Mass dependence of the effective action in gauge-theories

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Outline

General aspects, computation

The method

New results

The effective action in QED

For a classical $A_\mu(x)$ field, the effective action is given by :

- Spinor :

$$\Gamma = -i \ln \det(i\not{D} - m) \,,$$

- Scalar :

$$\Gamma = \frac{i}{2} \ln \det(-D_\mu^2 - m^2) \,,$$

where $\not{D} \equiv \gamma^\mu D_\mu = \gamma^\mu(\partial_\mu - ieA_\mu(x))$.

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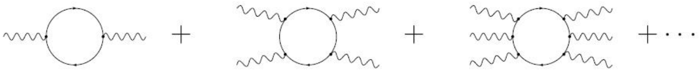
where $\not{D} \equiv \gamma^\mu D_\mu = \gamma^\mu(\partial_\mu - ieA_\mu(x))$.
These expressions correspond to a perturbative series :



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For a more general $F_{\mu\nu}$, the $m \to \infty$ limit can be systematically studied by means of the *heat-kernel* expansion :

$$\ln \det(\not{D}^2 - m^2) = \mathrm{Tr} \ln(\not{D}^2 - m^2) \sim - \int_0^\infty \frac{ds}{s} e^{-m^2 s} \mathrm{Tr} e^{-s(-\not{D}^2)}$$

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However, for $m \to 0$, there is no general approach.

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t'Hooft (1976)

$$m = 0 \quad : \quad \Gamma_{\rm spinor}(0) = \alpha(1/2) \approx 0.145873 + \mathcal{O}(m^2)$$

Kwon, Lee and Min (2000)

$$m \to 0 \quad : \quad \Gamma_{\rm spinor}(m) = \alpha(1/2) + \frac{m^2}{2}[\ln m + \gamma - \ln 2] + \mathcal{O}(m^4)$$

Outline

The Gel'Fand-Yaglom theorem

Lets calculate the determinant

$$\left[-\frac{d^2}{dx^2}+V(x)\right]\psi(x)=\lambda\psi(x) \qquad ; \qquad \psi(0)=\psi(L)=0$$

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$$\Rightarrow \quad \det\left[-\frac{d^2}{dx^2}+V(x)\right]=\psi(L)\,.$$

Example: The Helmholtz operator

$$\hat{H} = \left[-\frac{d^2}{dx^2} + m^2 \right] \quad ; \quad \lambda_n = m^2 + \left(\frac{n\pi}{L} \right)^2 ,$$

$$\frac{\det\left[-\frac{d^2}{dx^2} + m^2 \right]}{\det\left[-\frac{d^2}{dx^2} \right]} = \prod_{n=1}^{\infty} \frac{\left[m^2 + (\frac{n\pi}{L})^2 \right]}{\left[(\frac{n\pi}{L})^2 \right]} = \frac{\sinh(mL)}{mL}$$

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$$\Rightarrow \quad \frac{\det\left[-\frac{d^2}{dx^2} + m^2 \right]}{\det\left[-\frac{d^2}{dx^2} \right]} = \frac{\phi(L)}{\phi_0(L)} = \frac{\sinh(mL)}{mL}$$

Radially Symmetric Backgrounds

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We have an $O(2) \times O(3)$ symmetry and we can set up a *partial-wave decomposition* :

$$\ln\left[\frac{\det(\not{D}^2 - m^2)}{\det(\not{\partial}^2 - m^2)}\right] = \sum_{l=0}^{\infty} \Omega(l) \ln\left[\frac{\det(\mathcal{H}_l + m^2)}{\det(\mathcal{H}_l^0 + m^2)}\right]$$

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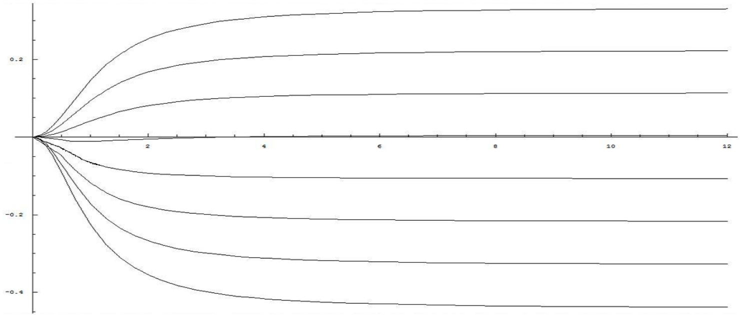
GY Theorem (initial value problem):

$$\frac{d^2 S_l(r)}{dr^2} + \left(\frac{dS_l(r)}{dr}\right)^2 + \left(\frac{1}{r} + 2m\frac{I'_{2l+1}(mr)}{I_{2l+1}(mr)}\right)\frac{dS_l(r)}{dr} = V(r)$$

$$\{\, S_l(0) = 0\,,\; S'_l(0) = 0\,\}$$

where $S_l(r) \equiv \ln \frac{\psi(r)}{\psi^0(r)}$ and $V(r)$ depends on $g(r)$.

We can find $S_l(r)$ numerically



- Example: $S_l(r)$, $\{l = 4, l_3 = -4, \cdots, 4, s_3 = 1/2\}$

Is that simple?

However,

$$\sum_{l=0}^{\infty} \Omega(l) \ln \left[\frac{\det(\mathcal{H}_l + m^2)}{\det(\mathcal{H}_l^0 + m^2)} \right] = \sum_{l=0}^{\infty} \Omega(l) S_l(\infty) \sim \infty \, !$$

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- Not really a surprise, in more than $1+1$ dimensions, we need renormalization.

The strategy

$$\sum_{l=0}^{\infty}\Omega(l)S_l(\infty)=\sum_{l=0}^{L}\Omega(l)S_l(\infty)+\sum_{l=L+1/2}^{\infty}\Omega(l)S_l(\infty)=\Gamma_{\mathrm{Low}}+\Gamma_{\mathrm{High}}$$

Low-modes : $\Rightarrow$ GY Theorem (numerical solution)
High-modes: $\Rightarrow$ WKB series (analytic calculation), perform renormalization.

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$$\frac{\det(\mathcal{H}_l + m^2)}{\det(\mathcal{H}_l^0 + m^2)} = -\int_0^{\infty} \frac{ds}{s}\, e^{-m^2 s} \int_0^{\infty} dr \{\Delta_l(r, r; s) - \Delta_l^0(r, r; s)\}$$

where $\Delta_l(r, r'; s) \equiv \langle r | e^{-s\mathcal{H}_l} | r' \rangle$.

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where $\Delta_l(r,r';s)\equiv\langle r|e^{-s\mathcal{H}_l}|r'\rangle$.

$$\Delta_l(r,r;s)=\frac{e^{-s\mathcal{V}_l(r)}}{\sqrt{4\pi s}}\left[1+\left(\frac{s^3}{12}(\mathcal{V}_l'(r))^2-\frac{s^2}{6}\mathcal{V}_l''(r)\right)+\cdots\right]$$

where $\mathcal{V}_l(r)$ includes a centrifugal term that depends on l.

The calculation

- First we perform the infinite sum over the angular momentum l. We use the Euler-Maclaurin formula for this:

$$\sum_{n=a}^{b} f(n) = \int_a^b f(x)dx + \frac{f(a)+f(b)}{2} + \cdots$$

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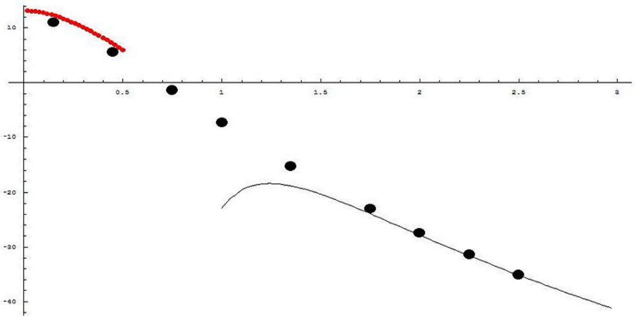
- We are left with an integral over dr

$$\Gamma^{\mathrm{ren}}_{\mathrm{High}} = \int_0^\infty dr \left(Q_{log}(r) \ln L + \sum_{n=0}^{2} Q_n(r) L^n + \sum_{n=1}^{N} Q_{-n}(r) \frac{1}{L^n} \right) + \mathcal{O}\left(\frac{1}{L^{N+1}}\right)$$

The result: $\Gamma^{\mathrm{ren}} = \Gamma_{\mathrm{Low}} + \Gamma^{\mathrm{ren}}_{\mathrm{High}} < \infty$



- This shows an example of $\Gamma^{\mathrm{ren}}(m)$ for Scalar QED, with

$$g(r) = B(1 - \mathrm{Tanh}[\beta\sqrt{B}r - \xi]) \quad ; \{B = 1,\ \beta = 1,\ \xi = 3\}$$

- The small-mass expansion (red), the large-mass expansion (line), our method (dots).

Outline

General aspects, computation

The method

New results

The spinor case

- The partial-wave decomposition is more subtle for Spinors.
- We must account for zero-modes.
- Our method allows $A_\mu(r) = \eta^3_{\mu\nu} x_\nu g(r)$ with arbitrary $g(r)$. The fall rate of $g(r)$ determines the existence of zero-modes.
- We can aim to find properties that are independent of the precise form of $g(r)$.

One application

$$
\begin{aligned}
\mathcal{L}^{(1)}_{\text{spinor}}(a,b) &= -\frac{1}{8\pi^2}\int_0^\infty \frac{ds}{s^3}\, e^{-m^2 s}\Big\{(ab)s^2 \coth(as)\coth(bs) \\
&- 1 - \frac{s^2}{3}(a^2 - b^2)\Big\}
\end{aligned}
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where $a^2 - b^2 = -\frac{1}{2}F_{\mu\nu}F^{\mu\nu}$ and $ab = -\frac{1}{4}F_{\mu\nu}\tilde{F}^{\mu\nu}$.

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- The *derivative expansion* gives ($m \to 0$):

$$\mathcal{L}^{(1)}_{\text{spinor}}(a,b) \sim \frac{1}{48\pi^2}[(a+b)^2 - 5(a-b)^2]\ln m + [\,\text{finite}\,] + [\,\text{terms that vanish as } m \to 0\,]$$

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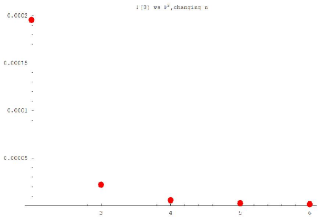
- However : $[\,\text{finite}\,] = f(a,b)$ remains unknown.

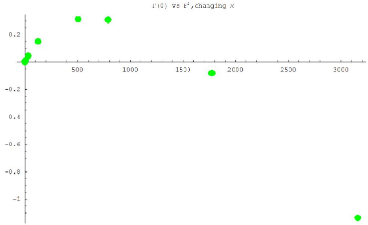
Some tests

- Set $m = 0$, and consider

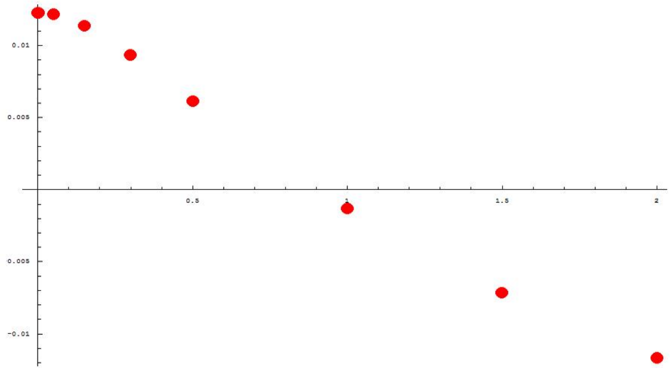
$$g(r) = \frac{\kappa}{(1 + r^2)^n}$$

- Now we may observe $\Gamma^{\text{ren}}(m = 0)$ as a function of n (red) or κ (green) :





Conclusion



- We can now accurately calculate $\Gamma(m)$ for spinor abelian theories, in radially symmetric backgrounds.
- There are interesting questions to investigate, now within our reach.
- Thanks ∎