Antiferromagnets at Low Temperatures

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Outline

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Motivation: Universality and efficiency

- Gerber and Leutwyler (1989): Low-temperature expansion of the QCD partition function
- Two-flavor QCD in the chiral limit:
  Spontaneous symmetry breaking of the chiral symmetry
  $SU(2) \times SU(2) \to SU(2) \;\equiv\; O(4) \to O(3)$
- Similar to antiferromagnets: $O(3) \to O(2)$
  $\Rightarrow$ Consider O(N) antiferromagnets: $O(N) \to O(N-1)$ and do the analysis both d=3+1 and d=2+1 space-time dimensions
- Universality of the effective Lagrangian method
- Systematic effective Lagrangian method is more powerful than conventional condensed matter methods

Outline

From the underlying theory to the effective theory

- Construction of effective theories via symmetry analysis, in particular of the spontaneously broken symmetry
- Weinberg (1979): If one writes down the most general possible Lagrangian, including all terms consistent with the assumed symmetries, and then calculates matrix elements with this Lagrangian to any given order of perturbation theory, the result will simply be the most general S-matrix consistent with analyticity, perturbative unitarity, cluster decomposition and the assumed symmetries
- The degrees of freedom in the effective Lagrangian are the Goldstone bosons

Spontaneous symmetry breaking in antiferromagnets

- Heisenberg model:

$$\mathcal{H} = -J \sum_{n.n.} \vec{S}_m \cdot \vec{S}_n \, , \qquad J = const.$$

- $J < 0$: Antiferromagnetic alignment of spins is preferred
- Spontaneous symmetry breaking: $O(3) \Rightarrow O(2)$
- Goldstone's theorem: 2 magnons or spin waves
- Antiferromagnetic magnons display a relativistic dispersion relation much like the pions

Magnon perturbation theory

Spontaneous global $O(3) \Rightarrow O(2)$ spin symmetry breaking:

- 2 Goldstone bosons (magnons) described by unit vector

$$U^i(x) = \left(U^a(x), U^3(x)\right) \in S^2 = O(3)/O(2)$$

with $x = (x_1, x_2, x_3, t)$ or $x = (x_1, x_2, t)$
$a = 1, 2\ (1, 2, \ldots, N-1)$, $i = 0, 1, 2\ (0, 1, 2, \ldots, N-1)$

- Low-energy magnon physics described by nonlinear σ-model

$$\mathcal{L} = \frac{\rho_s}{2}(\partial_r U^i \partial_r U^i + \frac{1}{c^2}\partial_0 U^i \partial_0 U^i) + \ldots$$

ρ_s: spin stiffness c: spin wave velocity
∂_0: Temporal derivative $\partial_r = \vec{\nabla}$

Effective Lagrangian for magnons up to order p^4

- The effective Lagrangian is organized according to the number of space and time derivatives

$$\begin{aligned}
\mathcal{L}_{eff} &= \mathcal{L}^2_{eff} + \mathcal{L}^4_{eff} \\
\mathcal{L}^2_{eff} &= \tfrac{1}{2} F^2 \partial_\mu U^i \partial_\mu U^i - \Sigma_s H^i U^i \\
\mathcal{L}^4_{eff} &= - e_1 (\partial_\mu U^i \partial_\mu U^i)^2 - e_2 (\partial_\mu U^i \partial_\nu U^i)^2 \\
&\quad + k_1 \frac{\Sigma_s}{F^2} (H^i U^i)(\partial_\mu U^k \partial_\mu U^k) - k_2 \frac{\Sigma_s^2}{F^4} (H^i U^i)^2 - k_3 \frac{\Sigma_s^2}{F^4} H^i H^i
\end{aligned}$$

- In a Lorentz-invariant framework ($c = v_s$) there are 7 unknown effective coupling constants up to order p^4 to be determined by experiment or simulation
- External field $H^i = (H, 0, \dots, 0)$
  Staggered magnetization Σ_s

Finite temperature

Partition function is represented as Euclidean functional integral

$$\mathrm{Tr}\,[\exp(-\mathcal{H}/T)] \;=\; \int [\mathrm{d}U]\,\exp\Big(-\int_{\mathcal{T}} \mathrm{d}^4x\,\mathcal{L}_{eff}\Big)\,,$$

where the integration is performed over all field configurations which are periodic in the Euclidean time direction: $U(\vec{x}, x_4+\beta) = U(\vec{x}, x_4)$, with $\beta \equiv 1/T$

The periodicity condition manifests itself in the thermal propagator

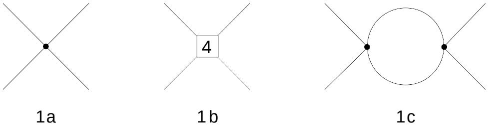
$$G(x) \;=\; \sum_{n=-\infty}^{\infty} \Delta(\vec{x}, x_4 + n\beta)$$

We work in the infinite volume limit

$$z \;=\; -\,T \lim_{L\to\infty} L^{-3}\,\ln\,[\mathrm{Tr}\exp(-\mathcal{H}/T)]$$

Momentum expansion and power counting

- Derivative expansion of the effective Lagrangian corresponds to an expansion in the momenta or temperature
- Example: Goldstone boson scattering



- Tree graph of order p^2 is finite
- Loops in d=3+1 are suppressed by two powers of momentum
- Divergences in one-loop graph 1c of order p^4 are absorbed into coupling constants of order p^4 graph 1b
- At a given order in the derivative expansion only a finite number of diagrams and coupling constants contribute

Remarks on loop suppression

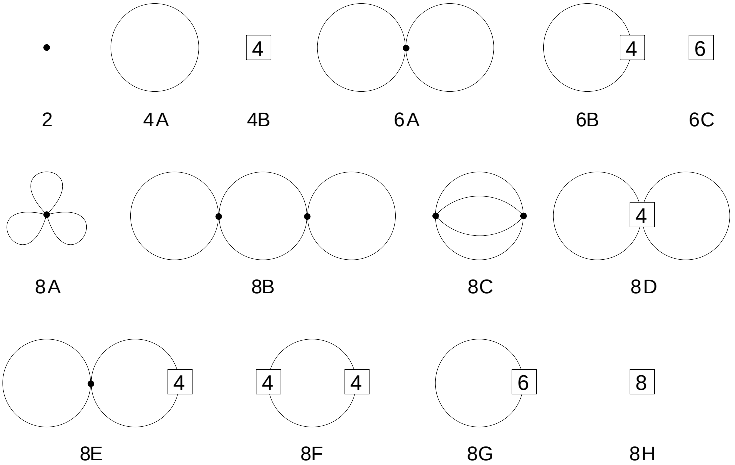
- Consider the Goldstone boson loop

$$\int \frac{d^{d_s+1}P}{P^2} = \int \frac{dE\, d^{d_s}p}{E^2+\vec{p}^2} \propto p^{d_s-1}$$

- Lorentz-invariant framework
  - d=3+1: Loops are suppressed by two powers of momentum
  - d=2+1: Loops are suppressed by one power of momentum
- Nonrelativistic framework: Ferromagnet with $E \propto \vec{p}^2$
  - d=3+1: Loops are suppressed by three powers of momentum
  - d=2+1: Loops are suppressed by two powers of momentum
- While for Lorentz-invariant theories the derivative expansion fails in d=1+1, ferromagnetic spin-chains, e.g., are accessible by effective field theory

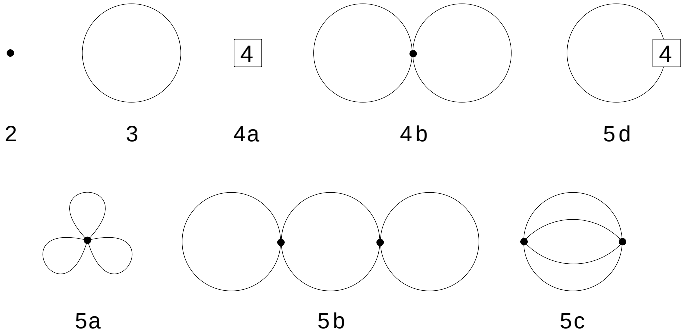
Feynman graphs: $O(N)$ antiferromagnet in $d=3+1$

Low-temperature expansion of the partition function up to three loop-order:



Feynman graphs: O(N) antiferromagnet in $d=2+1$

Low-temperature expansion of the partition function up to three loop-order:



- In 2+1 dimensions the symmetries are more restrictive as fewer effective coupling constants are needed

Pseudo-Lorentz-invariance at order p^2

- Cubic ($d_s = 3$) or square ($d_s = 2$) lattices: Four derivatives are needed to note the difference with respect to a spatially isotropic system
- The discrete 90 degrees rotation symmetry implies O(3) Euclidean space rotation symmetry of $\mathcal{L}^2_{eff}$
- $\mathcal{L}^2_{eff}$ can be brought to Pseudo-Lorentz-invariant form,

$$\tfrac{1}{2} F_1^2 \partial_0 U^i \partial_0 U^i + \tfrac{1}{2} F_2^2 \partial_r U^i \partial_r U^i \Rightarrow \tfrac{1}{2} F^2 \partial_\mu U^i \partial_\mu U^i$$

  with the velocity of light replaced by the spin-wave velocity $v_s = F_2 / F_1$
- Universality: $\mathcal{L}^2_{eff}$ for QCD exhibits the same structure, pions take the role of the magnons

Lorentz-noninvariant terms at order p^4

However, at order p^4 the lattice anisotropies allow the term

$$\sum_{s=1,2} \partial_s \partial_s U^i \, \partial_s \partial_s U^i \, ,$$

which is not O(3) rotation invariant

In a Lorentz-noninvariant framework there are further terms in $\mathcal{L}^4_{eff}$:

$$\Delta U^i \Delta U^i \, , \qquad H^i U^i \partial_r U^k \partial_r U^k$$

However, all these terms merely modify mass renormalization or the dispersion law through diagram 5d,

$$\omega(\vec{k}) = v|\vec{k}| + \mathcal{O}(\vec{k}^3)$$

- The magnon-magnon interaction will not be affected by Lorentz-non-invariance up to the order we are considering

Outline

Low-temperature expansion of the Free energy density

Free energy density of an O(N) antiferromagnet in d=2+1 up to and including three loops:

$$\begin{aligned} z &= -F^2M^2 - \tfrac{1}{2}(N-1)(4\pi)^{-3/2}\,\Gamma(-\tfrac{3}{2})\,M^3 - \tfrac{1}{2}(N-1)\,g_0(M,T) \\ &- (k_2+k_3)\,M^4 + \tfrac{1}{8}(N-1)(N-3)\frac{M^2}{F^2}\,(G_1)^2 \\ &+ \tfrac{1}{48}(N-1)(N-3)(3N-7)\frac{M^2}{F^4}\,(G_1)^3 \\ &- \tfrac{1}{16}(N-1)(N-3)^2\frac{M^4}{F^4}\,(G_1)^2\,G_2 + \tfrac{1}{48}(N-1)(N-3)\frac{M^4}{F^4}\,J_1 \\ &- \tfrac{1}{4}(N-1)(N-2)\frac{1}{F^4}\,J_2 + (N-1)(k_2-k_1)\frac{M^4}{F^2}\,G_1 + \mathcal{O}(p^6) \end{aligned}$$

- Note that the quantities above involve the bare Goldstone boson mass $M^2 = \Sigma_s H/F^2$

The functions g_0, G_1, G_2, J_1, and J_2

- Kinematical functions g_r are associated with the d-dimensional noninteracting Bose gas

$$g_r(M,T) = 2\int_0^\infty \frac{\mathrm{d}\rho}{(4\pi\rho)^{d/2}}\, \rho^{r-1} \exp(-\rho M^2) \sum_{n=1}^{\infty} \exp(-n^2/4\rho T^2)$$

- The quantities G_1 and G_2 are related to the thermal propagator at the origin

$$G_1 \equiv G(x)|_{x=0}\,, \qquad G_2 = -\frac{\mathrm{d}G_1}{\mathrm{d}M^2}$$

- The functions J_1 and J_2 involve integrals over products of four propagators

$$J_1 = \int_{\mathcal{T}} \mathrm{d}^d x \left\{ G(x) \right\}^4$$

$$J_2 = \int_{\mathcal{T}} \mathrm{d}^d x \left\{ \partial_\mu G(x)\, \partial_\mu G(x) \right\}^2$$

Dimensional regularization

Decompose the thermal propagator

$$G(x) = \sum_{n=-\infty}^{\infty} \Delta(\vec{x}, x_4 + n\beta)$$

into a temperature-independent and a temperature-dependent part

$$G(x) = \Delta(x) + \bar{G}(x)$$

In dimensional regularization the zero-temperature propagator reads

$$\begin{aligned} \Delta(x) &= (2\pi)^{-d} \int \mathrm{d}^d p \, e^{ipx} (M^2 + p^2)^{-1} \\ &= \int_0^\infty \mathrm{d}\rho \, (4\pi\rho)^{-d/2} \, e^{-\rho M^2 - x^2/4\rho} \end{aligned}$$

Dimensional regularization

At the origin we then have

$$\begin{aligned} G_1 &= 2M^2\lambda + g_1(M,T) \\ G_2 &= (2-d)\lambda + g_2(M,T) \end{aligned}$$

where the singularity is contained in λ

$$\lambda = \tfrac{1}{2}(4\pi)^{-d/2}\,\Gamma(1-\tfrac{d}{2})\,M^{d-4}$$

Remarkably, in d=2+1 the quantity λ is finite

$$\lambda = -\frac{1}{8\pi M}$$

Regularization of J_1 and J_2

Removing the singularities in the integrals J_1 and J_2:

$$\bar{J}_1 = J_1 - c_1 - c_2 \, g_1(M, T)$$
$$\bar{J}_2 = J_2 - c_3 - c_4 \, g_1(M, T)$$

where the counterterms c_i are singular functions of the space-time dimension d

- c_1 and c_3 renormalize the vacuum energy
- c_2 and c_4 renormalize the Goldstone boson mass

Renormalization of vacuum energy

Collect all contributions in the free energy density that are independent of the temperature:

$$\begin{aligned} z_0 &= -F^2M^2 - \tfrac{1}{12\pi}(N-1)M^3 - (k_2+k_3)\,M^4 \\ &+ \tfrac{1}{128\pi^2}(N-1)(N-3)\frac{M^4}{F^2} - \tfrac{1}{6144\pi^3}(N-1)(N-3)(9N-23)\frac{M^5}{F^4} \\ &+ \tfrac{1}{48}(N-1)(N-3)\frac{M^4}{F^4} - \tfrac{1}{4}(N-1)(N-2)\frac{1}{F^4}\,c_3 \\ &- \tfrac{1}{4\pi}(N-1)(k_2-k_1)\frac{M^5}{F^2} + \mathcal{O}(p^6) \end{aligned}$$

- z_0 renormalizes the vacuum energy

Mass renormalization

Collecting all terms linear in the kinematical functions g_r:

$$
\begin{aligned}
z^{\{1\}} &= -\tfrac{1}{2}(N-1)\,g_0(M,T) + \tfrac{1}{2}(N-1)(N-3)\,\frac{M^4}{F^2}\,\lambda\,g_1(M,T) \\
&- \tfrac{1}{4}(N-1)(N-3)^2\,\frac{M^8}{F^4}\,\lambda^2\,g_2(M,T) \\
&+ (N-1)(k_2-k_1)\,\frac{M^4}{F^2}\,g_1(M,T) \\
&+ \tfrac{1}{48}c_2(N-1)(N-3)\,\frac{M^4}{F^4}\,g_1(M,T) \\
&- \tfrac{1}{4}c_4(N-1)(N-2)\,\frac{1}{F^4}\,g_1(M,T) \\
&+ \tfrac{1}{2}(N-1)(N-3)(2N-5)\,\frac{M^6}{F^4}\,\lambda^2 g_1(M,T)
\end{aligned}
$$

Mass renormalization

Using the property

$$g_{r+1} = -\frac{\mathrm{d}g_r}{\mathrm{d}M^2} \,,$$

of the kinematical functions, one rewrites g_0 in terms of the physical mass, $g_0(M,T) \to g_0(M_\pi,T)$, with

$$M_\pi^2 = M^2 + (N-3)\,\lambda\,\frac{M^4}{F^2} + \left\{2(k_2-k_1) + \frac{b_1}{F^2} + \frac{b_2\,\lambda^2 M^2}{F^2}\right\}\frac{M^4}{F^2} + \mathcal{O}(M^5)$$

The coefficients b_1 and b_2

$$\begin{aligned} b_1 &= \tfrac{1}{24}(N-3)\,\gamma_2 - \tfrac{1}{2}(N-2)\,\gamma_4 \,, \\ b_2 &= (N-3)(2N-5) \,, \end{aligned}$$

involve γ_2 and γ_4 which are singular functions of the space-time dimension d and are related to the counterterms c_2 and c_4 via

$$c_2 = \gamma_2 M^{2d-6} \,, \quad c_4 = \gamma_4 M^{2d-2}$$

Free energy density in terms of renormalized mass M_π

$$
\begin{aligned}
z &= z_0 - \tfrac{1}{2}(N-1)\, g_0 + \tfrac{1}{8}(N-1)(N-3)\, \frac{M_\pi^2}{F^2}\, (g_1)^2 \\
&- \tfrac{1}{128\pi}(N-1)(N-3)(5N-11)\, \frac{M_\pi^3}{F^4}\, (g_1)^2 \\
&+ \tfrac{1}{48}(N-1)(N-3)(3N-7)\, \frac{M_\pi^2}{F^4}\, (g_1)^3 \\
&- \tfrac{1}{16}(N-1)(N-3)^2\, \frac{M_\pi^4}{F^4}\, (g_1)^2\, g_2 + \frac{Q}{F^4} + \mathcal{O}(p^6)\,,
\end{aligned}
$$

with $Q(M_\pi, T)$ defined by

$$
Q \equiv \tfrac{1}{48}(N-1)(N-3)\, M_\pi^4\, \overline{J}_1 - \tfrac{1}{4}(N-1)(N-2)\, \overline{J}_2
$$

- This expression for the free energy density is free of divergences and only involves the physical mass M_π

Free energy density in terms of h_r

For dimensional reasons, the thermodynamic functions are of the form $T^p f(\tau)$, where τ is the dimensionless ratio $\tau = T/M_\pi$

Explicitly, in d=2+1 we have

$$g_0 = T^3 \, h_0(\tau)\,, \quad g_1 = T \, h_1(\tau)\,, \quad g_2 = \frac{1}{T} \, h_2(\tau)\,, \quad Q = T^5 \, q(\tau)$$

The free energy density then reads

$$\begin{aligned} z &= z_0 - \tfrac{1}{2}(N-1) h_0(\tau)\, T^3 + \tfrac{1}{8}(N-1)(N-3)\frac{1}{F^2\tau^2} h_1(\tau)^2\, T^4 \\ &- \tfrac{1}{128\pi}(N-1)(N-3)(5N-11)\frac{1}{F^4\tau^3} h_1(\tau)^2\, T^5 \\ &+ \tfrac{1}{48}(N-1)(N-3)(3N-7)\frac{1}{F^4\tau^2} h_1(\tau)^3\, T^5 \\ &- \tfrac{1}{16}(N-1)(N-3)^2 \frac{1}{F^4\tau^4} h_1(\tau)^2 h_2(\tau)\, T^5 + \frac{1}{F^4}\, q(\tau)\, T^5 + \mathcal{O}(T^6) \end{aligned}$$

Cateye graph

Extraction of the singularities in J_2:

$$\overline{J}_2 = J_2 - c_3 - c_4\, g_1(M,T)$$

Method: Cut out a sphere sphere $\mathcal{S}$ around the origin of radius $|\mathcal{S}| \leq \beta/2$ and decompose J_2

$$J_2 = \int_{\mathcal{S}} \mathrm{d}^d x \left\{ \partial_\mu G(x)\, \partial_\mu G(x) \right\}^2 + \int_{\mathcal{T}\setminus\mathcal{S}} \mathrm{d}^d x \left\{ \partial_\mu G(x)\, \partial_\mu G(x) \right\}^2$$

- In the integral over the complement $\mathcal{T}\setminus\mathcal{S}$ of the sphere, the integrand is not singular and the limit $d \to 3$ can readily be taken
- In the integral over the sphere, insert the decomposition

$$G(x) = \Delta(x) + \bar{G}(x)$$

Cateye graph

$$\begin{aligned} J_2 &= \int_{\mathcal{S}} \mathrm{d}^d x \Bigg(\Big\{ \partial_\mu \bar{G}\, \partial_\mu \bar{G} \Big\}^2 + 4\partial_\mu \bar{G}\, \partial_\mu \bar{G}\, \partial_\nu \bar{G}\, \partial_\nu \Delta \\ &+ 4\partial_\mu \bar{G}\, \partial_\mu \Delta\, \partial_\nu \bar{G}\, \partial_\nu \Delta + 2\partial_\mu \bar{G}\, \partial_\mu \bar{G}\, \partial_\nu \Delta\, \partial_\nu \Delta \\ &+ 4\partial_\mu \bar{G}\, \partial_\mu \Delta\, \partial_\nu \Delta\, \partial_\nu \Delta + \Big\{ \partial_\mu \Delta\, \partial_\mu \Delta \Big\}^2 \Bigg) \end{aligned}$$

- In d=2+1 the first four terms are convergent. However, the last two terms, involving three and four non-thermal propagators, respectively, are divergent

Cateye graph

Disregarding derivatives for a moment, one shows that

$$\int_{\mathcal{S}} \mathrm{d}^d x \, \bar{G} \Delta^3 = g_1 \int_{\mathcal{S}} \mathrm{d}^d x \, ch(Mx_4) \, \Delta^3 \,,$$

and splits the integral over the sphere into two pieces,

$$\begin{aligned} & 4g_1 \int_{\mathcal{S}} \mathrm{d}^d x \, ch(Mx_4) \, \Delta^3 \\ = \quad & 4g_1 \int_{\mathcal{R}} \mathrm{d}^d x \, ch(Mx_4) \, \Delta^3 - 4g_1 \int_{\mathcal{R}\setminus\mathcal{S}} \mathrm{d}^d x \, ch(Mx_4) \, \Delta^3 \end{aligned}$$

- The singularity is now contained in the integral over all Euclidean space, in the form of the counterterm c_2:

$$c_2 = 4 \int_{\mathcal{R}} \mathrm{d}^d x \, ch(Mx_4) \, \Delta^3$$

Cateye graph

The same line of reasoning goes through for the expression $4\partial_\mu \bar{G} \partial_\mu \Delta \, \partial_\nu \Delta \partial_\nu \Delta$, where one ends up with the counterterm

$$c_4 = 4 \int_{\mathcal{R}} \mathrm{d}^d x \, \partial_\mu ch(Mx_4) \, \partial_\mu \Delta \, \partial_\nu \Delta \partial_\nu \Delta$$

As far as the term involving four non-thermal propagators is concerned, it suffices to subtract the temperature-independent integral of $\left\{ \partial_\mu \Delta(x) \partial_\mu \Delta(x) \right\}^2$ over all Euclidean space,

$$c_3 = \int_{\mathcal{R}} \mathrm{d}^d x \left\{ \partial_\mu \Delta \partial_\mu \Delta \right\}^2$$

Net result:

$$\bar{J}_2 = J_2 - c_3 - c_4 \, g_1(M, T)$$

Cateye graph

Collecting the various pieces, the renormalized integral in d=2+1 is

$$\begin{aligned}
\bar{J}_2 &= \int_{\mathcal{T}} d^3x \, T + \int_{\mathcal{T}\setminus\mathcal{S}} d^3x \, U - \int_{\mathcal{R}\setminus\mathcal{S}} d^3x \, \partial_\mu\Delta\partial_\mu\Delta \cdot W \\
T &= \left(\partial_\mu \bar{G} \, \partial_\mu \bar{G}\right)^2 + 4\partial_\mu \bar{G} \, \partial_\mu \bar{G} \, \partial_\nu \bar{G} \, \partial_\nu \Delta + 4\partial_\mu \bar{G} \, \partial_\mu \Delta \, \partial_\nu \bar{G} \, \partial_\nu \Delta \\
&\quad + 2\partial_\mu \bar{G} \, \partial_\mu \bar{G} \, \partial_\nu \Delta \, \partial_\nu \Delta \\
U &= 4\partial_\mu \bar{G} \partial_\mu \Delta \, \partial_\nu \Delta \partial_\nu \Delta + \partial_\mu \Delta \, \partial_\mu \Delta \, \partial_\nu \Delta \, \partial_\nu \Delta \\
W &= 4 g_1 \partial_\mu ch(M x_4) \, \partial_\mu \Delta + \partial_\mu \Delta \, \partial_\mu \Delta
\end{aligned}$$

- This expression involves ordinary, convergent integrals
- $\bar{G}(x)$ and $\Delta(x)$ only depend on $r = |\vec{x}|$ and on $t = x_4$, such that the integrals become two-dimensional $d^3x = 2\pi r \, dr \, dt$
- Numerical consistency check: $\bar{J}_2$ must be independent of the size of the sphere

Outline

Low-temperature expansion of the pressure

Remarkably, for N=3 – the Heisenberg antiferromagnet on a square lattice – most of the terms in the free energy density drop out:

$$z = z_0 - h_0(\tau)\, T^3 + \frac{1}{F^4}\, q(\tau)\, T^5 + \mathcal{O}(T^6) \qquad (N = 3)$$

The pressure is given by the temperature dependent part:

$$P = z_0 - z = h_0(\tau)\, T^3 - \frac{1}{F^4}\, q(\tau)\, T^5 + \mathcal{O}(T^6)$$

- The value $\tau = T/M_\pi$ can take any value, as long as both T and M_π are small compared to the intrinsic scale Λ (J) of the theory
- In particular, the limit $T \gg M_\pi$ is implemented by holding T fixed and sending M_π (or H) to zero
- Mermin-Wagner theorem: No spontaneous symmetry breaking at any finite temperature in the Heisenberg model

Low-temperature expansion of the pressure

The non-trivial dependence of P on the ratio $\tau = T/M_\pi$ is contained in the functions $h_0(\tau)$ and $q(\tau)$. In the limit $T \gg M_\pi$ we have

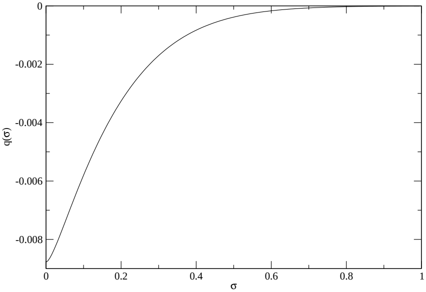
$$h_0^{d=3}(\tau) = \frac{1}{\pi}\Bigg[\zeta(3) - \frac{1}{4}\frac{M_\pi^2}{T^2} + \frac{1}{4}\frac{M_\pi^2}{T^2}\ln\frac{M_\pi^2}{T^2} - \frac{1}{6}\frac{M_\pi^3}{T^3} + \frac{1}{96}\frac{M_\pi^4}{T^4} + \mathcal{O}\Big(\frac{M_\pi}{T}\Big)^6\Bigg]$$

$$q(\tau) = q_1 + q_2\frac{M_\pi^2}{T^2} + \mathcal{O}\Big(\frac{M_\pi}{T}\Big)^4, \qquad \tau = \frac{T}{M_\pi}$$

$$q_1 = -0.008779$$

Low-temperature expansion of the pressure

The function $q(\sigma)$ for N=3 in terms of the dimensionless parameter $\sigma = M_\pi / 2\pi T = 1/2\pi\tau$:



$$P = \frac{\zeta(3)}{\pi} \, T^3 \left[1 - \frac{\pi q_1}{\zeta(3)} \, \frac{T^2}{F^4} + \mathcal{O}(T^3) \right]$$

Thermodynamic quantities

$$s = \frac{\partial P}{\partial T}, \qquad u = Ts - P, \qquad c_V = \frac{\partial u}{\partial T} = T \frac{\partial s}{\partial T}$$

$$\begin{aligned}
u &= \frac{2\,\zeta(3)}{\pi} T^3 \left[1 - \frac{2\pi q_1}{\zeta(3)} \frac{T^2}{F^4} + \mathcal{O}(T^3)\right] \\
&\approx 0.7653\, T^3 \left[1 + 0.04589 \frac{T^2}{F^4} + \mathcal{O}(T^3)\right] \\
s &= \frac{3\,\zeta(3)}{\pi} T^2 \left[1 - \frac{5\pi q_1}{3\zeta(3)} \frac{T^2}{F^4} + \mathcal{O}(T^3)\right] \\
&\approx 1.1479\, T^2 \left[1 + 0.03824 \frac{T^2}{F^4} + \mathcal{O}(T^3)\right],
\end{aligned}$$

Thermodynamic quantities

$$P = \frac{\zeta(3)}{\pi} T^3 \left[1 - \frac{\pi q_1}{\zeta(3)} \frac{T^2}{F^4} + \mathcal{O}(T^3)\right]$$

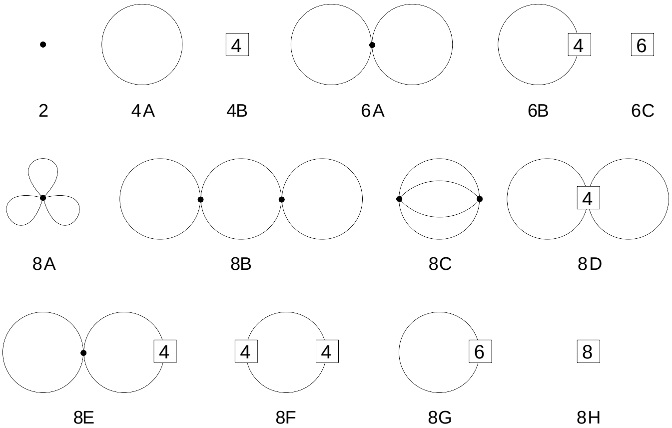
- T^3-term: Free Bose gas contribution (One-loop graph)
- Remarkably, the interaction only manifests itself through a term T^5 (Three loop-graph)
  - Coefficient q_1 is negative, such that interaction is repulsive at low temperatures
  - This statement is independent of the two-dimensional lattice
- No interaction term of order T^4: Two-loop contribution is proportional to N-3

Comparison with the condensed matter literature

- Inconsistencies between spin-wave theory, Schwinger boson mean field theory and Monte Carlo simulations already at leading order T^3 in the free energy density
- No calculation of order T^4 exists
- Calculation of order T^5 beyond the reach of any spin-wave calculation
- Conclusion: The systematic effective Lagrangian method clearly proves to be superior to conventional condensed matter methods, such as spin-wave theory

Power counting and loop suppression

Low-temperature expansion of the partition function up to three loop-order:



- Loops are suppressed by two momentum powers

Low-temperature expansion: General remarks

- $\mathcal{L}^8_{eff}$- and $\mathcal{L}^6_{eff}$-couplings merely renormalize the vacuum energy and the mass
- Effective couplings from the next-to-leading order Lagrangian $\mathcal{L}^4_{eff}$, however, are relevant for the magnon-magnon-interaction in d=3+1
- Loops are suppressed by two momentum powers: The low-temperature expansion of thermodynamic quantities is expected to proceed in steps of powers of T^2
- Logarithmic renormalization of low-energy coupling constants, such that *chiral logarithms* show up in the low-temperature expansion

Low-temperature expansion of the pressure

$$P = \tfrac{1}{2}(N-1)g_0 + 4\pi a\,(g_1)^2 + \pi g\left[b - \frac{j}{\pi^3 F^4}\right] + \mathcal{O}(p^{10}) \qquad (d = 3+1)$$

The temperature dependence is contained in the kinematical functions $g_r(M_\pi, T)$ and in $j(M_\pi, T)$. In the limit $H \to 0$ (or, equivalently, $T \gg M_\pi$), analytical expressions for g_0, g_1 and g can be provided:

$$g_1(M_\pi, T) = \tfrac{1}{12} T^2 \left[1 - \frac{3}{\pi}\frac{M_\pi}{T} + \mathcal{O}\left(\frac{M_\pi^2}{T^2}\ln\frac{M_\pi}{T}\right)\right],$$

$$g(M_\pi, T) = \tfrac{1}{675}\pi^4 T^8 \left[1 - \frac{15}{4\pi^2}\frac{M_\pi^2}{T^2} + \mathcal{O}\left(\frac{M_\pi}{T}\right)^3\right]$$

Low-temperature expansion of the pressure

$$P = \tfrac{1}{2}(N\!-\!1)g_0 + 4\pi a\,(g_1)^2 + \pi g\left[b - \frac{j}{\pi^3 F^4}\right] + \mathcal{O}(p^{10}) \qquad (d = 3{+}1)$$

The function j, containing the cateye graph, has to be evaluated numerically,

$$j = \nu \ln\frac{T}{M_\pi} + j_1 + j_2\,\frac{M_\pi^2}{T^2} + \mathcal{O}\left(\frac{M_\pi}{T}\right)^3, \quad \nu \equiv \frac{5(N-1)(N-2)}{48}$$

- The coefficients j_1 and j_2 are real numbers
- The function $j(\tau)$ diverges logarithmically in the limit $H \to 0$

$$a = -\frac{(N-1)(N-3)}{32\pi}\frac{\Sigma_s H}{F^4} - \frac{(N-1)^3}{256\pi^3}\frac{(\Sigma_s H)^2}{F^8}\ln\frac{H}{H_a},$$
$$b = -\frac{5(N-1)(N-2)}{96\pi^3 F^4}\ln\frac{H}{H_b}$$

Low-temperature expansion for of the pressure

Limit $H \to 0$:

$$P = \tfrac{1}{90}\pi^2(N{-}1)\, T^4 \left[1 + \frac{N-2}{72} \frac{T^4}{F^4} \ln \frac{T_p}{T} + \mathcal{O}(T^6) \right] \qquad (d = 3{+}1)$$

- T^4-contribution represents the free Bose gas term which originates from a one-loop graph
- Effective interaction among the Goldstone bosons, remarkably, only manifests itself through a term of order T^8
- This contribution contains a logarithm, characteristic of the effective Lagrangian method in four space-time dimensions, which involves a scale, T_p, related to H_b
- Divergences in the function j and in the constant cancel, such that the limit $H \to 0$ is well defined

Low-temperature expansion of the pressure

$$P = \tfrac{1}{90}\pi^2(N-1)\,T^4\left[1+\frac{N-2}{72}\,\frac{T^4}{F^4}\,\ln\frac{T_p}{T}+\mathcal{O}(T^6)\right] \qquad (d=3+1)$$

- At low temperatures, the logarithm $\ln[T_p/T]$ is positive, such that the interaction among the Goldstone bosons in d=3+1, in the absence of H, is repulsive, much like in d=2+1
- The symmetries in d=3+1, however, are less restrictive than in d=2+1, because next-to-leading order effective constants from $\mathcal{L}^4_{eff}$ do show up in the scale T_p.
- Still, the symmetry is also rather restrictive in d=3+1, as it unambiguously fixes the coefficient in front of the logarithm in terms of the coupling constant F
- No term of order T^6: Two-loop contribution $\propto a$

N=3: Comparing d=3+1 and d=2+1

$$P = \tfrac{1}{45}\pi^2 T^4 \left[1 + \frac{1}{72}\frac{T^4}{F^4}\ln\frac{T_p}{T} + \mathcal{O}(T^6)\right], \qquad (d = 3+1, N = 3)$$

$$s = \tfrac{4}{45}\pi^2 T^3 \left[1 + \frac{1}{288}\frac{T^4}{F^4}\left(8\ln\frac{T_p}{T} - 1\right) + \mathcal{O}(T^6)\right]$$

$$P = \frac{\zeta(3)}{\pi} T^3 \left[1 - \frac{\pi q_1}{\zeta(3)}\frac{T^2}{F^4} + \mathcal{O}(T^3)\right], \qquad (d = 2+1, N = 3),$$

$$s = \frac{3\,\zeta(3)}{\pi} T^2 \left[1 - \frac{5\pi q_1}{3\zeta(3)}\frac{T^2}{F^4} + \mathcal{O}(T^3)\right]$$

- T-series proceed in steps of T^2 and T, respectively
- No two-loop contribution in the limit $H \to 0$
- Absence of logarithmic scale in d=2+1

Outline

Summary

- Condensed matter systems exhibiting a spontaneously broken symmetry may be analyzed with the fully systematic effective Lagrangian method
- The low-temperature properties of O(N) antiferromagnets are determined by the Goldstone bosons
- The leading order effective Lagrangian is "Lorentz-invariant" – anisotropies only manifest themselves at order $\mathcal{L}^4_{eff}$
- The magnon-magnon interaction in the d=2+1 Heisenberg antiferromagnet is very weak and repulsive, manifesting itself through a term proportional to T^5 in the pressure – the coefficient of this term is completely determined by $\mathcal{L}^2_{eff}$
- The effective Lagrangian method is by far superior to conventional condensed matter methods as it adopts a unified and model-independent view based on symmetry

Outlook

- Incorporation of magnetic and external fields: Order parameter and susceptibilities
- Incorporation of magnetic-dipole and spin-orbit interactions
- Extraction of effective constants via experiment and simulation
- Condensed matter analog of baryon chiral perturbation theory: Doped antiferromagnets which are the precursors of high-temperature superconductors
- Low-energy physics of phonons and phonon-magnon interaction

Cateye graph

$$\bar{J}_2 = \int_{\mathcal{T}\setminus\mathcal{S}} \mathrm{d}^3x\, \widetilde{U} + \int_{\mathcal{S}} \mathrm{d}^3x\, \widetilde{V} - \int_{\mathcal{R}\setminus\mathcal{S}} \mathrm{d}^3x\, \partial_\mu\Delta\partial_\mu\Delta \cdot \widetilde{W}\,,$$

$$\widetilde{U} = \Big(\partial_\mu G\, \partial_\mu G\Big)^2,$$

$$\widetilde{V} = \Big(\partial_\mu \bar{G}\, \partial_\mu \bar{G}\Big)^2 + 4\partial_\mu \bar{G}\partial_\mu \bar{G}\, \partial_\nu \bar{G}\, \partial_\nu \Delta + 2Q_{\mu\mu}\, \partial_\nu\Delta\, \partial_\nu\Delta + 4Q_{\mu\nu}\, \partial_\mu$$

$$\widetilde{W} = \widetilde{w} + 4g_1\partial_\mu ch(Mx_4)\, \partial_\mu\Delta + \partial_\mu\Delta\, \partial_\mu\Delta\,,$$

with

$$\widetilde{w} = \frac{1}{x^2}\Big[(\tfrac{3}{2}x^4 - \tfrac{9}{2}x^2x_4^2 + 9x_4^4)\, g_0^2 + 12M^2x_4^4\, g_0\, g_1 + 2(2M^4x_4^4 + M^4$$

$$Q_{\mu\nu} = \partial_\mu \bar{G}(x)\, \partial_\nu \bar{G}(x) - \bar{G}_{\mu\alpha}\bar{G}_{\nu\beta}\, x_\alpha x_\beta\,,$$

$$\bar{G}_{\mu\nu} = -\tfrac{1}{2}\delta_{\mu\nu}\, g_0 + \delta^4_\mu\delta^4_\nu\, (\tfrac{3}{2}g_0 + M^2g_1)\,.$$