Hadron Physics and Continuum Strong QCD

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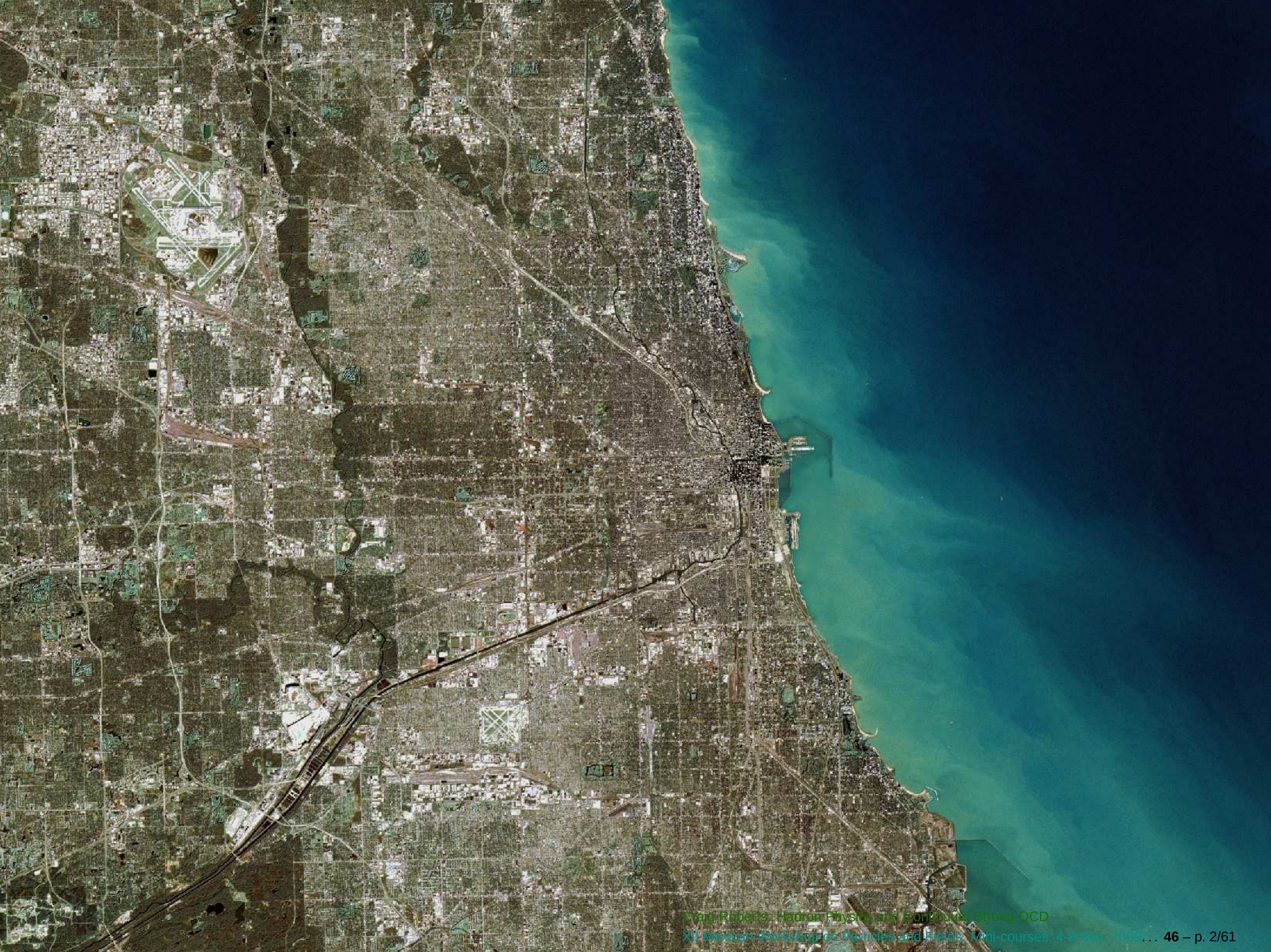
http://www.phy.anl.gov/theory/staff/cdr.html



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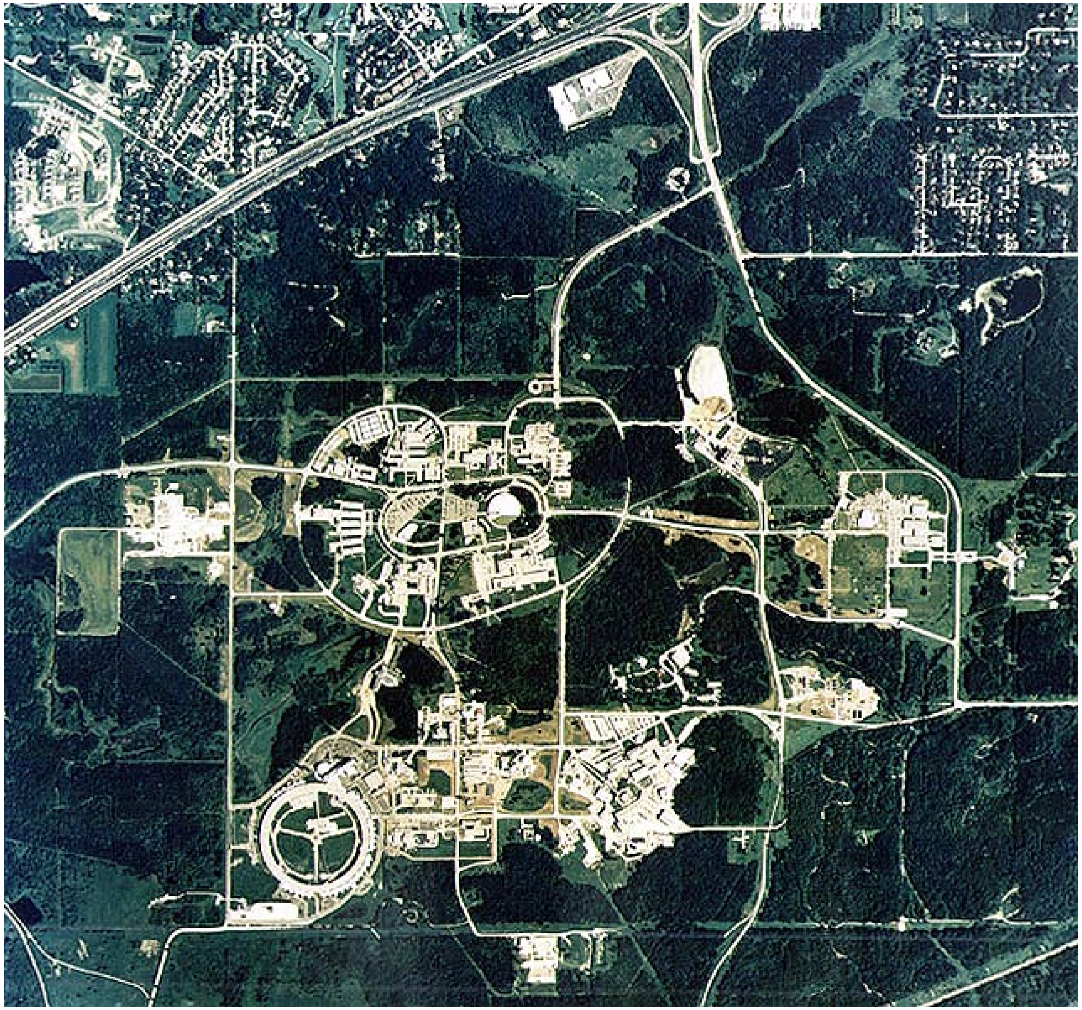

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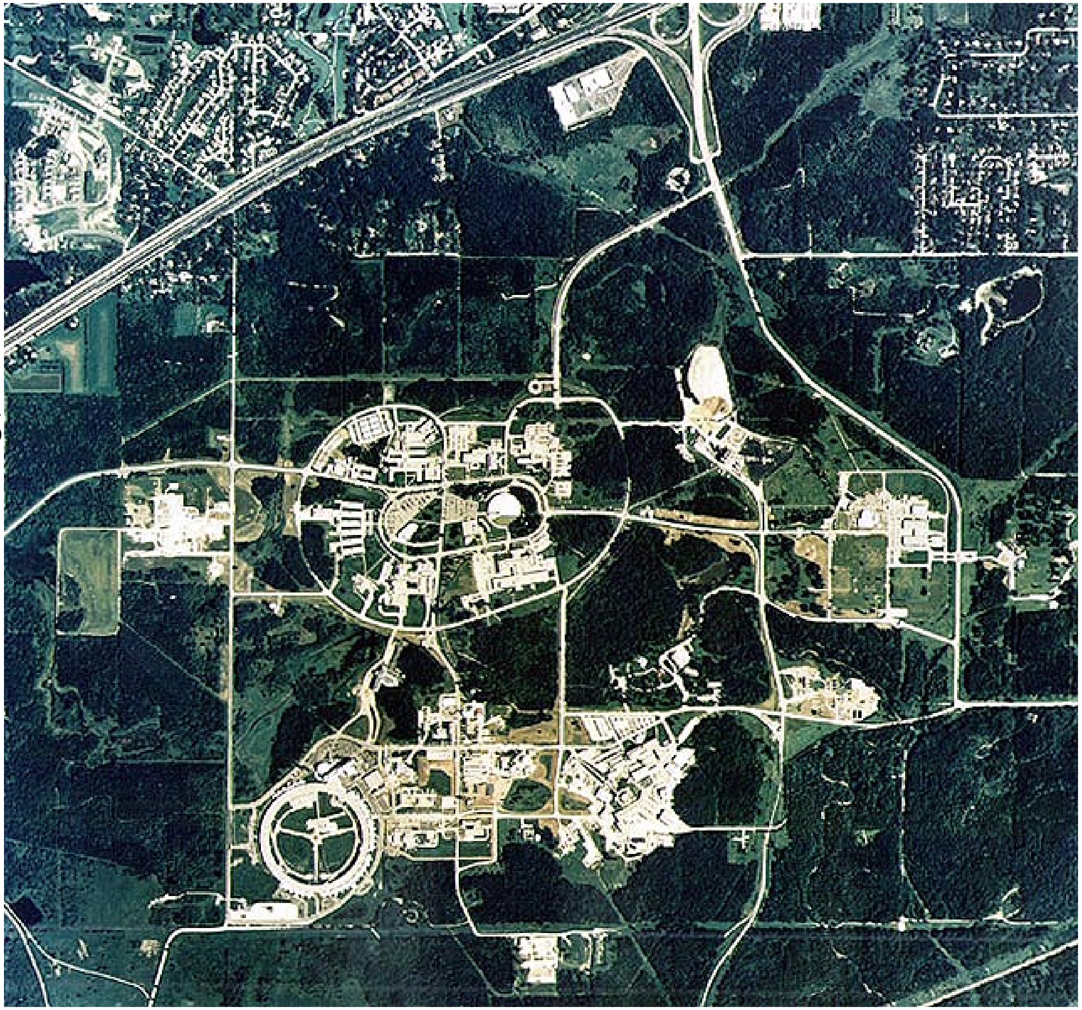

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Our research addresses the five key questions that comprise the USA's nuclear physics agenda. We place heavy emphasis on the prediction of phenomena accessible at Argonne's ATLAS facility, at JLab, and at other laboratories around the world; and on anticipating and planning for FRIB.

Our research explores problems in: theoretical and computational nuclear astrophysics; quantum chromodynamics and hadron physics; light-hadron reaction theory; ab-initio many-body calculations based on realistic two- and three-nucleon potentials; and coupled-channels calculations of heavy-ion reactions. Our programs provide much of the scientific basis for the drive to physics with rare isotopes. Additional research in the Group focuses on: atomic and neutron physics; fundamental quantum mechanics; quantum computing; and tests of fundamental symmetries and theories unifying all the forces of nature, and the search for a spatial or temporal variation in Nature's basic parameters. The pioneering development and use of massively parallel numerical simulations using hardware at Argonne and elsewhere is a major component of the Group's research.

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Recommended Reading and Reviews

- C.D. Roberts, M.S. Bhagwat, A. Höll and S.V. Wright, "Aspects of Hadron Physics," Eur. Phys. J. Special Topics 140 (2007) pp. 53-116
- A. Höll, C.D. Roberts and S.V. Wright, nucl-th/0601071, "Hadron Physics and Dyson-Schwinger Equations" (103 pages; http://www.phy.anl.gov/theory/z05HUGS/CDRoberts.pdf)
- "Primer for Quantum Field Theory in Hadron Physics" http://www.phy.anl.gov/ztfr/LecNotes.pdf
- P. Maris and C. D. Roberts, nucl-th/0301049, "Dyson-Schwinger equations: A tool for hadron physics," Int. J. Mod. Phys. E **12**, 297 (2003)
- R. Alkofer and L. von Smekal, hep-ph/0007355, "The infrared behavior of QCD Green's functions: Confinement, dynamical symmetry breaking, and hadrons as relativistic bound states," Phys. Rept. **353**, 281 (2001)
- C. D. Roberts and S. M. Schmidt, nucl-th/0005064, "Dyson-Schwinger equations: Density, temperature and continuum strong QCD," Prog. Part. Nucl. Phys. **45** (2000) S1
- C. D. Roberts and A. G. Williams, hep-ph/9403224,"Dyson-Schwinger equations and their application to hadronic physics," Prog. Part. Nucl. Phys. **33** (1994) 477

Relativistic Quantum Field Theory

- A theoretical understanding of the phenomena of Hadron Physics requires the use of the full machinery of relativistic quantum field theory. Relativistic quantum field theory is the ONLY way to reconcile quantum mechanics with special relativity.
- Relativistic quantum field theory is based on the relativistic quantum mechanics of Dirac.
- Relativistic quantum mechanics predicts the existence of antiparticles; i.e., the equations of relativistic quantum mechanics admit *negative energy* solutions. However, once one allows for particles with negative energy, then particle number conservation is lost:
  $E_{\rm system} = E_{\rm system} + (E_{p_1} + E_{\bar{p}_1}) + \ldots$ *ad infinitum*
- However, this is a fundamental problem for relativistic quantum mechanics – Few particle systems can be studied in relativistic quantum mechanics but the study of (infinitely) many bodies is difficult. No general theory currently exists.
- Not all Poincaré transformations commute with the Hamiltonian. Hence, a Poincaré transformation from one frame to another can change the number of particles. Therefore a solution of the N body problem in one frame is generally insufficient to study the typical scattering process encountered at modern facilities.

Relativistic Quantum Field Theory

- Relativistic quantum field theory is an answer. The fundamental entities are fields, which can simultaneously represent an uncountable infinity of particles.

e.g., neutral scalar: $$\phi(x) = \int \frac{d^3k}{(2\pi)^3 2\omega_k} \left[a(k)\mathrm{e}^{-ik\cdot x} + a^\dagger(k)\mathrm{e}^{ik\cdot x}\right] \quad (1)$$

Hence, the nonconservation of particle number is not a problem. This is crucial because key observable phenomena in hadron physics are essentially connected with the existence of *virtual* particles.

- Relativistic quantum field theory has its own problems, however. For the mathematician, the question of whether a given relativistic quantum field theory is rigorously well defined is *unsolved*.
- All relativistic quantum field theories admit analysis in perturbation theory. Perturbative renormalisation is a well-defined procedure and has long been used in Quantum Electrodynamics (QED) and Quantum Chromodynamics (QCD).
- A rigorous definition of a theory, however, means proving that the theory makes sense *nonperturbatively*. This is equivalent to proving that all the theory's renormalisation constants are nonperturbatively well-behaved.

Relativistic Quantum Field Theory

- Hadron Physics involves QCD. While it makes excellent sense perturbatively: Gross, Politzer, Wilczek – Nobel Prize 2004 – *Asymptotic Freedom*. It is not known to be a rigorously well-defined theory. Hence it cannot truly be said to be THE theory of the strong interaction (hadron physics).
- Nevertheless, physics does not wait on mathematics. Physicists make assumptions and explore their consequences. Practitioners assume that QCD is (somehow) well-defined and follow where that leads us.
- Experiment: explore and map the hadron physics landscape with well-understood probes, such as the electron at JLab and Mainz.
- Theory: employ established mathematical tools, and refine and invent others in order to use the Lagangian of QCD to predict what should be observable real-world phenomena.
- A key aim of worlds' hadron physics programmes in experiment & theory: determine whether there are any contradictions with what we can truly *prove* in QCD.

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- Interplay between Experiment and Theory is the engine of discovery and progress. The *Discovery Potential* of both is high. Much learnt in the last five years and I expect that many surprises remain in Hadron Physics.

Relativistic Quantum Field Theory

- In modern approaches to relativistic quantum field theory the primary elements are Green Functions or Schwinger Functions. These are the objects that naturally arise in the Functional Integral formulation of a theory.
- I will introduce this concept via the Green function of the Dirac operator from relativistic quantum mechanics. To understand this we're going to need some background on notation and conventions in relativistic quantum mechanics.
  - NB. Spin-$\frac{1}{2}$ particles do *not* have a classical analogue.
  - Pauli, in his paper on the exclusion principle:
    Quantum Spin is a "classically nondescribable two-valuedness."
    ... Problem: Configuration space of quantum spin is quite different to that accessible with a classical spinning top.

Notation

- *contravariant* four-vector:

$$x^\mu := (x^0, x^1, x^2, x^3) \equiv (t, x, y, z) \,. \tag{2}$$

$c = 1 = \hbar$, and the conversion between length and energy is just:

$$1\,\mathrm{fm} = 1/(0.197327\,\mathrm{GeV}) = 5.06773\,\mathrm{GeV}^{-1} \,. \tag{3}$$

- *covariant* four-vector is obtained by changing the sign of the spatial components of the contravariant vector:

$$x_\mu := (x_0, x_1, x_2, x_3) \equiv (t, -x, -y, -z) = g_{\mu\nu} x^\nu \,, \tag{4}$$

where the metric tensor is

$$g_{\mu\nu} = \left[\begin{array}{cccc} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{array}\right] \,. \tag{5}$$

Notation

- Contracted product of two four-vectors is

$$(a,b) := g_{\mu\nu} a^{\mu} b^{\nu} = a_{\mu} b^{\mu} \, ; \tag{6}$$

i.e, a contracted product of a covariant and contravariant four-vector.
The Poincaré-invariant length of any vector is $\boxed{x^2 := (x,x) = t^2 - \vec{x}^2}$.

- Momentum vectors are similarly defined:

$$p^{\mu} = (E, p_x, p_y, p_z) = (E, \vec{p}) \tag{7}$$

and

$$(p,k) = p_{\mu} k^{\mu} = E_p E_k - \vec{p} \cdot \vec{k} \, . \tag{8}$$

Likewise, a mixed coordinate-momentum contraction:

$$(x,p) = tE - \vec{x} \cdot \vec{p} \, . \tag{9}$$

Notation

- Momentum *operator*

$$p^\mu := i\frac{\partial}{\partial x_\mu} = (i\frac{\partial}{\partial t}, \frac{1}{i}\vec{\nabla}) =: i\nabla^\mu \tag{10}$$

transforms as a contravariant four-vector.
Poincaré invariant analogue of the Laplacian is

$$\partial^2 := -p^\mu p_\mu = \frac{\partial}{\partial x_\mu}\frac{\partial}{\partial x^\mu}\,. \tag{11}$$

- Contravariant four-vector associated with the electromagnetic field:

$$A^\mu(x) = (\Phi(x), \vec{A}(x)). \tag{12}$$

Electric and magnetic field strengths obtained from $\boxed{F^{\mu\nu} = \partial^\nu A^\mu - \partial^\mu A^\mu;}$ for example,

$$\vec{E}_i = F^{0i}\,;\ \text{i.e.,}\ \vec{E} = -\vec{\nabla}\Phi - \frac{\partial}{\partial t}\vec{A}\,. \tag{13}$$

Similarly, $B^i = \epsilon^{ijk}F^{jk}$, $j,k = 1,2,3$.
Analogous definitions hold in QCD for the chromomagnetic field strengths.

Dirac Matrices

- The Dirac matrices are *indispensable* in a manifestly Poincaré covariant description of particles with spin (intrinsic angular momentum).
- The Dirac matrices are *defined* by the Clifford Algebra (an identity matrix is implicit on the r.h.s.)
$$\{\gamma^\mu, \gamma^\nu\} = \gamma^\mu\gamma^\nu + \gamma^\nu\gamma^\mu = 2\, g^{\mu\nu}\,, \tag{14}$$
- One common 4×4 representation is [each entry represents a 2×2 matrix]

$$\gamma^0 = \begin{pmatrix} \mathbf{1} & \mathbf{0} \\ \mathbf{0} & -\mathbf{1} \end{pmatrix},\ \vec{\gamma} = \begin{bmatrix} \mathbf{0} & \vec{\boldsymbol{\sigma}} \\ -\vec{\boldsymbol{\sigma}} & \mathbf{0} \end{bmatrix}, \tag{15}$$

where $\vec{\sigma}$ are the usual Pauli matrices:

$$\sigma^1 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix},\ \sigma^2 = \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix},\ \sigma^3 = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, \tag{16}$$

and **1** $:= \mathrm{diag}[1,1]$. Clearly: $\gamma_0^\dagger = \gamma_0$; and $\vec{\gamma}^\dagger = -\vec{\gamma}$.
NB. These properties are *not* specific to this representation;
e.g., $\gamma^1\gamma^1 = -\mathbf{1}_{4\times 4}$, *for any* representation of the Clifford algebra.

Dirac Matrices

- In discussing spin, two combinations of Dirac matrices appear frequently:

$$\sigma^{\mu\nu} = \frac{i}{2}[\gamma^\mu, \gamma^\nu] = \frac{i}{2}\left(\gamma^\mu\gamma^\nu - \gamma^\nu\gamma^\mu\right)\,,\; \gamma^5 = i\gamma^0\gamma^1\gamma^2\gamma^3 = \gamma_5\,. \tag{17}$$

NB. $\boxed{\gamma^5\,\sigma^{\mu\nu} = \frac{i}{2}\,\epsilon^{\mu\nu\rho\sigma}\,\sigma_{\rho\sigma},}$ with $\epsilon^{\mu\nu\rho\sigma}$ the completely antisymmetric Lévi-Cività tensor: $\epsilon^{0123} = +1$, $\epsilon_{\mu\nu\rho\sigma} = -\epsilon^{\mu\nu\rho\sigma}$.

- The Dirac matrix γ_5 plays a special role in the discussion of parity and chiral symmetry, two key aspects of the Standard Model. In the representation we've already seen

$$\gamma^5 = \begin{bmatrix} \mathbf{0} & \mathbf{1} \\ \mathbf{1} & \mathbf{0} \end{bmatrix}. \tag{18}$$

Furthermore

$$\{\gamma_5, \gamma^\mu\} = 0 \Rightarrow \gamma_5\gamma^\mu = -\gamma^\mu\gamma_5 \quad \& \quad \gamma_5^\dagger = \gamma_5\,. \tag{19}$$

- Parity, Chiral Symmetry, and the relation between them, each play a special role in Hadron Physics.

Dirac Matrices

- The "slash" notation is a frequently used shorthand:

$$\gamma^\mu A_\mu \quad =: \quad \not{A} = \gamma^0 A^0 - \vec{\gamma}\cdot\vec{A}\,, \tag{20}$$

$$\gamma^\mu p_\mu \quad =: \quad \not{p} = \gamma^0 E - \vec{\gamma}\cdot\vec{p}\,, \tag{21}$$

$$\gamma^\mu \boldsymbol{p}_\mu \quad =: \quad i\not{\nabla} \equiv i\not{\partial} = i\gamma^0 \frac{\partial}{\partial t} + i\vec{\gamma}\cdot\vec{\nabla} = i\gamma^\mu \frac{\partial}{\partial x^\mu}\,. \tag{22}$$

- The following identities are important in evaluating the cross-sections.

$$\mathrm{tr}\,\gamma_5 = 0\,, \qquad \mathrm{tr}\,\mathbf{1} = 4\,, \tag{23}$$

$$\mathrm{tr}\,\not{a}\not{b} = 4(a,b)\,, \qquad \mathrm{tr}\,\not{a}_1\not{a}_2\not{a}_3\not{a}_4 = 4\left[(a_1,a_2)(a_3,a_4) - (a_1,a_3)(a_2,a_4) + (a_1,a_4)(a_2,a_3)\right], \tag{24}$$

$$\mathrm{tr}\,\not{a}_1 \ldots \not{a}_n = 0\,,\ \text{for n odd}\,, \qquad \mathrm{tr}\,\gamma_5 \not{a}\,\not{b} = 0\,, \tag{25}$$

$$\mathrm{tr}\,\gamma_5 \not{a}_1\not{a}_2\not{a}_3\not{a}_4 = 4i\epsilon_{\alpha\beta\gamma\delta}\, a^\alpha b^\beta c^\gamma d^\delta\,, \qquad \gamma_\mu \not{a} \gamma^\mu = -2\not{a}\,, \tag{26}$$

$$\gamma_\mu \not{a}\not{b}\gamma^\mu = 4(a,b)\,, \qquad \gamma_\mu \not{a}\not{b}\not{c}\gamma^\mu = -2\not{c}\not{b}\not{a}\,, \tag{27}$$

All follow from the fact that the Dirac matrices satisfy the Clifford algebra.

Exercises: Prove these relations using Eq. (14) and exploiting $\mathrm{tr}\,AB = \mathrm{tr}\,BA$.

Relativistic Quantum Mechanics

- Unification of special relativity (Poincaré covariance) and quantum mechanics took some time. Questions still remain as to a practical implementation of an Hamiltonian formulation of the relativistic quantum mechanics of interacting systems.
- Poincaré group has ten generators: the six associated with the Lorentz transformations (rotations and boosts) and the four associated with translations.
- Quantum mechanics describes the time evolution of a system with interactions. That evolution is generated by the Hamiltonian.
- However, if the theory is formulated with an interacting Hamiltonian then boosts will fail to commute with the Hamiltonian.
  Hence, the state vector calculated in one momentum frame will not be kinematically related to the state in another frame.
  That makes a new calculation necessary in every frame.
- Hence the discussion of scattering, which takes a state of momentum p to another state with momentum p', is problematic.
  (See, e.g., B.D. Keister and W.N. Polyzou (1991), "Relativistic Hamiltonian dynamics in nuclear and particle physics," *Adv. Nucl. Phys*. **20**, 225.)

Dirac Equation

- Dirac equation is starting point for Lagrangian formulation of quantum field theory for fermions. For a noninteracting fermion
$$[i\partial\!\!\!/ - m]\,\psi = 0\,, \tag{28}$$
where $\psi(x) = \begin{pmatrix} u_1(x) \\ u_2(x) \\ u_3(x) \\ u_4(x) \end{pmatrix}$ is the fermion's "spinor"
– a four component column vector, with each component spacetime dependent.

- In an external electromagnetic field the fermion's wave function obeys
$$[i\partial\!\!\!/ - eA\!\!\!/ - m]\,\psi = 0\,, \tag{29}$$
obtained, as usual, via "minimal substitution:" $\boldsymbol{p}^\mu \to \boldsymbol{p}^\mu - eA^\mu$ in Eq. (28).
The Dirac operator is a matrix-valued differential operator.

- These equations have a manifestly Poincaré covariant appearance. A proof of covariance is given in the early chapters of: Bjorken, J.D. and Drell, S.D. (1964), *Relativistic Quantum Mechanics* (McGraw-Hill, New York).

Free Particle Solutions

- Insert plane waves in free particle Dirac equation:
  $$\psi^{(+)}(x) = \mathrm{e}^{-i(k,x)}\, u(k)\,,\;\; \psi^{(-)}(x) = \mathrm{e}^{+i(k,x)}\, v(k)\,,$$
  and thereby obtain ...
  $$(\not k - m)\, u(k) = 0\,,\;\; (\not k + m)\, v(k) = 0\,. \tag{30}$$
  Here there are two qualitatively different types of solution, corresponding to positive and negative energy: k & $-k$.
  (Appreciation of physical reality of negative energy solutions led to prediction of antiparticles.)
- Assume particle's mass is nonzero; work in rest frame:
  $$(\gamma^0 - \mathbf{1})\, u(m,\vec{0}) = 0\,,\;\; (\gamma^0 + \mathbf{1})\, v(m,\vec{0}) = 0\,. \tag{31}$$
  There are clearly (remember the form of γ^0) two linearly-independent solutions of each equation:
  $$u^{(1),(2)}(m,\vec{0}) = \begin{pmatrix}1\\0\\0\\0\end{pmatrix}, \begin{pmatrix}0\\1\\0\\0\end{pmatrix},\;\; v^{(1),(2)}(m,\vec{0}) = \begin{pmatrix}0\\0\\1\\0\end{pmatrix}, \begin{pmatrix}0\\0\\0\\1\end{pmatrix}.$$

Positive Energy Free Particle

- Solution in arbitrary frame can be obtained via a Lorentz boost. However, simpler to observe that
$$(\not k - m)\,(\not k + m) = k^2 - m^2 = 0\,, \tag{33}$$
(The last equality is valid for real, on-shell particles.)
It follows that for arbitrary k^μ and positive energy ($E > 0$), the canonically normalised spinor is
$$u^{(\alpha)}(k) \;=\; \frac{\not k + m}{\sqrt{2m(m+E)}}\,u^{(\alpha)}(m,\vec{0}) = \left[\begin{array}{cc} \left(\dfrac{E+m}{2m}\right)^{1/2} & \phi^\alpha(m,\vec{0}) \\ \dfrac{\sigma\cdot k}{\sqrt{2m(m+E)}} & \phi^\alpha(m,\vec{0}) \end{array}\right], \tag{34}$$
with the two-component spinors, obviously to be identified with the fermion's spin in the rest frame (the only frame in which spin has its naive meaning)
$$\phi^{(1)} = \left(\begin{array}{c} 1 \\ 0 \end{array}\right),\; \phi^{(2)} = \left(\begin{array}{c} 0 \\ 1 \end{array}\right). \tag{35}$$

Negative Energy Free Particle

- For negative energy: $\hat{E} = -E > 0$,

$$v^{(\alpha)}(k) \;=\; \frac{-\not{k} + m}{\sqrt{2m(m+\hat{E})}}\, v^{(\alpha)}(m,\vec{0}) = \begin{pmatrix} \dfrac{\sigma \cdot k}{\sqrt{2m(m+\hat{E})}} & \chi^{\alpha}(m,\vec{0}) \\ \left(\dfrac{\hat{E}+m}{2m}\right)^{1/2} & \chi^{\alpha}(m,\vec{0}) \end{pmatrix}, \tag{36}$$

with $\chi^{(\alpha)}$ obvious analogues of $\phi^{(\alpha)}$ in Eq. (35).

- NB. For $\vec{k} \sim 0$ (rest frame) the lower component of the positive energy spinor is small, as is the upper component of the negative energy spinor $\Rightarrow$ Poincaré covariance, which requires the four component form, becomes important with increasing $|\vec{k}|$; indispensable for $|\vec{k}| \gtrsim m$.

NB. Solving $\vec{k} \neq 0$ equations this way works because it is clear that there are two, and only two, linearly-independent solutions of the momentum space free-fermion Dirac equations, Eqs. (30), and, for the homogeneous equations, any two covariant solutions with the correct limit in the rest-frame must give the correct boosted form.

Conjugate Spinor

- In quantum field theory, as in quantum mechanics, one needs a conjugate state to define an inner product.

  For fermions in Minkowski space that conjugate is $\boxed{\bar\psi(x) := \psi^\dagger(x)\gamma^0}$, and

$$\bar\psi(i \overleftarrow{\not\partial} + m) = 0\,. \tag{37}$$

- This yields the following free particle spinors in momentum space (using $\gamma^0(\gamma^\mu)^\dagger\gamma^0 = \gamma^\mu$, a relation that is particularly important in the discussion of intrinsic parity)

$$\bar u^{(\alpha)}(k) = \bar u^{(\alpha)}(m,\vec 0)\,\frac{\not k + m}{\sqrt{2m(m+E)}} \tag{38}$$

$$\bar v^{(\alpha)}(k) = \bar v^{(\alpha)}(m,\vec 0)\,\frac{-\not k + m}{\sqrt{2m(m+E)}}\,, \tag{39}$$

- Orthonormalisation

$$\begin{array}{ll} \bar u^{(\alpha)}(k)\, u^{(\beta)}(k) = \delta_{\alpha\beta} & \bar u^{(\alpha)}(k)\, v^{(\beta)}(k) = 0 \\ \bar v^{(\alpha)}(k)\, v^{(\beta)}(k) = -\delta_{\alpha\beta} & \bar v^{(\alpha)}(k)\, u^{(\beta)}(k) = 0 \end{array}\,. \tag{40}$$

Positive Energy Projection Operator

- Can now construct positive energy projection operators. Consider

$$\Lambda_+(k) \quad := \quad \sum_{\alpha=1,2} u^{(\alpha)}(k) \otimes \bar{u}^{(\alpha)}(k)\,. \tag{41}$$

  Plain from the orthonormality relations, Eqs. (40), that

$$\Lambda_+(k)\, u^{(\alpha)}(k) = u^{(\alpha)}(k)\,,\ \Lambda_+(k)\, v^{(\alpha)}(k) = 0\,. \tag{42}$$

- Now, since $\sum_{\alpha=1,2} u^{(\alpha)}(m,\vec{0}) \otimes \bar{u}^{(\alpha)}(m,\vec{0}) = \left(\begin{array}{cc} \mathbf{1} & 0 \\ 0 & 0 \end{array}\right) = \dfrac{\mathbf{1}+\gamma^0}{2}$, then

$$\Lambda_+(k) \quad = \quad \frac{1}{2m(m+E)}\,(\not k + m)\,\frac{\mathbf{1}+\gamma^0}{2}\,(\not k + m)\,. \tag{43}$$

- Noting that for $k^2 = m^2$; i.e., on shell,
  $(\not k + m)\,\gamma^0\,(\not k + m) = 2E\,(\not k + m)$, $(\not k + m)\,(\not k + m) = 2m\,(\not k + m)$,
  one finally arrives at the simple closed form:

$$\Lambda_+(k) = \frac{\not k + m}{2\,m}\,. \tag{44}$$

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Negative Energy Projection Operator

- The negative energy projection operator is

$$\Lambda_-(k) \quad := \quad -\sum_{\alpha=1,2} v^{(\alpha)}(k) \otimes \bar{v}^{(\alpha)}(k) = \frac{-\not k + m}{2\,m}\,. \tag{45}$$

- The projection operators have the following characteristic and important properties:

$$\Lambda_\pm^2(k) \quad = \quad \Lambda_\pm(k)\,, \tag{46}$$

$$\mathrm{tr}\,\Lambda_\pm(k) \quad = \quad 2\,, \tag{47}$$

$$\Lambda_+(k) + \Lambda_-(k) \quad = \quad \mathbf{1}\,. \tag{48}$$

Green-Functions/Propagators

- The Dirac equation is a partial differential equation.
  A general method for solving such equations is to use a Green function, which is the inverse of the differential operator that appears in the equation.
  The analogy with matrix equations is obvious and can be exploited heuristically.
- Dirac equation, Eq. (29): $\boxed{[i\not\partial_x - e\not A(x) - m]\,\psi(x) = 0\,,}$ yields the wave function for a fermion in an external electromagnetic field.
- Consider the operator obtained as a solution of the following equation

$$[i\not\partial_{x'} - e\not A(x') - m]\,S(x',x) = \mathbf{1}\,\delta^4(x'-x)\,. \tag{49}$$

- Obviously if, at a given spacetime point x, $\psi(x)$ is a solution of Eq. (29), then

$$\psi(x') := \int d^4x\,S(x',x)\,\psi(x) \tag{50}$$

is a solution of . . . $[i\not\partial_{x'} - e\not A(x') - m]\,\psi(x') = 0\,;$ (51)

i.e., $S(x',x)$ has *propagated* the solution at x to the point x'.

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i.e., $S(x',x)$ has *propagated* the solution at x to the point x'.

Analogue of Huygens Principle in Wave Mechanics

Green-Functions/Propagators

- This approach is practical because all physically reasonable external fields can only be nonzero on a compact subdomain of spacetime.
- Therefore the solution of the complete equation is transformed into solving for the Green function, which can then be used to propagate the free-particle solution, already found, to arbitrary spacetime points.
- However, obtaining the *exact* form of $S(x', x)$ is *impossible* for all but the simplest cases (see, e.g.,
  - Dittrich, W. and Reuter, M. (1985), *Effective Lagrangians in Quantum Electrodynamics* (Springer Verlag, Berlin);
  - Dittrich, W. and Reuter, M. (1985), *Selected Topics in Gauge Theories* (Springer Verlag, Berlin).)
- This is where and why perturbation theory so often rears its not altogether handsome head.

Free Fermion Propagator

- In the absence of an external field the Green Function equation, Eq. (49), becomes

$$[i\not\partial_{x'} - m]\, S(x',x) = \mathbf{1}\, \delta^4(x'-x)\,. \tag{55}$$

- Assume a solution of the form:

$$S_0(x',x) = S_0(x'-x) = \int \frac{d^4p}{(2\pi)^4}\, \mathrm{e}^{-i(p,x'-x)}\, S_0(p)\,, \tag{56}$$

so that substituting yields

$$(\not p - m)\, S_0(p) = \mathbf{1}\,;\ \text{i.e., } S_0(p) = \frac{\not p + m}{p^2 - m^2}\,. \tag{57}$$

- To obtain the result in configuration space one must adopt a prescription for handling the on-shell singularities in $S(p)$ at $p^2 = m^2$.
  - That convention is tied to the boundary conditions applied to Eq. (55).
  - An obvious and physically sensible definition of the Green function is that it should propagate
    - positive-energy-fermions and -antifermions forward in time but not backward in time,
    - and vice versa for negative energy states.

Feynman's Fermion Propagator

- The wave function for a positive energy free-fermion is

$$\psi^{(+)}(x) = u(p)\,\mathrm{e}^{-i(p,x)}\,. \tag{58}$$

The wave function for a positive-energy antifermion is the charge-conjugate of the negative-energy fermion solution ($C = i\gamma^2\gamma^0$ and $(\cdot)^{\mathrm{T}}$ denotes matrix transpose):

$$\psi_c^{(+)}(x) = C\,\gamma^0\,[v(p)\,\mathrm{e}^{i(p,x)}]^* = C\,\bar{v}(p)^{\mathrm{T}}\,\mathrm{e}^{-i(p,x)}\,, \tag{59}$$

- Follows from properties of spinors and projection operators that our physically sensible $S_0(x'-x)$ must contain only positive-frequency components for $t = x_0' - x_0 > 0$; i.e., in this case it must be proportional to $\Lambda_+(p)$.
  - Exercise: Verify this.
- Can ensure this via a small modification of the denominator of Eq. (57), with $\boldsymbol{\eta} \to 0^+$ at the end of all calculations:

$$S_0(p) = \frac{\not p + m}{p^2 - m^2} \to \frac{\not p + m}{p^2 - m^2 + i\boldsymbol{\eta}}\,. \tag{60}$$

(This prescription defines the **Feynman** propagator.)

Green Function: Interacting Theory

- Eq. (49), Green function for a fermion in an external electromagnetic field:

$$[i\not\partial_{x'} - e\not A(x') - m]\, S(x',x) = \mathbf{1}\, \delta^4(x'-x)\,, \tag{61}$$

A closed form solution of this equation is impossible in all but the simplest field configurations. Is there, nevertheless, a way to construct an approximate solution that can systematically be improved?

- One Answer: Perturbation Theory – rewrite the equation:

$$[i\not\partial_{x'} - m]\, S(x',x) = \mathbf{1}\, \delta^4(x'-x) + e\not A(x')\, S(x',x)\,, \tag{62}$$

which, as can easily be seen by substitution (Verify This), is solved by

$$\begin{aligned} S(x',x) &= S_0(x'-x) + e\int d^4y\, S_0(x'-y)\not A(y)\, S(y,x) \\ &= S_0(x'-x) + e\int d^4y\, S_0(x'-y)\not A(y)\, S_0(y-x) \\ &\quad + e^2 \int d^4y_1 \int d^4y_2\, S_0(x'-y_1)\not A(y_1)\, S_0(y_1-y_2)\not A(y_2)\, S_0(y_2-x) \\ &\quad + \ldots \end{aligned} \tag{63}$$

Green Function: Interacting Theory

- This perturbative expansion of the full propagator in terms of the free propagator provides an archetype for perturbation theory in quantum field theory.
  - One obvious application is the scattering of an electron/positron by a Coulomb field, which is an example explored in Sec. 2.5.3 of Itzykson, C. and Zuber, J.-B. (1980), *Quantum Field Theory* (McGraw-Hill, New York).
  - Equation (63) is a first example of a Dyson-Schwinger equation.
- This Green function has the following interpretation
  1. It creates a positive energy fermion (antifermion) at spacetime point x;
  2. Propagates the fermion to spacetime point x'; i.e., forward in time;
  3. Annihilates this fermion at x'.

  The process can equally well be viewed as

  1. The creation of a negative energy antifermion (fermion) at spacetime point x';
  2. Propagation of the antifermion to the spacetime point x; i.e., backward in time;
  3. Annihilation of this antifermion at x.

  Other propagators have similar interpretations.

Anything Troubling You?

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Functional Integrals

- Local gauge theories are the keystone of contemporary hadron and high-energy physics. QCD is a local gauge theory.
- Such theories are difficult to quantise because the gauge dependence is an extra non-dynamical degree of freedom that must be dealt with.
- The modern approach is to quantise the theories using the method of functional integrals. Good references:
  - Itzykson, C. and Zuber, J.-B. (1980), *Quantum Field Theory* (McGraw-Hill, New York);
  - Pascual, P. and Tarrach, R. (1984), Lecture Notes in Physics, Vol. **194**, *QCD: Renormalization for the Practitioner* (Springer-Verlag, Berlin).

  Functional Integration replaces canonical second-quantisation.
- NB. In general mathematicians do not regard local gauge theory functional integrals as well-defined.

Dyson-Schwinger Equations

- It has long been known that from the field equations of quantum field theory one can derive a system of coupled integral equations interrelating all of a theory's Green functions:
  - Dyson, F.J. (1949), "The S Matrix In Quantum Electrodynamics," *Phys. Rev*. **75**, 1736.
  - Schwinger, J.S. (1951), "On The Green's Functions Of Quantized Fields: 1 and 2," *Proc. Nat. Acad. Sci*. **37** (1951) 452; *ibid* 455.
- This collection of a countable infinity of equations is called the complex of Dyson-Schwinger equations (DSEs).
- It is an intrinsically nonperturbative complex, which is vitally important in proving the renormalisability of quantum field theories. At its simplest level the complex provides a generating tool for perturbation theory.
- In the context of quantum electrodynamics (QED) we will illustrate a nonperturbative derivation of one equation in this complex. The derivation of others follows the same pattern.

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Photon Vacuum Polarisation

NB. This is one part of the Lamb Shift

- Action for QED with N_f flavours of electromagnetically active fermions:

$$S[A_\mu, \psi, \bar{\psi}] = \int d^4x \left[\sum_{f=1}^{N_f} \bar{\psi}^f \left(i\partial\!\!\!/ - m_0^f + e_0^f A\!\!\!/ \right) \psi^f - \frac{1}{4} F_{\mu\nu} F^{\mu\nu} - \frac{1}{2\lambda_0} \partial^\mu A_\mu(x)\, \partial^\nu A_\nu(x) \right] \quad (64)$$

- Manifestly Poincaré covariant action:
  - $\bar{\psi}^f(x)$, $\psi^f(x)$ are elements of Grassmann algebra that describe the fermion degrees of freedom;
  - m_0^f are the fermions' bare masses and e_0^f, their charges;
  - and $A_\mu(x)$ describes the gauge boson [photon] field, with $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$, and λ_0 the bare gauge fixing parameter. (NB. To describe an electron the physical charge $e_f < 0$.)

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QED Generating Functional

- The Generating Functional is defined via the QED action

$$
\begin{aligned}
W[J_\mu,\xi,\bar\xi] = & \int [\mathcal{D}A_\mu]\,[\mathcal{D}\psi][\mathcal{D}\bar\psi] \\
& \times \exp\left\{ i\int d^4x \left[-\frac{1}{4} F^{\mu\nu}(x)\,F_{\mu\nu}(x) - \frac{1}{2\lambda_0}\,\partial^\mu A_\mu(x)\,\partial^\nu A_\nu(x) \right.\right. \\
& \quad + \sum_{f=1}^{N_f} \bar\psi^f \left(i\partial\!\!\!/ - m_0^f + e_0^f A\!\!\!/ \right) \psi^f \\
& \quad \left.\left. + J^\mu(x) A_\mu(x) + \bar\xi^f(x)\psi^f(x) + \bar\psi^f(x)\xi^f(x) \right] \right\} ,
\end{aligned} \tag{65}
$$

  - simple interaction term: $\boxed{\bar\psi^f e_0^f A\!\!\!/ \psi^f}$
  - J_μ is an external source for the electromagnetic field
  - ξ^f , $\bar\xi^f$ are external sources for the fermion field that, of course, are elements in the Grassmann algebra.
- The Generating Functional expresses ***every*** feature of the theory.

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Functional Field Equations

- Advantageous to work with the generating functional of connected Green functions; i.e., $Z[J_\mu, \bar{\xi}, \xi]$ defined via

$$W[J_\mu, \xi, \bar{\xi}] =: \exp\left\{ iZ[J_\mu, \xi, \bar{\xi}] \right\}. \tag{64}$$

- Derivation of a DSE follows simply from observation that the integral of a total derivative vanishes, given appropriate boundary conditions; e.g.,

$$\begin{aligned} 0 &= \int_{-\infty}^{\infty} dx \, \boxed{\frac{d}{dx} f(x)} \\ &= f(\infty) - f(-\infty) \end{aligned}$$

so long as $f(\infty) = f(-\infty)$, which includes $f(\infty) = f(-\infty) = 0$; viz., the case of "fields" that vanish far from the interaction domain, centred on $x = 0$.

Functional Field Equations

- Advantageous to work with the generating functional of connected Green functions; i.e., $Z[J_\mu,\bar\xi,\xi]$ defined via

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- Derivation of a DSE follows simply from observation that the integral of a total derivative vanishes, given appropriate boundary conditions; e.g.,

$$\begin{aligned} 0 &= \int [\mathcal{D}A_\mu]\,[\mathcal{D}\psi][\mathcal{D}\bar\psi]\,\frac{\delta}{\delta A_\mu(x)}\,\mathrm{e}^{i\left(S[A_\mu,\psi,\bar\psi]+\int d^4x\,[\bar\psi^f\xi^f+\bar\xi^f\psi^f+A_\mu J^\mu]\right)} \\ &= \int [\mathcal{D}A_\mu]\,[\mathcal{D}\psi][\mathcal{D}\bar\psi]\left\{\frac{\delta S}{\delta A_\mu(x)}+J_\mu(x)\right\} \\ &\quad\times\exp\left\{i\left(S[A_\mu,\psi,\bar\psi]+\int d^4x\left[\bar\psi^f\xi^f+\bar\xi^f\psi^f+A_\mu J^\mu\right]\right)\right\} \\ &= \left\{\frac{\delta S}{\delta A_\mu(x)}\left[\frac{\delta}{i\delta J},\frac{\delta}{i\delta\bar\xi},-\frac{\delta}{i\delta\xi}\right]+J_\mu(x)\right\}W[J_\mu,\xi,\bar\xi]\,, \end{aligned} \tag{65}$$

where the last line has meaning as a functional differential operator acting on the generating functional.

Functional Field Equations

- Differentiate Eq. (64) to obtain

$$\frac{\delta S}{\delta A_\mu(x)} = \left[\partial_\rho \partial^\rho g_{\mu\nu} - \left(1 - \frac{1}{\lambda_0}\right)\partial_\mu \partial_\nu\right] A^\nu(x) + \sum_f e_0^f\, \overline{\psi}^f(x)\gamma_\mu \psi^f(x)\,, \quad (66)$$

- Equation (65) then becomes

$$-J_\mu(x) = \left[\partial_\rho \partial^\rho g_{\mu\nu} - \left(1 - \frac{1}{\lambda_0}\right)\partial_\mu \partial_\nu\right] \frac{\delta Z}{\delta J_\nu(x)}$$
$$+ \sum_f e_0^f \left(- \frac{\delta Z}{\delta \xi^f(x)} \gamma_\mu \frac{\delta Z}{\delta \bar{\xi}^f(x)} + \frac{\delta}{\delta \xi^f(x)}\right] \gamma_\mu \frac{\delta\, iZ}{\delta \bar{\xi}^f(x)}\right]\Bigg), \quad (67)$$

where we have divided through by $W[J_\mu, \xi, \bar{\xi}]$.

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Functional Field Equations

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$$-J_\mu(x) = \left[\partial_\rho \partial^\rho g_{\mu\nu} - \left(1 - \frac{1}{\lambda_0}\right)\partial_\mu \partial_\nu\right] \frac{\delta Z}{\delta J_\nu(x)}$$
$$+ \sum_f e_0^f \left(- \frac{\delta Z}{\delta \xi^f(x)} \gamma_\mu \frac{\delta Z}{\delta \bar{\xi}^f(x)} + \frac{\delta}{\delta \xi^f(x)} \right] \gamma_\mu \frac{\delta\, iZ}{\delta \bar{\xi}^f(x)} \right] \Bigg), \quad (67)$$

where we have divided through by $W[J_\mu, \xi, \bar{\xi}]$.

- Equation (67) represents a compact form of the nonperturbative equivalent of Maxwell's equations.

One-particle Irreducible Green Function

- Introduce generating functional for one-particle-irreducible (1PI) Green functions:

$$\Gamma[A_\mu, \psi, \bar\psi] \qquad (68)$$

- Obtained from $Z[J_\mu, \xi, \bar\xi]$ via a Legendre transformation; namely,

$$Z[J_\mu, \xi, \bar\xi] = \Gamma[A_\mu, \psi, \bar\psi] + \int d^4x \left[\overline{\psi}^f \xi^f + \bar\xi^f \psi^f + A_\mu J^\mu \right]. \qquad (69)$$

NB. On the right-hand-side, $\boldsymbol{A_\mu}$, $\boldsymbol{\psi}$, $\boldsymbol{\bar\psi}$ are functionals of the sources.

- One-particle-irreducible n-point function or "proper vertex" contains no contributions that become disconnected when a single connected m-point Green function is removed; e.g., via functional differentiation.
  - No diagram representing or contributing to a given proper vertex separates into two disconnected diagrams if only one connected propagator is cut. (Detailed explanation: Itzykson, C. and Zuber, J.-B. (1980), *Quantum Field Theory* (McGraw-Hill, New York), pp. 289-294.)

Implications of Legendre Transformation

- It is plain from the definition of the Generating Functional, Eq. (65), that

$$\frac{\delta Z}{\delta J^\mu(x)} = A_\mu(x)\,,\ \frac{\delta Z}{\delta \bar\xi(x)} = \psi(x)\,,\ \frac{\delta Z}{\delta \xi(x)} = -\bar\psi(x)\,, \tag{70}$$

where here the external sources are **nonzero**.

- Hence Γ in Eq. (69) must satisfy

$$\frac{\delta \Gamma}{\delta A^\mu(x)} = -J_\mu(x)\,,\ \frac{\delta \Gamma}{\delta \bar\psi^f(x)} = -\xi^f(x)\,,\ \frac{\delta \Gamma}{\delta \psi^f(x)} = \bar\xi^f(x)\,. \tag{71}$$

(NB. Since the sources are not zero then, e.g.,

$$A_\rho(x) = A_\rho(x;[J_\mu,\xi,\bar\xi]) \ \Rightarrow\ \frac{\delta A_\rho(x)}{\delta J^\mu(y)} \neq 0\,, \tag{72}$$

with analogous statements for the Grassmannian functional derivatives.)

- NB. It is easy to see that setting $\bar\psi = 0 = \psi$ after differentiating Γ gives zero *unless* there are equal numbers of $\bar\psi$ and ψ derivatives. (Integrand is odd under $\psi \to -\psi$.)

Green Function's Inverse

- Consider the operator and matrix product (with spinor labels r, s, t)

$$-\int d^4z \, \frac{\delta^2 Z}{\delta\xi_r^f(x)\bar{\xi}_t^h(z)} \, \frac{\delta^2\Gamma}{\delta\psi_t^h(z)\overline{\psi}_s^g(y)}\Bigg|_{\substack{\xi=\bar{\xi}=0\\ \psi=\overline{\psi}=0}} \,. \tag{73}$$

- Using Eqs. (70), (71), this simplifies as follows:

$$= \int d^4z \, \frac{\delta\psi_t^h(z)}{\delta\xi_r^f(x)} \, \frac{\delta\xi_s^g(y)}{\delta\psi_t^h(z)}\Bigg|_{\substack{\xi=\bar{\xi}=0\\ \psi=\overline{\psi}=0}} = \frac{\delta\xi_s^g(y)}{\delta\xi_r^f(x)}\Bigg|_{\psi=\overline{\psi}=0} = \delta_{rs}\,\delta^{fg}\,\delta^4(x-y)\,. \tag{74}$$

- Back in Eq. (67), setting $\bar{\xi}=0=\xi$ one obtains

$$\frac{\delta\Gamma}{\delta A^\mu(x)}\Bigg|_{\psi=\overline{\psi}=0} = \left[\partial_\rho\partial^\rho g_{\mu\nu} - \left(1-\frac{1}{\lambda_0}\right)\partial_\mu\partial_\nu\right]A^\nu(x) - i\sum_f e_0^f \mathrm{tr}\left[\gamma_\mu S^f(x,x;[A_\mu])\right], \tag{75}$$

Identification: $S^f(x,y;[A_\mu]) = -\dfrac{\delta^2 Z}{\delta\xi^f(y)\bar{\xi}^f(x)} = \dfrac{\delta^2 Z}{\delta\bar{\xi}^f(x)\xi^f(y)}$ (no summation on f), (76)

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Green Function's Inverse

- As a direct consequence of Eq. (73) the inverse of this Green function is given by

$$S^f(x,y;[A])^{-1} = \left.\frac{\delta^2\Gamma}{\delta\psi^f(x)\delta\bar{\psi}^f(y)}\right|_{\psi=\bar{\psi}=0} . \quad (77)$$

General property: functional derivatives of the generating functional for 1PI Green functions are related to the associated propagator's inverse.

- Clearly, the in-vacuum fermion propagator or, another name, the connected fermion 2-point function is

$$S^f(x,y) := S^f(x,y;[A_\mu = 0])\,. \quad (78)$$

Such vacuum Green functions are keystones in quantum field theory.

- To continue, differentiate Eq. (75) with respect to $A_\nu(y)$ and set $J_\mu(x)=0$:

$$\left.\frac{\delta^2\Gamma}{\delta A^\mu(x)\delta A^\nu(y)}\right|_{\substack{A_\mu=0\\ \psi=\bar{\psi}=0}} = \left[\partial_\rho\partial^\rho g_{\mu\nu} - \left(1-\frac{1}{\lambda_0}\right)\partial_\mu\partial_\nu\right]\delta^4(x-y)$$
$$-i\sum_f e_0^f \mathrm{tr}\left[\gamma_\mu \frac{\delta}{\delta A_\nu(y)}\left(\left.\frac{\delta^2\Gamma}{\delta\psi^f(x)\delta\bar{\psi}^f(x)}\right|_{\psi=\bar{\psi}=0}\right)^{-1}\right] . \quad (79)$$

Inverse of Photon Propagator

- l.h.s. is easily understood – Eqs. (77), (78) define the inverse of the fermion propagator. Hence, l.h.s. must be the inverse of the photon propagator:

$$(D^{-1})^{\mu\nu}(x,y) := \left.\frac{\delta^2\Gamma}{\delta A^\mu(x)\delta A^\nu(y)}\right|_{\substack{A_\mu = 0 \\ \psi=\overline{\psi}=0}} . \tag{80}$$

- r.h.s., however, must be simplified and interpreted. First observe that

$$-\frac{\delta}{\delta A_\nu(y)}\left(\left.\frac{\delta^2\Gamma}{\delta\psi^f(x)\delta\bar\psi^f(x)}\right|_{\psi=\overline{\psi}=0}\right)^{-1} = \int d^4u d^4w \; \ldots \tag{81}$$

$$\left(\left.\frac{\delta^2\Gamma}{\delta\psi^f(x)\delta\bar\psi^f(w)}\right|_{\psi=\overline{\psi}=0}\right)^{-1} \frac{\delta}{\delta A_\nu(y)}\frac{\delta^2\Gamma}{\delta\psi^f(u)\delta\bar\psi^f(w)}\left(\left.\frac{\delta^2\Gamma}{\delta\psi^f(w)\delta\bar\psi^f(x)}\right|_{\psi=\overline{\psi}=0}\right)^{-1},$$

  - Analogue of result for finite dimensional matrices:

$$\frac{d}{dx}\left[A(x)A^{-1}(x) = \boldsymbol{I}\right] \quad = 0 = \quad \frac{dA(x)}{dx}\,A^{-1}(x) + A(x)\,\frac{dA^{-1}(x)}{dx}$$

$$\Rightarrow \quad \frac{dA^{-1}(x)}{dx} = -A^{-1}(x)\,\frac{dA(x)}{dx}\,A^{-1}(x)\,. \tag{82}$$

Proper Fermion-Photon Vertex

- Equation (81) involves the 1PI 3-point function (no summation on f)

$$e_0^f \Gamma_\mu^f(x,y;z) := \frac{\delta}{\delta A_\nu(z)} \frac{\delta^2 \Gamma}{\delta \psi^f(x) \delta \bar{\psi}^f(y)} \,. \tag{83}$$

This is the proper fermion-gauge-boson vertex.

- At leading order in perturbation theory

$$\Gamma_\nu^f(x,y;z) = \gamma_\nu \, \delta^4(x-z) \, \delta^4(y-z) \,, \tag{84}$$

Result can be obtained via explicit calculation of functional derivatives in Eq. (83).

Photon Vacuum Polarisation

- Define the gauge-boson *vacuum polarisation*:

$$\Pi_{\mu\nu}(x,y) = i\sum_f (e_0^f)^2 \int d^4z_1\, d^4z_2\, \mathrm{tr}[\gamma_\mu S^f(x,z_1)\Gamma_\nu^f(z_1,z_2;y)S^f(z_2,x)], \quad (85)$$

- Gauge-boson vacuum polarisation, or "photon self-energy,"
  - Describes modification of gauge-boson's propagation characteristics owing to the presence of virtual particle-antiparticle pairs in quantum field theory.
  - In particular, the photon vacuum polarisation is an important element in the description of a process such as $\rho^0 \to e^+e^-$.
- Eq. (79) may now be expressed as

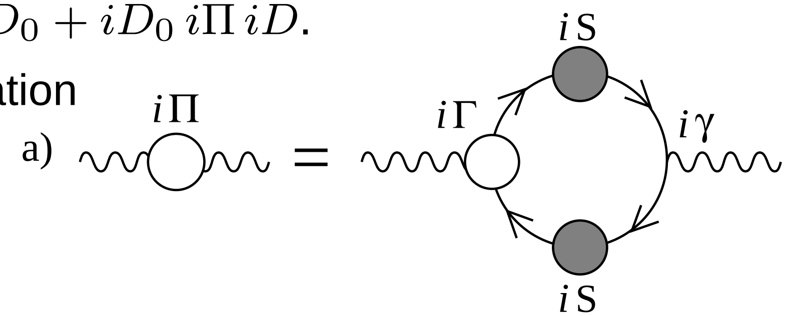
$$(D^{-1})^{\mu\nu}(x,y) = [\partial_\rho\partial^\rho g_{\mu\nu} - (1 - \frac{1}{\lambda_0})\partial_\mu\partial_\nu]\delta^4(x-y) + \Pi_{\mu\nu}(x,y)\,. \quad (86)$$

- The propagator for a free gauge boson is [use $\Pi_{\mu\nu}(x,y) \equiv 0$ in Eq. (86)]

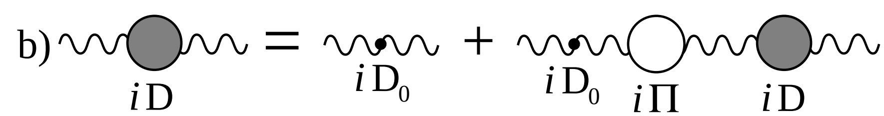
$$D_0^{\mu\nu}(q) = \frac{-g^{\mu\nu} + (q^\mu q^\nu/[q^2+i\eta])}{q^2+i\eta} - \lambda_0 \frac{q^\mu q^\nu}{(q^2+i\eta)^2}\,, \quad (87)$$

DSE for Photon Propagator

- Then Eq. (86) can be written $iD = iD_0 + iD_0\, i\Pi\, iD$.
  - This is a Dyson-Schwinger Equation



- In presence of interactions;
  i.e., for $\Pi_{\mu\nu} \neq 0$ in Eq. (86),



$$D^{\mu\nu}(q) = \frac{-g^{\mu\nu} + (q^\mu q^\nu/[q^2+i\eta])}{q^2+i\eta}\,\frac{1}{1+\Pi(q^2)} - \lambda_0 \frac{q^\mu q^\nu}{(q^2+i\eta)^2}\,, \qquad (88)$$

Used the "Ward-Takahashi identity:" $\boxed{q_\mu\, \Pi_{\mu\nu}(q) = 0 = \Pi_{\mu\nu}(q)\, q_\nu}$,

$$\Rightarrow \Pi^{\mu\nu}(q) = (-g^{\mu\nu} q^2 + q^\mu q^\nu)\, \Pi(q^2)\,. \qquad (89)$$

- $\Pi(q^2)$ is the polarisation scalar. Independent of the gauge parameter, λ_0, in QED.
  - $\lambda_0 = 1$ is called "Feynman gauge." Useful in perturbative calculations because it simplifies the $\Pi(q^2) = 0$ gauge boson propagator enormously.
  - In nonperturbative applications, however, $\lambda_0 = 0$, "Landau gauge," is most useful because it ensures that the gauge boson propagator is itself transverse.

Ward-Takahashi Identities

- Ward-Takahashi identities (WTIs) are relations satisfied by n-point Green functions, relations which are an essential consequence of a theory's local gauge invariance; i.e., local current conservation.
- They can be proved directly from the generating functional and have physical implications. For example, Eq. (89) ensures that the photon remains massless in the presence of charged fermions.
- A discussion of WTIs can be found in
  - Bjorken, J.D. and Drell, S.D. (1965), *Relativistic Quantum Fields* (McGraw-Hill, New York), pp. 299-303,
  - Itzykson, C. and Zuber, J.-B. (1980), *Quantum Field Theory* (McGraw-Hill, New York), pp. 407-411;
- Their generalisation to non-Abelian theories as "Slavnov-Taylor" identities is described in

  Pascual, P. and Tarrach, R. (1984), Lecture Notes in Physics, Vol. **194**, *QCD: Renormalization for the Practitioner* (Springer-Verlag, Berlin), Chap. 2.

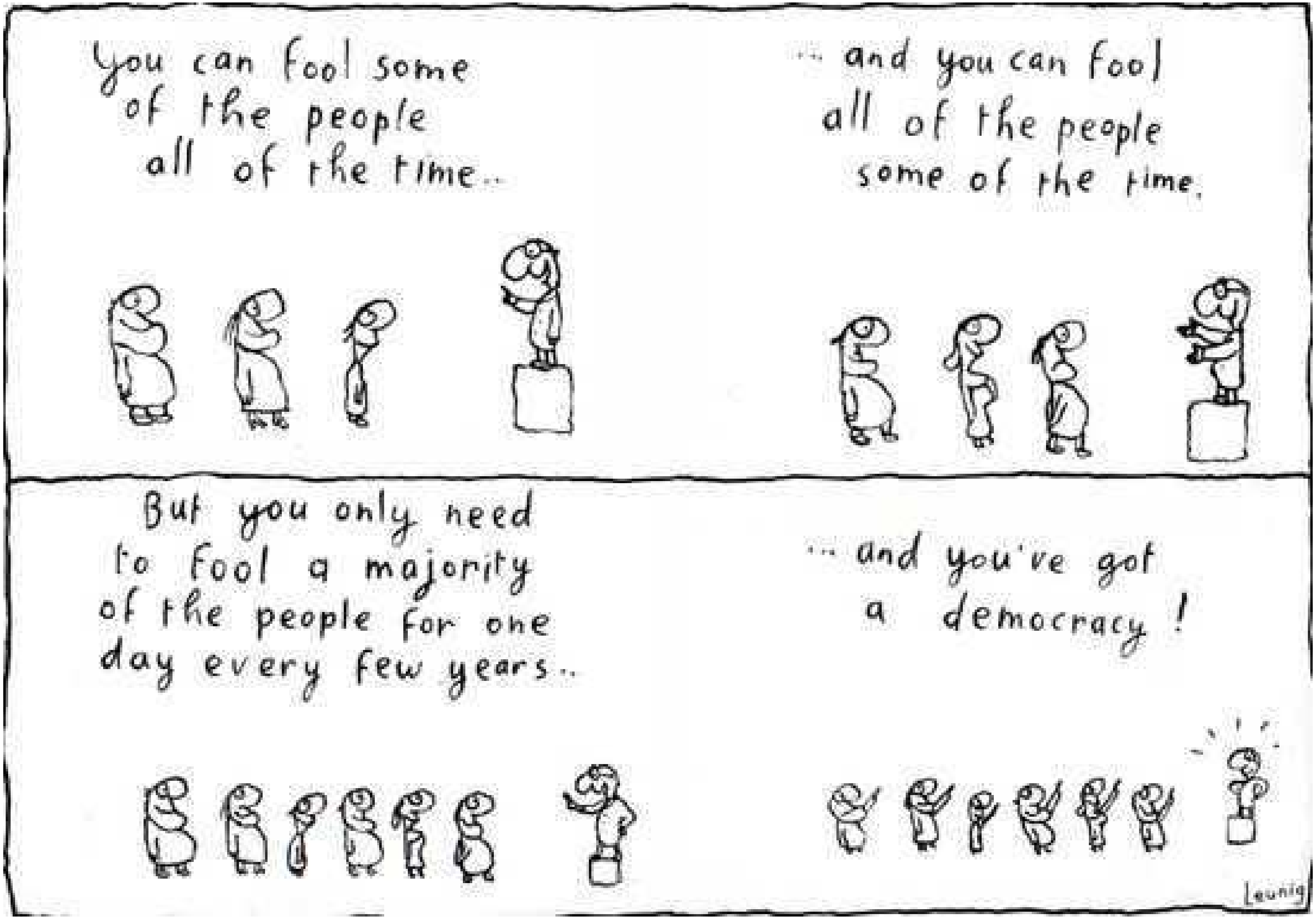
Vacuum Polarisation in Momentum Space

- In absence of external sources, Eq. (85) can easily be represented in momentum space, because then the 2- and 3-point functions appearing therein must be translationally invariant and hence they can be expressed simply in terms of Fourier amplitudes; i.e., we have

$$i\Pi_{\mu\nu}(q) = -\sum_f (e_0^f)^2 \int \frac{d^4\ell}{(2\pi)^4} \mathrm{tr}[(i\gamma_\mu)(iS^f(\ell))(i\Gamma^f(\ell,\ell+q))(iS(\ell+q))]\,. \quad (90)$$

  The reduction to a single integral makes momentum space representations most widely used in continuum calculations.

- QED: the vacuum polarisation is directly related to the running coupling constant, which is a connection that makes its importance obvious.
- QCD: connection not so direct but, nevertheless, the polarisation scalar is a key component in the evaluation of the strong running coupling.
- Observed: second derivatives of the generating functional, $\Gamma[A_\mu,\psi,\bar\psi]$, give the inverse-fermion and -photon propagators; third derivative gave the proper photon-fermion vertex. In general, all derivatives of $\Gamma[A_\mu,\psi,\bar\psi]$, higher than two, produce a proper vertex, number and type of derivatives give the number and type of proper Green functions that it can connect.



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Any Questions?

Functional Dirac Equation

- Equation (67) is a nonperturbative generalisation of Maxwell's equation in quantum field theory. Its derivation provides the archetype by which one can obtain an equivalent generalisation of Dirac's equation:

$$\begin{aligned}
0 &= \int [\mathcal{D}A_\mu]\,[\mathcal{D}\psi][\mathcal{D}\bar\psi]\,\frac{\delta}{\delta\bar\psi^f(x)}\,\mathrm{e}^{\,i(S[A_\mu,\psi,\bar\psi]+\int d^4x\,[\overline{\psi}^g\xi^g+\bar\xi^g\psi^g+A_\mu J^\mu])} \\
&= \int [\mathcal{D}A_\mu]\,[\mathcal{D}\psi][\mathcal{D}\bar\psi]\left\{\frac{\delta S}{\delta\bar\psi^f(x)}+\xi^f(x)\right\} \\
&\quad \times\exp\left\{i(S[A_\mu,\psi,\bar\psi]+\int d^4x\,[\overline{\psi}^g\xi^g+\bar\xi^g\psi^g+A_\mu J^\mu])\right\} \\
&= \left\{\frac{\delta S}{\delta\bar\psi^f(x)}\left[\frac{\delta}{i\delta J},\frac{\delta}{i\delta\bar\xi},-\frac{\delta}{i\delta\xi}\right]+\eta^f(x)\right\}W[J_\mu,\xi,\bar\xi] \qquad (91)\\
0 &= \left[\xi^f(x)+\left(i\partial\!\!\!/-m_0^f+e_0^f\gamma^\mu\,\frac{\delta}{i\delta J^\mu(x)}\right)\frac{\delta}{i\delta\bar\xi^f(x)}\right]W[J_\mu,\xi,\bar\xi]. \qquad (92)
\end{aligned}$$

- The last line furnishes a nonperturbative functional equivalent of Dirac's equation.

Functional Green Function

- Next step ... a functional derivative with respect to ξ^f: $\delta/\delta\xi^f(y)$, yields

$$\delta^4(x-y)W[J_\mu] - (i\partial\!\!\!/ - m_0^f + e_0^f\gamma^\mu \frac{\delta}{i\delta J^\mu(x)})W[J_\mu]\, S^f(x,y;[A_\mu]) = 0\,, \tag{93}$$

after setting $\xi^f = 0 = \bar{\xi}^f$, where $W[J_\mu] := W[J_\mu, 0, 0]$ and $S(x,y;[A_\mu])$ is defined in Eq. (76).

- Now, using Eqs. (64), (71), this can be rewritten

$$\delta^4(x-y) - [i\partial\!\!\!/ - m_0^f + e_0^f A\!\!\!/(x;[J]) + e_0^f\gamma^\mu \frac{\delta}{i\delta J^\mu(x)}]\, S^f(x,y;[A_\mu]) = 0\,, \tag{94}$$

which defines the nonperturbative connected 2-point fermion Green function

- NB. This is clearly the functional equivalent of Eq. (61):

$$[i\partial\!\!\!/_{x'} - eA\!\!\!/(x') - m]\, S(x',x) = \mathbf{1}\, \delta^4(x'-x)\,. \tag{95}$$

namely, Differential Operator Green Function for the Interacting Dirac Theory.

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DSE for Fermion Propagator

- The electromagentic four-potential vanishes in the absence of an external source; i.e., $A_\mu(x;[J=0])=0$
- Remains only to exhibit the content of the remaining functional differentiation in Eq. (94), which can be accomplished using Eq. (81):

$$\begin{aligned}\frac{\delta}{i\delta J^\mu(x)}\,S^f(x,y;[A_\mu]) = \int d^4z\,\frac{\delta A_\nu(z)}{i\delta J^\mu(x)}\,\frac{\delta}{\delta A_\nu(z)}\left(\left.\frac{\delta^2\Gamma}{\delta\psi^f(x)\delta\bar\psi^f(y)}\right|_{\psi=\bar\psi=0}\right)^{-1}\\ = \quad -e_0^f\int d^4z\,d^4u\,d^4w\,\frac{\delta A_\nu(z)}{i\delta J_\mu(x)}\,S^f(x,u)\,\Gamma_\nu(u,w;z)\,S(w,y)\\ = \quad -e_0^f\int d^4z\,d^4u\,d^4w\,iD_{\mu\nu}(x-z)\,S^f(x,u)\,\Gamma_\nu(u,w;z)\,S(w,y)\,,\end{aligned} \tag{96}$$

In the last line, we have set $J=0$ and used Eq. (80).

- Hence in the absence of external sources Eq. (94) is equivalent to

$$\begin{aligned}\delta^4(x-y) = (i\not\partial - m_0^f)\,S^f(x,y)\\ - i\,(e_0^f)^2\int d^4z\,d^4u\,d^4w\,D^{\mu\nu}(x,z)\,\gamma_\mu\,S(x,u)\,\Gamma_\nu(u,w;z)\,S(w,y)\,.\end{aligned} \tag{97}$$

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Fermion Self Energy

- Photon vacuum polarisation was introduced to re-express the DSE for the gauge boson propagator, Eq. (85). Analogue, one can define a fermion self-energy:

$$\Sigma^f(x,z) = i(e_0^f)^2 \int d^4u\, d^4w\; D^{\mu\nu}(x,z)\, \gamma_\mu\, S(x,u)\, \Gamma_\nu(u,w;z)\,, \tag{98}$$

so that Eq. (97) assumes the form

$$\int d^4z\, [(i\partial\!\!\!/_x - m_0^f)\delta^4(x-z) - \Sigma^f(x,z)]\, S(z,y) = \delta^4(x-y)\,. \tag{99}$$

- Using property that Green functions are translationally invariant in the absence of external sources:

$$-i\Sigma^f(p) = (e_0^f)^2 \int \frac{d^4\ell}{(2\pi)^4}\, [iD^{\mu\nu}(p-\ell)]\, [i\gamma_\mu]\, [iS^f(\ell)]\, [i\Gamma^f_\nu(\ell,p)]\,. \tag{100}$$

Now follows from Eq. (99) that connected fermion 2-point function in momentum space is

$$S^f(p) = \frac{1}{p\!\!\!/ - m_0^f - \Sigma^f(p) + i\eta^+}\,. \tag{101}$$

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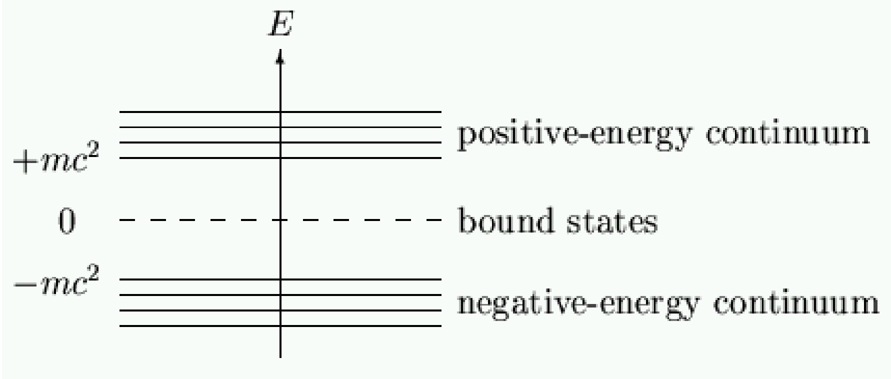
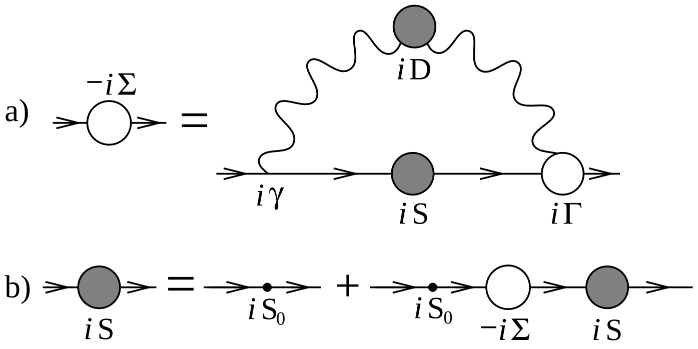
Gap Equation

- Equation (100) is the **exact** *Gap Equation*.



- Describes manner in which propagation characteristics of a fermion moving through ground state of QED (the QED vacuum) is altered by the repeated emission and reabsorption of virtual photons.



- Equation can also describe the real process of Bremsstrahlung. Furthermore, solution of analogous equation in QCD provides information about dynamical chiral symmetry breaking and also quark confinement.

Perturbative Calculation of Gap

- Keystone of strong interaction physics is dynamical chiral symmetry breaking (DCSB). In order to understand DCSB one must first come to terms with explicit chiral symmetry breaking. Consider then the DSE for the quark self-energy in QCD:

$$-i\,\Sigma(p) = -g_0^2 \int \frac{d^4\ell}{(2\pi)^4}\, D^{\mu\nu}(p-\ell)\, \frac{i}{2}\lambda^a \gamma_\mu\, S(\ell)\, i\Gamma^a_\nu(\ell,p)\,, \tag{102}$$

where the flavour label is suppressed.

- Form is precisely the same as that in QED, Eq. (100) but . . .
  - colour (Gell-Mann) matrices: $\{\lambda^a; a = 1,\ldots,8\}$ at the fermion-gauge-boson vertex
  - $D^{\mu\nu}(\ell)$ is the connected gluon 2-point function
  - $\Gamma^a_\nu(\ell,\ell')$ is the proper quark-gluon vertex
- One-loop contribution to quark's self-energy obtained by evaluating r.h.s. of Eq. (102) using the free quark and gluon propagators, and the quark-gluon vertex:

$$\Gamma^{a\,(0)}_\nu(\ell,\ell') = \frac{1}{2}\lambda^a\gamma_\nu\,. \tag{103}$$

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Explicit Leading-Order Calculation

- $$-\,i\,\Sigma^{(2)}(p) \;=\; -g_0^2 \int \frac{d^4k}{(2\pi)^4}\left[-\,g^{\mu\nu} + (1-\lambda_0)\frac{k^\mu k^\nu}{k^2+i\eta^+}\right]\frac{1}{k^2+i\eta^+} \times \frac{i}{2}\lambda^a\gamma_\mu\,\frac{1}{\not k + \not p - m_0 + i\eta^+}\,\frac{i}{2}\lambda^a\gamma_\mu\,. \quad (104)$$

- To proceed, first observe that Eq. (104) can be re-expressed as

$$-i\,\Sigma^{(2)}(p) = -g_0^2\,C_2(R)\int\frac{d^4k}{(2\pi)^4}\,\frac{1}{(k+p)^2-m_0^2+i\eta^+}\,\frac{1}{k^2+i\eta^+} \times\left\{\gamma^\mu\,(\not k+\not p+m_0)\,\gamma_\mu - (1-\lambda_0)\,(\not k-\not p+m_0) - 2\,(1-\lambda_0)\,\frac{(k,p)\,\not k}{k^2+i\eta^+}\right\}, \quad (105)$$

where we have used $\boxed{\frac{1}{2}\lambda^a\,\frac{1}{2}\lambda^a = C_2(R)\,\boldsymbol{I}_c\,;\; C_2(R) = \frac{N_c^2-1}{2N_c}}$, with N_c the number of colours ($N_c = 3$ in QCD), and $\boldsymbol{I}_c$ is the identity matrix in colour space.

Explicit Leading-Order Calculation

- Now note that $2\,(k,p) = [(k+p)^2 - m_0^2] - [k^2] - [p^2 - m_0^2]$ and hence

$$\begin{aligned} -\,i\,\Sigma^{(2)}(p) &= -g_0^2\,C_2(R)\int\frac{d^4k}{(2\pi)^4}\,\frac{1}{(k+p)^2 - m_0^2 + i\eta^+}\,\frac{1}{k^2+i\eta^+} \\ &\quad \Big\{\gamma^\mu\,(\not k + \not p + m_0)\,\gamma_\mu + (1-\lambda_0)\,(\not p - m_0) \\ &\quad + (1-\lambda_0)\,(p^2 - m_0^2)\,\frac{\not k}{k^2+i\eta^+} \\ &\quad - (1-\lambda_0)\,[(k+p)^2 - m_0^2]\,\frac{\not k}{k^2+i\eta^+}\Big\}. \end{aligned} \tag{106}$$

- Focus on the last term:

$$\begin{aligned} &\int\frac{d^4k}{(2\pi)^4}\,\frac{1}{(k+p)^2 - m_0^2 + i\eta^+}\,\frac{1}{k^2+i\eta^+}\,[(k+p)^2 - m_0^2]\,\frac{\not k}{k^2+i\eta^+} \\ =\; &\int\frac{d^4k}{(2\pi)^4}\,\frac{1}{k^2+i\eta^+}\,\frac{\not k}{k^2+i\eta^+} = 0 \end{aligned} \tag{107}$$

because the integrand is odd under $k \to -k$, and so this term in Eq. (106) vanishes.

Explicit Leading-Order Calculation

- $$-i\,\Sigma^{(2)}(p) = -g_0^2\,C_2(R)\int\frac{d^4k}{(2\pi)^4}\,\frac{1}{(k+p)^2-m_0^2+i\eta^+}\,\frac{1}{k^2+i\eta^+}$$
$$\left\{\gamma^\mu\,(\not k+\not p+m_0)\,\gamma_\mu+(1-\lambda_0)\,(\not p-m_0)+(1-\lambda_0)\,(p^2-m_0^2)\,\frac{\not k}{k^2+i\eta^+}\right\}.$$

- Consider the second term:

$$(1-\lambda_0)\,(\not p-m_0)\int\frac{d^4k}{(2\pi)^4}\,\frac{1}{(k+p)^2-m_0^2+i\eta^+}\,\frac{1}{k^2+i\eta^+}\,.$$

In particular, focus on the behaviour of the integrand at large k^2:

$$\frac{1}{(k+p)^2-m_0^2+i\eta^+}\,\frac{1}{k^2+i\eta^+}\overset{k^2\to\pm\infty}{\sim}\frac{1}{(k^2-m_0^2+i\eta^+)\,(k^2+i\eta^+)}\,. \qquad (108)$$

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Wick Rotation

- Integrand has poles in the second and fourth quadrant of the complex-k_0-plane but vanishes on any circle of radius $R \to \infty$ in this plane. That means one may rotate the contour anticlockwise to find

$$\begin{aligned}
&\int_0^\infty dk^0 \, \frac{1}{(k^2 - m_0^2 + i\eta^+)\,(k^2 + i\eta^+)} \\
&= \int_0^{i\infty} dk^0 \, \frac{1}{([k^0]^2 - \vec{k}^2 - m_0^2 + i\eta^+)([k^0]^2 - \vec{k}^2 + i\eta^+)} \\
&\overset{k^0 \to ik_4}{=} i\int_0^\infty dk_4 \, \frac{1}{(-k_4^2 - \vec{k}^2 - m_0^2)\,(-k_4^2 - \vec{k}^2)} \,.
\end{aligned} \tag{109}$$

- Performing a similar analysis of the $\int_{-\infty}^0$ part, one obtains the complete result:

$$\begin{aligned}
&\int \frac{d^4k}{(2\pi)^4} \, \frac{1}{(k^2 - m_0^2 + i\eta^+)\,(k^2 + i\eta^+)} \\
&= i \int \frac{d^3k}{(2\pi)^3} \int_{-\infty}^{\infty} \frac{dk_4}{2\pi} \, \frac{1}{(-\vec{k}^2 - k_4^2 - m_0^2)\,(-\vec{k}^2 - k_4^2)} \,.
\end{aligned} \tag{110}$$

These two steps constitute what is called a *Wick rotation*.

Euclidean Integral

- The integral on the r.h.s. is defined in a four-dimensional Euclidean space; i.e., $k^2 := k_1^2 + k_2^2 + k_3^2 + k_4^2 \geq 0$, with k^2 nonnegative.
- A general vector in this space can be written in the form:

$$(k) = |k| \, (\cos\phi \, \sin\theta \, \sin\beta, \sin\phi \, \sin\theta \, \sin\beta, \cos\theta \, \sin\beta, \cos\beta) \, ; \tag{111}$$

i.e., using hyperspherical coordinates, and clearly $k^2 = |k|^2$.

- In this Euclidean space using these coordinates the four-vector measure factor is

$$\begin{aligned} &\int d_E^4 k \, f(k_1, \ldots, k_4) \\ &= \quad \frac{1}{2} \int_0^\infty dk^2 \, k^2 \int_0^\pi d\beta \, sin^2\beta \int_0^\pi d\theta \sin\theta \int_0^{2\pi} d\phi \, f(k, \beta, \theta, \phi) \, . \end{aligned} \tag{112}$$

Euclidean Integral

- Returning to Eq. (108) and making use of the material just introduced, the large k^2 behaviour of the integral can be determined via

$$\begin{aligned}
&\int \frac{d^4k}{(2\pi)^4}\, \frac{1}{(k+p)^2 - m_0^2 + i\eta^+}\, \frac{1}{k^2 + i\eta^+} \\
&\approx \quad \frac{i}{16\pi^2} \int_0^\infty dk^2 \frac{1}{(k^2 + m_0^2)} \\
&= \quad \frac{i}{16\pi^2} \lim_{\Lambda\to\infty} \int_0^{\Lambda^2} dx\, \frac{1}{x + m_0^2} \\
&= \quad \frac{i}{16\pi^2} \lim_{\Lambda\to\infty} \ln(1 + \Lambda^2/m_0^2) \to \infty\,; \qquad (113)
\end{aligned}$$

- After all this work, the result is meaningless: the one-loop contribution to the quark's self-energy is divergent!

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Regularisation and Renormalisation

- Such "ultraviolet" divergences, and others which are more complicated, arise whenever loops appear in perturbation theory. (The others include "infrared" divergences associated with the gluons' masslessness; e.g., consider what would happen in Eq. (113) with $m_0 \to 0$.)
- In a *renormalisable* quantum field theory there exists a well-defined set of rules that can be used to render perturbation theory sensible.
  - First, however, one must *regularise* the theory; i.e., introduce a cutoff, or use some other means, to make finite every integral that appears. Then each step in the calculation of an observable is rigorously sensible.
  - Renormalisation follows; i.e, the absorption of divergences, and the redefinition of couplings and masses, so that finally one arrives at $\mathcal{S}$-matrix amplitudes that are finite and physically meaningful.
- The regularisation procedure must preserve the Ward-Takahashi identities (the Slavnov-Taylor identities in QCD) because they are crucial in proving that a theory can be sensibly renormalised.
- A theory is called renormalisable if, and only if, number of different types of divergent integral is finite. Then only finite number of masses & couplings need to be renormalised; i.e., *a priori* the theory has only a finite number of undetermined parameters that must be fixed through comparison with experiments.

Renormalised One-Loop Result

- Don't have time to explain and illustrate the procedure. Interested? Read . . . Pascual, P. and Tarrach, R. (1984), Lecture Notes in Physics, Vol. **194**, *QCD: Renormalization for the Practitioner* (Springer-Verlag, Berlin).
- Answer, in Momentum Subtraction Scheme:

$$\Sigma_R^{(2)}(\not p) = \Sigma_{VR}^{(2)}(p^2)\,\not p + \Sigma_{SR}^{(2)}(p^2)\,\mathbf{1}_D;$$

$$\Sigma_{VR}^{(2)}(p^2;\zeta^2) = \frac{\alpha(\zeta)}{\pi}\,\lambda(\zeta)\,\frac{1}{4}\,C_2(R)\left\{-m^2(\zeta)\left(\frac{1}{p^2}+\frac{1}{\zeta^2}\right)\right.$$
$$\left. + \left(1-\frac{m^4(\zeta)}{p^4}\right)\ln\left(1-\frac{p^2}{m(\zeta)^2}\right) - \left(1-\frac{m^4(\zeta)}{\zeta^4}\right)\ln\left(1+\frac{\zeta^2}{m^2(\zeta)}\right)\right\},$$

$$\Sigma_{SR}^{(2)}(p^2;\zeta^2) = m(\zeta)\,\frac{\alpha(\zeta)}{\pi}\,\frac{1}{4}\,C_2(R)\left\{-\left[3+\lambda(\zeta)\right]\right.$$
$$\left.\times\left[\left(1-\frac{m^2(\zeta)}{p^2}\right)\ln\left(1-\frac{p^2}{m^2(\zeta)}\right) - \left(1+\frac{m^2(\zeta)}{\zeta^2}\right)\ln\left(1+\frac{\zeta^2}{m^2(\zeta)}\right)\right]\right\},$$

where the renormalised quantities depend on the point at which the renormalisation has been conducted;
e.g., $\alpha(\zeta)$ is the running coupling, $m(\zeta)$ is the running quark mass.

Observations on Quark Self Energy

- QCD is Asymptotically Free. Hence, at some large spacelike $p^2 = \zeta^2$ the propagator is exactly the free propagator *except* that the bare mass is replaced by the renormalised mass.
- At one-loop order, the vector part of the dressed self energy is proportional to the running gauge parameter. In Landau gauge, that parameter is zero. Hence, the vector part of the renormalised dressed self energy vanishes at one-loop order in perturbation theory.
- The scalar part of the dressed self energy is proportional to the renormalised current-quark mass.
  - This is true at one-loop order, and at every order in perturbation theory.
  - Hence, if current-quark mass vanishes, then $\Sigma_{SR} \equiv 0$ in perturbation theory. That means if one starts with a chirally symmetric theory, one ends up with a chirally symmetric theory:

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Observations on Quark Self Energy

- QCD is Asymptotically Free. Hence, at some large spacelike $p^2 = \zeta^2$ the propagator is exactly the free propagator *except* that the bare mass is replaced by the renormalised mass.
- At one-loop order, the vector part of the dressed self energy is proportional to the running gauge parameter. In Landau gauge, that parameter is zero. Hence, the vector part of the renormalised dressed self energy vanishes at one-loop order in perturbation theory.
- The scalar part of the dressed self energy is proportional to the renormalised current-quark mass.
  - This is true at one-loop order, and at every order in perturbation theory.
  - Hence, if current-quark mass vanishes, then $\Sigma_{SR} \equiv 0$ in perturbation theory. That means if one starts with a chirally symmetric theory, one ends up with a chirally symmetric theory:

NO DCSB
in perturbation theory.

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