Noncommutative Gravity from Gauge Theory

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Motivation

space-time deformation: \([x^\mu, x^\nu] = \theta^{\mu\nu}\)

This deformation induce a deformation of the algebra of fields

\[\phi \psi \rightarrow \phi \star \psi\]

where

\[(\Phi \star \Psi)(x) = \Phi(x)e^{\frac{i}{2}\theta^{\mu\nu}\overrightarrow{\partial}_\mu \overrightarrow{\partial}_\nu} \Psi(x)\]

so, for instance

\[S^\theta = \int d^4x \left[ \frac{1}{2} \overrightarrow{\partial}_\mu \Phi \star \partial_\mu \Phi - \frac{1}{2} m^2 \Phi \star \Phi - \frac{\lambda}{4!} \Phi \star \Phi \star \Phi \star \Phi \star \Phi \right]\]
Context of Noncommutative deformations of gravity

- Use Moyal quantization and replace the usual Algebra of functions with a star Algebra

- Deform de co-product of the bi-Algebra by an abelian twist. Recover tensor calculus.

- Gauging the noncommutative version of SO(4,1). Use SW map and contract to ISO(3,1)

- Restrict diffeomorphism to a subgroup. Unimodular Gravity. Symplectomorphism. Volume preserving transformations

Star products are NOT invariant under diffeomorphism transformations
Context of Noncommutative deformations of gravity

- Teleparallel version of Gravity
- Dimensional reduction of noncommutative YM theory
- Gauging the twist-deformed Poincaré symmetry
- Position dependent noncommutative parameter
- Extract General Relativity from the dynamics of gauge fields
- Matrix Models, emergent Gravity

Sazbo, Landsmann...
Wess, Chaichian, Calmet......
Harikumar, Rivelles, Aschieri ......
Steinacker, Grosse,..
Central Idea

Two deformations:

A) Consistent deformation of a Gauge Theory $\kappa$

B) Noncommutative deformation $\theta$

Gauge Theory $\rightarrow$ A def $\rightarrow$ DGT $\rightarrow$ Gravity

Gauge Theory $\rightarrow$ B def $\rightarrow$ NCFT $\rightarrow$ A def $\rightarrow$ NCG

NCFT are trivial A def (SW map)

Promote the global translation symmetry to a local symmetry (teleparallel gravity)  

Cho, Hehl...
New deformation of Gravity

Consistent deformation of YM theory: non-polynomial in the gauge fields

merging properties that comes from the original gauge symmetries and space-time symmetries related with the idea of emergent gravity

general relativity itself will emerge from our deformation as a gauge theory of space-time translations: teleparallel gravity

deformation constructed using a killing vector of the flat space background metric

NC deformation enters here through the Seiberg-Witten map of the standard Yang-Mills action.
New deformation of Gravity

- The deformed action can be cast in the form of a Yang-Mills action in a curved space-time whose metric depends in a given way on the gauge fields and the killing vectors of the background metric.

- Minimal construction: gauge the translation symmetry of the background

- Constructed from a standard noncommutative deformation of a Gauge Theory

- It is not clear that the resulting action can be reorganized in the form of a Moyal star product deformation
Toy Model: Maxwell Case

**FIRST DEFORMATION**

**Action:**

\[
S^{(0)} = -\frac{1}{4} \int d^4 x F_{\mu\nu} F^{\mu\nu},
\]

Local

\[
\delta^{(0)}_\lambda A_\mu = \partial \lambda,
\]

Global

\[
\delta A_\mu = \xi^\nu F_{\nu\mu},
\]

Killing eq.:

\[
\partial_\mu \xi_\nu + \partial_\nu \xi_\mu = \frac{1}{2} \eta_{\mu\nu} \partial_\rho \xi^\rho,
\]

Start deformation from:

\[
S^{(1)} = \int d^4 x A_\mu \xi^\nu \left(-\frac{1}{4} \delta^\mu_\nu F^\rho_\sigma F^{\rho\sigma} + F_{\nu\rho} F^{\mu\rho}\right).
\]
Toy Model: Maxwell Case

FIRST DEFORMATION

Consistent deformation condition:

$$\delta^{(0)} \lambda S^{(1)} + \delta^{(1)} \lambda S^{(0)} = 0$$

Solution:

$$\delta^{(1)} \lambda A_\mu = \lambda \xi^\nu F_{\nu\mu},$$

Deformation can be constructed to any order in the deformation parameter!!

$$L = -\frac{1}{4} (1 + \xi^\rho A_\rho) \hat{F}_{\mu\nu} \hat{F}^{\mu\nu}, \quad \delta \lambda A_\mu = \partial_\mu \lambda + \lambda \xi^\nu \hat{F}_{\nu\mu},$$

where:

$$\hat{F}_{\mu\nu} = E^\rho_\mu E^\sigma_\nu F_{\rho\sigma}, \quad E^\rho_\mu = \delta^\rho_\mu - \frac{\xi^\rho A_\mu}{(1 + \xi \cdot A)}.$$
Toy Model: Maxwell Case

FIRST DEFORMATION

“Geometric version”:

\[ L = -\frac{1}{4} \sqrt{-g} g^{\mu \rho} g^{\nu \sigma} F_{\mu \nu} F_{\rho \sigma} \]

where:

\[ g_{\mu \nu} = \eta_{\alpha \beta} e_{\mu}^{\alpha} e_{\nu}^{\beta}, \quad e_{\mu}^{\alpha} = \delta_{\mu}^{\alpha} + \xi^{\alpha} A_{\mu}, \]

Symmetries: \((\varepsilon^{\mu} = \omega \xi^{\mu})\)

\[ \delta_{\omega} A_{\mu} = \partial_{\mu} \omega + \mathcal{L}_{\varepsilon} A_{\mu} = \partial_{\mu} \omega + \varepsilon^{\nu} \partial_{\nu} A_{\mu} + A_{\nu} \partial_{\mu} \varepsilon^{\nu}, \]

\[ \delta_{\omega} g_{\mu \nu} = \mathcal{L}_{\varepsilon} g_{\mu \nu} = \varepsilon^{\rho} \partial_{\rho} g_{\mu \nu} + g_{\rho \nu} \partial_{\mu} \varepsilon^{\rho} + g_{\mu \rho} \partial_{\nu} \varepsilon^{\rho} \]

\[ \delta_{\omega} e_{\mu}^{\alpha} = \varepsilon^{\rho} \partial_{\rho} e_{\mu}^{\alpha} + (\partial_{\mu} \varepsilon^{\rho}) e_{\rho}^{\alpha} + \frac{1}{2} \omega \Lambda_{\rho}^{\alpha} e_{\mu}^{\rho} \]
Deformation of YM

\[ S_{YM} = \int d^4 x F_{\mu\nu}^A F_{\mu\nu}^A. \]

\[ \delta A_\mu^A = D_\mu \omega^A + \varepsilon^\nu \partial_\nu A_\mu^A + A_\nu^A \partial_\mu \varepsilon^\nu = D_\mu \omega^A + \mathcal{L}_\varepsilon A_\mu^A, \]

\[ \varepsilon^\mu = \omega^A \xi^\mu_A. \]

\[ L = -\frac{1}{4} \sqrt{-g} g^{\mu\rho} g^{\nu\sigma} F_{\mu\nu}^A F_{\rho\sigma}^A = -\frac{1}{4} (1 + \xi^\rho_A A_\rho^A) \hat{F}_{\mu\nu}^A \hat{F}_{\mu\nu}^A, \]

\[ A_\mu^\nu = \xi^\nu_A A_\mu^A, \quad \hat{F}_{\mu\nu}^A = E_\mu^\rho E_\nu^\sigma F_{\rho\sigma}^A, \quad e_\mu^\nu = \delta_\mu^\nu + A_\mu^\nu. \]

\[ g_{\mu\nu} = \eta_{\alpha\beta} e_\mu^\alpha e_\nu^\beta = \eta_{\mu\nu} + A_{\mu\nu} + A_{\nu\mu} + A_{\mu\rho} A_{\nu}^\rho, \]

\[ \delta g_{\mu\nu} = \mathcal{L}_\varepsilon g_{\mu\nu} !!!! \]
Relation with standard Gravity

Consider only the translation subgroup $T_4$

Define:

$$F_{\mu\nu}^\rho = \xi_A^\rho F_{\mu\nu}^A = \partial_\mu A_\nu^\rho - \partial_\nu A_\mu^\rho = \partial_\mu e_\nu^\rho - \partial_\nu e_\mu^\rho.$$  

Relate them with the Ricci Rotation coefficients

$$T_{\mu\nu}^\rho \sim \Omega_{\mu\nu}^\rho \quad \Omega_{\sigma\kappa}^\rho = \hat{F}_{\sigma\kappa}^\rho = E_\sigma^\mu E_\kappa^\nu F_{\mu\nu}^\rho.$$  

$$L = -\frac{1}{4} (1 + \xi_A^\rho A_\rho^A) \hat{F}_{\mu\nu}^A \hat{F}_{\mu\nu}^A, \quad \rightarrow \quad L = -\frac{1}{4} e \Omega_{\mu\nu}^\rho \Omega_{\mu\nu}^\rho.$$
Relation with standard Gravity

Invariants:

\[ I_1 = \Omega_{\mu\nu\rho}\Omega^{\mu\nu\rho}, \quad I_2 = \Omega_{\mu\nu\rho}\Omega^{\rho\mu\nu}, \quad I_3 = \Omega_{\mu\rho}\Omega^{\mu\sigma}\Omega^{\rho\sigma}. \]

Pellegrini-Plebanski Lagrangian

\[ L = ec^i I_i \]

Recover full Einstein Gravity for:

\[ c_1 = 1, c_2 = 2, c_3 = -4 \]

\[ L = e(\Omega_{\mu\nu\rho}\Omega^{\mu\nu\rho} + 2\Omega_{\mu\nu\rho}\Omega^{\rho\mu\nu} - 4\Omega_{\mu\rho}\Omega^{\mu\sigma}\Omega^{\rho\sigma}) \]
NC deformed gauge theory

**NC-YM Lagrangian**

\[ L = -\frac{1}{4} \epsilon^{\mu\rho\sigma\nu} F_{\mu\nu}^A \star F_{\rho\sigma}^A \]

\[ F_{\mu\nu}^A = \partial_\mu A_\nu^A - \partial_\nu A_\mu^A - [A_\mu, A_\nu]^A \]

**Seiberg-Witten map:**

\[ F_{\mu\nu}^C \rightarrow F_{\mu\nu}^C + \frac{1}{2} \theta^{\alpha\beta} d^{ABC} \left( F_{\mu\alpha}^A F_{\nu\beta}^B - A_\alpha^A \partial_\beta F_{\mu\nu}^B + \frac{1}{2} f_{BDE} A_\alpha^A A_\beta^E F_{\mu\nu}^D \right) \]

**Seiberg-Witten Lagrangian up to first order:**

\[ L = -\frac{1}{4} \left( \text{Tr}(F_{\mu\nu} F^{\mu\nu}) + \theta^{\alpha\beta} d_{ABC} F^{A\mu\nu} \left( \frac{1}{4} F_{\beta\alpha}^B F_{\mu\nu}^C + F_{\mu\alpha}^B F_{\nu\beta}^C \right) \right) \]
NC deformed gauge theory

NC action ansatz:

\[
L = -\frac{1}{4} (1 + \xi_A A^A) \left( \text{Tr}(\hat{F}_{\mu\nu} \hat{F}^{\mu\nu}) + \theta^{\alpha\beta} d_{ABC} \hat{F}^A_{\mu\nu} \left( \frac{1}{4} \hat{F}^B_{\beta\alpha} \hat{F}^C_{\mu\nu} + \hat{F}^B_{\mu\alpha} \hat{F}^C_{\nu\beta} \right) \right),
\]

Symmetries:

\[
\delta_w A^A_\mu = D_\mu \omega^A + \omega^B \varepsilon^{\nu}_{\beta} \hat{F}^A_{\nu\mu},
\]

\[
\delta A^A_\mu = D_\mu \omega^A + \varepsilon^{\nu}_{\beta} \partial_\nu A^A_\mu + A^A_\nu \partial_\mu \varepsilon^{\nu} = D_\mu \omega^A + \mathcal{L}_\varepsilon A^A_\mu
\]

where

\[
\varepsilon^\mu = \omega^A \xi^\mu_A
\]

condition:

\[
\omega^A \mathcal{L}_{\xi_A} \theta^{\mu\nu} = 0
\]

Rigid frames

\[
\delta e^{\mu\nu}_\mu = \varepsilon^\rho \partial_\rho e^{\mu\nu}_\nu + e^{\nu}_{\rho} \partial_\mu \varepsilon^\rho
\]
NC Gravity from NC deformed Gauge Theory

NC action ansatz:

\[ L = - \frac{1}{4} (1 + \xi_A^\rho A^A_\rho) \left( \text{Tr}(\hat{F}_{\mu\nu}\hat{F}^{\mu\nu}) + \theta^{\alpha\beta} d_{ABC} \hat{F}^{A\mu\nu} \left( \frac{1}{4} \hat{F}^B_\beta \hat{F}^C_{\mu\nu} + \hat{F}^B_{\mu\alpha} \hat{F}^C_{\nu\beta} \right) \right) , \]

Dictionary:

\[ \mathcal{F}_{\mu\nu}^\rho = \xi_A^\rho F_{\mu\nu}^A = \partial_\mu A_\nu^\rho - \partial_\nu A_\mu^\rho = \partial_\mu e_\nu^\rho - \partial_\nu e_\mu^\rho \]

\[ \Omega_{\sigma\kappa}^\rho = \hat{F}_{\sigma\kappa}^\rho = E_\sigma^\mu E_\kappa^\nu \mathcal{F}_{\mu\nu}^\rho \]

Lagrangian (invariant \( I_1 \))

\[ L_1 = e \left( \Omega_{\mu\nu}^\rho \Omega^{\mu\nu}_\rho + \theta^{\alpha\beta} d_{\rho\sigma\delta} \Omega^{\rho\mu\nu} \left( \frac{1}{4} \Omega_\beta^\sigma \Omega_\mu^\delta + \Omega_\mu^\sigma \Omega_\nu^\delta \right) \right) , \]
NC Gravity from NC deformed Gauge Theory

To construct the other invariants:

$$\Omega_{\mu\nu}^\rho \rightarrow \Omega_{\mu\nu}^\rho + (\Omega_{NC})_{\mu\nu}^\rho$$

$$(\Omega_{NC})_{\mu\nu}^\rho = \theta^{\alpha\beta} d_{\sigma\delta}^\rho \left( \frac{1}{4} \Omega_{\beta\alpha}^\sigma \Omega_{\mu\nu}^\delta + \Omega_{\mu\alpha}^\sigma \Omega_{\nu\beta}^\delta \right)$$

Then:

$$L_2 = e \left( \Omega_{\mu\nu\rho} \Omega_{\rho\mu\nu} + \Omega_{\rho\mu\nu} (\Omega_{NC})_{\mu\nu\rho} + \Omega_{\mu\nu\rho} (\Omega_{NC})_{\rho\mu\nu} \right).$$

and the trace:

$$L_3 = e \left( \Omega_{\mu} \Omega_{\mu} + 2\Omega_{\mu} (\Omega_{NC})_{\mu} \right).$$

$$L_{NCG} = L + 2e \left( \Omega_{\rho\mu\nu} (\Omega_{NC})_{\rho\mu\nu} + \Omega_{\rho\mu\nu} (\Omega_{NC})_{\mu\nu\rho} + \Omega_{\mu\nu\rho} (\Omega_{NC})_{\rho\mu\nu} - 4\Omega_{\mu} (\Omega_{NC})_{\mu} \right).$$
CONCLUSIONS

- NC Gravity from a Gauge theory
- NC come from standard SW map
- Deformed NC: formal series in $\theta^{\mu\nu}$
- The result could not be recast as the know star product

Future work:
- Linearized theory, FP Lagrangian
- Phenomenological implications???
- Are the vertices of this NCG the same that come from String Theory???