HEAVY QUARK MESONS FROM A NON RELATIVISTIC FRAMEWORK

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IN COLLABORATION WITH

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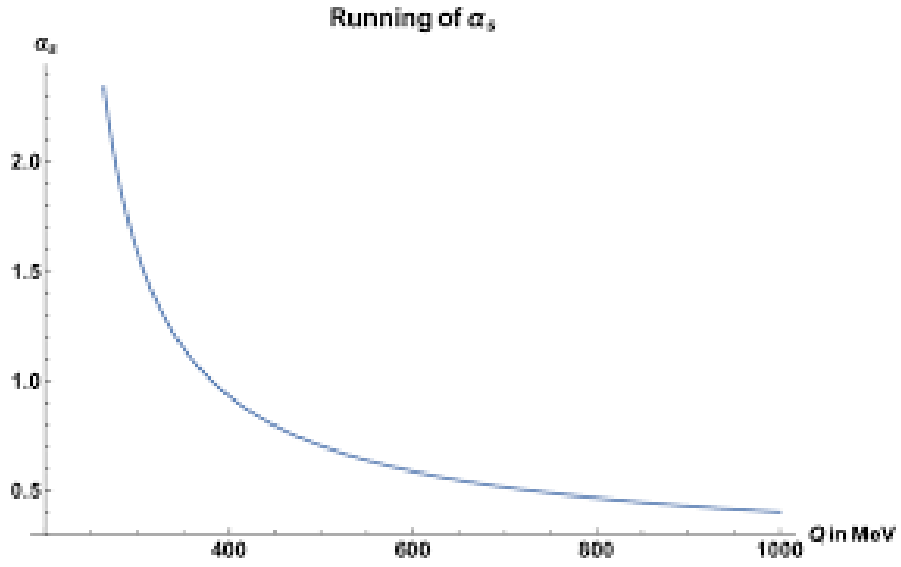
- Special thanks to Marco Antonio Bedolla, Javier Cobos and others

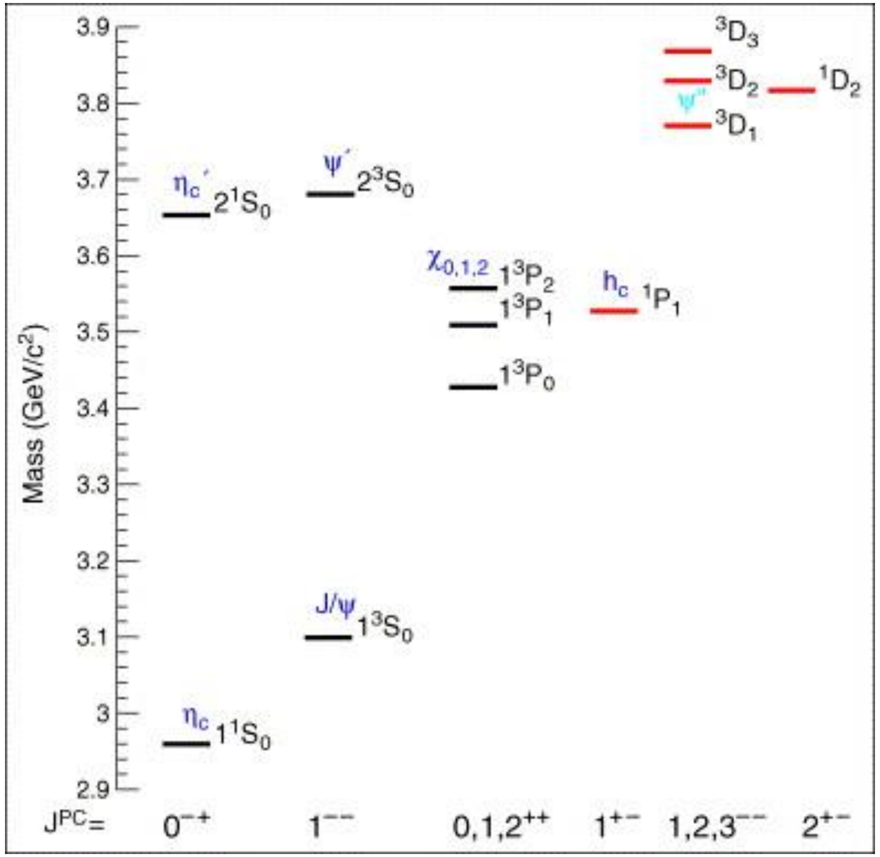
PRELIMINARIES

- QCD is the theory of quarks, gluons and their interactions
- Strong interactions atract lots of interest
- The behavior of the running coupling naturally sets up three scenarios for QCD interactions
  - Light sector: Chiral symmetry breaking and confinement
  - Heavy sector: Asymptotic freedom, non relativistic description
  - Transition region
- Simplest bound states: Mesons



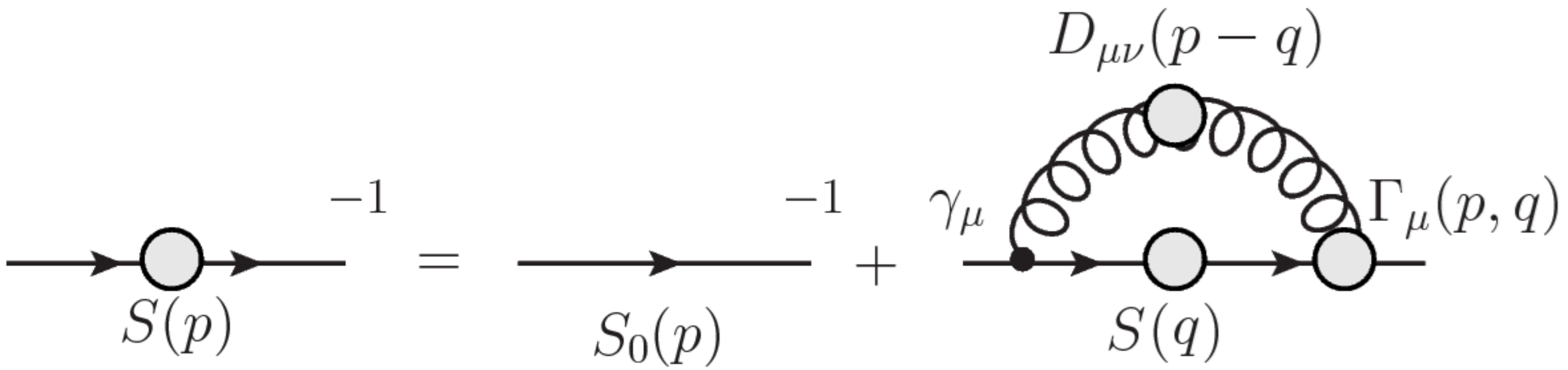
PRELIMINARIES

- Experimental measurements
- Lattice simulations
- Continuum studies
    - Schwinger-Dyson equations
    - Effective models
    - Etc



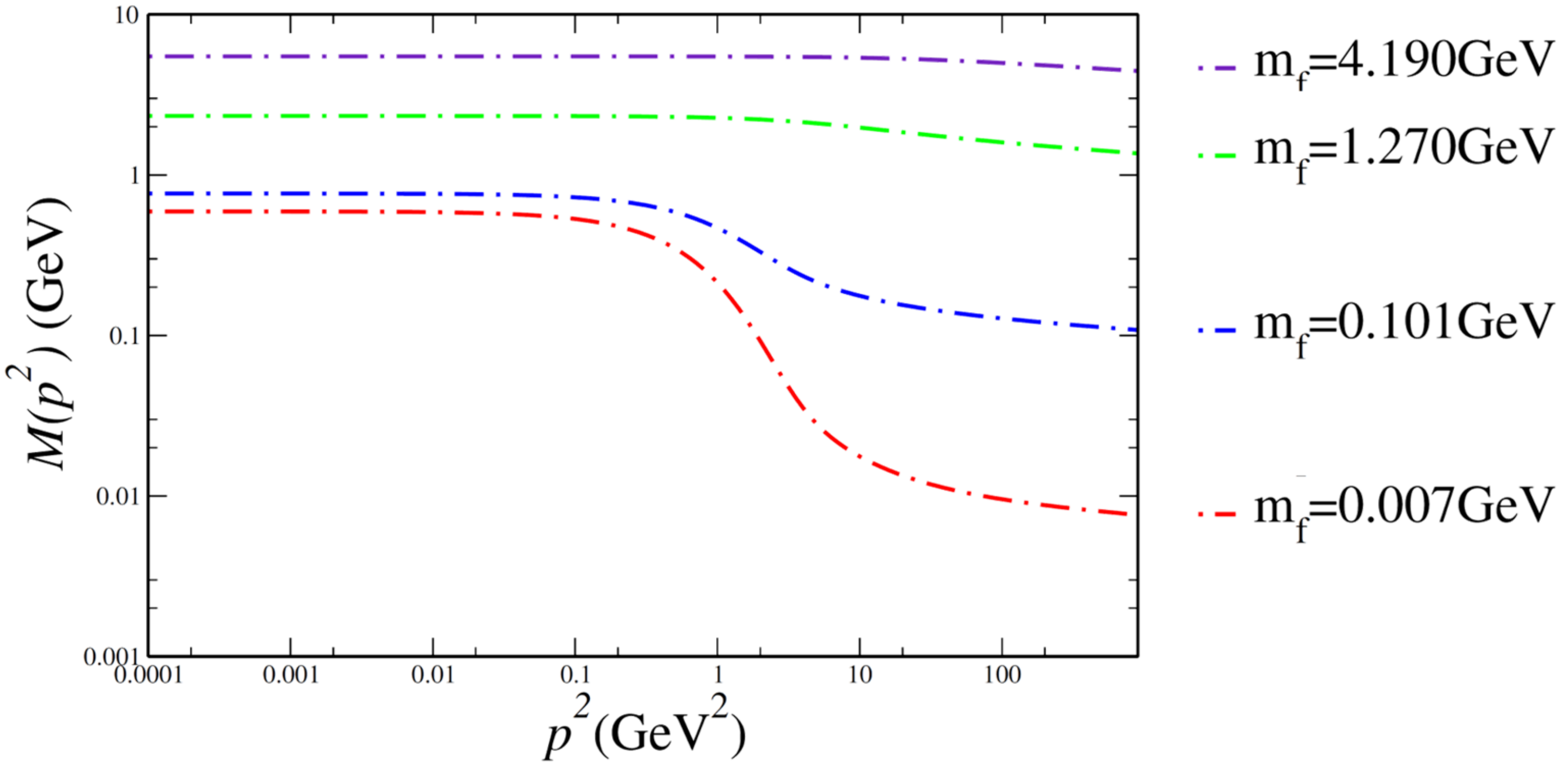
PRELIMINARIES

- Schwinger-Dyson equations



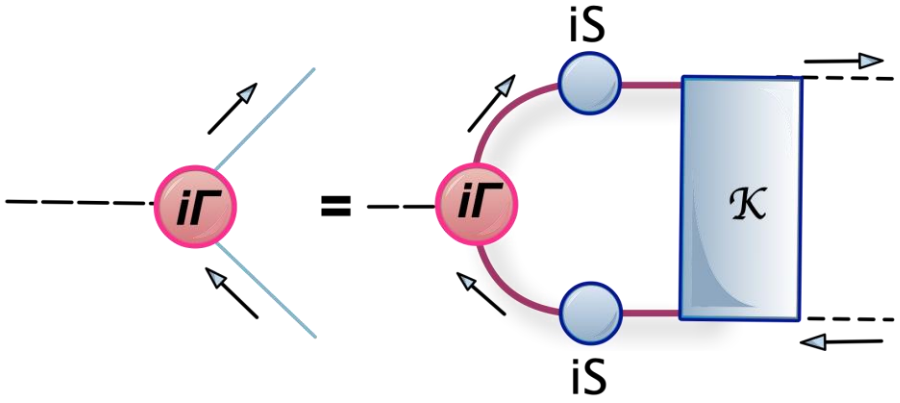
PRELIMINARIES

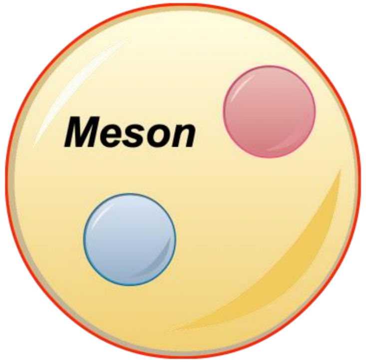
- Schwinger-Dyson equations



PRELIMINARIES

- Schwinger-Dyson equations

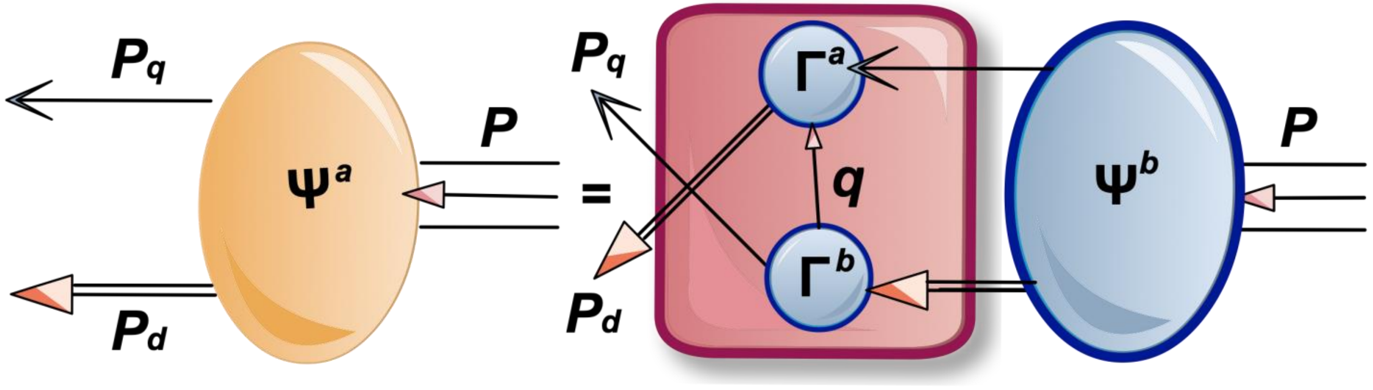




- Mesons appear as poles in the Bethe-Salpeter kernel
- Diquarks 1/2

PRELIMINARIES

- Schwinger-Dyson equations



- Barions: Quark-Diquark system with one-quark-exchange
- Tetraquarks: Diquark molecules

NON-RELATIVISTIC FRAMEWORK

- For mesons with heavy quarks, a natural starting point to study their spectra is the one body reduced Schrödinger stationary equation

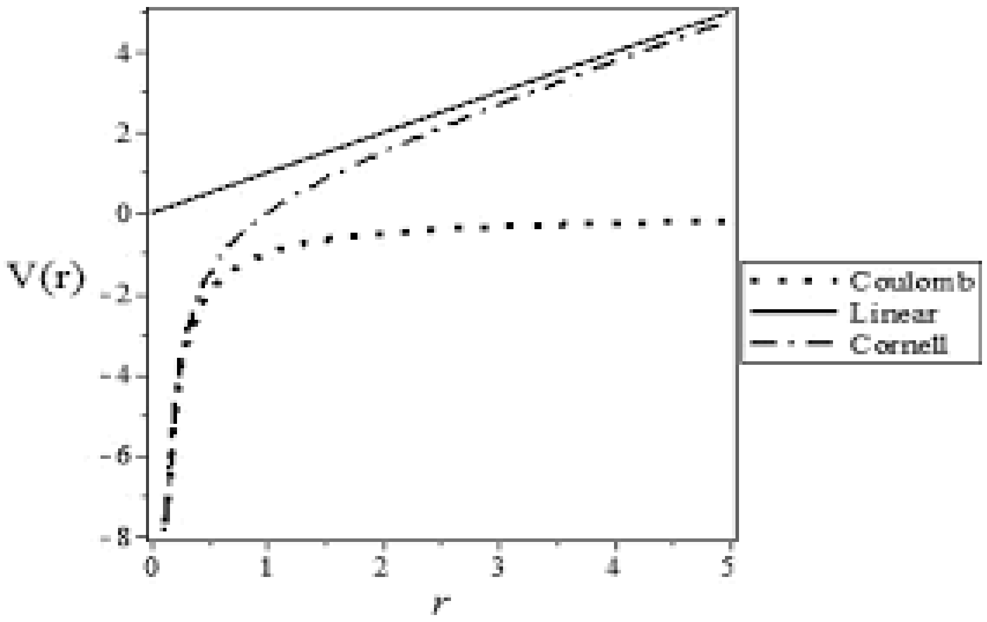
$$H\varphi = E\varphi$$

- With $H = K + V$; $\qquad K = m_{\bar{q}} + m_Q + \frac{1}{2\mu}\vec{p}^{\,2}$;

$$V = V(r) + V_S + V_T + \cdots$$

NON-RELATIVISTIC FRAMEWORK

- Cornell Potential



$$V_{\text{Cornell}} = -\frac{4}{3}\frac{\alpha_s}{r} + br$$

NON-RELATIVISTIC FRAMEWORK

- $V_s = \frac{32\pi\alpha_s}{9m_{\bar{Q}}m_Q}\delta_\sigma(r)S_Q \cdot S_{\bar{Q}}$

$$\delta_\sigma(r) = \left(\frac{\sigma}{\sqrt{\pi}}\right)^3 e^{-\sigma^2 r^2}, \quad S_Q \cdot S_{\bar{Q}} = \frac{s(s+1)}{2} - \frac{3}{4},$$

CHARMONIUM

- Schrödinger equation

$$\left(m_1 + m_2 - \frac{1}{2\mu}\frac{d^2}{dr^2} + \frac{\ell(\ell+1)}{2\mu r^2} + V(r)\right)\psi(r) = M\psi(r)$$

$$-\frac{1}{2\mu}A_{N,N}B_{N,N}^{-1}\psi_i + \left[V_N(r) + \frac{l(l+1)}{2\mu r^2}\right]\psi_i = \Delta\psi_i \,;$$

$$V_N(r) \quad = \quad \mathrm{diag}(\ldots, V_{i-1}, V_i, V_{i+1}, \ldots) \qquad A_{N,N} = \frac{I_{-1} - 2I_0 + I_1}{d^2}\,, \quad B_{N,N} = \frac{I_{-1} + 10I_0 + I_1}{12}$$

CHARMONIUM

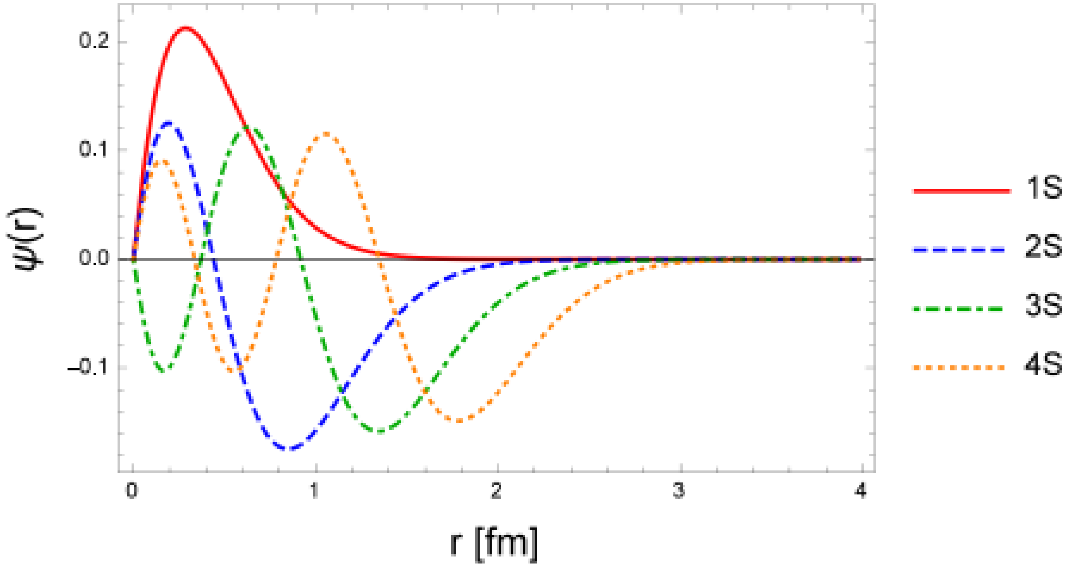


Table 1: Mass spectra in GeV for $c\bar{c}$ mesons in S- and P-states.

State	Name	Exp. value	V_1	V_2
$1\,^3S_1$	J/ψ	3.09687 ± 0.0004	3.05027	3.07834
$1\,^1S_0$	$\eta_c(1S)$	2.9792 ± 0.0013	3.05027	2.94518
$2\,^3S_1$	$\psi'(2s)$	3.68609 ± 0.0004	3.65176	3.66535
$2\,^1S_0$	$\eta'_c(2S)$	3.637 ± 0.004	3.65176	3.60943
$3\,^3S_1$	$\psi(3S)$	4.039 ± 0.001	4.06608	4.07606
$3\,^1S_0$	$\eta_c(3S)$	–	4.06608	4.03636
$4\,^3S_1$	$\psi(4S)$	4.421 ± 0.004	4.41315	4.42131
$4\,^1S_0$	$\eta_c(4S)$	–	4.41315	4.38935
$1\,^3P_2$	$\chi_2(1P)$	3.55620 ± 0.00009	3.50078	3.52544
$1\,^3P_1$	$\chi_1(1P)$	3.51066 ± 0.00007	3.50078	3.52544
$1\,^3P_0$	$\chi_0(1P)$	3.41475 ± 0.00031	3.50078	3.52544
$1\,^1P_1$	$h_c(1P)$	3.52541 ± 0.00016	3.50078	3.51837

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CHARMONIUM

$$r_{\rm rms}^2 = \int_0^\infty dr\, r^2 |\psi(r)|^2$$

$$\beta = \sqrt{2(n-1)+l+\frac{3}{2}}\,\frac{1}{r_{\rm rms}} \equiv \frac{\delta}{r_{\rm rms}}$$

Table 2: $r_{\rm rms}$ and β for $c\bar{c}$ mesons in S- and P-states.

State	Name	$r_{\rm rms}$ [fm]		β	
		V_1	V_2	V_1	V_2
$1\,^3S_1$	J/ψ	0.417	0.432	0.5874	0.5666
$1\,^1S_0$	$\eta_c(1S)$	0.417	0.369	0.5874	0.6642
$2\,^3S_1$	$\psi'(2s)$	0.87	0.88	0.4248	0.4249
$2\,^1S_0$	$\eta'_c(2S)$	0.87	0.84	0.4299	0.4456
$3\,^3S_1$	$\psi(3S)$	1.238	1.246	0.3787	0.3763
$3\,^1S_0$	$\eta_c(3S)$	1.238	1.216	0.3787	0.3858
$4\,^3S_1$	$\psi(4S)$	1.559	1.565	0.3513	0.3499
$4\,^1S_0$	$\eta_c(4S)$	1.559	1.54	0.3513	0.3555
$1\,^3P_2$	$\chi_2(1P)$	0.6912	0.7025	0.3544	0.3487
$1\,^3P_1$	$\chi_1(1P)$	0.6912	0.7025	0.3544	0.3487
$1\,^3P_0$	$\chi_0(1P)$	0.6912	0.7025	0.3544	0.3487
$1\,^1P_1$	$h_c(1P)$	0.6912	0.6928	0.3544	0.3535

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CHARMONIUM

- Variational & Uncertainty Principles

$$\Delta p_x \Delta x \geq \frac{1}{2}$$

$$p_x \equiv \tilde{\beta} = 1/(2\bar{x})$$

$$H = \frac{p_x^2}{2m} + \frac{p_y^2}{2m} + \frac{p_z^2}{2m} + V(\sqrt{x^2+y^2+z^2})$$
$$= \frac{1}{8m^2\bar{x}^2} + \frac{1}{8m^2\bar{y}^2} + \frac{1}{8m^2\bar{z}^2} + V(\sqrt{\bar{x}^2+\bar{y}^2+\bar{z}^2})$$

$$\frac{\partial H}{\partial \bar{x}} = -\frac{1}{4m\bar{x}^3} + V_{\bar{x}}(\sqrt{\bar{x}^2+\bar{y}^2+\bar{z}^2}) \equiv 0$$

$$\bar{x} = \bar{y} = \bar{z} = x_{\min},$$

$$\Delta = 2m_c + \frac{1}{2\mu}\beta^2 + \frac{l(l+1)}{2\mu r_{\rm rms}^2} + V_{1,2}(r_{\rm rms})$$

Table 3: Mass spectra for $c\bar{c}$-mesons in S- and P-states from the variational constraint (17).

$c\bar{c}$ state	Name	Δ [GeV]		Error [%]	
		V_1	V_2	V_1	V_2
$1\,^3S_1$	J/ψ	3.10915	3.11616	1.93	1.24
$1\,^1S_0$	$\eta_c(1S)$	3.10915	3.09165	1.93	4.99
$2\,^3S_1$	$\psi'(2s)$	3.51506	3.52171	3.74	3.91
$2\,^1S_0$	$\eta'_c(2S)$	3.51506	3.49553	3.74	3.15
$3\,^3S_1$	$\psi(3S)$	3.81042	3.81574	6.29	6.38
$3\,^1S_0$	$\eta_c(3S)$	3.81042	3.79502	6.29	5.98
$4\,^3S_1$	$\psi(4S)$	4.06008	4.06453	8.0	8.07
$4\,^1S_0$	$\eta_c(4S)$	4.06008	4.04732	8.0	7.79
$1\,^3P_2$	$\chi_2(1P)$	3.41341	3.48758	4.02	1.93
$1\,^3P_1$	$\chi_1(1P)$	3.41341	3.48758	2.77	0.66
$1\,^3P_0$	$\chi_0(1P)$	3.41341	3.48758	0.04	2.13
$1\,^1P_1$	$h_c(1P)$	3.41341	3.48469	3.18	1.16

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SONG-LIN POTENTIAL

$$V(r) = -\frac{b}{\sqrt{r}} + a\sqrt{r},$$

Leptonic widths

Flux tube

$$a = 0.7011\ \text{GeV}^{\frac{3}{2}}, \quad b = 0.8912\ \text{GeV}^{\frac{1}{2}}$$

$$m_c = 1.2\ \text{GeV}, \quad m_b = 4.668\ \text{GeV}$$

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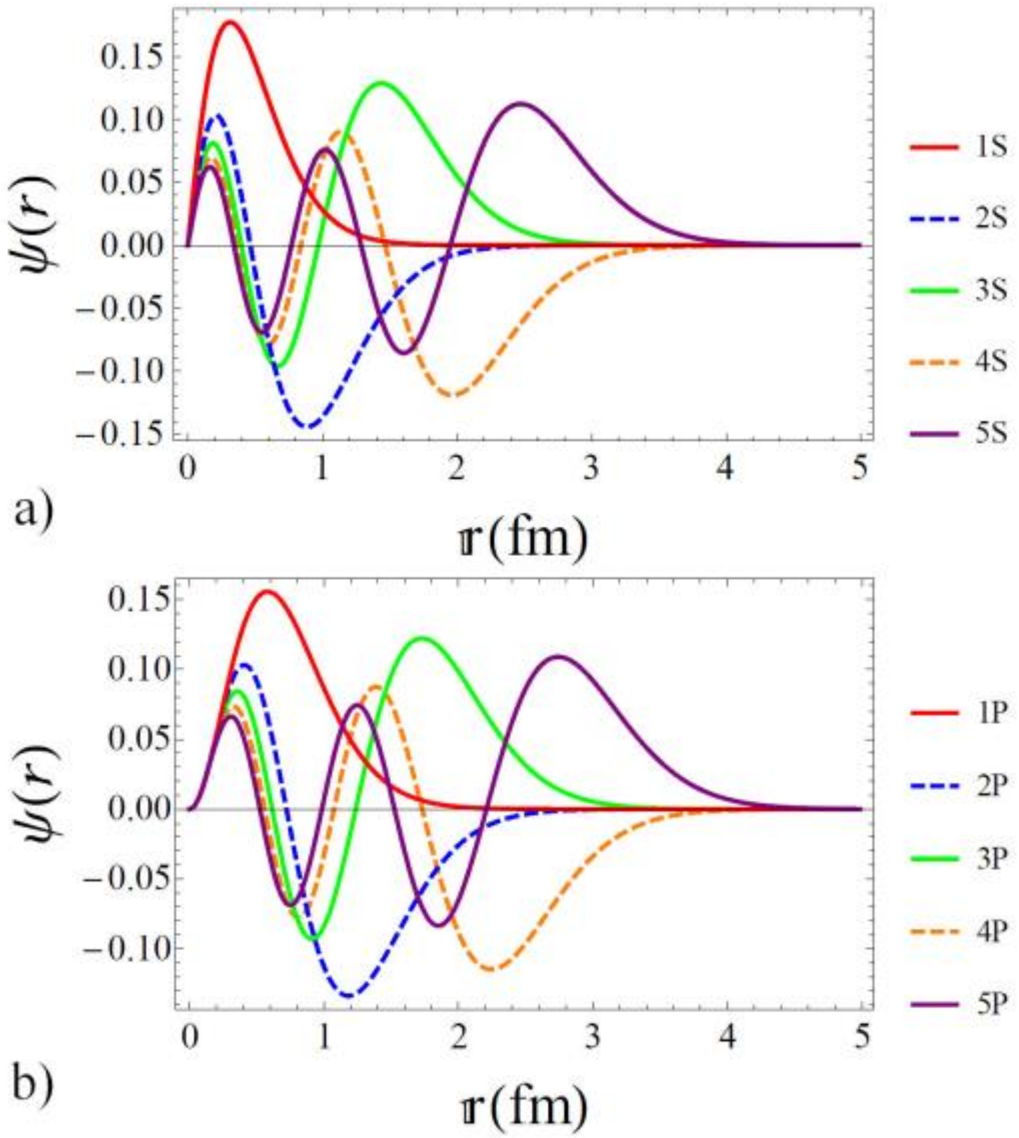

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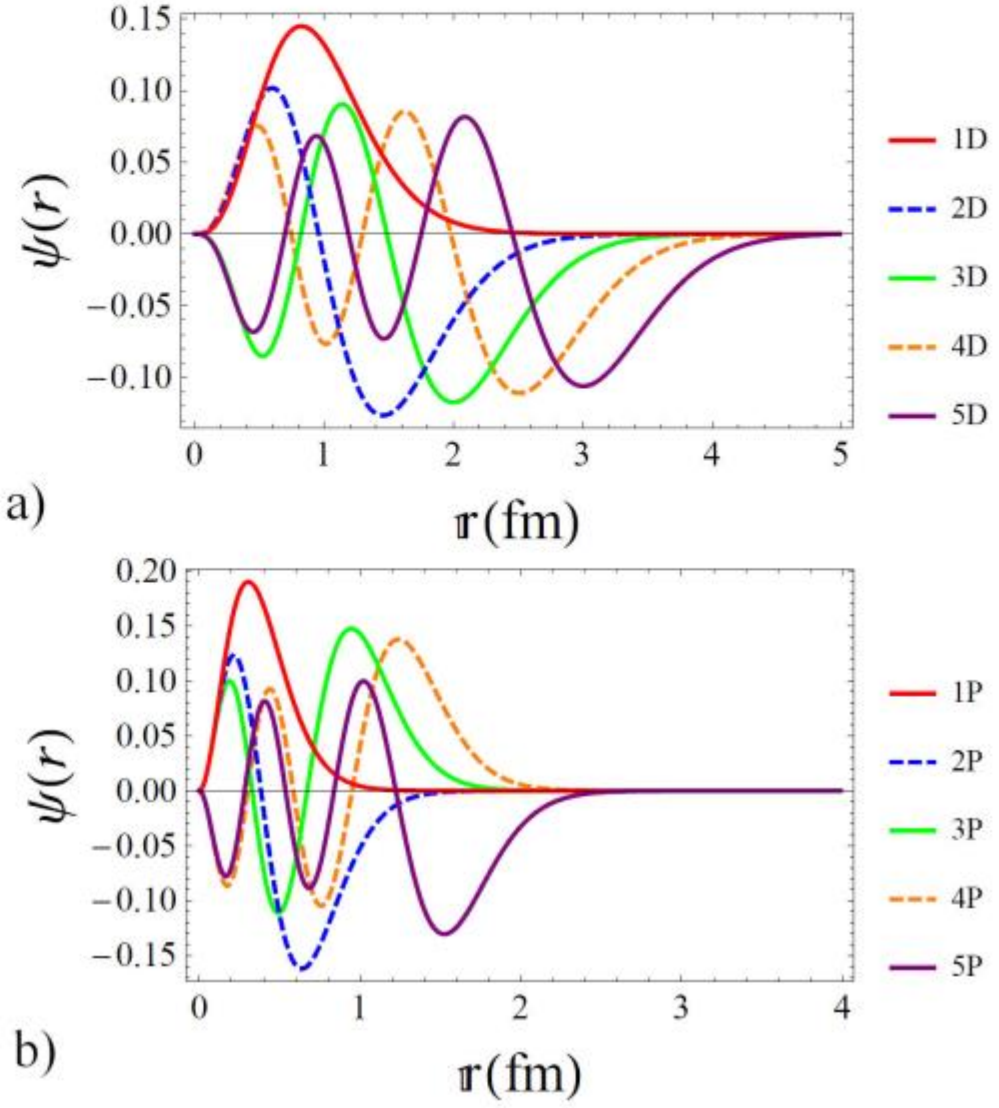

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SONG-LIN POTENTIAL

TABLE I. Masses [GeV] for $c\bar{c}$ spin-averaged S, P and D States. We calculated spin averaged masses using $(\sum_J(2J+1)\Delta_J/\sum_J(2J+1))$ from Ref. [62] for Cornell potential model in Refs. [27,49] for purpose of comparison with our predicted 5P and 4D spin averaged states. In the last column, N stands for "from Numerov Method", E for "from Experimental mass of the meson" and VP is for "from our variational principle" (Sec. 4).

nL	Exp. Masses [33]	$m_c = 1.8$ GeV [22]	$m_c = 1.8$ GeV [26]	$m_c = 1.2$ GeV [Our work]	Our QIE [MeV] N/E/VP
$1S$	3.067	3.104	3.097	3.066	666/ 660/ 662
$2S$	3.649	3.703	3.673	3.764	1364/ 1250/ 1211
$3S$	4.040	4.090	4.017	4.208	1808/ 1640/ 1624
$4S$	4.415	4.375	4.276	4.545	2145/ 2015/ 1937
$5S$	—	4.692	4.487	4.824	2424/ ——/ 2194
$1P$	3.525	3.572	3.524	3.566	1166/ 1125/ 1042
$2P$	—	3.986	3.907	4.055	1655/ ——/ 1480
$3P$	—	4.280	4.186	4.418	2018/ ——/ 1813
$4P$	—	4.580	4.410	4.714	2314/ ——/ 2086
$5P$	—	5.034 [27]	—	4.969	2569/ ——/ 2320
$1D$	3.769	3.806	3.791	3.902	1502/ 1369/ 1388
$2D$	4.159	4.185	4.090	4.292	1892/ 1759/ 1713
$3D$	—	4.474	4.328	4.605	2205/ ——/ 1991
$4D$	—	4.898 [49]	—	4.871	2471/ ——/ 2231
$5D$	—	—	—	5.107	2707 ——/ 2444

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SONG-LIN POTENTIAL

TABLE II. Masses [Gev] for $b\bar{b}$ spin-averaged S -,P -,D States. We calculated spin averaged masses using $(\sum_J (2J+1)\Delta_J / \sum_J (2J+1))$ from Ref. [62] for Cornell potential model in Ref. [48] for purpose of comparison with our predicted 5P, 4D and 5D spin averaged states. In the last column, N stands for "from Numerov Method", E for "from Experimental mass of the meson" and VP is for "from our variational principle" (Sec. 4).

nL	Exp. Masses [33]	m_b = 5.2 GeV [22]	m_b = 5.2 GeV [26]	m_b = 4.668 GeV [Our work]	Our QIE [MeV] N/E/VP
1S	9.444	9.473	9.460	9.444	110/ 108/ 122
2S	10.023	10.024	10.034	10.098	764/ 687/ 651
3S	10.355	10.327	10.356	10.482	1148/ 1019/ 1017
4S	10.579	10.593	10.589	10.766	1432/ 1243/ 1283
5S	10.865	10.788	10.776	10.998	1664/ 1529/ 1498
1P	9.900	9.912	9.902	9.930	596/ 564/ 496
2P	10.260	10.275	10.261	10.358	1024/ 924/ 890
3P	——	10.580	10.512	10.665	1331/ ——/ 1178
4P	——	10.703	10.711	10.911	1577/ ——/ 1408
5P	——	11.013 [48]	—	11.119	1785/ ——/ 1602
1D	10.161	10.156	10.162	10.234	900/ 825/ 810
2D	——	10.434	10.433	10.565	1231/ ——/ 1093
3D	——	10.625	10.643	10.825	1491/ ——/ 1328
4D	——	10.934 [48]	—	11.043	1709/ ——/ 1529
5D	——	11.143 [48]	—	11.232	1898/ ——/ 1703

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SONG-LIN POTENTIAL

TABLE III. Radii $r^{c\bar{c}}_{\rm rms}$ of $c\bar{c}$ and $r^{b\bar{b}}_{\rm rms}$ of $b\bar{b}$ states in [fm] and corresponding momentum width $\beta^{c\bar{c}}$ and $\beta^{b\bar{b}}$ in [GeV] in spin averaged S -,P- and D quarkonia states.

nL	$r^{c\bar{c}}_{\rm rms}(SWE)/r^{c\bar{c}}_{\rm min}(VP)$/Ref. [63]	$\beta^{c\bar{c}}_{SWE}/\beta^{c\bar{c}}_{VP}$	$r^{b\bar{b}}_{rms}(SWE)/r^{b\bar{b}}_{min}(VP)$/Ref. [64]	$\beta^{b\bar{b}}_{SWE}/\beta^{b\bar{b}}_{VP}$
1S	0.439/ 0.3533/ 0.41	0.682/ 0.693	0.225/ 0.179 / 0.233	1.335/ 1.369
2S	0.915/ 0.922/ 0.91	0.765/ 0.760	0.488/ 0.501/ 0.545	1.434/ 1.397
3S	1.352/ 1.363/ 1.38	0.813/ 0.810	0.737/ 0.755/ 0.805	1.493/ 1.457
4S	1.762/ 1.773/ 1.87	0.851/ 0.846	0.972/ 0.991/ 1.030	1.544/ 1.514
5S	2.151/ 2.161/ 2.39	0.882/ 0.879	1.200/ 1.216/ 1.232	1.588/ 1.562
1P	0.697/ 0.775/ 0.71	0.717/ 0.645	0.370/ 0.416/ 0.435	1.350/ 1.202
2P	1.155/ 1.195/ 1.19	0.779/ 0.753	0.628/ 0.658/ 0.711	1.432/ 1.368
3P	1.577/ 1.601/ 1.67	0.824/ 0.812	0.869/ 0.893/ 0.945	1.496/ 1.456
4P	1.976/ 1.992/ –	0.860/ 0.853	1.097/ 1.118/ 1.154	1.549/ 1.521
5P	2.343/ 2.367/ –	0.891/ 0.887	1.317/ 1.336/ 1.346	1.594/ 1.572
1D	0.936/ 1.097/ 0.96	0.748/ 0.638	0.507/ 0.601/ 0.593	1.379/ 1.165
2D	1.380/ 1.472/ 1.44	0.797/ 0.747	0.760/ 0.818/ –	1.448/ 1.345
3D	1.791/ 1.850/ 1.94	0.838/ 0.811	0.994/ 1.036/ –	1.508/ 1.448
4D	2.178/ 2.220/ –	0.871/ 0.856	1.218/ 1.251/ –	1.559/ 1.560
5D	2.497 /2.581/ –	0.900/ 0.891	1.434/ 1.460/ –	1.604/ 1.575

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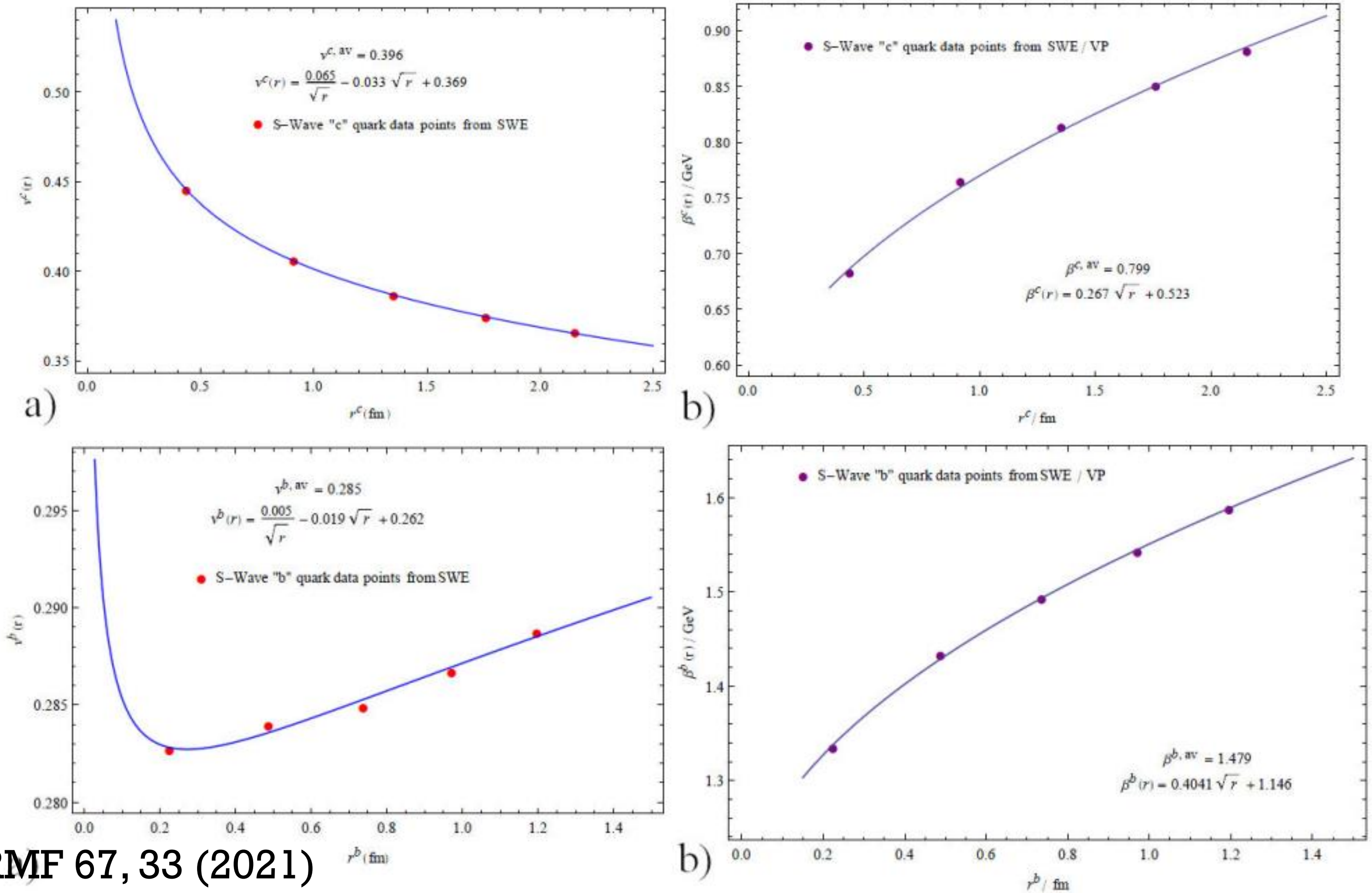
SONG-LIN POTENTIAL

- Quark velocities

$$v_S = 2\frac{\beta}{\Delta}$$

$$v_S(r) = \frac{\sigma}{\sqrt{r}} + \varepsilon\sqrt{r} + v_S^0,$$

$$\beta(r) = \rho\sqrt{r} + \beta^0,$$



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SONG-LIN POTENTIAL

$$\Gamma(nS \to e^- e^+) = \frac{|R_{ns}(0)|^2}{\Delta_{ns}^2} 4e_Q^2\alpha^2 \left(1 - \frac{16\alpha_s}{3\pi}\right),$$

$$\Gamma(nS \to 3g) = \frac{|R_{ns}(0)|^2}{m_Q^2} \frac{10(\pi^2-9)}{81\pi}\alpha_s^3 \times \left(1 - \frac{4.9\alpha_s}{\pi}\right),$$

$$\Gamma(nS \to \gamma\gamma\gamma) = \frac{|R_{ns}(0)|^2}{m_Q^2} \frac{4(\pi^2-9)}{3\pi} e_Q^6\alpha^3 \times \left(1 - \frac{12.6\alpha_s}{\pi}\right),$$

$$\Gamma(nS \to \gamma gg) = \frac{|R_{ns}(0)|^2}{m_Q^2} \frac{8(\pi^2-9)}{9\pi}\alpha\alpha_s^2 e_Q^2 \times \left(1 - \frac{6.7\alpha_s}{\pi}\right),$$

$$\Gamma(nS \to gg) = \frac{|R_{ns}(0)|^2}{m_Q^2} \frac{2\alpha_s^2}{3}, \left(1 + \frac{4.8\alpha_s}{\pi}\right),$$

$$\Gamma(nS \to \gamma\gamma) = \frac{|R_{ns}(0)|^2}{m_Q^2} \frac{3e_Q^4\alpha^2}{1} \left(1 - \frac{3.4\alpha}{\pi}\right),$$

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SONG-LIN POTENTIAL

TABLE VII. Leptonic decay widths [keV] of spin averaged S-wave $c\bar{c}$ states.

State	Our $\Gamma(nS)/\Gamma^c(nS)$	Exp. [38]	[39]	[40]	$(\Gamma(nS- > e^+e^-)/\Gamma(1S- > e^+e^-))$ our/Exp. [26]
1S	5.55	5.55	5.63	3.112	1.00/1.00
2S	2.18	2.33	2.19	2.197	0.39/(0.45±0.08)
3S	1.34/0.91	0.86	1.20	1.701	0.16/(0.16±0.04)
4S	0.97/0.63	0.58	0.63	-	0.11/(0.11±0.04)
5S	0.76/0.47	-	0.24	-	0.08/–

TABLE VIII. Three-gluon decay widths [keV] of spin averaged S-wave $c\bar{c}$ states.

State	Our $\Gamma(nS)/\Gamma^c(nS)$	Exp. [38]	[41]	[42]	$(\Gamma(nS- > ggg)/\Gamma(1S- > ggg))$ our/Exp.
1S	59.45	59.55	269.06	52.8±5	1.00/1.00
2S	35.26/30.48	31.38	112.03	23±2.6	0.51/0.53
3S	27.14/23.52	-	94.57	-	0.40/–
4S	22.85/19.85	-	88.44	-	0.33/–
5S	20.12/17.38	-	85.30	-	0.29/–

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SONG-LIN POTENTIAL

- Variational mass formula

$$\Delta_{VP} = (m_1 + m_2) + \frac{\delta^2 + \ell(\ell+1)}{6\mu}\frac{1}{x_{\min}^2} - \frac{b}{\sqrt{\sqrt{3}x_{\min}}} + a\sqrt{\sqrt{3}x_{\min}}.$$

TABLE XXIII. Masses [GeV] for $c\bar{c}$ and $b\bar{b}$ spin-averaged S, P and D states from our VP compared with those from SWE and from the experiments.

nL	$\text{Exp}_{c\bar{c}}$. [33]	$(\text{Numerov})_{c\bar{c}}$	Our VP	$\text{Exp}_{b\bar{b}}$. [33]	$(\text{Numerov})_{b\bar{b}}$	Our VP
$1S$	3.067	3.066	3.062	9.444	9.444	9.458
$2S$	3.649	3.764	3.611	10.023	10.098	9.987
$3S$	4.040	4.208	4.024	10.355	10.482	10.353
$4S$	4.415	4.545	4.337	10.597	10.766	10.619
$5S$	4.487 [26]	4.824	4.595	10.865	10.998	10.834
$1P$	3.525	3.566	3.442	9.900	9.930	9.832
$2P$	3.907 [26]	4.055	3.879	10.260	10.358	10.226
$3P$	4.186 [26]	4.418	4.213	10.512 [57]	10.665	10.514
$4P$	4.409 [26]	4.714	4.486	10.711 [57]	10.911	10.744
$5P$	4.807	4.969	4.720	10.014	11.119	10.938
$1D$	3.769	3.902	3.788	10.161	10.234	10.146
$2D$	4.159	4.292	4.113	10.432 [57]	10.565	10.429
$3D$	4.328 [26]	4.605	4.391	10.643 [57]	10.825	10.664
$4D$	4.520	4.871	4.631	11.011	11.043	10.865
$5D$	4.885	5.107	4.844	11.389	11.232	11.039

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SONG-LIN POTENTIAL

- Beyond spin-averaged approximation

$$\left(m_1 + m_2 - \frac{1}{2\mu}\frac{d^2}{dr^2} + \frac{\ell(\ell+1)}{2\mu r^2} + V(r)\right)\psi(r) = M\psi(r).$$

$$V_{NR}(r) = V(r) + S(r)\ ,\ V(r) = -\frac{a}{\sqrt{r}}\ ,\ \ S(r) = b\sqrt{r},$$

$$\begin{aligned}
V(r) &= V_{NR}(r) + V_{SS}(r) + V_{SL}(r) + V_{TT}(r)\ , \\
V_{SS}(r) &= \frac{1}{6m_1m_2}\left(\frac{s(s+1)}{2} - \frac{3}{4}\right)\frac{a}{r^{5/2}}\ , \\
V_{SO}(r) &= \frac{\lambda}{4m_1m_2}\left(3\frac{a}{\sqrt{r}} - b\sqrt{r}\right)\frac{1}{r^2}\ , \\
V_{TT}(r) &= \frac{5}{12m_1m_2}\left(s(s+1)\ell(\ell+1) - \frac{3}{2}\lambda(2\lambda+1)\right) \\
&\quad \times \frac{1}{(2\ell+3)(2\ell-1)}\frac{a}{r^{5/2}}\ .
\end{aligned}$$

$$a = 0.891\ \mathrm{GeV}^{\frac{3}{2}}\ ,\ \ b = 0.701\ \mathrm{GeV}^{\frac{1}{2}}\ ,$$
$$m_c = 1.20\ \mathrm{GeV}\ ,\ \ m_b = 4.663\ \mathrm{GeV}\ .$$

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SONG-LIN POTENTIAL

$M(n^{2s+1}L_j)$ denotes the mass of the state $n^{2s+1}L_j$.

Hyperfine splitting

$$\delta_{hf} = \Delta M_{n\ell} := M(n^1L_\ell) - \langle M(n^3L)\rangle$$
$$\langle M(n^3L)\rangle := \frac{1}{3\mathbf{L}^0}[\mathbf{L}^- M(n^3L_{\ell-1}) + \mathbf{L}^0 M(n^3L_\ell) + \mathbf{L}^+ M(n^3L_{\ell+1})]$$

$\langle M(n^3L)\rangle$ defines the center of gravity for the corresponding spin-triplet states;

$\mathbf{L}^0 = 2\ell + 1$

$$\Delta M_{n\ell} := M(n^1P_1) - \frac{1}{9}[(n^3P_0) + 3M(n^3P_1) + 5M(n^3P_2)]$$

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SONG-LIN POTENTIAL

- For convenience notation

$$M({}^1L_\ell) - \delta_{hf} = \frac{1}{3\mathbf{L}^0}[\mathbf{L}^- M({}^3L_{\ell-1}) + \mathbf{L}^0 M({}^3L_\ell) + \mathbf{L}^+ M({}^3L_{\ell+1})]$$

$$(1+\delta_\xi)M({}^3L_\ell) = M({}^1L_\ell).$$

$$M({}^3L_\ell)\left(\frac{2}{3}+\delta_\xi\right) - \delta_{hf} = \frac{\mathbf{L}^- M({}^3L_{\ell-1}) + \mathbf{L}^+ M({}^3L_{\ell+1})}{3\mathbf{L}^0}$$

$$[M({}'L_\ell) + M(L_\ell)]\left(\frac{2+3\delta_\xi}{6+3\delta_\xi}\right) - \delta_{hf} = \frac{\mathbf{L}^- M({}^3L_{\ell-1}) + \mathbf{L}^+ M({}^3L_{\ell+1})}{3\mathbf{L}^0}.$$

$$\mathbf{L}^0 = \mathbf{L}^\pm \mp 2.$$

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SONG-LIN POTENTIAL

Table 3 Numerical results of the $c\bar{c}$ sector: singlet and center of gravity masses, hyperfine and δ_ξ splittings. Bracketed quantities correspond to the RPP results [31], without experimental uncertainties. Mass units in MeV

(ℓ, n)	$M(^1L_\ell)$	$\langle M(n^3L)\rangle$	$-\delta_{hf}$	$10^3\delta_\xi$
(1,1)	3556.96 [3525.38]	3562.42 [3525.29]	5.46 [−0.09]	0.02 [4.19]
(1,2)	4048.82	4052.69	3.67	0.35
(1,3)	4413.28	4416.41	3.13	0.44
(2,1)	3898.99	3902.21	3.22	−1.70
(2,2)	4289.71	4292.04	2.33	−1.09
(2,3)	4603.23	4605.07	1.84	−0.77
(3,1)	4163.24	4165.1	1.86	−1.48
(3,2)	4494.14	4495.62	1.48	−1.07
(3,3)	4772.12	4773.36	1.24	−0.82

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SONG-LIN POTENTIAL

Table 4 Numerical results of the $b\bar{b}$ sector: singlet and center of gravity masses, hyperfine and δ_ξ splittings. Bracketed quantities correspond to the RPP results [31], without experimental uncertainties. Mass units in MeV

(ℓ, n)	$M(^1L_\ell)$	$\langle M(n^3L)\rangle$	$-\delta_{hf}$	$10^3\delta_\xi$
(1,1)	9918.22 [9899.30]	9921.07 [9899.87]	2.85 [0.57]	0.43 [0.66]
(1,2)	10347.3 [10259.8]	10349.1 [10260.2]	1.86 [0.44]	0.29 [0.43]
(1,3)	10655.2	10656.6	1.42	0.22
(2,1)	10224.8	10225.9	1.09	0.02
(2,2)	10555.7	10556.5	0.81	0.02
(2,3)	10816.1	10816.7	0.66	0.02
(3,1)	10454.5	10455.1	0.55	−0.04
(3,2)	10729.6	10730.1	0.44	−0.03
(3,3)	10957.7	10958.1	0.38	−0.02

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SONG-LIN POTENTIAL

Table 5 Numerical results of the B_c sector: singlet and center of gravity masses, hyperfine and δ_ξ splittings. Mass units in MeV

(ℓ, n)	$M(^1L_\ell)$	$\langle M(n^3L)\rangle$	$-\delta_{hf}$	$10^3\delta_\xi$
(1,1)	6824.09	6827.68	3.59	0.31
(1,2)	7289.03	7291.36	2.33	0.26
(1,3)	7630.62	7632.39	1.77	0.23
(2,1)	7148.70	7150.25	1.55	−0.22
(2,2)	7515.39	7516.56	1.17	−0.14
(2,3)	7808.04	7808.99	0.95	−0.10
(3,1)	7398.29	7399.10	0.81	−0.24
(3,2)	7707.30	7707.96	0.66	−0.18
(3,3)	7965.92	7966.48	0.56	−0.14

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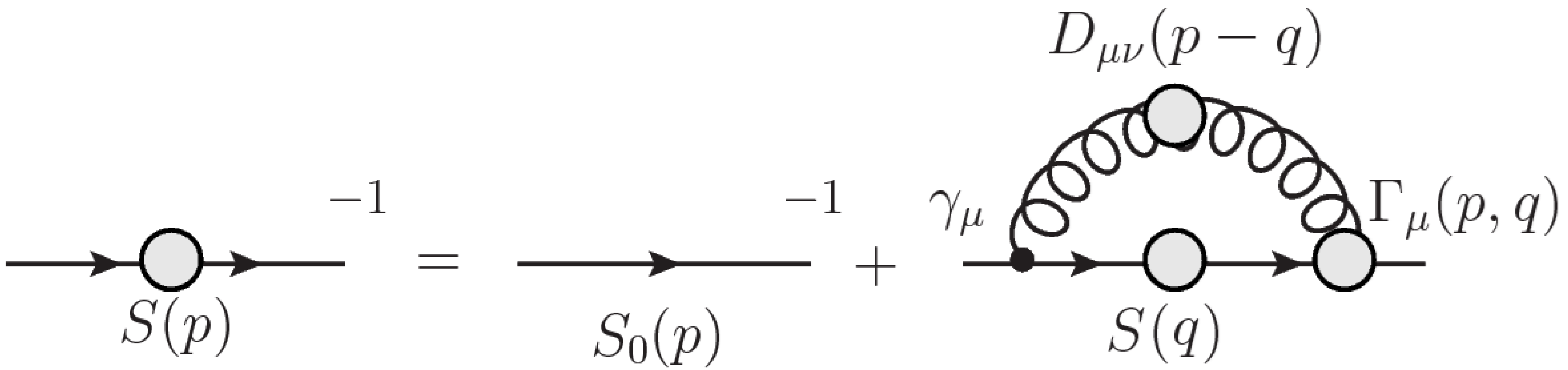
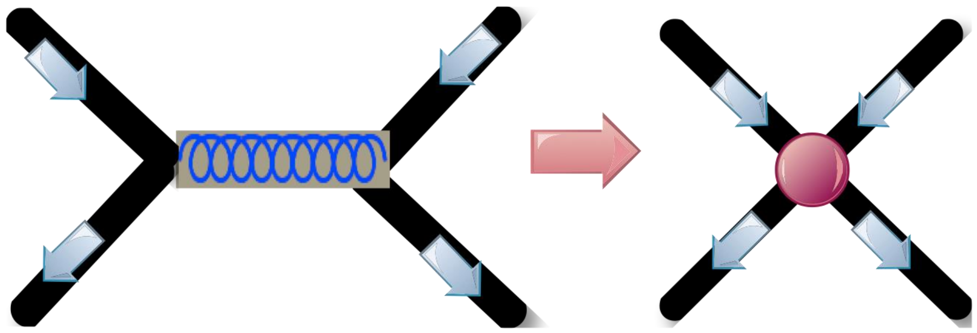
CONTACT INTERACTION POTENTIAL





$$\Delta_{\mu\nu}(k) = \frac{1}{m_G^2}\delta_{\mu\nu}\Theta(\Lambda^2 - k^2)\Theta(k^2 - \lambda^2)$$

CONTACT INTERACTION POTENTIAL

- Static potential

$$V_0(\mathbf{r}) = \int \frac{d^3k}{(2\pi)^3}\, e^{ir\cdot k}\Delta_{00}(\mathbf{k})$$

$$V_0(r) = \frac{1}{2\pi^2 m_G^2 r^3}(\lambda r\cos(\lambda r) - \Lambda r\cos(\Lambda r) + \sin(\Lambda r) - \sin(\lambda r))$$

$$\boxed{\begin{array}{cccc} M_u = 0.40 & M_s = 0.53 & M_c = 1.48 & M_b = 4.73 \\ \Lambda = 0.905 & m_g = 0.5 & \lambda = 0.24 & \alpha = 0.93\pi \end{array}}\,.$$

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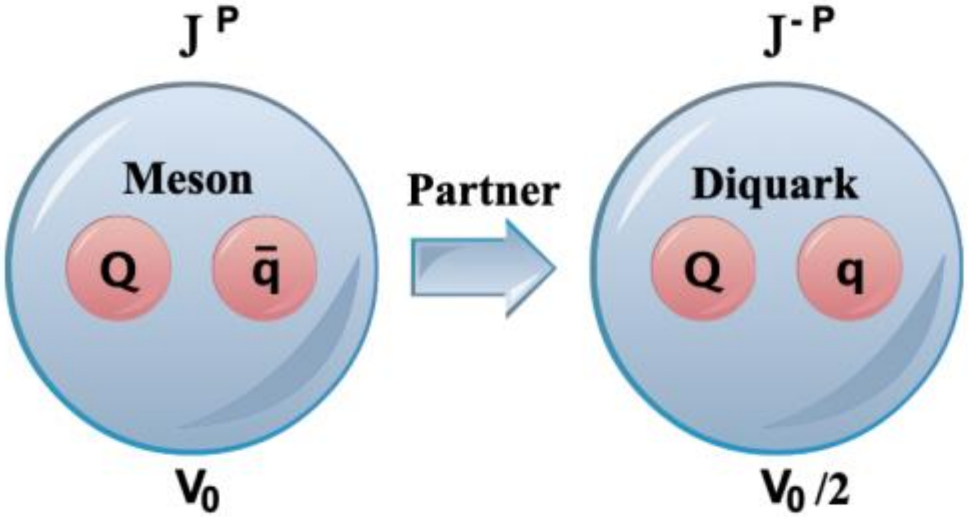
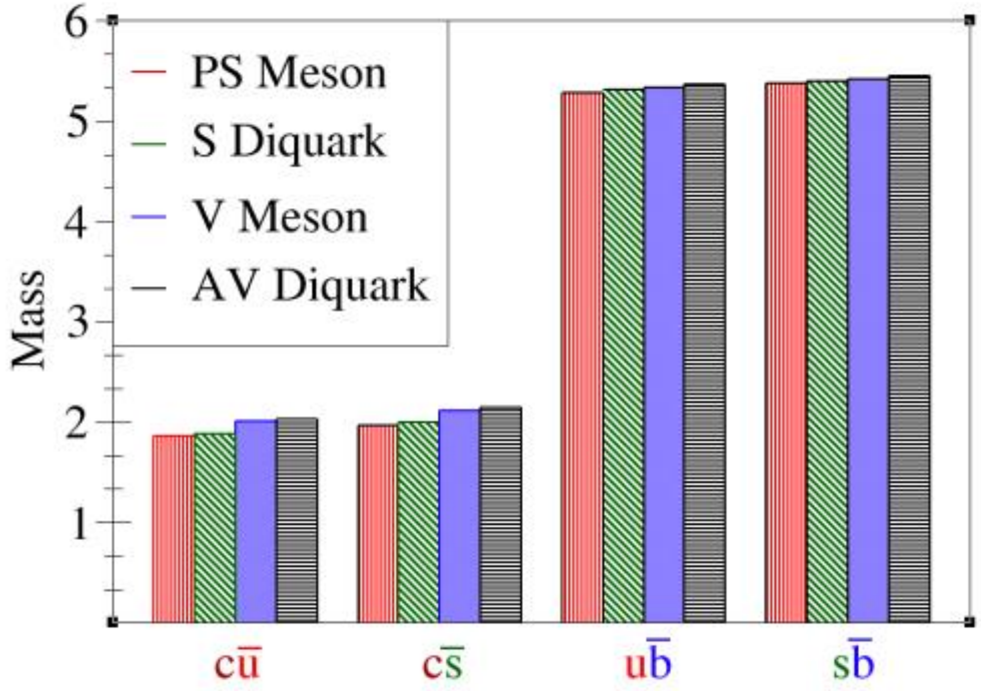
- Mesons and diquarks





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CONTACT INTERACTION MODEL POTENTIAL

Pseudoscalar				Scalar					
Mesons	Exp.	V_0	V_{0S}	Diquark	V_0	V_{0S}	V_F	$\langle H_{SS}\rangle$	$\langle H_{SS0}\rangle$
$D^0(c\bar{u})$	1.86	2.06	1.98	$[cu]$	2.07	2.00	1.88	-0.080	-0.19
$D_s^+(c\bar{s})$	1.97	2.14	2.08	$[cs]$	2.17	2.11	2.0	-0.056	-0.17
$B^+(u\bar{b})$	5.28	5.27	5.25	$[ub]$	5.30	5.27	5.31	-0.025	0.01
$B_s^0(s\bar{b})$	5.37	5.35	5.33	$[sb]$	5.38	5.37	5.40	-0.017	0.02
Vector				Vector–axial					
Mesons	Exp.	V_C	V_{0S}	Diquark		V_{0S}	V_F	$\langle H_{SS}\rangle$	$\langle H_{SS0}\rangle$
$D^{*0}(c\bar{u})$	2.01	2.02	2.08	$\{cu\}$		2.10	2.03	0.026	-0.045
$D_s^*(cs)$	2.11	2.57	2.16	$\{cs\}$		2.18	2.14	0.018	-0.025
$B^{+*}(u\bar{b})$	5.33	5.66	5.28	$\{ub\}$		5.30	5.36	0.008	0.06
$B_s^{0*}(s\bar{b})$	5.42	5.38	5.36	$\{sb\}$		5.39	5.45	0.005	0.07

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Table 2. Diquark differences with other models.

	$[cu]$	$[cs]$	$[ub]$	$[sb]$	$\{cu\}$	$\{cs\}$	$\{ub\}$	$\{sb\}$
Our	1.88	2.0	5.31	5.40	2.03	2.14	5.36	5.45
Ref. 44	2.01	2.13	5.23	5.34	2.09	2.19	5.26	5.36
Diff.	6.46%	6.10%	1.53%	1.12%	2.87%	2.28%	1.90%	1.67%
Ref. 41	2.15	2.26	5.51	5.60	2.24	2.34	5.53	5.62
Diff.	12.55%	11.50%	3.63%	3.57%	9.37%	8.55%	3.07%	3.02%

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Table 5. Heavy mesons masses (in GeV) for potential V_0 compared to the experimental masses from PDG. By taking $N = 100$ $r_{\rm max} = 10$ fm.

State	Exp. (Ref. 62)	V_0	Diquark	State	Exp. (Ref. 62)	V_F	Diquark
$1\,^1S_0(c\bar{c})$	2.98	2.98	3.14	$1\,^3S_1(c\bar{c})$	3.10	3.10	3.15
$2\,^1S_0(c\bar{c})$	3.64	3.45	3.65	$2\,^3S_1(c\bar{c})$	3.69	3.57	3.51
$3\,^1S_0(c\bar{c})$	—	3.98	4.20	$3\,^3S_1(c\bar{c})$	4.04	4.10	3.92
$4\,^1S_0(c\bar{c})$	—	4.55	4.89	$4\,^3S_1(c\bar{c})$	4.42	4.67	4.39
$1\,^1S_0(b\bar{b})$	9.40	9.38	9.45	$1\,^3S_1(b\bar{b})$	9.46	9.46	9.53
$2\,^1S_0(b\bar{b})$	9.99	9.60	9.62	$2\,^3S_1(b\bar{b})$	10.02	9.68	9.70
$3\,^1S_0(b\bar{b})$	—	9.85	9.81	$3\,^3S_1(b\bar{b})$	10.36	9.93	9.89
$4\,^1S_0(b\bar{b})$	—	10.13	10.03	$4\,^3S_1(b\bar{b})$	10.58	10.21	10.11
State	Ref. 9	V_F	Diquark	State	Ref. 9	V_F	Diquark
$1\,^1S_0(c\bar{b})$	6.27	6.27	6.33	$2\,^3S_1(c\bar{b})$	6.33	6.33	6.38
$2\,^1S_0(c\bar{b})$	6.87	6.62	6.60	$2\,^3S_1(c\bar{b})$	6.89	6.68	6.66
$3\,^1S_0(c\bar{b})$	7.24	7.03	6.92	$3\,^3S_1(c\bar{b})$	7.25	7.09	6.97
$4\,^1S_0(c\bar{b})$	7.54	7.48	7.25	$4\,^3S_1(c\bar{b})$	7.55	7.54	7.30

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CONTACT INTERACTION POTENTIAL

Table 6. Results for diquark masses (in GeV). Second and fifth rows contain the difference between the masses obtained herein and those obtained in Refs. 41 and 44.

	$[cc]$	$\{cc\}$	$[bb]$	$\{bb\}$	$[cb]$	$\{cb\}$
Herein	3.14	3.15	9.45	9.53	6.33	6.38
Ref. 44	3.11	3.12	9.53	9.53	6.31	6.31
Diff.	0.96%	0.96%	0.83%	0%	0.31%	1.10%
Ref. 41	—	3.30	—	9.68	6.48	6.50
Diff.	—	4.54%	—	1.54%	2.31%	1.84%

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SUMMARY

- Potential models allow simple calculations with high degree of accuracy in the predictions of mass of heavy quarks (~10%)
- Other physicsl observables can be obtained directly
- Variational techniques render the calculations even simpler without compromising accuracy

SUMMARY

- Cornell potential, even though the most popular choice, demands tuning of parameters for different heavy meson families
- Song-Lin potential requires a single set of parameters to describe the spectra of charmonium, bottomonium and Bc mesons
- Additional phenemenology can be straightforwardly obtained
- Contact interaction potential allows direct comparison against field theoretical predictions

UNDER CONSIDERATION

- Radii, masses and decay widths from variational considerations
- Tetraquarks and other exotic states
- Baryons
- Full comparison agains SDE-BSE

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