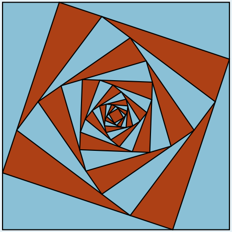

The local structure of extra dimensional theories

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Extra dimensional field theory

We live in $3 + 1$ dimensions

A salient feature of the universe

- What fixes $D = 3$? could it be otherwise? $\Rightarrow$ presumably, a complete theory of quantum gravity needed to answer.
- How to find out if there are XD?
- Energy conservation in $3 + 1$ to high accuracy: either the XD are curled up to minuscule scales $(\sim m_P)$ or the Standard Model particles are not free to traverse them,
- Possible exceptions: gravitons and right neutrinos, (SM singlets).

The main idea

At some small scale we can find ourselves a flat patch of spacetime, where the symmetry group is the $(D+1)$–dimensional Poincaré. Under the (unknown, but presumibly unitary) dynamics of the full theory, this patch contracts to a product with local structure

$$\mathcal{M}_{D+1} \Rightarrow \mathcal{M}_{3+1} \times \mathcal{K}_{D-3}.$$

A possible program for *top-down* model–building

- Construct the higher-dimensional quantum field theory.
- Correlate its structure with a $3+1$ QFT plus suppressed higher dimensional objects.
- Search for imprints of the extra dimensions in small corrections in the contraction parameters.

The main idea

To construct a QFT we impose

- Poincaré invariance of the scattering matrix.
- Cluster decomposition.

This implies the following general form for the fields:

$$\psi_l(x) = \int d\Gamma(\kappa e^{ip\cdot x} u_l(\Gamma) a^\dagger(\Gamma) + \lambda e^{-ip\cdot x} v_l(\Gamma) a(\Gamma)),$$

with $a^\dagger(\Gamma)$ and $a(\Gamma)$ creation and annihilation operators for particles in the state $|\Gamma\rangle$.

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The main idea

Symmetries of the construction

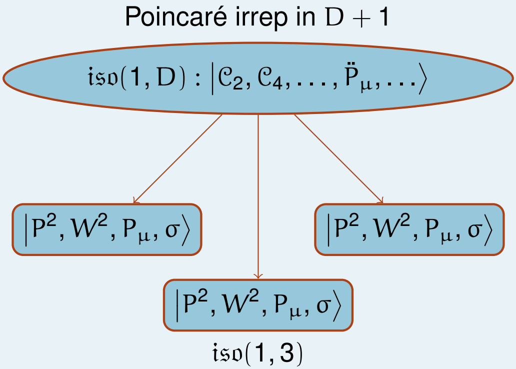
- $\psi_l(x)$ must transform irreducibly under Poincaré.
- $a^\dagger(\Gamma)$ and $a(\Gamma)$ transform like the free states they create and destroy.
- $u_l(\Gamma)$ y $v_l(\Gamma)$ transform in the irreducible representations of the corresponding Lorentz algebra.

To fully determine the fields we build the Poincaré irreducible representations using the Lorentz algebra representations.

The main idea

A roadmap

- Build the Poincaré irreps using the Lorentz irreps in $D+1$ dimensions.
- Formulate the free field theory and try to construct interactions.
- Relate this construction to a $3+1$ dimensional theory.



Some notation

We will be continuously talking about $3+1$ and $D+1$ dimensional objects. I haven't found yet a simple, logical, uncluttered notation for this.

- We will use lowercase Greek letters for spacetime indexes in $3+1$ and lowercase Latin letters for purely spatial indexes :

$(\lambda\mu\nu\rho\sigma\tau)$	run from	0	to	3
$(lmnrst)$	run from	1	to	3

- We will use dotted indexes for the equivalent quantities in $D+1$:

$(\ddot{\lambda}\ddot{\mu}\ddot{\nu}\ddot{\rho}\ddot{\sigma}\ddot{\tau})$	run from	0	to	D
$(\ddot{l}\ddot{m}\ddot{n}\ddot{r}\ddot{s}\ddot{t})$	run from	1	to	D

- Frequently we will dot the operators themselves; this means that all indexes are dotted.

$$\ddot{M}_{\mu}{}^{\nu} \equiv M_{\ddot{\mu}}{}^{\ddot{\nu}}$$

Classification of the Poincaré irreps

The Poincaré algebra (the semidirect product $\mathfrak{t}_D \ltimes \mathfrak{so}(1, D)$) is

$$[\ddot{M}_{\mu\nu}, \ddot{P}_\rho] = -i(\ddot{\eta}_{\mu\rho}\ddot{P}_\nu - \ddot{\eta}_{\nu\rho}\ddot{P}_\mu) \qquad [\ddot{P}_\mu, \ddot{P}_\nu] = 0$$
$$[\ddot{M}_{\mu\nu}, \ddot{M}_{\rho\sigma}] = -i(\ddot{\eta}_{\mu\rho}\ddot{M}_{\nu\sigma} - \ddot{\eta}_{\mu\sigma}\ddot{M}_{\nu\rho} - \ddot{\eta}_{\nu\rho}\ddot{M}_{\mu\sigma} + \ddot{\eta}_{\nu\sigma}\ddot{M}_{\mu\rho})$$

- The generalization of the Poincaré algebra to an arbitrary number of dimensions is direct. $\ddot{P}^2$ is an invariant for every D; the idea of mass makes sense across dimensions.
- The classification of orbits is radically different.
- Free states and well-defined quantum numbers are also different.

Classification of the Poincaré irreps

Because the Poincaré algebra is not semisimple, we have to use the little group method to study its irreps. In $\mathrm{D}+1$ dimensions the algebras for the little groups are as shown in the table.

Little algebras

Orbits	(short) Little algebra
$\ddot{\mathrm{P}}^2 > 0$	$\mathfrak{so}(\mathrm{D})$
$\ddot{\mathrm{P}}^2 = 0$	$\mathfrak{so}(\mathrm{D}-1)$
$\ddot{\mathrm{P}}^2 < 0$	$\mathfrak{so}(\mathrm{D}-1,1)$
$\ddot{\mathrm{P}}^\mu = 0$	$\mathfrak{so}(\mathrm{D},1)$

Table: Little group structure in $\mathrm{D}+1$ dimensions

In $3+1$, the little group for massive particles is $\mathfrak{so}(3)$ which we identify with *spin*.

In $\mathrm{D}+1$ dimensions the little algebra is generated by

$$\ddot{W}_{\lambda\mu\nu} = \sum_{\mathcal{P}(\ddot{\lambda}\ddot{\mu}\ddot{\nu})} \ddot{P}_{\lambda}\ddot{W}_{\mu\nu},$$

the Pauli-Lubanski tensor.

The relevant commutation rules are

$$[\ddot{P}_{\rho}, \ddot{W}_{\lambda\mu\nu}] = 0,$$

$$\begin{aligned}
[\ddot{M}_{\alpha\beta}, \ddot{W}_{\lambda\mu\nu}] &= \eta_{\ddot{\alpha}\ddot{\lambda}}\ddot{W}_{\beta\mu\nu} - \eta_{\ddot{\beta}\ddot{\lambda}}\ddot{W}_{\alpha\mu\nu} + \eta_{\ddot{\alpha}\ddot{\mu}}\ddot{W}_{\lambda\beta\nu} \\
&- \eta_{\ddot{\beta}\ddot{\mu}}\ddot{W}_{\lambda\alpha\nu} + \eta_{\ddot{\alpha}\ddot{\nu}}\ddot{W}_{\lambda\mu\beta} - \eta_{\ddot{\beta}\ddot{\nu}}\ddot{W}_{\lambda\mu\alpha}
\end{aligned}$$

and

$$[\ddot{W}_{\lambda\mu\nu}, \ddot{W}_{\rho\alpha\beta}] = \frac{6}{3!3!} \sum_{\mathcal{P}(\ddot{\lambda}\ddot{\mu}\ddot{\nu})} \sum_{\mathcal{P}(\ddot{\rho}\ddot{\alpha}\ddot{\beta})} \eta_{\ddot{\beta}\ddot{\lambda}} \ddot{P}_{\rho}\ddot{W}_{\alpha\mu\nu}.$$

Invariants of the Poincaré algebra

In all dimensions there are two readily constructed invariants of the algebra:

$$\mathcal{C}_2 = P_{\ddot{\mu}} P^{\ddot{\mu}} \qquad \mathcal{C}_4 = \frac{1}{2}\left(P^{\ddot{\rho}} P_{\ddot{\rho}} M_{\ddot{\mu}\ddot{\nu}} M^{\ddot{\mu}\ddot{\nu}}\right) - P^{\ddot{\mu}} M_{\ddot{\mu}\ddot{\nu}} P_{\ddot{\rho}} M^{\ddot{\rho}\ddot{\nu}}.$$

Depending on the dimension, there can be more invariants.

- The number of independent invariants of an algebra is called the *rank*.

$$\mathrm{rank}(\mathfrak{iso}(1, D)) = \frac{(D+1) + \mathrm{mod}_2(D+1)}{2}.$$

The extra dimensional Lorentz algebra

Independent of dimension we can always construct an invariant of the Lorentz algebra,

$$\mathcal{S} = M_{\ddot{\mu}\ddot{\nu}} M^{\ddot{\mu}\ddot{\nu}}.$$

There can as well be more invariants. For example in $3 + 1$ there is also the invariant

$$\mathcal{S}' = M_{\ddot{\mu}\ddot{\nu}} \widetilde{M}^{\ddot{\mu}\ddot{\nu}}.$$

A tilde means contraction of all indexes with the Levi–Civita tensor.

- The *rank* of the Lorentz algebra $\mathfrak{so}(1, D)$ is

$$\mathrm{rank}(\mathfrak{so}(1, D)) = \frac{(D + 1) - \mathrm{mod}_2(D + 1)}{2}.$$

The procedure to follow

How do we find out the possible values of the invariant operators like $\mathcal{C}_2$, $\mathcal{C}_2$ and $\mathcal{S}$? In other words, how do we construct all representations of the Poincaré and Lorentz algebras?

The Cartan-Weyl decomposition

- This is a standard construction for all finite–dimensional irreducible representations of a semisimple algebra $\mathfrak{g}$, that we can apply to the Lorentz algebra (see the book by Fuchs.)
- For the Poincaré algebra we have to use the little group method. This entails constructing all the representations of the little algebra $\mathfrak{so}(1, \mathrm{D})$ by the previous method.

The Cartan-Weyl algorithm

- Find the maximal abelian subalgebra $\mathfrak{h} = H^i$, (Cartan subalgebra). The dimension of this subalgebra is the *rank* r of $\mathfrak{g}$.
- Diagonalize $\mathfrak{h}$ in the adjoint and find eigenvectors E such that $\forall h \in \mathfrak{h}$,

$$[h, E] = \alpha_E(h)E.$$

- The eigenvalues are r-dimensional vectors useful to label the ladder operators E:

$$[H^i, E^\alpha] = \alpha^i E^\alpha.$$

A simple example: $\mathfrak{sl}(2) \sim \mathfrak{su}(2)$

- The algebra has generators (E^0, E^+, E^-), which satisfy

$$[E^+, E^-] = 2E^0 \quad [E^0, E^\pm] = \pm E^\pm.$$

- The Cartan subalgebra is $\mathfrak{h} = \{E^0\}$ and $E^\pm$ are already eigenvectors. (Equivalent to ladder operators in the angular momentum algebra).
- We are already labeling $E^\pm$ by their eigenvalues 1 and -1.

The Cartan-Weyl algorithm

- Each α is known as a root of $\mathfrak{g}$; the set of all roots is called the root system Φ. Many properties of the algebra are summed in Φ.
- The vector space spanned by the roots is called the root space.We can arbitrarily separate the root system in two pieces, that we will call the positive roots Φ^+ and the negative roots Φ^-.

A simple example: $\mathfrak{sl}(2) \sim \mathfrak{su}(2)$

- The root system is then $\Phi = (1, -1)$. The root space is the line

- We choose $\Phi^+ = (1)$ and $\Phi^- = (-1)$.

The Cartan-Weyl algorithm

- The irreps are constructed from a maximal weight state $|\Lambda\rangle$ such that

$$E^{\alpha}|\Lambda\rangle = 0 \quad \forall \alpha \in \Phi^{+}$$

  and then applying E^{α} with $\alpha \in \Phi^{-}$.

- For each root there is an element H^{α} of $\mathfrak{h}$ such that $(E^{\pm\alpha}, H^{\alpha})$ satisfy a $\mathfrak{sl}()_{\alpha}$ subalgebra:

$$[E^{\alpha}, E^{-\alpha}] = 2H(\alpha),$$

$$[H(\alpha), E^{\pm\alpha}] = \pm E^{\pm\alpha}.$$

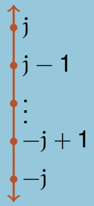
  This is enough to fix the representation of the $E^{\pm\alpha}$.

A simple example: $\mathfrak{sl}(2) \sim \mathfrak{su}(2)$

- The maximal weights states are $|j\rangle$ with j integer or half-intenger. They satisfy

$$E^{+}|j\rangle = 0.$$

- The representations are chains of $2j+1$ states



Fields in $4 + 1$

What this all means for $3+1$

- *Irreps* are labeled by the invariants $\mathcal{C}_2$ and $\mathcal{C}_4$.
- In $3+1$ there are no further invariants.
- $\mathcal{C}_4$ is customarily written in terms of the dual vector to $W_{\lambda\mu\nu}$, the Pauli-Lubanski vector W_μ:

$$W_\mu = \frac{1}{2}\epsilon_{\mu\rho\sigma\tau}P^\rho M^{\sigma\tau}$$

This vector generates the $\mathfrak{so}(3)$ spin algebra in the rest frame.

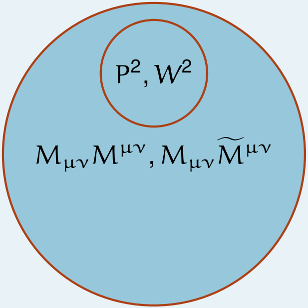


Figure: The labelling quantum numbers

Conventional $3+1$ spacetime

- The Cartan subalgebra is

$$\mathfrak{h}_{1,2} = \frac{1}{2}\left(M_{12} \pm i M_{03}\right).$$

- The root system consists of the vectors

$$\Phi = \{\pm(1,0), \pm(0,1)\}$$

- We can separate the positive roots as

$$\Phi^+ = \{(1,0),(0,1)\}.$$

- Maximal weights (Λ_1, Λ_2) label the irreps.

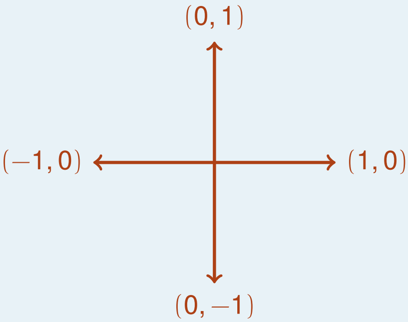


Figure: Root system for $\mathfrak{so}(1,3)$

Conventional $3+1$ spacetime

All of this is exactly the traditional decomposition

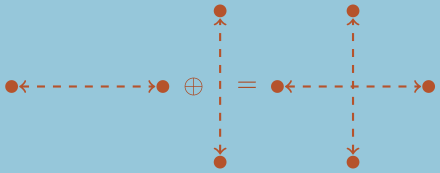
$$H^1 \equiv A_3 = \frac{1}{2}\left(M_{12} + iM_{03}\right); \quad H^2 \equiv B_3 = \frac{1}{2}\left(M_{12} - iM_{03}\right)$$

together with

$$\begin{aligned} A^{\pm} &= \frac{1}{2}\left(M_{23} + iM_{01} \pm M_{02} \pm iM_{31}\right) \\ B^{\pm} &= \frac{1}{2}\left(M_{23} - iM_{01} \mp M_{02} \pm iM_{31}\right). \end{aligned}$$

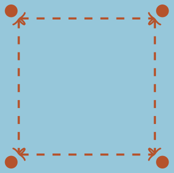
Some important representations of $\mathfrak{so}(1,3)$:

$$\left(\frac{1}{2},0\right)\oplus\left(0,\frac{1}{2}\right)$$



The Dirac representation

$$\left(\frac{1}{2},\frac{1}{2}\right)$$



The vector representation

The algebraic structure of $4+1$ spacetime

In $4+1$ the Poincaré algebra has three invariants. Just as in the conventional $3+1$ case it proves useful to use the Pauli-Lubanski four–vector here we will work with the dual Pauli-Lubanski tensor

$$\ddot{W}_{\mu\nu} = \frac{1}{2}\epsilon_{\ddot{\mu}\ddot{\nu}\ddot{\lambda}\ddot{\sigma}\ddot{\rho}}\ddot{P}^{\lambda}\ddot{M}^{\sigma\rho}.$$

- This antisymmetric tensor has ten independent components.
- Four are zero when acting on rest frame states,

$$\{W_{01}, W_{02}, W_{03}, W_{04}\}$$

- The other six become the $so(4)$ generators

$$W_{\ddot{m}\ddot{n}} = P^{0}\widetilde{M}_{\ddot{m}\ddot{n}}.$$

The algebraic structure of $4+1$ spacetime

Irreducible representations are labeled by the invariant operators $(\mathcal{C}_2, \mathcal{C}_3, \mathcal{C}_4)$.

$$\mathcal{C}_2 = P_{\ddot{\mu}} P^{\ddot{\mu}} \quad \mathcal{C}_3 = \ddot{W}_{\mu\nu} \ddot{M}^{\mu\nu} \quad \mathcal{C}_4 = \ddot{W}_{\mu\nu} \ddot{W}^{\mu\nu}.$$

In terms of $\ddot{P}$ and $\ddot{M}$ these last two are

$$\mathcal{C}_3 = \frac{1}{2} \epsilon_{\ddot{\lambda}\ddot{\mu}\ddot{\nu}\ddot{\rho}\ddot{\tau}} P^{\ddot{\lambda}} M^{\ddot{\mu}\ddot{\nu}} M^{\ddot{\rho}\ddot{\tau}}$$

and

$$\mathcal{C}_4 = \frac{1}{2} \left(P^{\ddot{\rho}} P_{\ddot{\rho}} M_{\ddot{\mu}\ddot{\nu}} M^{\ddot{\mu}\ddot{\nu}} - 2 P^{\ddot{\mu}} M_{\ddot{\mu}\ddot{\nu}} P_{\ddot{\rho}} M^{\ddot{\rho}\ddot{\nu}} \right)$$

One–particle states are labeled by these invariants, plus the Cartan subalgebra eigenvalues; they are of the form

$$|\Gamma\rangle = \left| \mathcal{C}_2, \mathcal{C}_3, \mathcal{C}_4, P^{\ddot{\mu}}, l \right\rangle .$$

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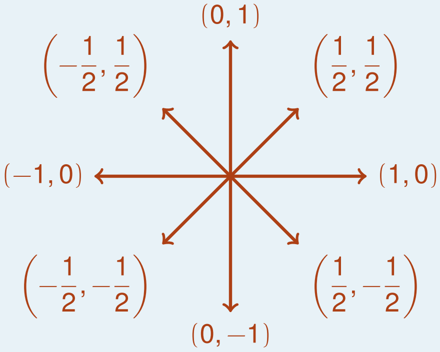
$\mathfrak{so}(1,4)$

The Cartan-Weyl decomposition



Root system of $\mathfrak{so}(1,4)$

- The Cartan subalgebra

$$H^{1,2} = \frac{1}{2}\left(M_{12} \pm i M_{03}\right)$$

- We choose the positive roots to be

$$\left\{(1,0), (0,1), \left(\frac{1}{2}\,\frac{1}{2}\right), \left(\frac{1}{2}, -\frac{1}{2}\right)\right\}$$

The algebraic structure of $4+1$ spacetime

This is the traditional Cartan subalgebra

$$H^1 \equiv A_3 = \frac{1}{2}\left(M_{12} + iM_{03}\right); \quad H^2 \equiv B_3 = \frac{1}{2}\left(M_{12} - iM_{03}\right)$$

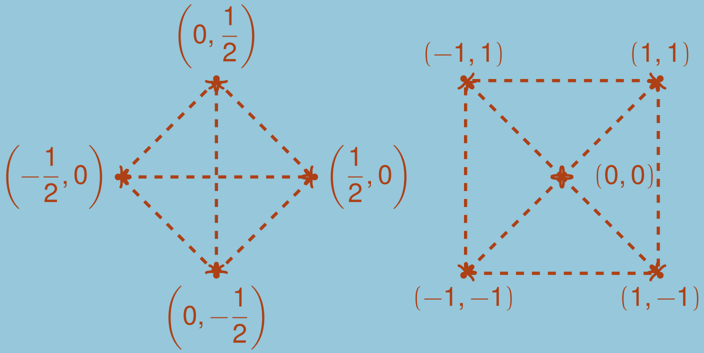
together with the conventional ladder operators

$$\begin{aligned} E^{\pm(1,0)} = A^{\pm} &= \frac{1}{2}\left(M_{13} - iM_{02} \pm iM_{23} \mp M_{01}\right) \\ E^{\pm(0,1)} = B^{\pm} &= \frac{1}{2}\left(M_{23} - iM_{01} \pm M_{02} \mp iM_{13}\right) \end{aligned}$$

plus the extradimensional operators

$$\begin{aligned} E^{\pm(1/2,1/2)} &= C^{\pm} = \sqrt{2}\left(M_{24} \mp iM_{14}\right) \\ E^{\pm(1/2,-1/2)} &= D^{\pm} = \sqrt{2}\left(M_{34} \pm M_{04}\right). \end{aligned}$$

Spinor and vector representations of $\mathfrak{so}(1,4)$



The $\mathbf{4}$ and $\mathbf{5}$ representations

- The lowest–dimensional irreducible representations have dimension 4 and 5, corresponding to the highest weights $(0, 1/2)$ and $(1/2, 1/2)$.
- The dimension 4 representation with highest weight $(0, 1/2)$, in particular, corresponds upon projection to $3+1$ dimensions to the Dirac representation.

The explicit matricial representation with our conventions is

$$(M_{23}, M_{31}, M_{12}) = \frac{1}{2}\begin{pmatrix} \vec{\sigma} & 0 \\ 0 & \vec{\sigma} \end{pmatrix} \quad (M_{01}, M_{02}, M_{03}) = -\frac{i}{2}\begin{pmatrix} \vec{\sigma} & 0 \\ 0 & -\vec{\sigma} \end{pmatrix}$$

$$M_{04} = \frac{1}{2}\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \quad (M_{14}, M_{24}, M_{34}) = \frac{1}{2}\begin{pmatrix} 0 & \vec{\sigma} \\ \vec{\sigma} & 0 \end{pmatrix}.$$

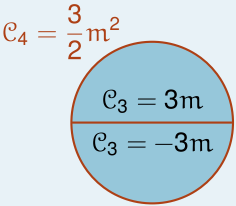
The spinor $\mathbf{4}$

- We have an explicit representation; we can calculate $\mathcal{C}_3$ and $\mathcal{C}_4$.
- The second one is trivial:

$$\mathcal{C}_4 = -\frac{3}{16}\delta_{\ddot{\mu}\ddot{\nu}\ddot{\rho}\ddot{\sigma}\ddot{\tau}\ddot{\lambda}} P^{\ddot{\mu}} P^{\ddot{\nu}} [\gamma^{\ddot{\rho}}, \gamma^{\ddot{\sigma}}][\gamma^{\ddot{\tau}}, \gamma^{\ddot{\lambda}}]$$
$$\Rightarrow \mathcal{C}_4 = \frac{3}{2}\mathcal{C}_2 \mathbf{1}_{4\times 4}.$$

- On the other hand, the cubic invariant is

$$\mathcal{C}_3 = -\frac{1}{32}\epsilon_{\ddot{\lambda}\ddot{\mu}\ddot{\nu}\ddot{\rho}\ddot{\sigma}} P^{\ddot{\lambda}} [\gamma^{\ddot{\mu}}, \gamma^{\ddot{\nu}}][\gamma^{\ddot{\rho}}, \gamma^{\ddot{\sigma}}]$$
$$\Rightarrow \mathcal{C}_3 = 3\ddot{\not{P}}.$$



Poincaré content of $\mathbf{4}$

Poincaré content

We have two species of particles in this representation;

$$\mathcal{C}_3 \left|\Psi^+\right\rangle = 3m \left|\Psi^+\right\rangle \quad \text{and} \quad \mathcal{C}_3 \left|\Psi^-\right\rangle = -3m \left|\Psi^-\right\rangle$$

They both satisfy

$$\mathcal{C}_4 \left|\Psi^\pm\right\rangle = \frac{3}{2}\mathcal{C}_2 \left|\Psi^\pm\right\rangle$$

These are not chiral states, but they are good-parity states. They correspond to the solutions of the $4+1$ Dirac equation

$$(\ddot{\not{P}} \pm m)\psi = 0.$$

The vector 5

- We likewise can calculate $\mathcal{C}_3$ and $\mathcal{C}_4$; we have a representation

$$\ddot{M}_{\mu\nu}{}^{\alpha}{}_{\beta} = i(\ddot{\eta}_{\mu}{}^{\alpha}\ddot{\eta}_{\nu\beta} - \ddot{\eta}_{\nu}{}^{\alpha}\ddot{\eta}_{\mu\beta})$$

- The cubic invariant is trivial

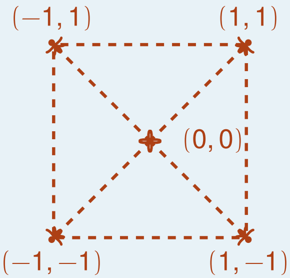
$$\mathcal{C}_3{}^{\alpha}{}_{\beta} = -\tfrac{1}{2}\ddot{P}^{\ddot{\lambda}}\epsilon_{\ddot{\lambda}\ddot{\mu}\ddot{\nu}\ddot{\rho}\ddot{\sigma}}\ddot{M}_{\mu\nu}{}^{\alpha}{}_{\gamma}\ddot{M}_{\mu\nu}{}^{\gamma}{}_{\beta}$$
$$\Rightarrow \mathcal{C}_3 = 0.$$

- On the other hand, the quartic invariant is

$$\mathcal{C}_4{}^{\alpha}{}_{\beta} = 3(\ddot{P}^2\ddot{\eta}^{\alpha}{}_{\beta} - \ddot{P}^{\alpha}\ddot{P}_{\beta})$$

  This has eigenvalues

$$(-3m^2, -3m^2, -3m^2, -3m^2, 0).$$



The 5 representation

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The way back from higher dimensions

To adequately compare $4+1$ and $3+1$ quantities we need to decompose the fields in terms of $3+1$ quantities. We define the operators

$$Q = P^4 \qquad N^\mu = M^{4\mu}.$$

The $4+1$ Poincaré algebra seen from $3+1$

Besides the conventional $3+1$ Poincaré algebra

$$[M_{\mu\nu}, P_\rho] = -i(\eta_{\mu\rho}P_\nu - \eta_{\nu\rho}P_\mu) \quad [P_\mu, P_\nu] = 0$$
$$[M_{\mu\nu}, M_{\rho\sigma}] = -i(\eta_{\mu\rho}M_{\nu\sigma} - \eta_{\mu\sigma}M_{\nu\rho} - \eta_{\nu\rho}M_{\mu\sigma} + \eta_{\nu\sigma}M_{\mu\rho})$$

we get for the (N_μ, Q) commutation rules

$$[M_{\mu\nu}, Q] = 0 \quad [P_\mu, Q] = 0 \quad [N_\mu, Q] = iP_\mu \quad [N_\mu, N_\nu] = iM_{\mu\nu}$$
$$[M_{\mu\nu}, N_\rho] = -i(\eta_{\mu\rho}N_\nu - \eta_{\nu\rho}N_\mu) \quad [P_\mu, N_\nu] = -i\eta_{\mu\nu}Q$$

The most well–known consequence of extra dimensions is the presence of a Kaluza-Klein tower of massive states.

- $\ddot{P}^2$ is an invariant for any dimension.
- The extra dimensions are supposed to be compact; translations are quantized.
- For an additional dimension, P^4 takes discrete values (say)

$$Q = \frac{n}{R}$$

Then the invariant $\mathcal{C}_2$

$$\mathcal{C}_2 = \ddot{P}_\mu \ddot{P}^\mu = P^\mu P_\mu - Q^2 \Rightarrow P^2 = \mathcal{C}_2 + \frac{n^2}{R^2}.$$

Quantization of the momenta has many consequences.

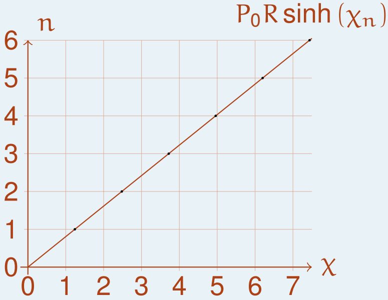
- Boost angles, coming from a rest state, are restricted by:

$$\bar{Q} = \frac{\bar{n}}{R} = -P_0 \sinh \chi$$

- States transform according to

$$\psi_0(p) \Rightarrow \exp\left(-iN_\mu \chi_n^\mu\right) \psi_n(\bar{p})$$

where χ_n is a vector with allowed angles.



Quantization of rapidity

Is there a tower of spin states?

We want the $4+1$ invariants $\mathcal{C}_3$ and $\mathcal{C}_4$ in terms of (M, P, N, Q)

- The cubic invariant is

$$\mathcal{C}_3 = -QM_{\mu\nu}\widetilde{M}^{\mu\nu} + 4N_\mu\widetilde{M}^{\mu\nu}P_\nu$$

- The cuartic invariant is

$$\begin{aligned}\mathcal{C}_4 &= W^2 - \frac{1}{2}Q^2M_{\mu\nu}M^{\mu\nu} + [QN_\mu, P_\nu M^{\nu\mu}] \\ &+ P^\mu P_\mu N^\nu N_\nu - P_\mu N_\mu P^\nu N^\nu + Q[N_\mu, Q]N^\mu.\end{aligned}$$

- For the zero mode ($Q \to 0$) the invariant $\mathcal{C}_4$ must produce the conventional squared Pauli Lubanski. We can write

$$W^2 = W_0^2 + \frac{1}{2} Q^2 M_{\mu\nu} M^{\mu\nu} - [Q N_\mu, P_\nu M^{\nu\mu}] - Q [N_\mu, Q] N^\mu$$

- Equivalently, for the quantized momentum component,

$$W^2 = W_0^2 + \frac{1}{2} Q^2 M_{\mu\nu} M^{\mu\nu} - 3iQ P_\mu N^\mu$$

- For example, in the $\mathbf{4}$ representation this boils down to

$$W_{\mathbf{4}}^2 = \frac{3}{2} P^2 + \frac{3}{2} Q^2 - \frac{3}{2} Q \gamma_5 \not{P}$$

What did we expect?

If the Kaluza-Klein tower was the whole story,the eigenvalues of $\mathcal{W}^2$ should be

$$\mathcal{W}_n^2 \left|\Psi\right\rangle = (\mathcal{C}_2 + Q^2) s(s+1) \left|\Psi\right\rangle$$

- For the fermion representation $\mathbf{4}$ we get instead

$$\mathcal{W}_n^2 \left|\Psi\right\rangle = \left(\frac{3}{4} m^2 + \frac{3}{2} \frac{n^2}{R^2} \pm \frac{3}{2} \frac{nm}{R} \right) \left|\Psi\right\rangle$$

The field theory in extra dimensions

Our expansion for the field has become

$$\psi_l(x) = \int d^4p\ dQ \left\{ e^{ip_\mu x^\mu} e^{iQy} u_l a^\dagger(p,Q) + e^{-ip_\mu x^\mu} e^{-iQy} v_l a(p,Q) \right\}$$

for some field $(\mathcal{C}_2, \mathcal{C}_3, \mathcal{C}_4)$.

The creation/annihilation operators are

$$a^\dagger(\ddot{p}) \left|0\right\rangle = \left|\mathcal{C}_2, \mathcal{C}_3, \mathcal{C}_4 : \ddot{p}_\mu, h^+, h^-\right\rangle$$

with (h^+, h^-) the eigenvalues of the Cartan subalgebra of the little algebra.

We need to be able to relate the $4+1$ states

$$\left|\mathcal{C}_2, \mathcal{C}_3, \mathcal{C}_4 : \ddot{p}_\mu, h^+, h^-\right\rangle$$

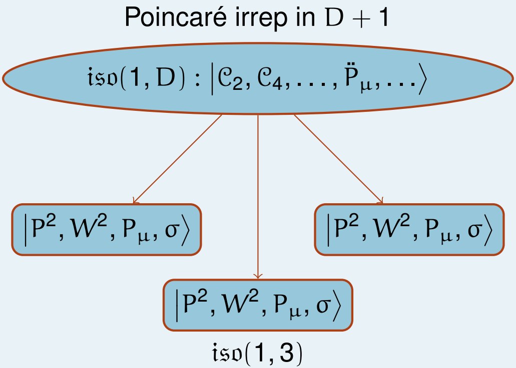
with

$$a^\dagger(p, Q)\,|0\rangle = |\mathfrak{C}_2, \mathfrak{C}_4 : p_\mu, \sigma\rangle$$

where

$$\mathfrak{C}_2 = P_\mu P^\mu \qquad \mathfrak{C}_4 = W_\mu W^\mu.$$

the $3+1$ invariant operators.



Conclusions

- We can learn a lot about the $4+1$ field theory.
- The additional structure given looks promising. Two similar constructions: Snyder spacetime and Vector Supersymmetry (VSUSY).
- We need a better understanding of the $4+1$ quantum numbers.
- There is much to do!