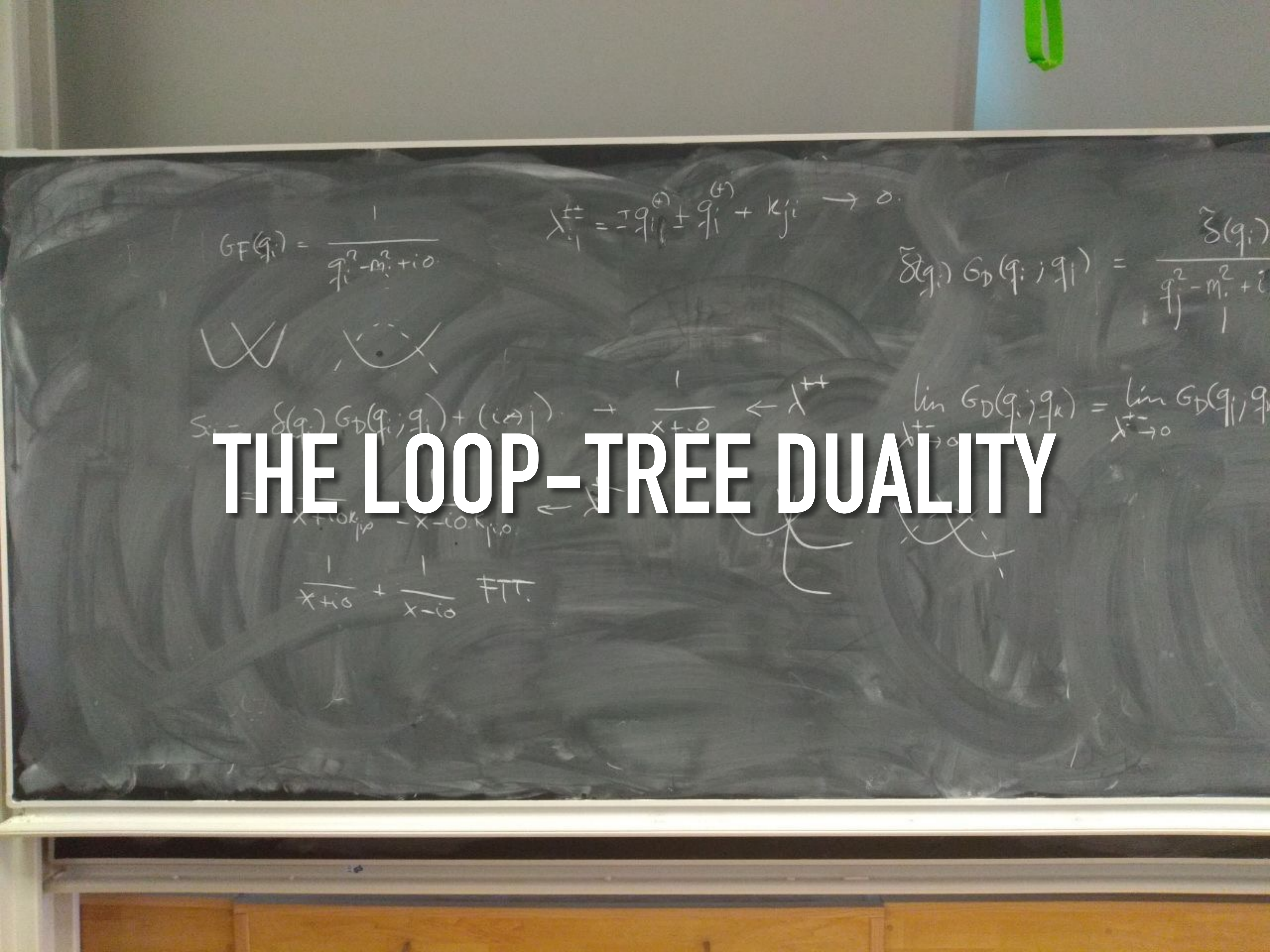








- A causal divergence is the kind of divergence that depends only on the configuration of external momenta.

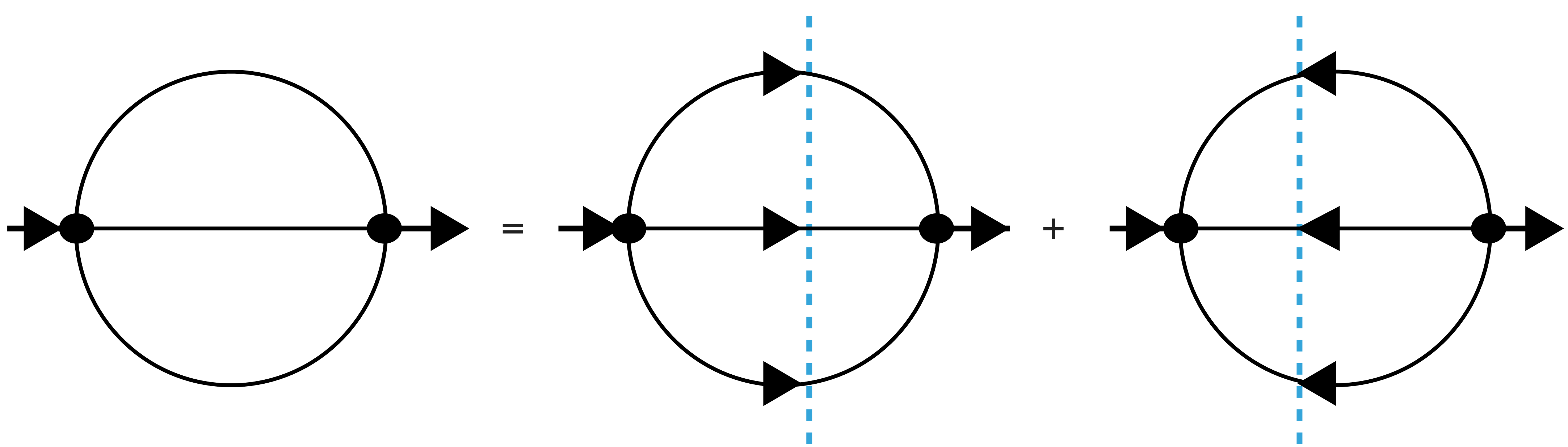


THE LOOP-TREE DUALITY

HIGHER PERTURBATIVE ORDERS

- An algebraic simplification of the nested residues to the scalar sunrise integral leads to **causal divergences only**.

$$L^{(2)}(p) = -\int_{\vec{q}_1,\vec{q}_2} \frac{1}{\prod_{i=1}^{3} 2q_{i,0}^{(+)}} \left(\frac{1}{q_{1,0}^{(+)} + q_{2,0}^{(+)} + q_{3,0}^{(+)} + p_0} + \frac{1}{q_{1,0}^{(+)} + q_{2,0}^{(+)} + q_{3,0}^{(+)} - p_0} \right)$$

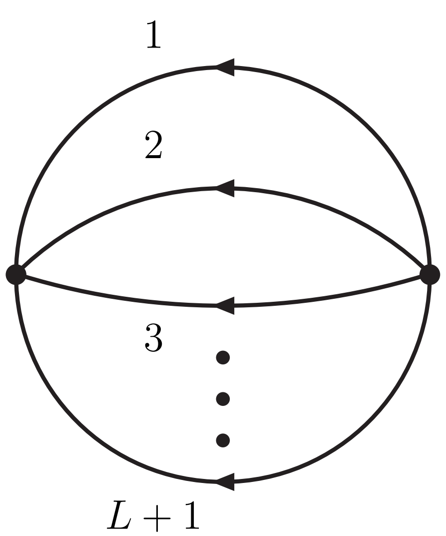


Aguilera-Verdugo, et al, Phys. Rev. Lett. 124 211602

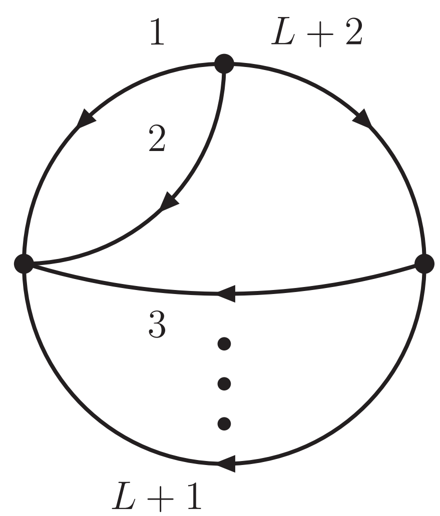
TOPOLOGICAL FAMILIES

Feynman diagrams can be classified by the number n of vertex of the diagrams, the **topological families**. The number $\tau = n - 1$ is called **topological complexity**.

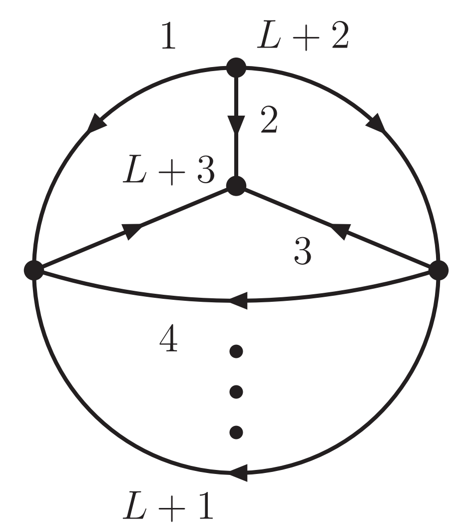
- The simplest topological family (2 vertices) form the Maximal Loop Topology (MLT) family.
- Diagrams with 3 vertices form the Next-to Maximal Loop Topology (NMLT) family.
- Diagrams with n vertices form the Next-to-...-Next-to Maximal Loop Topology (N^{n-2}MLT) family.



MLT diagram with L loops



NMLT diagram with L loops



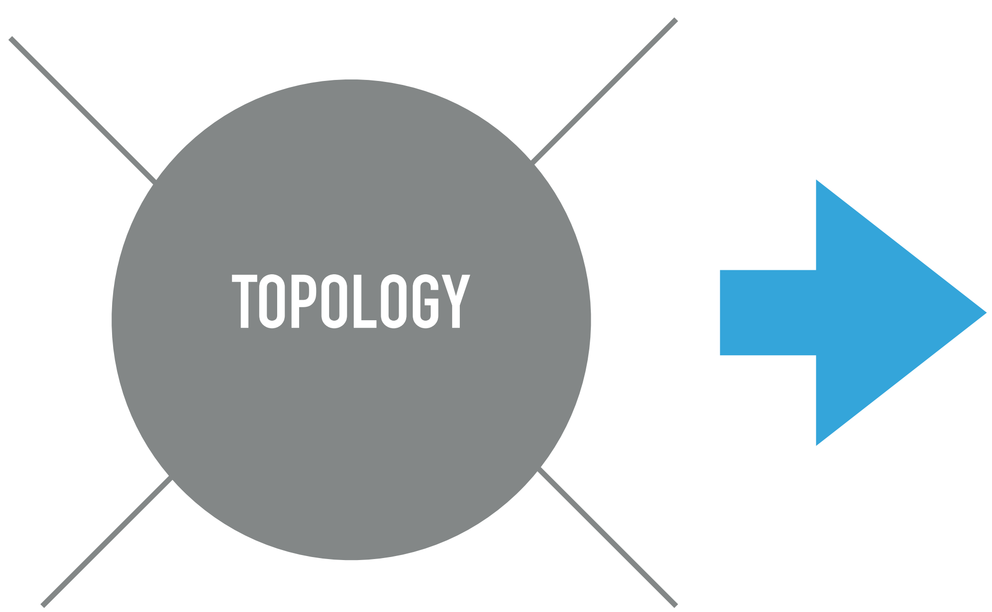
N^{2}MLT diagram with L loops

Aguilera-Verdugo, et al, Phys. Rev. Lett. 124 211602

TOPOLOGICAL FAMILIES

The topological classification of Feynman diagrams showed a great power when studying arbitrary L-loop diagrams. The topology of the diagrams is encoded within the relations of the internal momenta with the loop momenta.

SBORLINI'S TALK



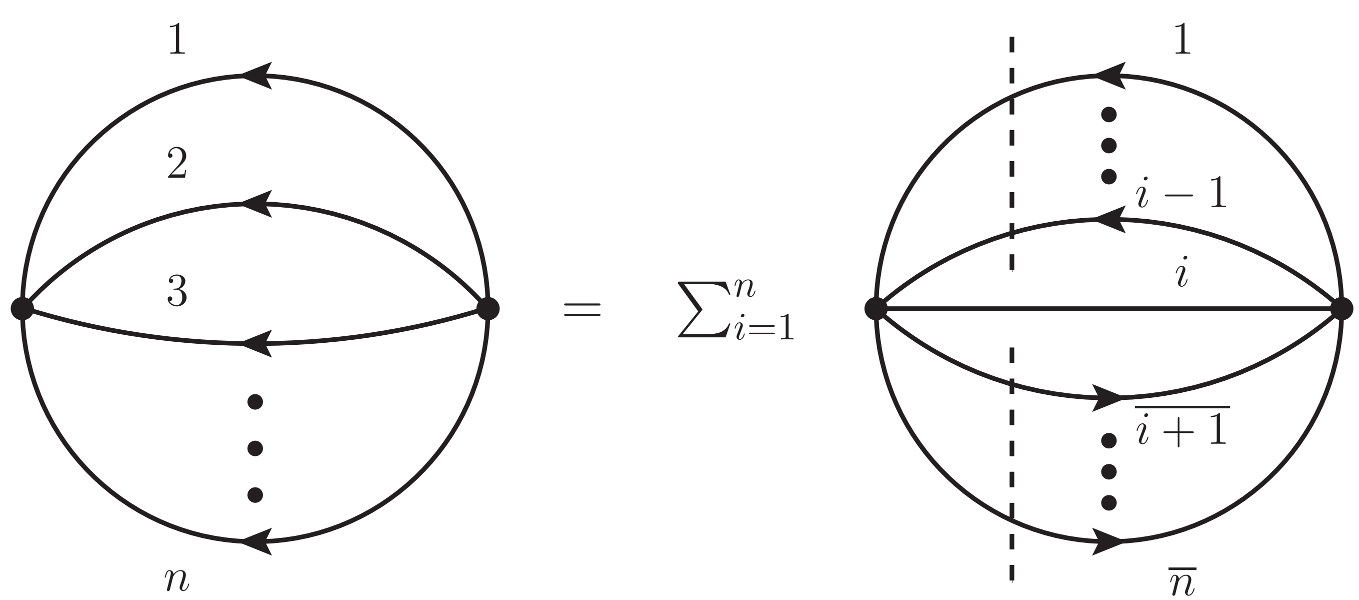
INTERNAL MOMENTA

$$q_{L+i} = \sum_{j=1}^{L} \alpha_{ij}\; q_j + \sum_{j=1}^{N} \beta_{ij}\; p_j$$

Aguilera-Verdugo, et al, Phys. Rev. Lett. 124 211602

MAXIMAL LOOP TOPOLOGY

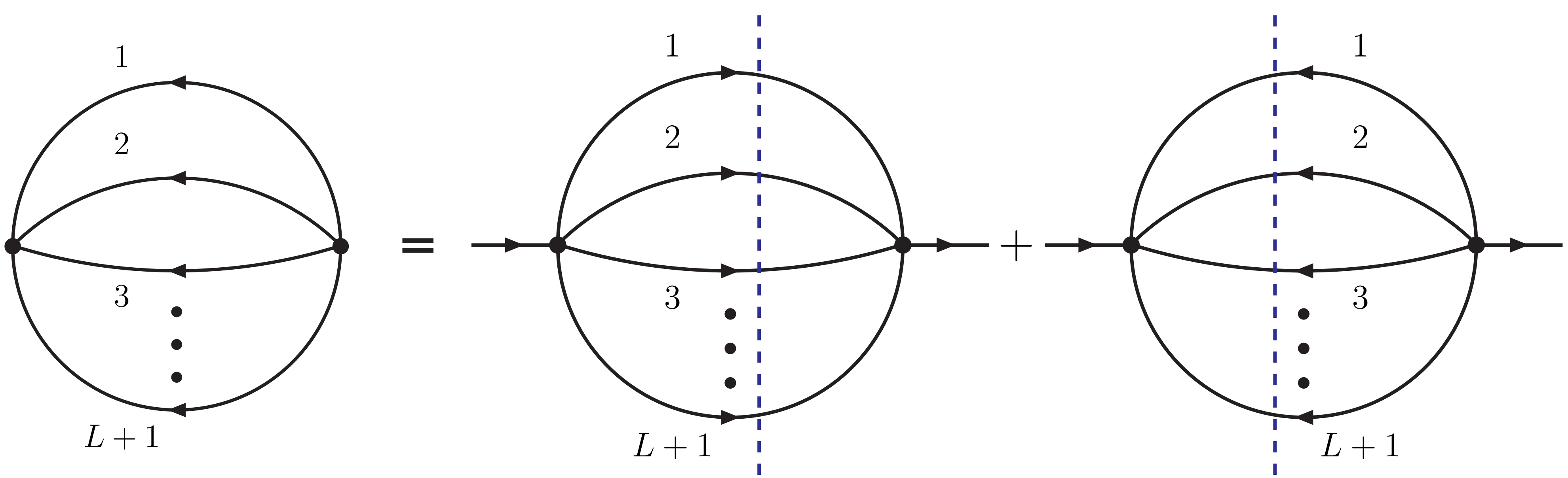
After applying the nested residues to the scalar MLT(L) diagram with and $n = L + 1$ internal particles, a term related with each spanning tree is obtained.



Aguilera-Verdugo, et al, Phys. Rev. Lett. 124 211602

MAXIMAL LOOP TOPOLOGY

As the MLT(L) diagram is the natural generalization of the scalar 2-point 1-loop diagram and the scalar sunrise diagram with respect to the number of loops, it is possible to obtain the causal representation,



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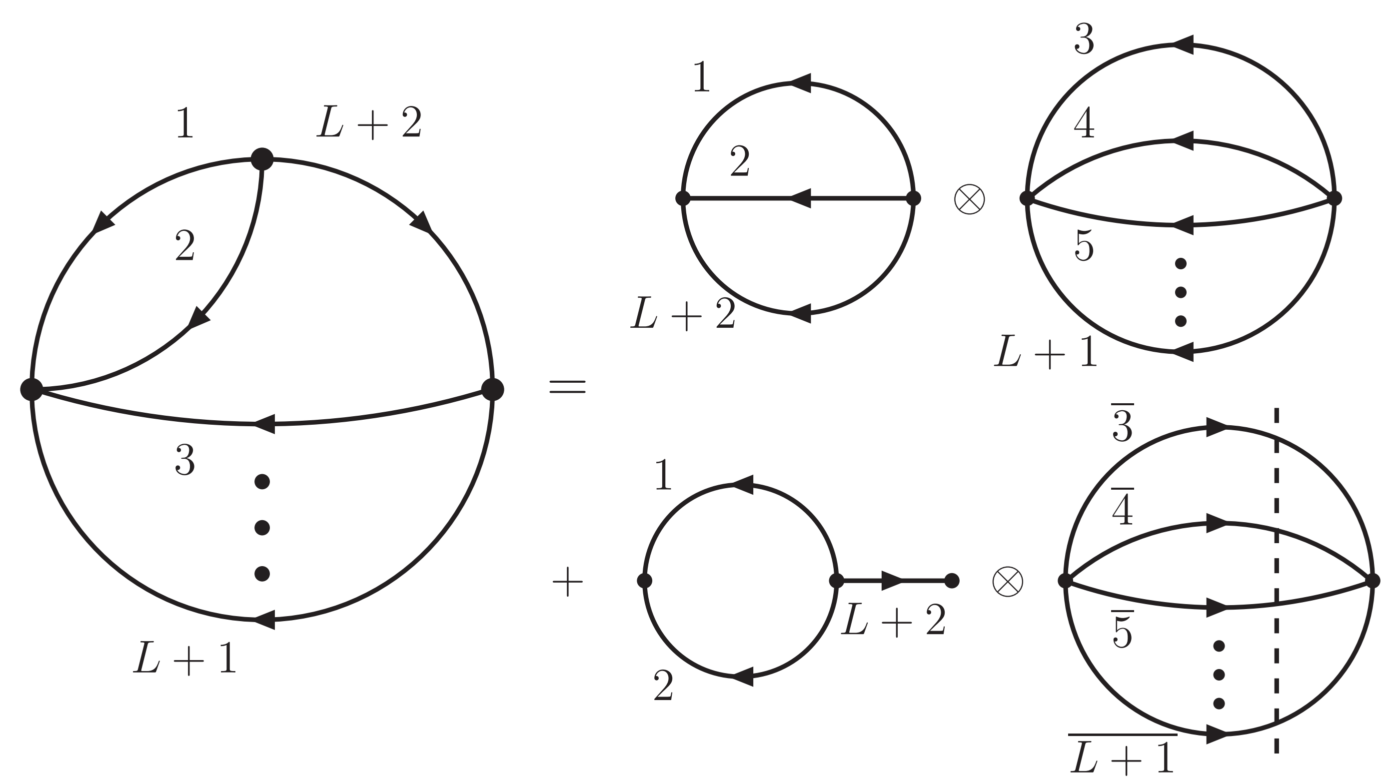
NEXT-TO MAXIMAL LOOP TOPOLOGY

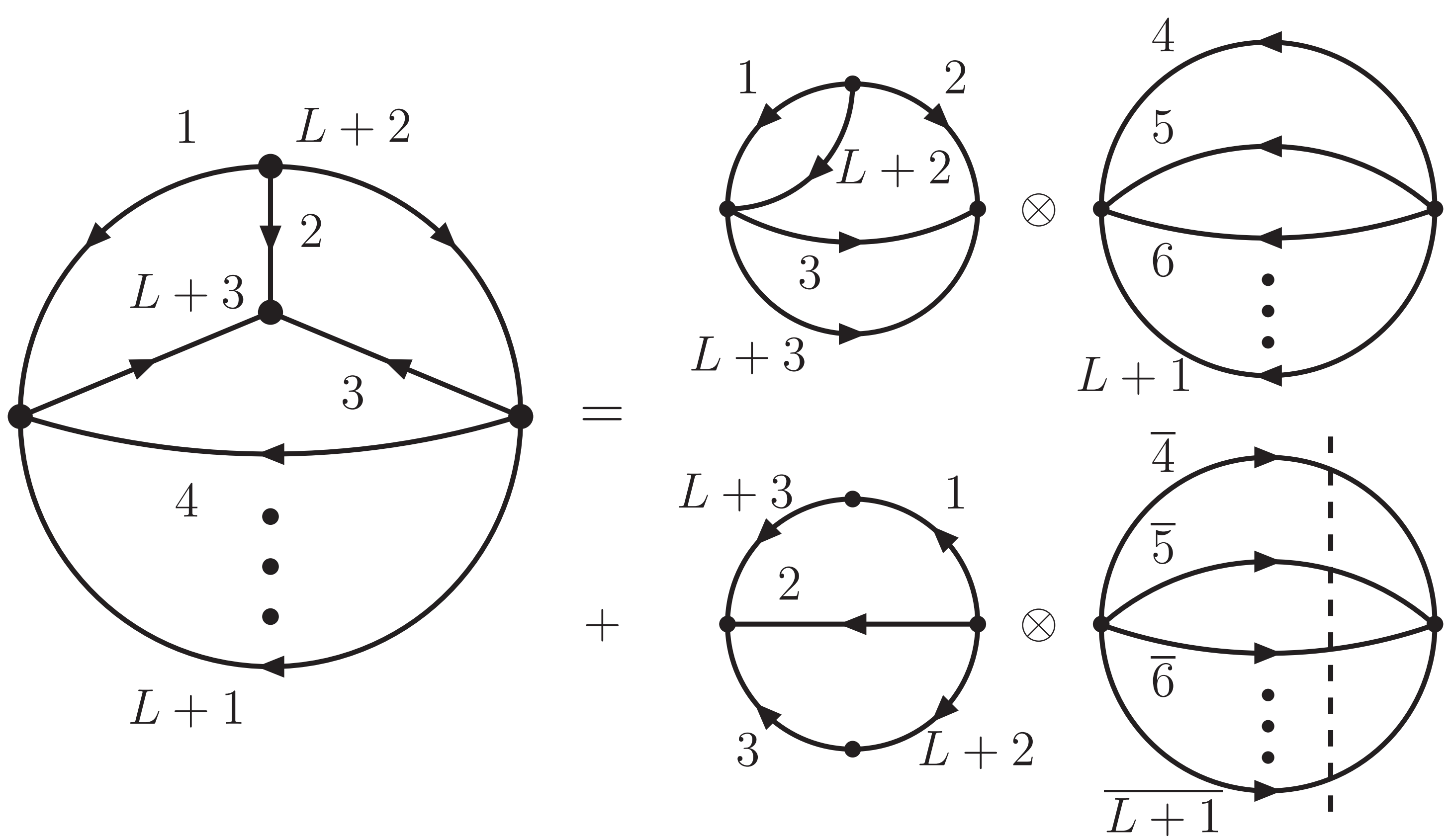
The case of the scalar NMLT(L) diagram with $n = L + 2$ internal particles, the nested residue leads to a non trivial combination of tree-level contributions, which can be expressed with convolutions.



Aguilera-Verdugo, et al, Phys. Rev. Lett. 124 211602

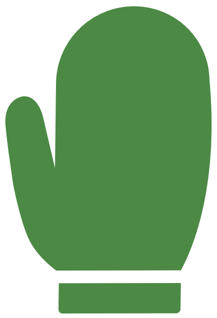
NEXT-TO-NEXT-TO MAXIMAL LOOP TOPOLOGY

For a scalar N$^{(2)}$MLT(L), the nested residues lead to quite more intricate combination of tree-level contributions. This combination can be expressed as some other convolutions.

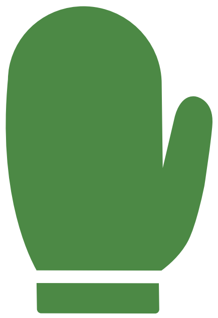


CONVOLUTION RELATIONS

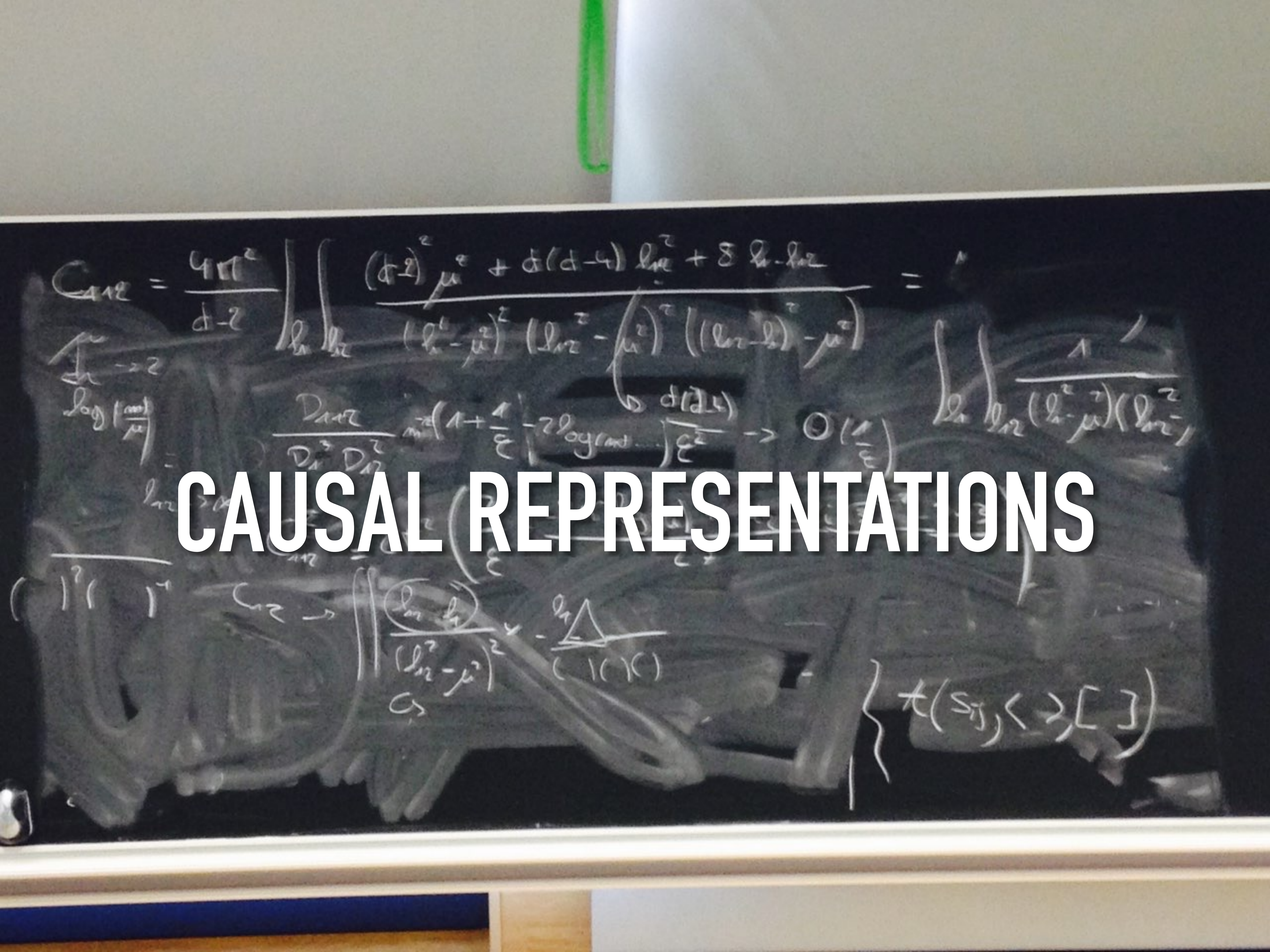
Nested residues lead to a combination of contributions associated with each spanning tree of the underlying graph, expressed as convolution relations.



Each convolution relation of a diagram with topological complexity τ expresses the integrand in terms of diagrams with lower topological complexity.



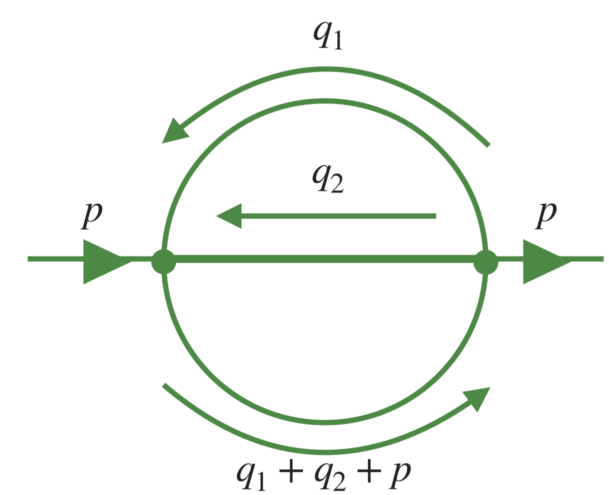
Nested residues could lead to causal representations of Feynman diagrams (iterating the convolution relations). ?



CAUSAL REPRESENTATIONS

Aguilera-Verdugo, Hernandez-Pinto, Rodrigo, Sborlini, Torres Bobadilla, JHEP02 (2021) 069

SCALAR SUNRISE DIAGRAM: CAUSAL REPRESENTATION



- Scalar sunrise: $G_F(1,2,3)$
  - After nested residues

$$G_F(1,2,3) \to \frac{1}{4q_{1,0}^{(+)}q_{2,0}^{(+)}} \frac{1}{\left(q_{1,0}^{(+)} + q_{2,0}^{(+)} + p_0\right)^2 - \left(q_{3,0}^{(+)}\right)^2} + \frac{1}{4q_{1,0}^{(+)}q_{3,0}^{(+)}} \frac{1}{\left(q_{1,0}^{(+)} - q_{3,0}^{(+)} + p_0\right)^2 - \left(q_{2,0}^{(+)}\right)^2} + \frac{1}{4q_{2,0}^{(+)}q_{3,0}^{(+)}} \frac{1}{\left(q_{2,0}^{(+)} + q_{3,0}^{(+)} - p_0\right)^2 - \left(q_{3,0}^{(+)}\right)^2}$$

  - Causal representation

$$G_F(1,2,3) \to \frac{-1}{8q_{1,0}^{(+)}q_{2,0}^{(+)}q_{3,0}^{(+)}} \left(\frac{1}{q_{1,0}^{(+)} + q_{2,0}^{(+)} + q_{3,0}^{(+)} + p_0} + \frac{1}{q_{1,0}^{(+)} + q_{2,0}^{(+)} + q_{3,0}^{(+)} - p_0} \right)$$

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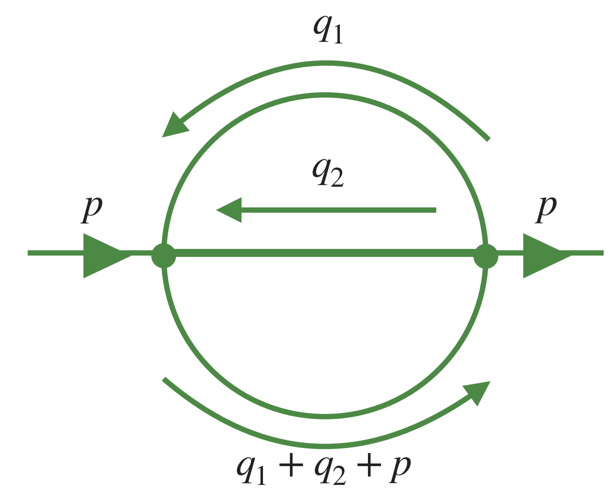
SCALAR SUNRISE DIAGRAM: CAUSAL REPRESENTATION



- Scalar sunrise: $G_F(1,2,3)$
  - After nested residues

$$G_F(1,2,3) \to \frac{1}{4q_{1,0}^{(+)}q_{2,0}^{(+)}} \frac{1}{\left(q_{1,0}^{(+)} + q_{2,0}^{(+)} + p_0\right)^2 - \left(q_{3,0}^{(+)}\right)^2} + \frac{1}{4q_{1,0}^{(+)}q_{3,0}^{(+)}} \frac{1}{\left(q_{1,0}^{(+)} - q_{3,0}^{(+)} + p_0\right)^2 - \left(q_{2,0}^{(+)}\right)^2} + \frac{1}{4q_{2,0}^{(+)}q_{3,0}^{(+)}} \frac{1}{\left(q_{2,0}^{(+)} + q_{3,0}^{(+)} - p_0\right)^2 - \left(q_{3,0}^{(+)}\right)^2}$$

  - Causal representation

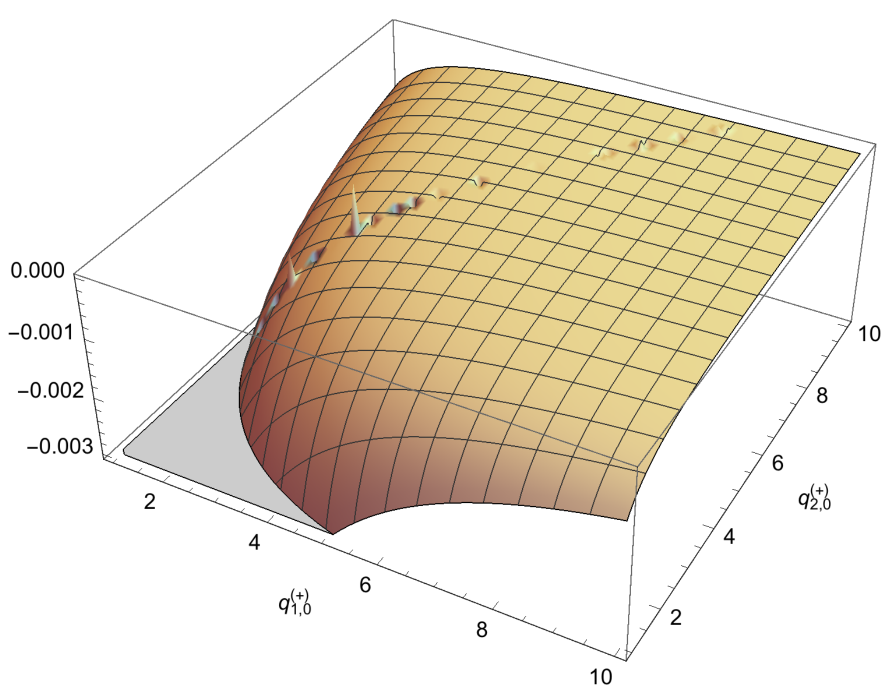
$$G_F(1,2,3) \to \frac{-1}{8q_{1,0}^{(+)}q_{2,0}^{(+)}q_{3,0}^{(+)}} \left(\frac{1}{q_{1,0}^{(+)} + q_{2,0}^{(+)} + q_{3,0}^{(+)} + p_0} + \frac{1}{q_{1,0}^{(+)} + q_{2,0}^{(+)} + q_{3,0}^{(+)} - p_0} \right)$$

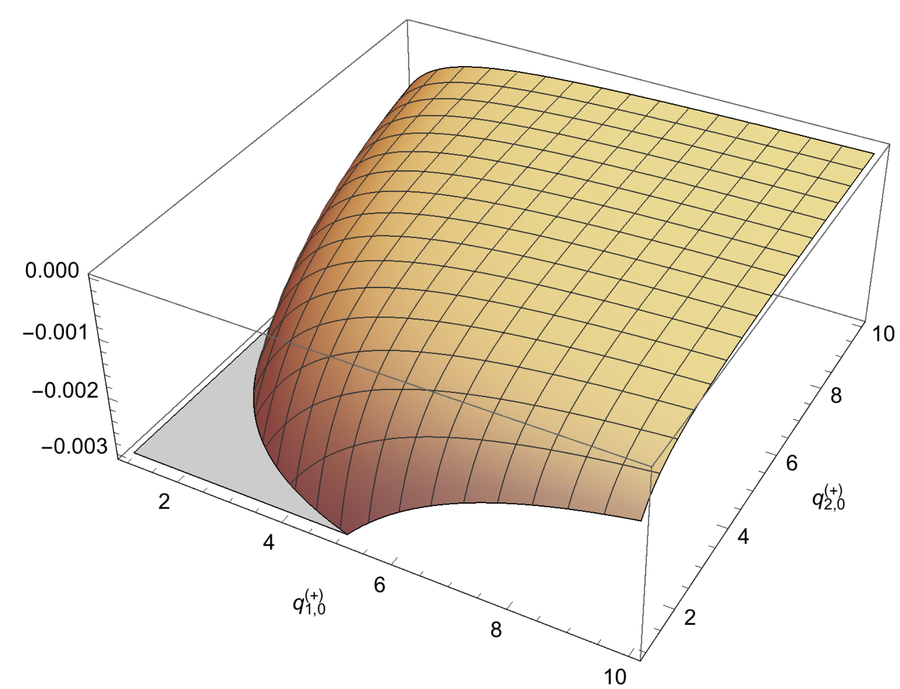
$$G_F(1,2,3) \to -\frac{1}{x_3} \left(\frac{1}{\lambda_1^+} + \frac{1}{\lambda_1^-} \right)$$

Aguilera-Verdugo, Hernandez-Pinto, Rodrigo, Sborlini, Torres Bobadilla, JHEP02 (2021) 069

SCALAR SUNRISE DIAGRAM: CAUSAL REPRESENTATION

- Scalar sunrise: $G_F(1,2,3)$
   - After nested residues
   - Causal representation





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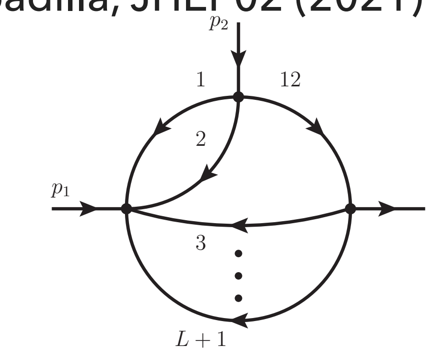
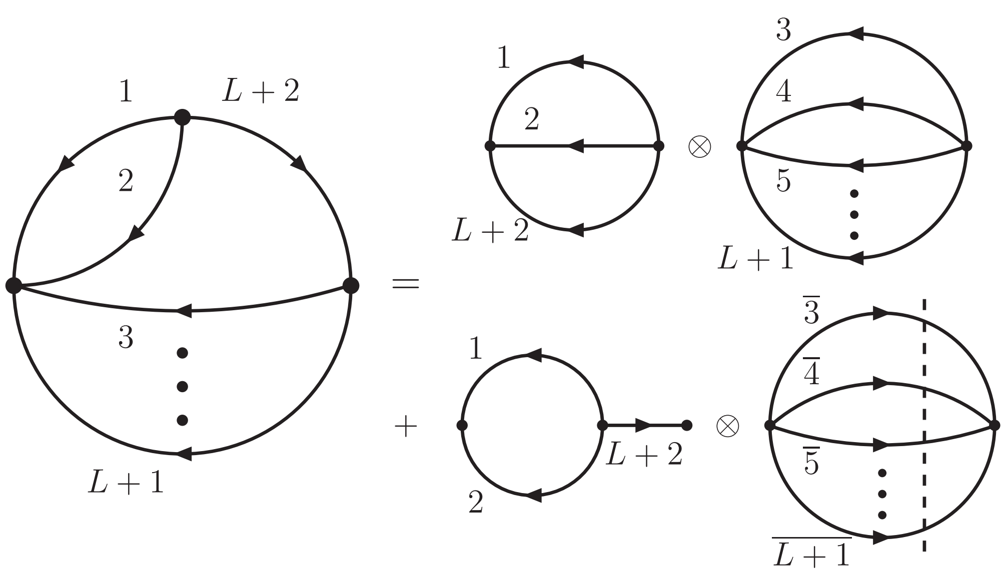
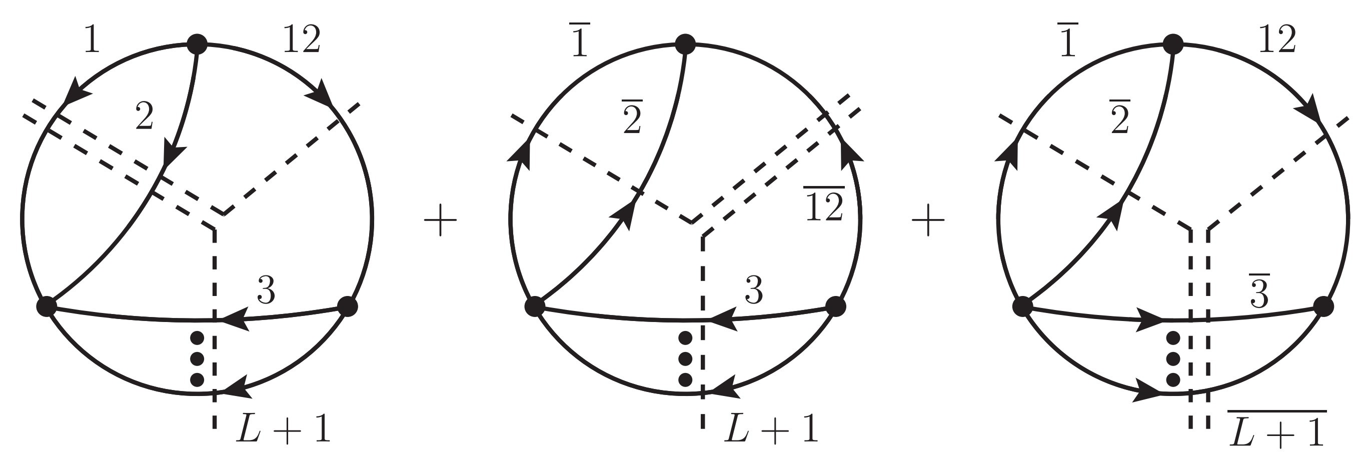
SCALAR NMLT DIAGRAM: CAUSAL REPRESENTATION



- Scalar NMLT integrand: $G_F(1,\ldots,L+2) = G_F(1,\ldots,L,1\ldots L,12)$

  - After nested residues
  - Causal representation





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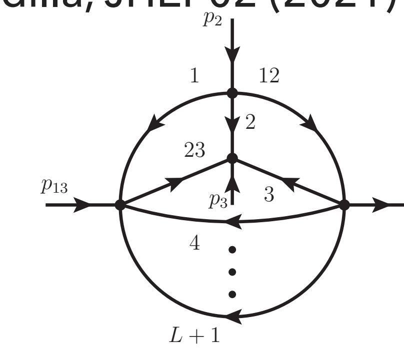
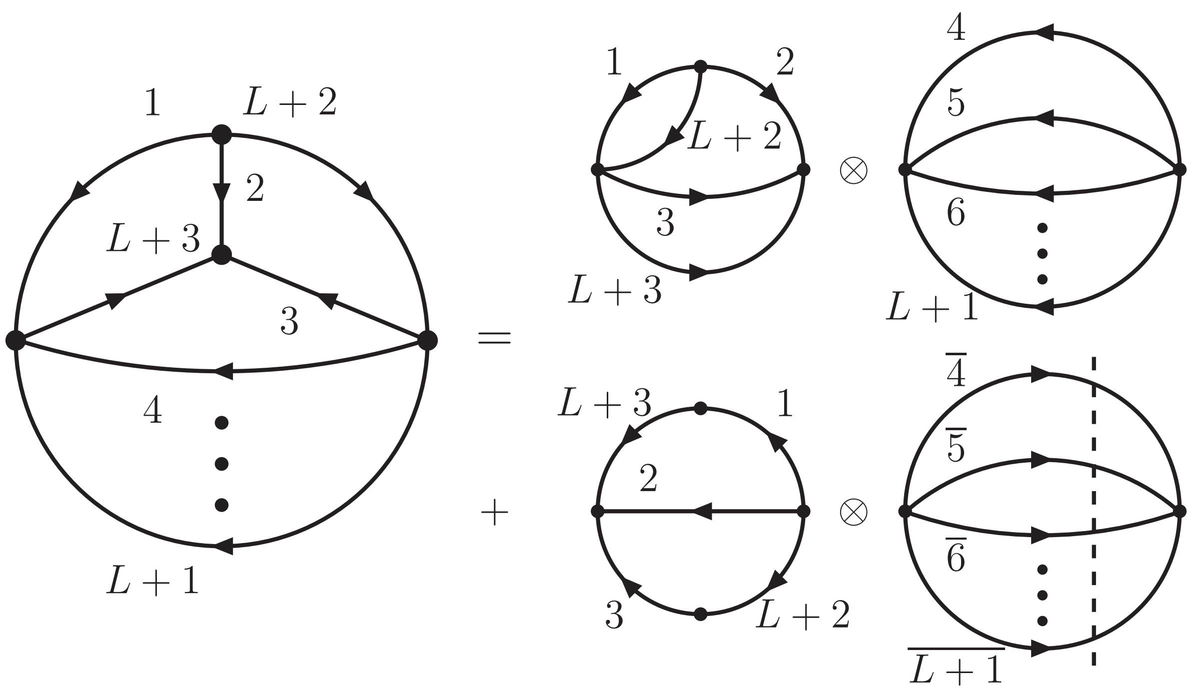
SCALAR N2MLT DIAGRAM: CAUSAL REPRESENTATION



- Scalar N^{2}MLT integrand: $G_F(1,\ldots,L+3) = G_F(1,\ldots,L,1\ldots L,12,23)$

- After nested residues



- Causal representation

$$G_F(1,\ldots,L+2) \to \frac{-1}{x_{L+2}}\left(\boxed{\frac{1}{\lambda_1^+}\left(\frac{1}{\lambda_2^-}+\frac{1}{\lambda_3^-}\right)\left(\frac{1}{\lambda_4^+}+\frac{1}{\lambda_5^+}\right)}\right.$$
$$+\frac{1}{\lambda_6^+}\left(\frac{1}{\lambda_3^-}+\frac{1}{\lambda_5^-}\right)\left(\frac{1}{\lambda_2^+}+\frac{1}{\lambda_4^+}\right)$$
$$\left.+\frac{1}{\lambda_7^+}\left(\frac{1}{\lambda_3^-}+\frac{1}{\lambda_4^-}\right)\left(\frac{1}{\lambda_2^+}+\frac{1}{\lambda_5^+}\right)+\left(\lambda_i^+ \leftrightarrow \lambda_i^-\right)\right)$$

- $\lambda_4^\pm = q_{2,0}^{(+)} + q_{3,0}^{(+)} + q_{L+3,0}^{(+)} \pm p_{3,0}$
- $\lambda_5^\pm = q_{1,0}^{(+)} + q_{L+3,0}^{(+)} + \sum_{i=4}^{L+1} q_{i,0}^{(+)} \pm p_{1,0}$
- $\lambda_6^\pm = q_{1,0}^{(+)} + q_{3,0}^{(+)} + q_{L+2,0}^{(+)} + q_{L+3,0}^{(+)} \pm (p_{2,0} + p_{3,0})$
- $\lambda_7^\pm = q_{2,0}^{(+)} + \sum_{i=4}^{L+3} q_{i,0}^{(+)} \pm (p_{1,0} + p_{2,0})$

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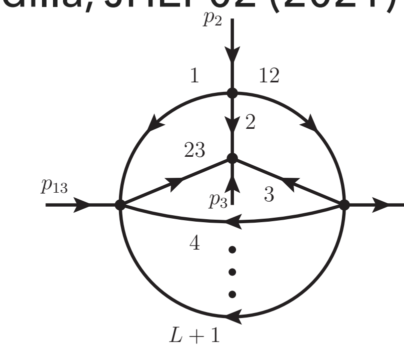
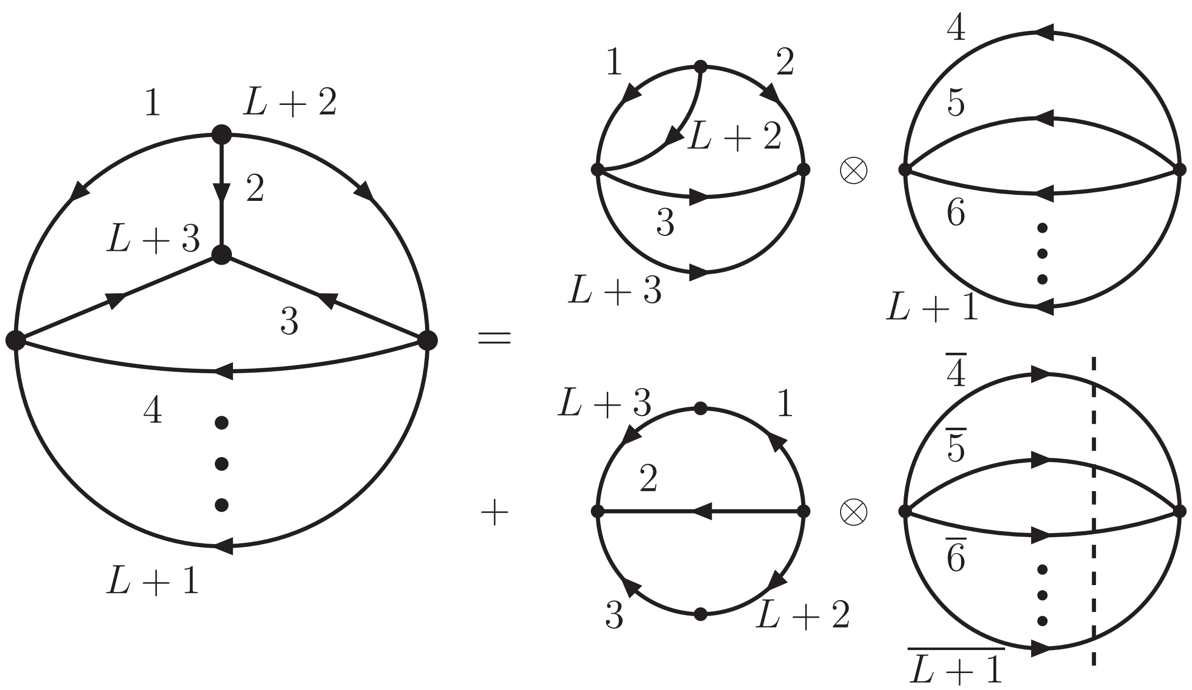
SCALAR N2MLT DIAGRAM: CAUSAL REPRESENTATION

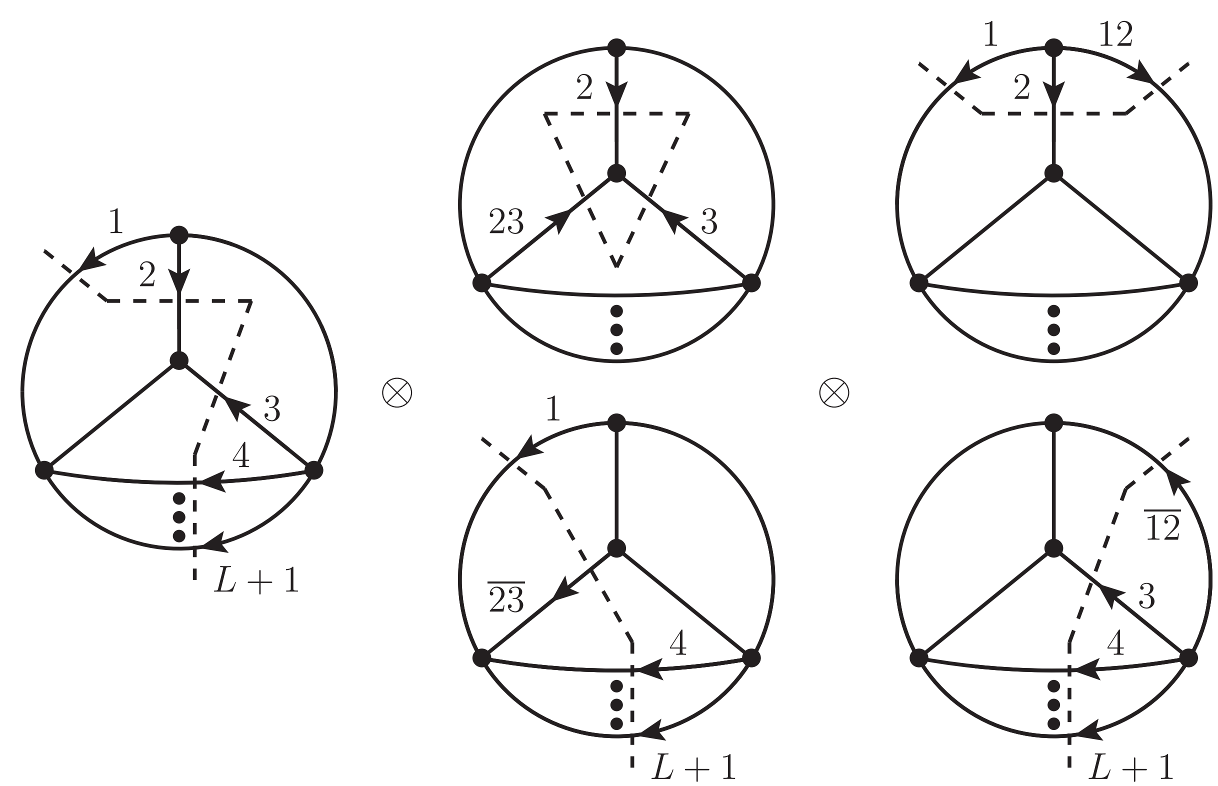


- Scalar N^{2}MLT integrand: $G_F(1,\ldots,L+3) = G_F(1,\ldots,L,1\ldots L,12,23)$

- After nested residues



- Causal representation



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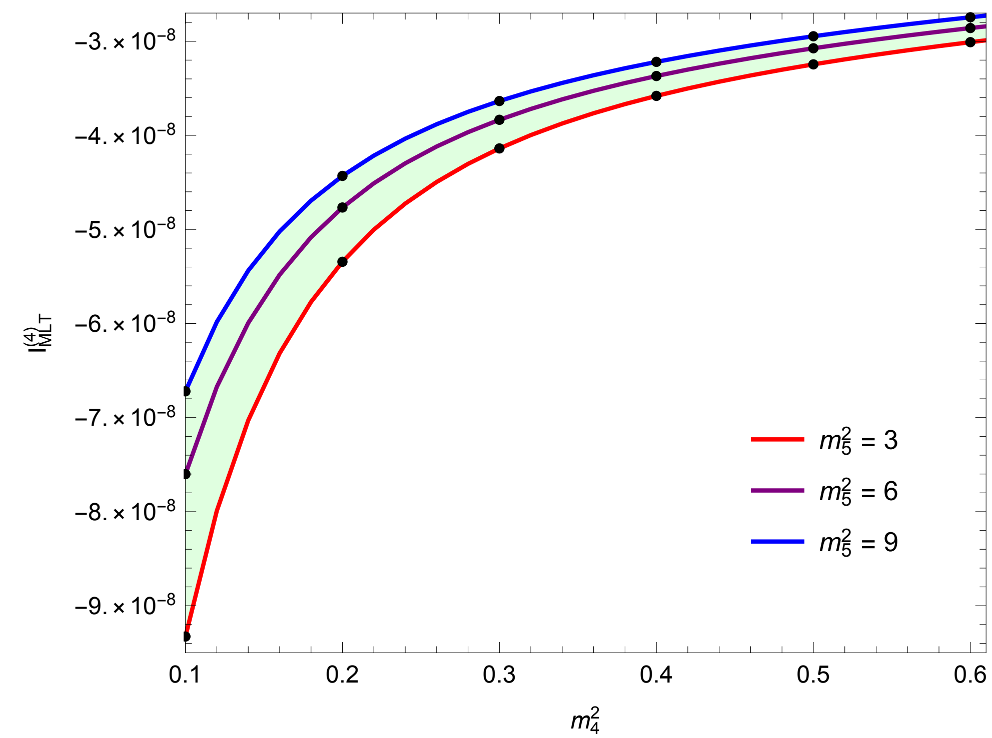
NUMERICAL EVALUATION

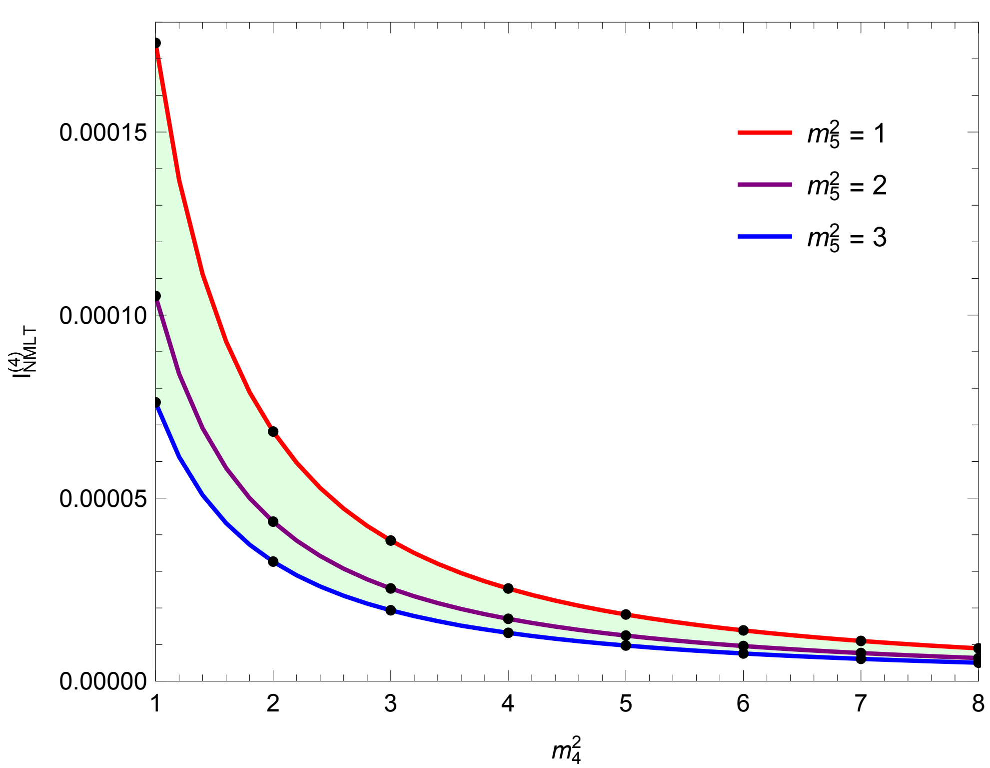
- Internal momenta $\{1,\ldots,L\}$ have mass m_4^2.
- Other particles have mass m_5^2.



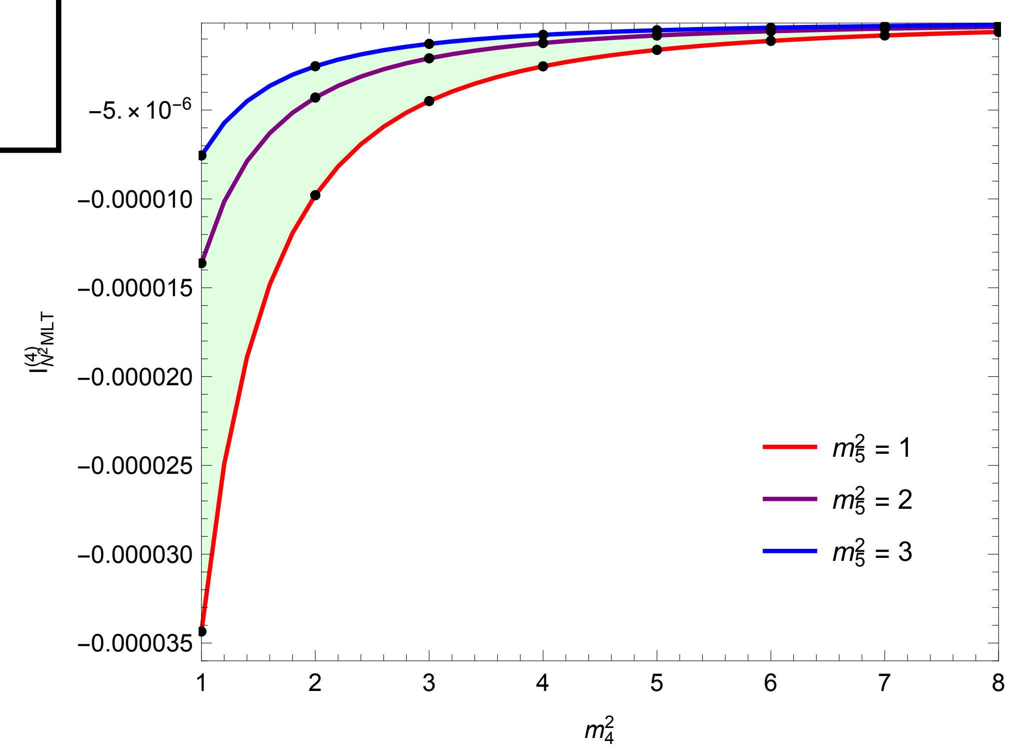
Scalar MLT(4) integral.

LTD: Solid lines

FIESTA: Dots



Scalar NMLT(4) integral.



Scalar $\mathrm{N^2MLT}(4)$ integral.

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ENTANGLED CAUSAL THRESHOLDS

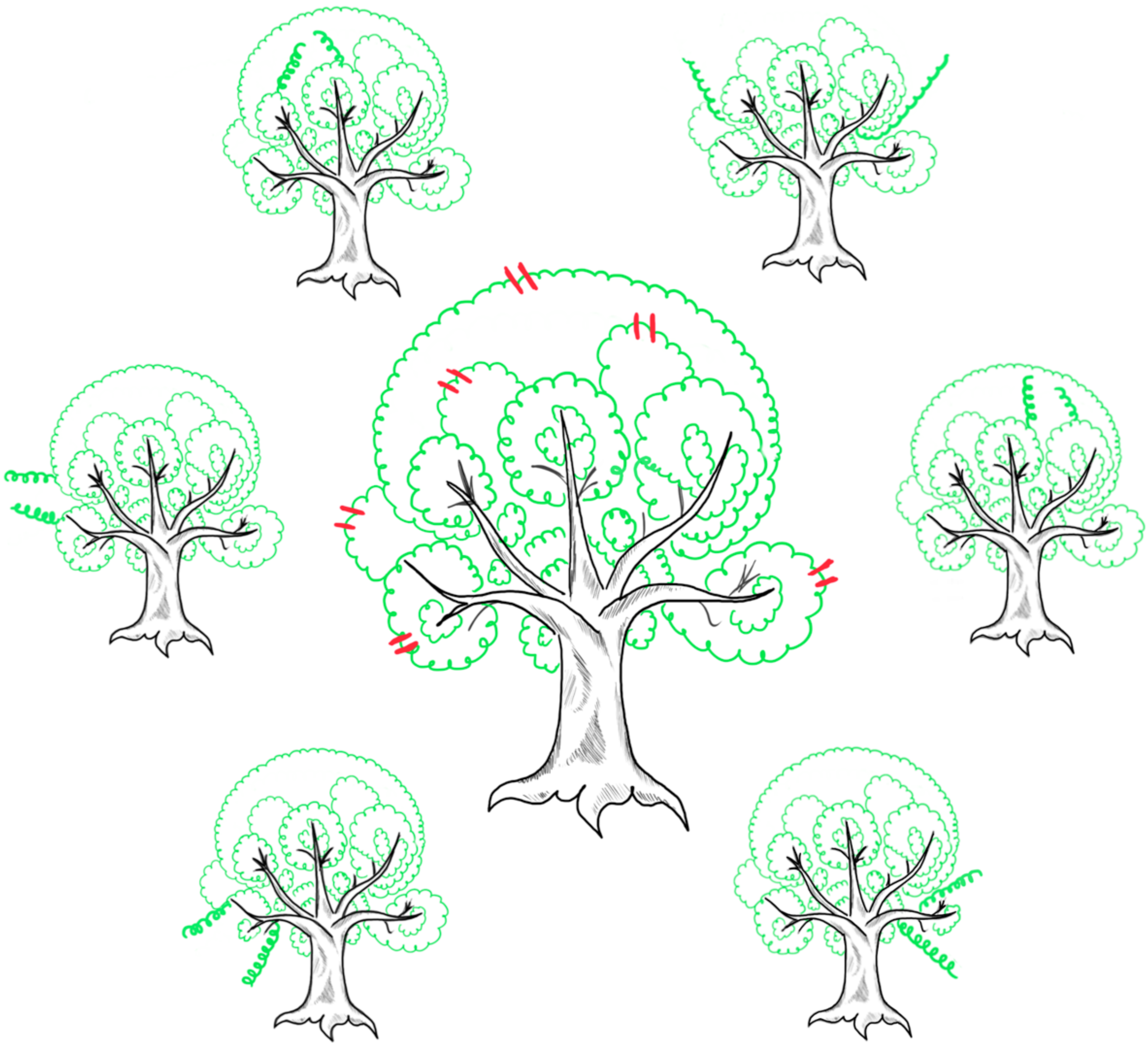
- Causal representations of topological families with positive topological complexity demand specific properties to the causal thresholds appearing in each term.
- For the N$^{\tau-1}$MLT(L), its causal representation involves terms with τ causal thresholds each. These thresholds are not arbitrary, but they should satisfy:
  - All internal particles become on-shell.
  - Causal thresholds do not intersect.
  - Compatible momentum flow.

SBORLINI'S TALK

ENTANGLED CAUSAL THRESHOLDS

SUMMARY

- The contribution to the residues of the displaced poles cancel.
- The causal structure of the scalar MLT(L) diagram is naturally obtained and is independence of the order of integration.
- Factorization formulae to NMLT and N^{2}MLT topological families were found.
- Analytic reconstructions can be used to obtain the causal structures of the NMLT and N^{2}MLT topological families.
- We have studied the stability of the causal structures obtained through the LTD, obtaining good agreement with numerical approach.



Thank you!