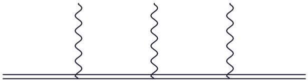
Worldline master formulas for dressed electron propagator in constant external fields



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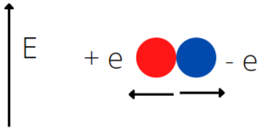
When we introduce an external field to the QED vacuum,

we can generate new physical effects

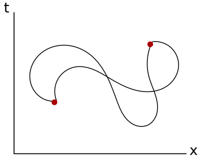
Example. Schwinger effect



To explore the physics of the QED processes in the presence of an external field, we can use the worldline formalism

Worldline formalism:

Based on first-quantized relativistic particle path integrals



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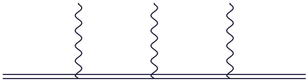


Worldline fermion propagator in an Abelian field

Worldline master formulas for the doubly dressed fermion propagator

Application. Landau levels

The fermion propagator in an Abelian field

can be obtained from a relativistic path integral

$$S^{x'x} = \langle x'| \left[m - i\not{D}\right]^{-1} |x\rangle$$

$$D_\mu = \partial_\mu + ieA_\mu$$

The fermion propagator in an Abelian field

can be obtained from a relativistic path integral

$$S^{x'x} = (m + i\not{D}_{x'})K^{x'x}$$

$$K^{x'x} = 2^{-\frac{D}{2}} \int_0^\infty dT \, e^{-m^2 T} \int_{x(0)=x}^{x(T)=x'} \mathcal{D}x e^{-\int_0^T d\tau \, \left(\frac{1}{4}\dot{x}^2 + ie\dot{x}\cdot A\right)}$$
$$\times \mathrm{symb}^{-1} \int_{\psi(0)+\psi(T)=0} D\psi \, e^{-\int_0^T d\tau \, \left[\frac{1}{2}\psi_\mu \dot{\psi}^\mu - ie(\psi+\eta)^\mu F_{\mu\nu}(\psi+\eta)^\nu\right]}$$

Proper time integral

The fermion propagator in an Abelian field

can be obtained from a relativistic path integral

$$S^{x'x} = (m + i\not{D}_{x'})K^{x'x}$$

$$K^{x'x} = 2^{-\frac{D}{2}} \int_0^\infty dT\, e^{-m^2T} \int_{x(0)=x}^{x(T)=x'} \mathcal{D}x e^{-\int_0^T d\tau\, \left(\frac{1}{4}\dot{x}^2 + ie\dot{x}\cdot A\right)}$$
$$\times \text{symb}^{-1} \int_{\psi(0)+\psi(T)=0} D\psi\, e^{-\int_0^T d\tau\, \left[\frac{1}{2}\psi_\mu \dot{\psi}^\mu - ie(\psi+\eta)^\mu F_{\mu\nu}(\psi+\eta)^\nu\right]}$$

Path integral over the worldlines of the fermion particle

The fermion propagator in an Abelian field

can be obtained from a relativistic path integral

$$S^{x'x} = (m + i\not{D}_{x'})K^{x'x}$$

$$K^{x'x} = 2^{-\frac{D}{2}} \int_0^\infty dT\, e^{-m^2 T} \int_{x(0)=x}^{x(T)=x'} \mathcal{D}x e^{-\int_0^T d\tau\, \left(\frac{1}{4}\dot{x}^2 + ie\dot{x}\cdot A\right)}$$
$$\times \mathsf{symb}^{-1} \int_{\psi(0)+\psi(T)=0} D\psi\, e^{-\int_0^T d\tau\, \left[\frac{1}{2}\psi_\mu \dot{\psi}^\mu - ie(\psi+\eta)^\mu F_{\mu\nu}(\psi+\eta)^\nu\right]}$$

Path integral representing the spin interaction

The fermion propagator in an Abelian field

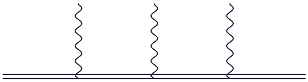
can be obtained from a relativistic path integral

$$K^{x'x} = 2^{-\frac{D}{2}} \int_0^\infty dT\, e^{-m^2 T} \int_{x(0)=x}^{x(T)=x'} \mathcal{D}x e^{-\int_0^T d\tau\, \left(\frac{1}{4}\dot{x}^2 + ie\dot{x}\cdot A\right)}$$
$$\times \mathsf{symb}^{-1} \int_{\psi(0)+\psi(T)=0} D\psi\, e^{-\int_0^T d\tau\, \left[\frac{1}{2}\psi_\mu \dot{\psi}^\mu - ie(\psi+\eta)^\mu F_{\mu\nu}(\psi+\eta)^\nu\right]}$$

$$\mathsf{symb}\left(\gamma^{\alpha_1\alpha_2\cdots\alpha_n}\right) \equiv (-i\sqrt{2})^n \eta^{\alpha_1}\eta^{\alpha_2}\ldots\eta^{\alpha_n}$$

$$\gamma^{\alpha_1\alpha_2\cdots\alpha_n} \equiv \frac{1}{n!}\sum_{\pi\in S_n} \mathsf{sign}(\pi)\gamma^{\alpha_{\pi(1)}}\gamma^{\alpha_{\pi(2)}}\cdots\gamma^{\alpha_{\pi(n)}}$$

Worldline master formulas for dressed electron propagator in constant external fields



Worldline fermion propagator in an Abelian field

Worldline master formulas for the doubly dressed fermion propagator

Application. Landau levels

To derive the master formulas, we split the Abelian field A

into two parts: $A = A_{\mathsf{ext}} + A_{\mathsf{phot}}$

$$A_{\mathsf{phot}} = \sum_{i=1}^{N} \varepsilon_i \; \mathrm{e}^{ik_i x}$$

$$A_{\mathsf{ext}}(y) = -\frac{1}{2} \; F^{\mu\nu}(y-x)^{\nu}$$ Fock-Schwinger gauge

To derive the master formulas, we split the Abelian field A into two parts: $A = A_{\mathsf{ext}} + A_{\mathsf{phot}}$

$$
\begin{aligned}
K_N^{x'x} \quad = \quad & (-ie)^N 2^{-\frac{D}{2}} \int_0^\infty dT\, e^{-m^2 T} \int \mathcal{D}x e^{-\int_0^T d\tau\, \left(\frac{1}{4}\dot{x}^2 + ie\dot{x}\cdot A_{\mathsf{ext}}\right)} \\
& \times \mathsf{symb}^{-1} \int D\psi\, V_\eta^{x'x}[k_1,\varepsilon_1]\dots V_\eta^{x'x}[k_N,\varepsilon_N] \\
& \quad \times \mathrm{e}^{-\int_0^T d\tau\, \left[\frac{1}{2}\psi_\mu\dot{\psi}^\mu - ie(\psi+\eta)^\mu F_{\mu\nu}(\psi+\eta)^\nu\right]}
\end{aligned}
$$

$$
V_\eta^{x'x}[k,\varepsilon] = \int_0^T d\tau\, \left[\varepsilon_\mu \dot{x}^\mu - i(\psi+\eta)^\mu f_{\mu\nu}(\psi+\eta)^\nu\right] \mathrm{e}^{ik\cdot x}
$$

One approach to compute the path integrals

is the spin-orbit decomposition

$$
\begin{aligned}
V_\eta^{x'x}[k,\varepsilon] \quad &= \quad \int_0^T d\tau \ \left[\varepsilon_\mu \dot{x}^\mu - i(\psi+\eta)^\mu f_{\mu\nu}(\psi+\eta)^\nu\right] \ \mathrm{e}^{ik\cdot x} \\
&= \quad \int_0^T d\tau \ \varepsilon_\mu \dot{x}^\mu \, \mathrm{e}^{ik\cdot x} + V_\eta^{\mathsf{spin}}[k,\varepsilon] \, \mathrm{e}^{ik\cdot x}
\end{aligned}
$$

One approach to compute the path integrals

is the spin-orbit decomposition

$$
\begin{aligned}
K_N^{x'x} &= \sum_{S=0}^{N} K_{NS}^{x'x} \\
K_{NS}^{x'x} &= \sum_{\{i_1 \ldots i_S\}} K_{NS}^{x'x\,\{i_1 \ldots i_S\}}
\end{aligned}
$$

One approach to compute the path integrals

is the spin-orbit decomposition

$$K_N^{x'x} = \sum_{S=0}^{N} K_{NS}^{x'x}$$

$$K_{NS}^{x'x} = \sum_{\{i_1 \dots i_S\}} K_{NS}^{x'x\ \{i_1 \dots i_S\}}$$

Example: $K_2^{x'x} = K_{20}^{x'x\ \{\}} + K_{21}^{x'x\ \{1\}} + K_{21}^{x'x\ \{2\}} + K_{22}^{x'x\ \{12\}}$

After we perform the path integrals and Fourier transform $K_{NS}^{x'x\,\{i_1\dots i_S\}}$, we obtain the momentum space worldline master formula

$$
\begin{aligned}
K_{NS}^{p'p\,\{i_1\dots i_S\}} \quad = \quad & (2\pi)^D \delta(p' + p + \sum_{i=1}^{N} k_i)\,(-ie)^N (-i)^N \int_0^\infty dT\, e^{-m^2 T} \\
& \times \prod_{i=1}^{N} \int_0^T d\tau_i\, e^{\sum_{i,j=1}^{N} k_i \underset{\smile}{\Delta}_{ij} k_j - T\, b_0 \frac{\tan \mathcal{Z}}{\mathcal{Z}} b_0} \\
& \times \bar{\mathcal{P}}_{NS}^{\{i_1\dots i_S\}}\, \mathsf{symb}^{-1}\; e^{i\eta \tan \mathcal{Z} \eta}\, \mathcal{W}_\eta(k_{i_1}, \varepsilon_{i_1}; \dots; k_{i_S}, \varepsilon_{i_S})
\end{aligned}
$$

After we perform the path integrals and Fourier transform $K_{NS}^{x'x\,\{i_1\dots i_S\}}$, we obtain the momentum space worldline master formula

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K_{NS}^{p'p\,\{i_1\dots i_S\}} \quad = \quad & (2\pi)^D \delta(p'+p+\sum_{i=1}^{N} k_i)\,(-ie)^N(-i)^N \int_0^\infty dT\, e^{-m^2 T} \\
& \times \prod_{i=1}^{N} \int_0^T d\tau_i\, e^{\sum_{i,j=1}^{N} k_i \underset{\smile}{\Delta}_{ij} k_j - T\, b_0 \frac{\tan \mathcal{Z}}{\mathcal{Z}} b_0} \\
& \times \bar{\mathcal{P}}_{NS}^{\{i_1\dots i_S\}}\, \mathsf{symb}^{-1}\; e^{i\eta \tan \mathcal{Z} \eta}\, \mathcal{W}_\eta(k_{i_1}, \varepsilon_{i_1}; \dots; k_{i_S}, \varepsilon_{i_S})
\end{aligned}
$$

Momentum conservation

After we perform the path integrals and Fourier transform $K_{NS}^{x'x\,\{i_1\dots i_S\}}$, we obtain the momentum space worldline master formula

$$
\begin{aligned}
K_{NS}^{p'p\,\{i_1\dots i_S\}} &= (2\pi)^D\delta(p'+p+\sum_{i=1}^N k_i)\,(-ie)^N(-i)^N\int_0^\infty dT\,e^{-m^2T}\\
&\times\prod_{i=1}^N\int_0^T d\tau_i\,e^{\sum_{i,j=1}^N k_i \underset{\smile}{\Delta}_{ij} k_j - T\,b_0\,\frac{\tan\mathcal{Z}}{\mathcal{Z}}\,b_0}\\
&\times\bar{\mathcal{P}}_{NS}^{\{i_1\dots i_S\}}\,\mathsf{symb}^{-1}\;e^{i\eta\tan\mathcal{Z}\eta}\,\mathcal{W}_\eta(k_{i_1},\varepsilon_{i_1};\dots;k_{i_S},\varepsilon_{i_S})
\end{aligned}
$$

$\mathcal{Z}=eTF$

After we perform the path integrals and Fourier transform $K^{x'x\,\{i_1\ldots i_S\}}_{NS}$, we obtain the momentum space worldline master formula

$$
\begin{aligned}
K^{p'p\,\{i_1\ldots i_S\}}_{NS} \quad = \quad & (2\pi)^D \delta(p' + p + \sum_{i=1}^{N} k_i)\,(-ie)^N (-i)^N \int_0^\infty dT\, e^{-m^2 T} \\
& \times \prod_{i=1}^{N} \int_0^T d\tau_i\, e^{\sum_{i,j=1}^{N} k_i \underset{\smile\, ij}{\Delta} k_j - T\, b_0 \frac{\tan \mathcal{Z}}{\mathcal{Z}} b_0} \\
& \times \bar{\mathcal{P}}^{\{i_1\ldots i_S\}}_{NS}\, \mathsf{symb}^{-1}\; e^{i\eta \tan \mathcal{Z} \eta}\, \mathcal{W}_\eta(k_{i_1}, \varepsilon_{i_1}; \ldots; k_{i_S}, \varepsilon_{i_S})
\end{aligned}
$$

$$\underset{\smile}{\Delta}(\tau, \tau')$$

Green's function of the operator

$$\frac{d^2}{d\tau^2} - 2ieF\frac{d}{d\tau}$$

with Dirichlet boundary conditions

$$\underset{\smile}{\Delta}(0, \tau') = \underset{\smile}{\Delta}(T, \tau') = \underset{\smile}{\Delta}(\tau, 0) = \underset{\smile}{\Delta}(\tau, T) = 0$$

After we perform the path integrals and Fourier transform $K_{NS}^{x'x\;\{i_1\dots i_S\}}$, we obtain the momentum space worldline master formula

$$
\begin{aligned}
K_{NS}^{p'p\;\{i_1\dots i_S\}} &= (2\pi)^D \delta(p'+p+\sum_{i=1}^N k_i)\,(-ie)^N(-i)^N \int_0^\infty dT\, e^{-m^2 T} \\
&\times \prod_{i=1}^N \int_0^T d\tau_i\, e^{\sum_{i,j=1}^N k_i \underset{\smile}{\Delta}_{ij} k_j - T\, b_0 \frac{\tan\mathcal{Z}}{\mathcal{Z}} b_0} \\
&\times \bar{\mathcal{P}}_{NS}^{\{i_1\dots i_S\}}\, \mathsf{symb}^{-1}\; e^{i\eta\tan\mathcal{Z}\eta}\, \mathcal{W}_\eta(k_{i_1},\varepsilon_{i_1};\dots;k_{i_S},\varepsilon_{i_S})
\end{aligned}
$$

$$\bar{\mathcal{P}}_{NS}^{\{i_1 \ldots i_S\}} \quad = \quad \left. \left. \mathrm{e}^{f(k_i,\, \varepsilon_i,\, F,\, b_0,\, \underset{\smile}{\Delta}_{ij})} \right|_{\varepsilon_{i_1} = \cdots = \varepsilon_{i_S} = 0} \right|_{\varepsilon_{i_{S+1}} \ldots \varepsilon_{i_N}}$$

$$b_0 = p' + \frac{1}{T} \sum_{i=1}^{N} \left(\tau_i - 2ieF^{\circ} \underset{\smile}{\Delta}_i \right) k_i$$

After we perform the path integrals and Fourier transform $K_{NS}^{x'x\,\{i_1\dots i_S\}}$, we obtain the momentum space worldline master formula

$$
\begin{aligned}
K_{NS}^{p'p\,\{i_1\dots i_S\}} \quad = \quad & (2\pi)^D \delta(p' + p + \sum_{i=1}^{N} k_i)\,(-ie)^N (-i)^N \int_0^\infty dT\, e^{-m^2 T} \\
& \times \prod_{i=1}^{N} \int_0^T d\tau_i\, e^{\sum_{i,j=1}^{N} k_i \underset{\smile}{\Delta}_{ij} k_j - T\, b_0 \frac{\tan \mathcal{Z}}{\mathcal{Z}} b_0} \\
& \times \bar{\mathcal{P}}_{NS}^{\{i_1\dots i_S\}}\, \mathsf{symb}^{-1}\; e^{i\eta \tan \mathcal{Z} \eta}\, \mathcal{W}_\eta(k_{i_1}, \varepsilon_{i_1}; \dots ; k_{i_S}, \varepsilon_{i_S})
\end{aligned}
$$

$$\mathcal{W}_\eta(k_1,\varepsilon_1;\dots;k_S,\varepsilon_S) = \sum_{\text{partitions}} (-1)^{cy}\mathcal{G}_F(i_1 i_2\dots i_{m_1})\cdots\mathcal{G}_F(i_{m_1+\dots+m_{cy-1}+1}\dots i_{m_1+\dots+m_{cy}}) \times\mathcal{G}_F|i_{m_1+\dots+m_{cy}+1}\dots i_{m_1+\dots+m_{cy}+n_1}|\cdots\mathcal{G}_F|i_{m_1+\dots+m_{cy}+n_1+\dots n_{ch-1}+1}\dots i_S|$$

$$\begin{aligned}\mathcal{G}_F(i) &= \frac{1}{2}\,\mathrm{Tr}\,\left(f_i\mathcal{G}_{F\,ii}\right)\\ \mathcal{G}_F(i_1 i_2) &= \frac{1}{2}\,\mathrm{Tr}\,\left(f_{i_1}\mathcal{G}_{Fi_1i_2}f_{i_2}\mathcal{G}_{Fi_2i_1}\right)\\ \mathcal{G}_F(i_2\dots i_n) &= \mathrm{Tr}\,\left(f_{i_1}\mathcal{G}_{Fi_1i_2}\dots f_{i_n}\mathcal{G}_{Fi_ni_1}\right), \qquad (n>2)\end{aligned}$$

$\mathcal{G}_F(\tau,\tau')$

Green function for the operator $\frac{d}{d\tau}-2ieF$ with antiperiodic boundary conditions

$$
\begin{aligned}
&\mathcal{W}_\eta(k_1,\varepsilon_1;\ldots;k_S,\varepsilon_S) = \\
&\qquad \sum_{\text{partitions}} (-1)^{cy}\mathcal{G}_F(i_1 i_2 \ldots i_{m_1})\cdots\mathcal{G}_F(i_{m_1+\ldots+m_{cy-1}+1}\ldots i_{m_1+\ldots+m_{cy}}) \\
&\qquad \times\mathcal{G}_F|i_{m_1+\ldots+m_{cy}+1}\ldots i_{m_1+\ldots+m_{cy}+n_1}|\cdots\mathcal{G}_F|i_{m_1+\ldots+m_{cy}+n_1+\ldots n_{ch-1}+1}\ldots i_S|
\end{aligned}
$$

$$
\begin{aligned}
\mathcal{G}_F|i| &= \eta\,(1+ieF^{\,\circ}\mathcal{G}_{F\,i})\,f_i\,(1+ie\mathcal{G}^{\circ}_{F\,i})\,\eta, \\
\mathcal{G}_F|i_1\ldots i_n| &= 2\,\eta\,(1+ieF^{\,\circ}\mathcal{G}_{F\,i_1})\,f_{i_1}\mathcal{G}_{F i_1 i_2}f_{i_2}\ldots f_{i_{n-1}}\mathcal{G}_{F i_{n-1}i_n}f_{i_n}\left(1+ie\mathcal{G}^{\circ}_{F\,i_n}\right)\eta,
\end{aligned}
$$

Using the Kernel in momentum space, we can write the worldline version of the doubly dressed fermion propagator in momentum space

$$\begin{aligned} S_N^{p'p}(F;k_1,\varepsilon_1;\dots;k_N,\varepsilon_N) &= \left[m + \not{p}' + i\frac{e}{2}\gamma^\mu F^{\mu\nu}\left(\frac{\partial}{\partial p'_\nu} - \frac{\partial}{\partial p_\nu}\right)\right] \\ &\quad \times K_N^{p'p}(F;k_1,\varepsilon_1;\dots;k_N,\varepsilon_N) \\ &\quad - e\sum_i \not{\varepsilon}_i K_{N-1}^{p'+k_i,p}(F;k_1,\varepsilon_1;\dots;\hat{k}_i,\hat{\varepsilon}_i;\dots;k_N,\varepsilon_N) \end{aligned}$$

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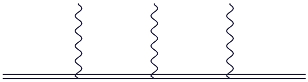

Worldline fermion propagator in an Abelian field

Worldline master formulas for the doubly dressed fermion propagator

Application. Landau levels

The relativistic Landau levels can be obtained by rewriting the exact fermion propagator in its spectral representation

$$S^{x'x} = \int dp \sum_n \frac{1}{E - E_{p,n}} \chi_{n,p}(x') \bar{\chi}_{n,p}(x)$$

We need to rewrite the worldline version of the propagator with $N = 0$ to have a similar structure to the spectral representation

$$S^{p'p} = \left[m + \not{p}' + \frac{ieB}{2} \left(\gamma^1 \frac{\partial}{\partial p_2'} - \gamma^2 \frac{\partial}{\partial p_1'} \right) \right] K$$

$$F = B \begin{pmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

We need to rewrite the worldline version of the propagator with $N = 0$ to have a similar structure to the spectral representation

$$S^{p'p} = \left[m + \not{p}' + \frac{ieB}{2} \left(\gamma^1 \frac{\partial}{\partial p'_2} - \gamma^2 \frac{\partial}{\partial p'_1} \right) \right] K$$

$$K = \int_0^\infty dT \, e^{-T\left(m^2 + p'^{\,2}_{\parallel} + \frac{\tanh(z)}{z} p'^{\,2}_{\perp}\right)} \left[1 - i \tanh(z) \sigma^{12} \right]$$

$z = eTB \qquad p'^{\,2}_{\parallel} = (p'^3)^2 - (p'^0)^2 \qquad p'^{\,2}_{\perp} = (p'^1)^2 + (p'^2)^2$

Using the generating function of the Laguerre polynomials, we can integrate the expression for K

$$K = 2e^{-\frac{p'^{\,2}_{\perp}}{eB}} \sum_{n=0}^{\infty} \frac{1}{m^2 + p'^{\,2}_{\parallel} + 2neB} M_n \left(\frac{2\, p'^{\,2}_{\perp}}{eB} \right)$$

We can obtain the energy levels from the poles of the Kernel K

Poles: $m^2 + {p'_{\|}}^2 + 2neB$

Thus,

$$E_{p,n} = \sqrt{m^2 + (p^3)^2 + 2neB}, \qquad n = 0, 1, \ldots$$

Two key ideas of the world line formalism for the fermion propagator...

- Rewrite the fermion propagator in an Abelian field into a path integral representation

- Perform the fermionic and bosonic path integrals to obtain the master formulas

References

Part I. JHEP 2008 (2020) 049, arXiv:2004.01391

Part II. JHEP 01 (2022) 050, arXiv:2107.00199

Part III. In preparation