Lattice QCD: From Action to Hadrons

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– QCDSF Collaboration –

QCDSF Collaboration

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Outline

Objective

Lattice

Vacuum

Hadron Spectrum

Nucleon Structure

Conclusions

Objective

- Understanding how the **spectrum** and **structure of hadrons** emerge from QCD is one of the central challenges of Lattice QCD

- Among the key quantities to be studied are
  - Resonances — phase shift analysis
  - Generalized form factors — GFFs
  - Generalized parton distributions — GPDs
  - Distribution amplitudes — DAs
  - Higher twist — Lattice OPE

- Since the cost of full QCD computations in a volume large enough to contain the pion grows with a large inverse power of the pion mass, initial calculations were restricted to relatively heavy pions

- In order for lattice calculations to capture the physics of quarks and gluons in captivity, and reach the needed accuracy requested by the experiments, simulations at physical quark masses, on large volumes and at small lattice spacings are required

- In this talk I shall report on recent progress made in developing a quantitative understanding of nucleon structure

Lattice

Action

$$\mathcal{L}_{QCD} = -\frac{1}{g^2}\mathrm{Tr}\, F_{\mu\nu}(x)F_{\mu\nu}(x) + \sum_f \bar{\psi}_f(x)(\not{D} + m)\psi_f(x) \qquad S_{QCD} = \int \mathrm{d}^4x\, \mathcal{L}_{QCD}$$

Scale invariant at $m = 0$ But the world is not!

The theory needs to be regularized: Regulate the high frequency modes by introducing a momentum cut-off a^{-1}. Then remove the cut-off again by renormalization: $a \longrightarrow 0$ while varying g, m, $\cdots$ so as to keep the low-energy physics constant:

$$\mathcal{L}_{QCD} \;\longrightarrow\; \mathcal{L}_{QCD} \;+\; \mathcal{O}(a)$$

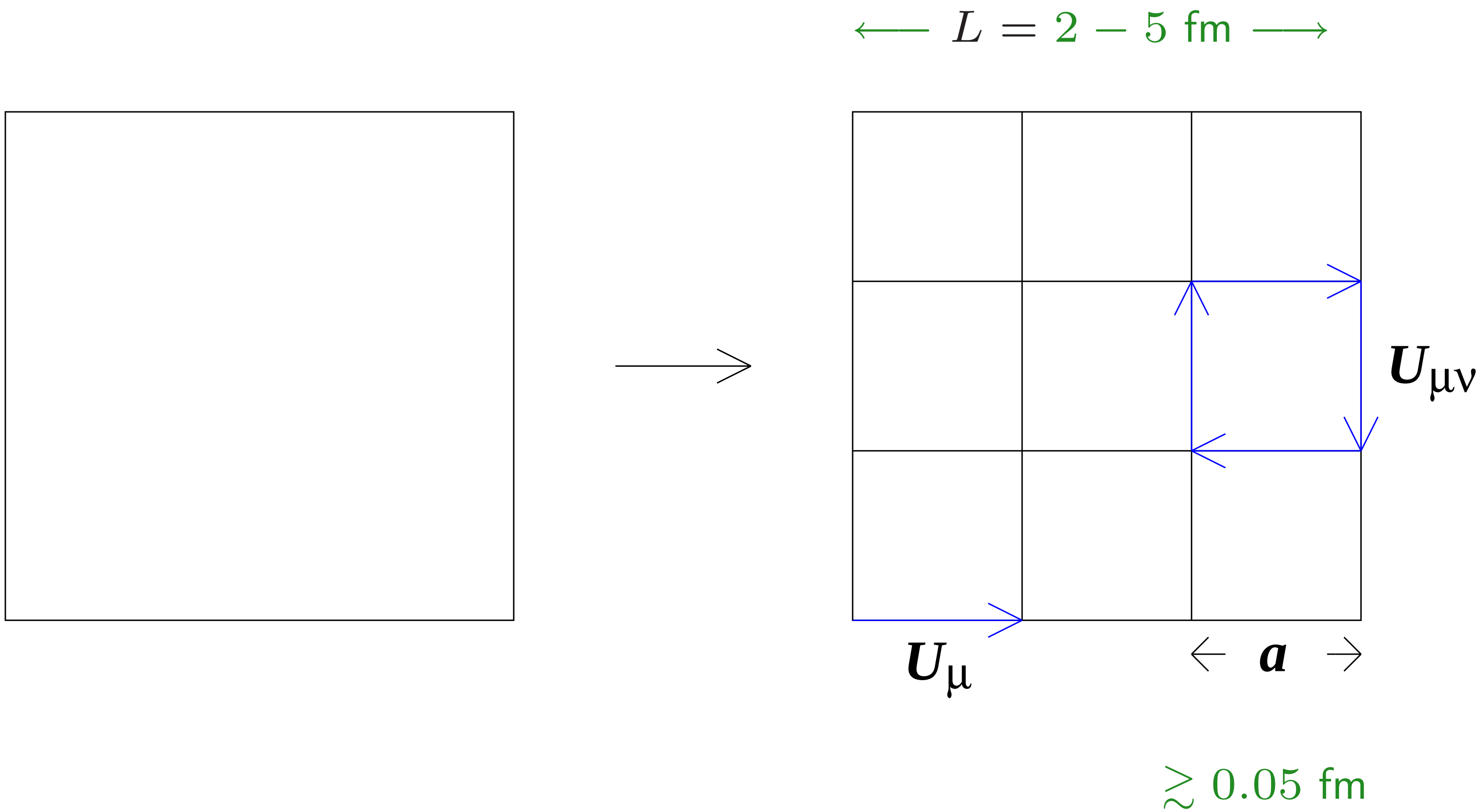
$\uparrow$ log's $\uparrow$ counterterms

a: lattice spacing

This introduces a mass scale   Mass gap

$$\Lambda_L \simeq \frac{1}{a}\exp\Big(-\frac{1}{2b_0\, g^2(a)}\Big), \quad g^2(a) \simeq -\frac{1}{2b_0 \ln(a\Lambda_L)}$$

$$\boxed{m_N, \cdots, \Lambda_{\overline{MS}} \propto \Lambda_L}$$



Counterterms partially being taken into account by improving the action: $S_{QCD} = S_G + S_F$

Remove Lattice at the end of the calculation by extrapolating to $L \longrightarrow \infty$ and $a \longrightarrow 0$ (continuum)

The simulation

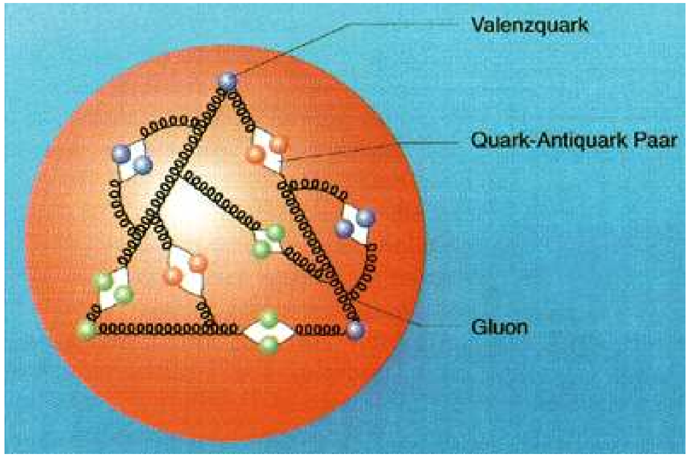
- Generate sequence of configurations $\{U_\mu^{(i)} | i = 1, \cdots, N\}$ with probability (R)HMC

$$\mathcal{P}\{U_\mu^{(i)}\} \propto \int \prod_x \mathcal{D}\bar{\psi}(x)\mathcal{D}\psi(x)\, \exp\{-S_F - S_G\} = \det\left[\not{D}(U_\mu^{(i)}) + am\right]\, \exp\{-S_G\}$$

$$(12\, L^3 T) \times (12\, L^3 T)$$

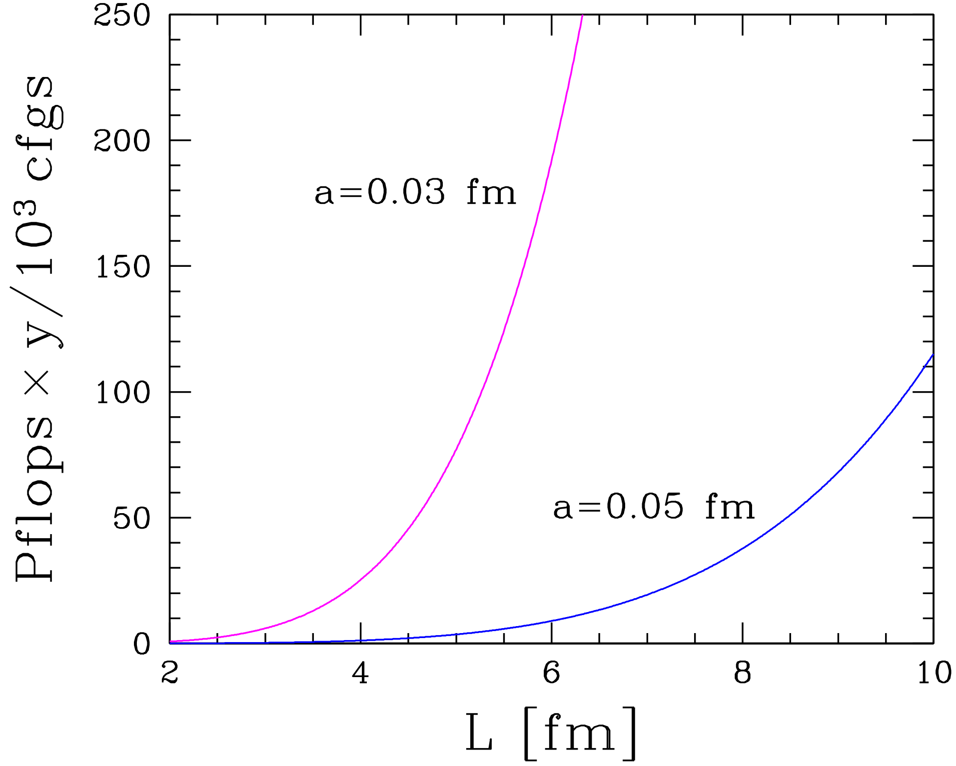
- Compute observable

$$\langle \mathcal{O} \rangle = \frac{1}{N} \sum_{i=1}^{N} \mathcal{O}(U_\mu^{(i)})$$



Costs

$m_\pi = 140$ MeV



10^3 independent configurations

QPACE

$$(4+4) \times 52 \text{ TFlops} = 416 \text{ TFlops} \quad \text{(SP)}$$

Vacuum

Vacua – distinguished by integer valued topological charge

$$Q = \frac{1}{16\pi^2} \int \mathrm{d}^4x \, F_{\mu\nu}(x) \widetilde{F}_{\mu\nu}(x) \, , \quad \widetilde{F}_{\mu\nu} = \frac{1}{2} \epsilon_{\mu\nu\rho\sigma} F_{\rho\sigma} \qquad Q \in \mathbb{Z}$$

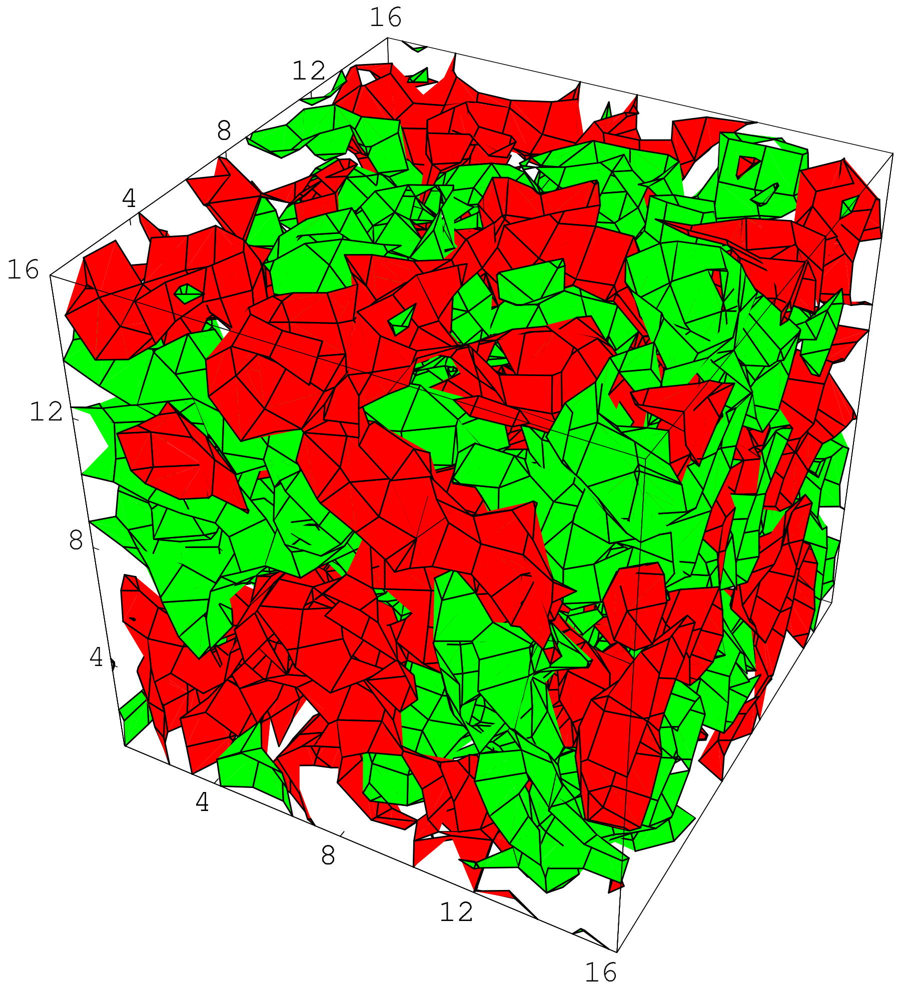
Topological charge density breaks chiral $U_A(1)$ symmetry of the classical action

$$\partial_\mu J^5_\mu(x) = -\frac{N_f}{8\pi^2} F_{\mu\nu}(x) \widetilde{F}_{\mu\nu}(x) \qquad \text{ABJ Anomaly}$$

Consequences (selective)

- $U_A(1)$ problem $\quad m^2_{\eta'} + m^2_\eta - m^2_K = \frac{6}{f^2_\pi} \chi_t \, , \quad \chi_t = \frac{\langle Q^2 \rangle}{V}$
- Nucleon spin $\quad \Delta\Sigma \equiv \Delta u + \Delta d + \Delta s \; = \; \lim_{\vec{p} \to 0} \mathrm{i} \frac{|\vec{s}|}{\vec{p}\vec{s}} \langle \vec{p}, s | \frac{1}{2\pi^2} F_{\mu\nu} \widetilde{F}_{\mu\nu} | 0, s \rangle$

Quenched overlap fermions



Isosurfaces of positive (red) and negative (green) topological charge density

$$Q = \sum_x q(x)$$

$$q(x) = \sum_\lambda \left(1 - \frac{\lambda}{2}\right) \psi_\lambda^\dagger(x)\gamma_5\psi_\lambda(x)$$

Horvath et al.

DIK

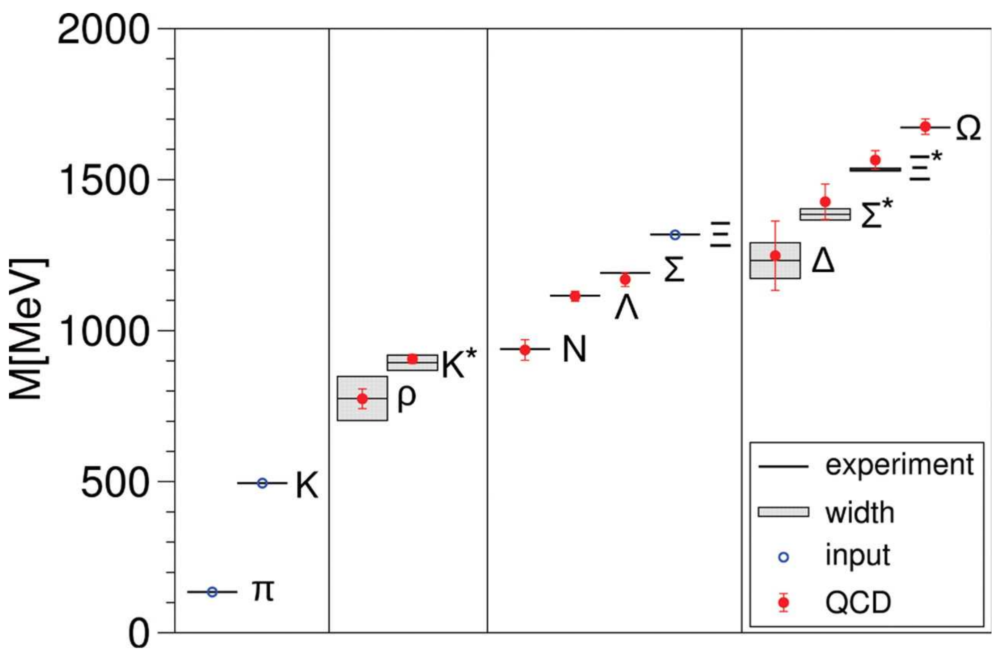
ITEP

QCDSF

Paper ball

Hadron Spectrum

Dürr, Fodor, Lippert, et al.



Science 21 (2008)

Hadrons of most interest (from spectroscopy point of view) are resonances

$$\rho(770) \longrightarrow \pi\pi \qquad \text{Benchmark}$$

$$f_0(600) \longrightarrow \pi\pi$$

$$\vdots$$

$$N(1440) \longrightarrow N\pi \qquad \text{Roper}$$

$$\Delta\pi$$

$$N\eta$$

$$N^*(1535) \longrightarrow N\pi$$

$$N\eta$$

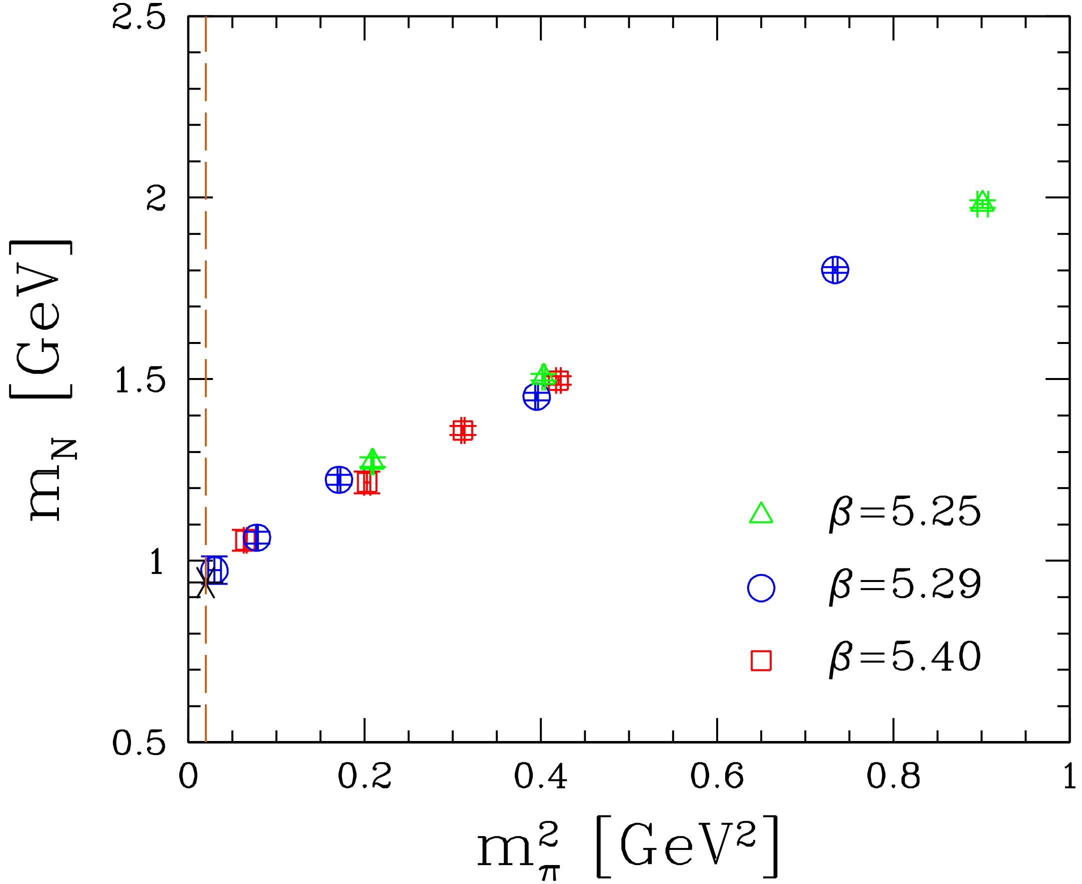
$$\Delta(1232) \longrightarrow N\pi$$

$$\vdots$$

which are not accessible by analytic extrapolation from below threshold, but require a separate analysis

Nucleon

Approaching the chiral limit



Scale: $r^2 \left.\frac{\partial V(r)}{\partial r}\right|_{r=r_0} = 1.65$ $\quad r_0 = 0.467(15)$ fm

$\rho \longrightarrow \pi\pi$

The ρ meson is practically a two-pion resonance. It has isospin 1, and the two pions form a p-wave state

We denote the pion momentum in the center-of-mass frame by $k = |\vec{k}|$. Phenomenologically, the scattering phase shift $\delta_{11}(k)$ is very well described by the effective range formula

$$\frac{k^3}{E} \cot\, \delta_{11}(k) = \frac{24\pi}{g^2_{\rho\pi\pi}} \left(k^2_\rho - k^2\right)$$

where $E = 2\sqrt{k^2 + m^2_\pi}$ and $k_\rho = \frac{1}{2}\sqrt{m^2_\rho - 4m^2_\pi}$. The width of the ρ is given by

$$\Gamma_\rho = \frac{g^2_{\rho\pi\pi}}{6\pi} \frac{k^3_\rho}{m^2_\rho}$$

Experimentally, $\Gamma_\rho = 146\,\text{MeV}$, which translates into

$$g_{\rho\pi\pi} \;=\; 5.9$$

The physical ρ mass (at any given m_π) is obtained from the momentum k, at which the phase shift $\delta_{11}(k)$ passes through $\pi/2$

In the case of noninteracting pions, the possible energy levels in a periodic box of length L are given by

$$E = 2\sqrt{k^2 + m_\pi^2} \qquad\qquad k = \frac{2\pi|\vec{n}|}{L}\ , \quad \vec{n} \in \mathbb{N}^3$$

In the interacting case, k is the solution of a nonlinear equation involving the phase shift

$$\delta_{11}(k) = \arctan\left\{\frac{\pi^{3/2} q}{\mathcal{Z}_{00}(1, q^2)}\right\} \text{ mod } \pi\ , \quad q = \frac{kL}{2\pi}$$

Task

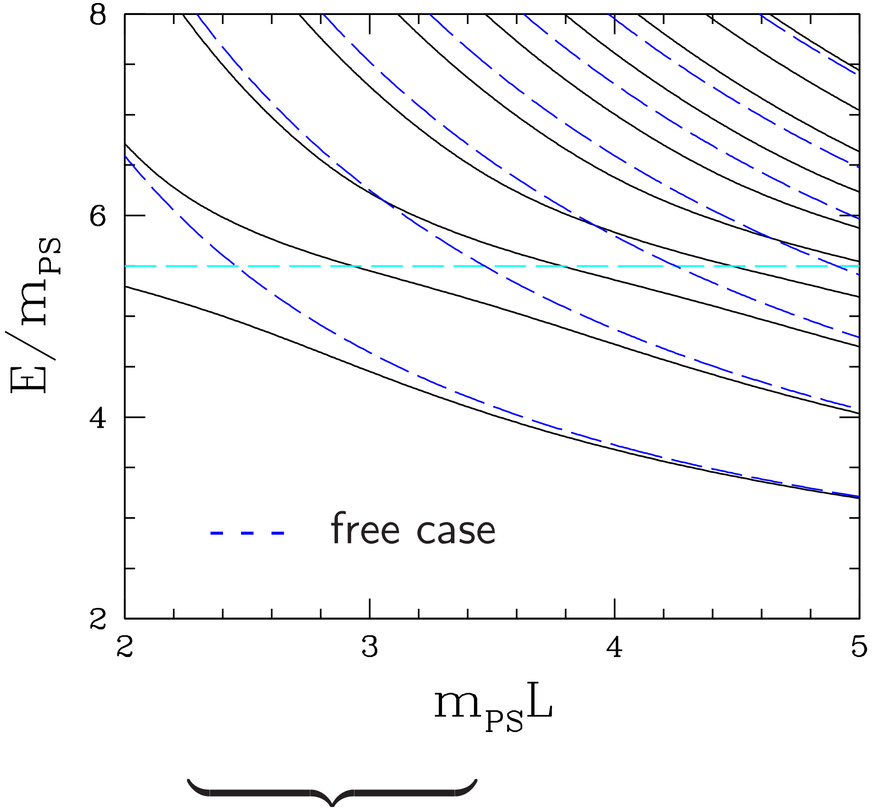
$$E\,|_{m_\pi, L} \quad \longrightarrow \quad k \quad \longrightarrow \quad \delta_{11}(k) \quad \longrightarrow \quad m_\rho\ ,\ \Gamma_\rho$$

by fitting $\delta_{11}(k)$
to effective range
formula

Lüscher, Wiese

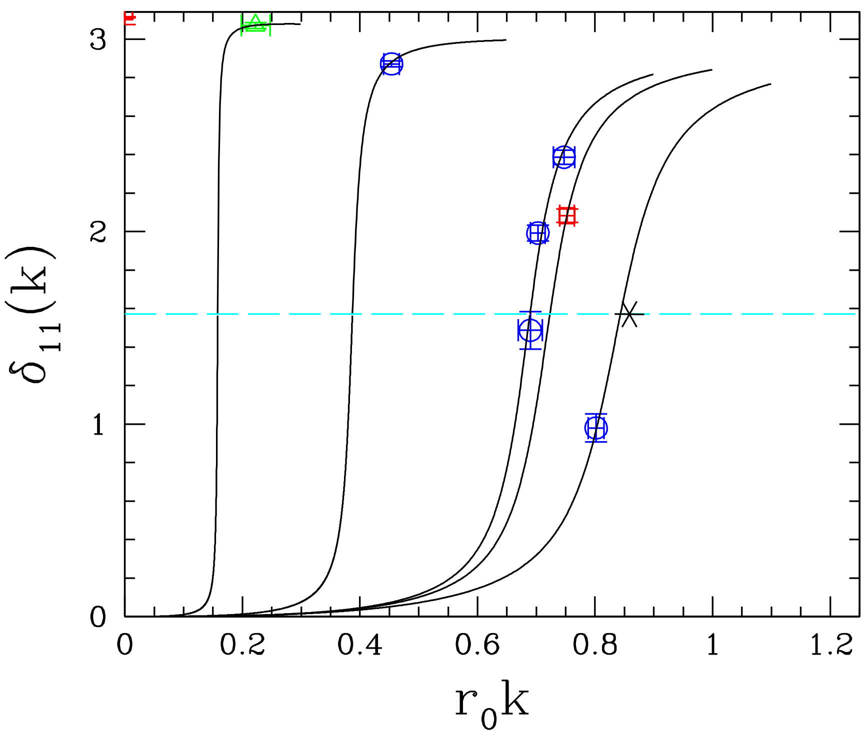
Energy Levels

Physical m_π, m_ρ and Γ_ρ



Useful region

Phase Shift



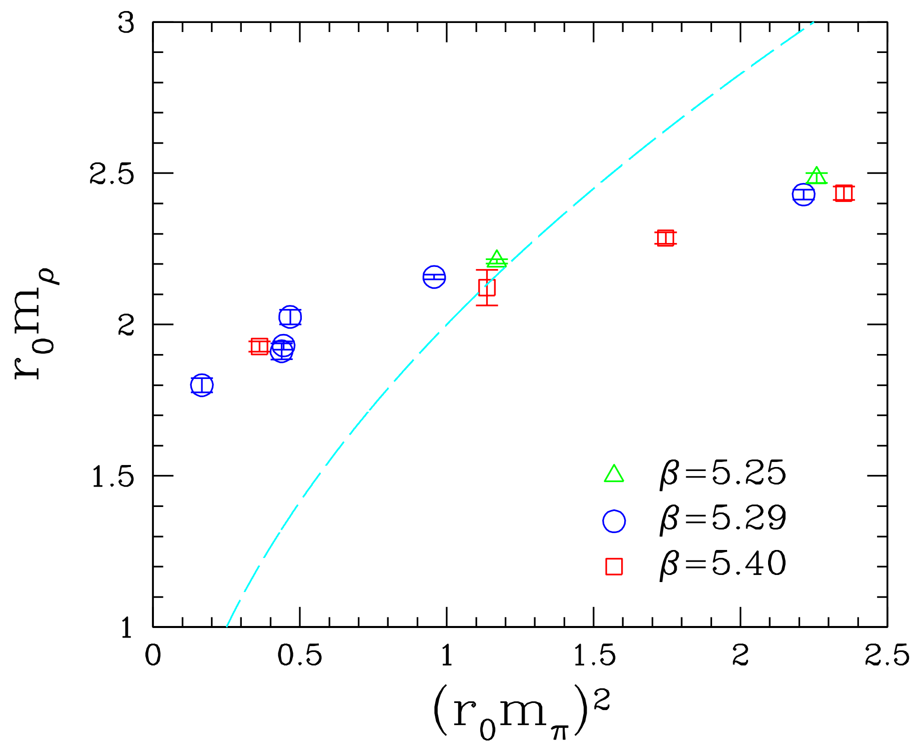
$m_\pi =$ 430 390 250 150 MeV
240

↑
Fit

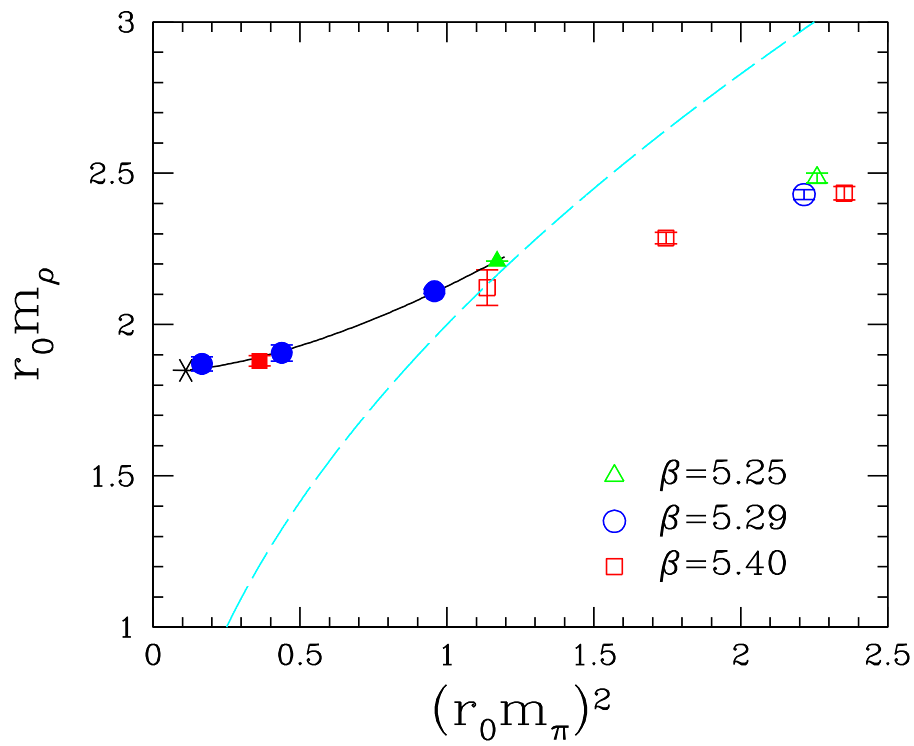
$g_{\rho\pi\pi} = 5.1 \pm 0.4$

Rho Mass

Lowest energy levels



True ρ mass



Chiral fit: $m_\rho = m_\rho^0 + c_1 m_\pi^2 + c_2 m_\pi^3 + c_3 m_\pi^4 \ln(m_\pi^2)$

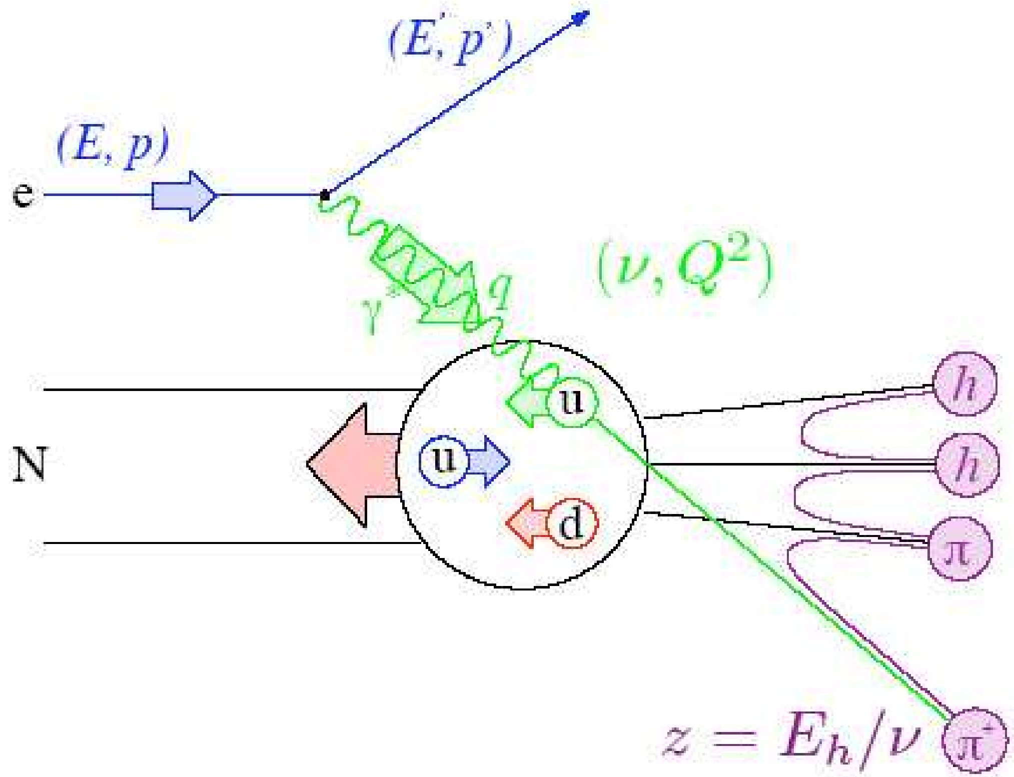
Kink ? Bruns & Meißner

Hadron Structure

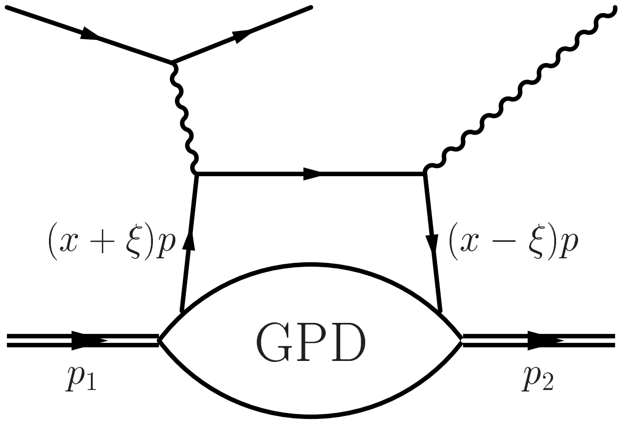
- Inclusive Processes
  - Leading Twist
  - Higher Twist
- Exclusive Processes

Benchmark calculations of $\langle r_2^2 \rangle^{u-d}$, $\langle x \rangle^{u-d}$, g_A

Inclusive



OPE



$p = \frac{1}{2}(p_1+p_2),\ \ \Delta = p_2 - p_1, \quad q = \frac{1}{2}(q_1+q_2)$

$\xi = 0$: Momentum transfer of the struck parton purely transverse, i.e. $\Delta = \boldsymbol{\Delta}_\perp$

$$J(q)\,J(-q) \;=\; \sum_n c_n \;\;\times\;\; \left\{ \begin{array}{l} \mathcal{O}^q_{\mu_1\cdots\mu_n} = \left(\dfrac{i}{2}\right)^{n-1} \bar{q}\gamma_{\mu_1}\overleftrightarrow{D}_{\mu_2}\cdots\overleftrightarrow{D}_{\mu_n}q \\[2ex] \mathcal{O}^{5q}_{\sigma\mu_1\cdots\mu_n} = \left(\dfrac{i}{2}\right)^{n} \bar{q}\gamma_\sigma\gamma_5\overleftrightarrow{D}_{\mu_1}\cdots\overleftrightarrow{D}_{\mu_n}q \\[2ex] \mathcal{O}^{Tq}_{\mu\nu\mu_1\cdots\mu_n} = \left(\dfrac{i}{2}\right)^{n} \bar{q}\sigma_{\mu\nu}\gamma_5\,\overleftrightarrow{D}_{\mu_1}\cdots\overleftrightarrow{D}_{\mu_n}q \end{array} \right.$$

Leading Twist

$$\langle p_1, s|\, \mathcal{O}^q_{\{\mu_1\cdots\mu_n\}}\, |p_2, s\rangle = \bar{u}(p_1, s)\Big[A^q_n(\Delta^2)\ \gamma_{\{\mu_1} + B^q_n(\Delta^2)\ \frac{\mathrm{i}\Delta^\alpha}{2m_N}\sigma_{\alpha\{\mu_1}\Big] p_{\mu_2}\cdots p_{\mu_n\}}\, u(p_2, s)\ +\ \cdots$$

$$\langle p_1, s|\mathcal{O}^{5q}_{\{\mu\mu_1\cdots\mu_n\}}|p_2, s\rangle = \bar{u}(p_1, s)\Big[\tilde{A}^q_{n+1}(\Delta^2)\ \gamma_{\{\mu}\gamma_5 p_{\mu_1}\cdots p_{\mu_n\}}\Big] u(p_2, s)\ +\ \cdots$$

$$\langle p_1, s|\mathcal{O}^{Tq}_{\mu\{\nu\mu_1\cdots\mu_n\}}|p_2, s\rangle = \bar{u}(p_1, s)\Big[A^{Tq}_{n+1}(\Delta^2)\ \sigma_{\mu\{\nu}\gamma_5 - \tilde{A}^{Tq}_{n+1}(\Delta^2)\big(\frac{\Delta^2}{2m_N^2}\sigma_{\mu\{\nu} - \frac{\Delta_\mu\Delta_\alpha}{2m_N^2}\sigma_{\alpha\{\nu}\big)\gamma_5 + \bar{B}^{Tq}_{n+1}(\Delta^2)\ \epsilon_{\alpha\beta\mu\{\nu}\frac{\Delta_\alpha\gamma_\beta}{2m_N}\Big]\ p_{\mu_1}\cdots p_{\mu_n\}}u(p_2, s)\ +\ \cdots$$

Requires to compute $O(1000)$ Matrix Elements $+$ Renormalization Constants

$$A_n^q(\Delta^2) = \int_0^1 dx\, x^{n-1} H^q(x, \Delta^2)$$

$$H^q(x, 0) = q(x)$$

$$B_n^q(\Delta^2) = \int_0^1 dx\, x^{n-1} E^q(x, \Delta^2)$$

$$\tilde{A}_n^q(\Delta^2) = \int_0^1 dx\, x^{n-1} \tilde{H}^q(x, \Delta^2)$$

$$\tilde{H}^q(x, 0) = \Delta q(x)$$

$$A_n^{Tq}(\Delta^2) = \int_0^1 dx\, x^{n-1} H^{Tq}(x, \Delta^2)$$

$$H^{Tq}(x, 0) = \delta q(x)$$

↑ GFFs (under A_n^q, left-hand side) ↑ GPDs (under the integrand)

$$\boxed{\frac{1}{2}\left(A_2^q(0) + B_2^q(0)\right) = J^q}$$

Ji

$$A_1^q\ (\Delta^2) = F_1^q(\Delta^2)$$

$$B_1^q\ (\Delta^2) = F_2^q(\Delta^2)$$

$$\tilde{A}_1^q\ (\Delta^2) = g_A^q\ (\Delta^2)$$

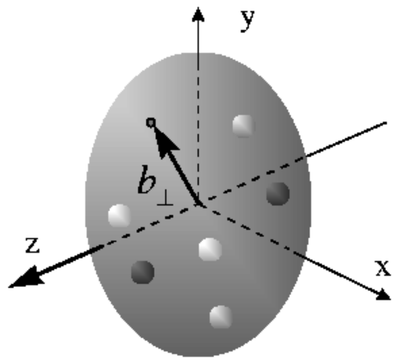
$$A_1^{Tq}(\Delta^2) = g_T^q\ (\Delta^2)$$

$$\Delta^2 = t = -Q^2$$

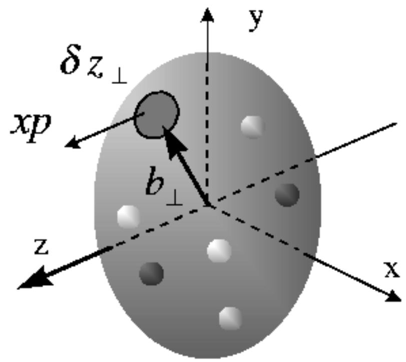
Impact Parameter Space

Generically

$$A_n^q(\mathbf{b}_\perp^2) = \int \frac{d^2\mathbf{\Delta}_\perp}{(2\pi)^2}\, \mathrm{e}^{\,\mathrm{i}\,\mathbf{b}_\perp \mathbf{\Delta}_\perp} A_n^q(\mathbf{\Delta}_\perp^2)$$



$$H^q(x, \mathbf{b}_\perp^2) = \int \frac{d^2\mathbf{\Delta}_\perp}{(2\pi)^2}\, \mathrm{e}^{\,\mathrm{i}\,\mathbf{b}_\perp \mathbf{\Delta}_\perp} H^q(x, \mathbf{\Delta}_\perp^2)$$



Probability interpretation

Not directly accessible by experiment!

Burkardt

$$H^q(x,\Delta^2) = \int_x^1 \frac{dy}{y} C\left(\frac{x}{y},\Delta^2\right) q(y)$$

Similarly for $\tilde{H}^q$ and $H^{T\,q}$

$$\underbrace{\int_0^1 dx\, x^n C(x,\Delta^2)}_{\text{By inverse Mellin transform}} = \frac{A_{n+1}(\Delta^2)}{A_{n+1}(0)}$$

$$1 + \frac{1}{6} r^2 \Delta^2 + O(\Delta^4)$$

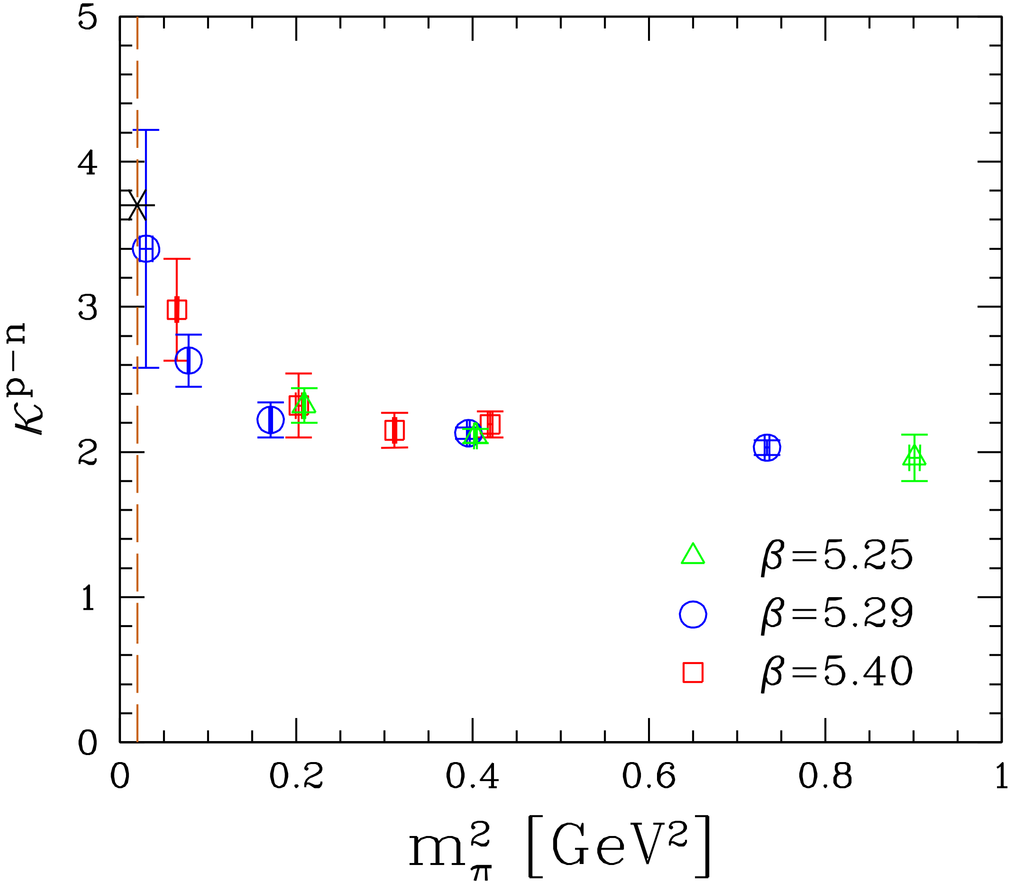
By inverse Mellin transform

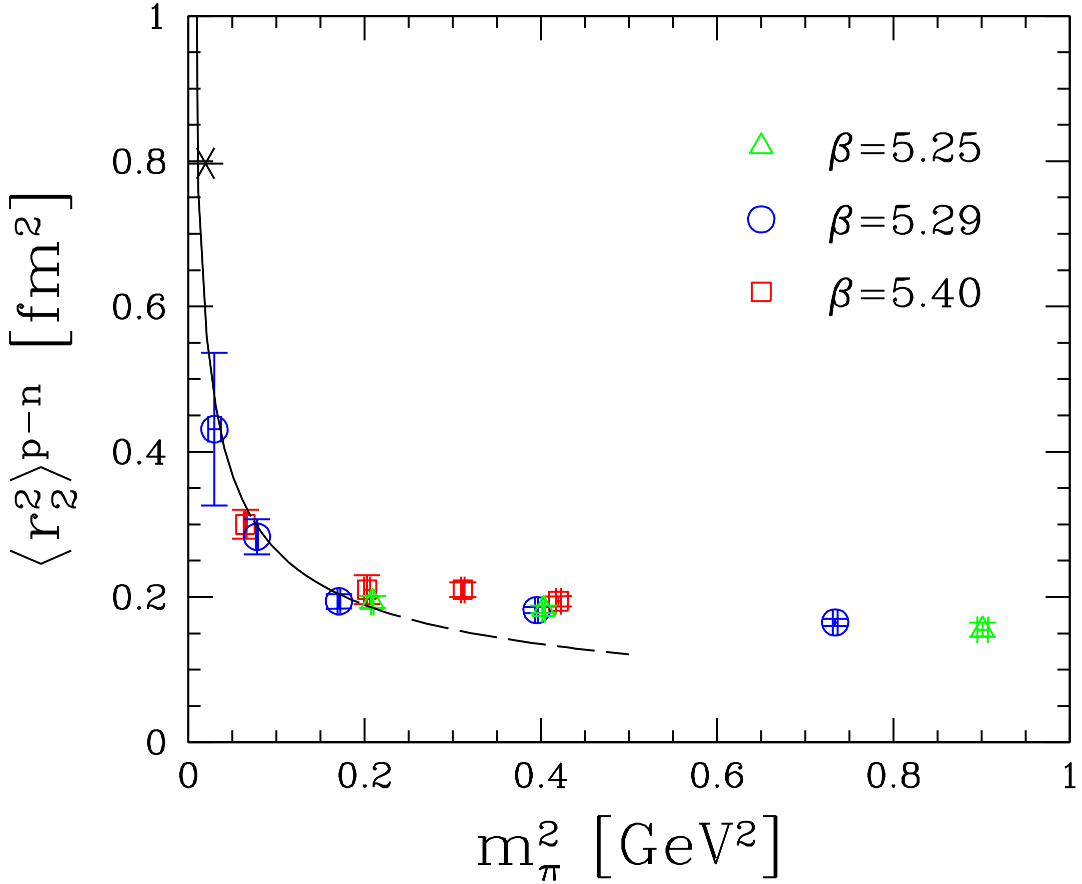
$$H^q(x,\mathbf{b}_\perp^2) = \int_x^1 \frac{dy}{y} C\left(\frac{x}{y},\mathbf{b}_\perp^2\right) q(y)$$

$$C(x,\mathbf{b}_\perp^2) = \int \frac{d^2\mathbf{\Delta}_\perp}{(2\pi)^2} e^{i\,\mathbf{b}_\perp \mathbf{\Delta}_\perp} C(x,\mathbf{\Delta}_\perp^2)$$

Form Factor

$$F_{1,2}(Q^2) = F_{1,2}(0)\left(1 - \frac{1}{6} r_{1,2}^2 Q^2 + O(Q^4)\right) \quad ; \quad F_1^N(0) = e^N \, , \; F_2(0) = \kappa^N$$





Fit to ChPT

ChPT

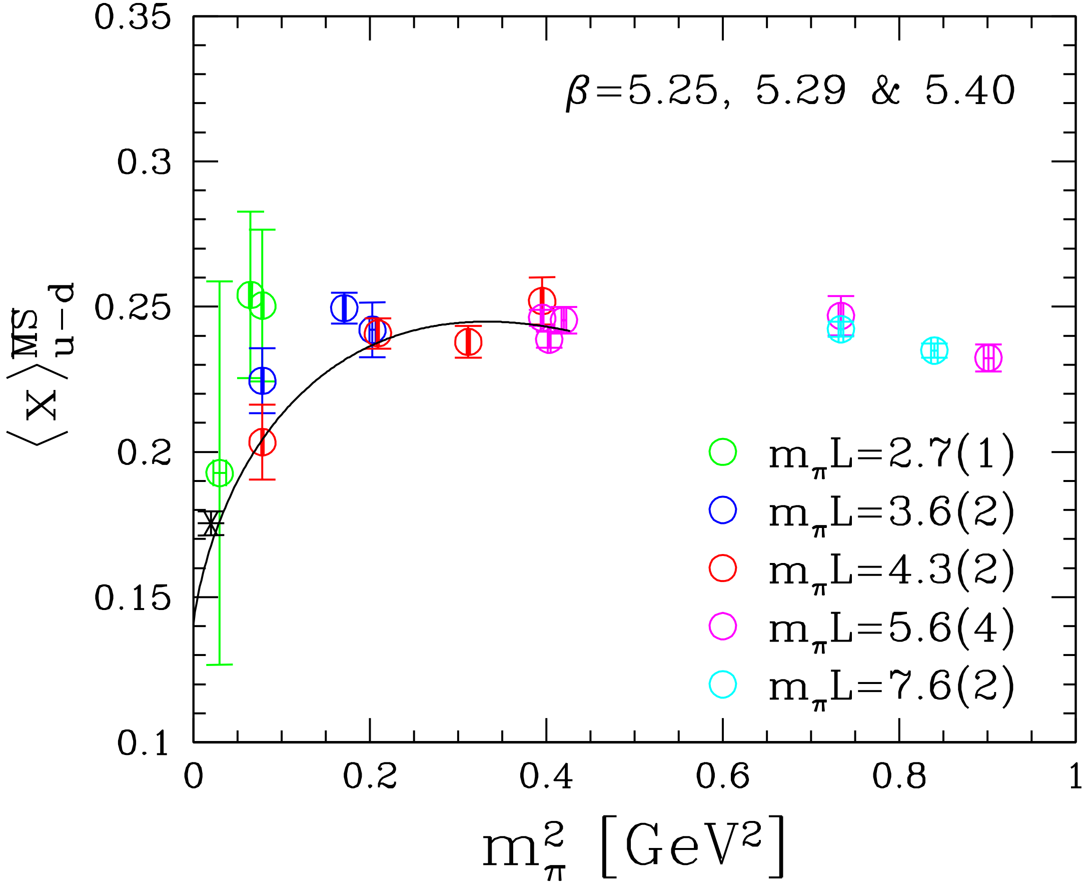
$$r_1^2 = -\frac{1}{(4\pi f_\pi)^2}\left\{1 + 7g_A^2 + (10g_A^2 + 2)\log\left[\frac{m_\pi}{\lambda}\right]\right\} - \frac{12B_{10}^{(r)}(\lambda)}{(4\pi f_\pi)^2}$$
$$+ \frac{c_A^2}{54\pi^2 f_\pi^2}\left\{26 + 30\log\left[\frac{m_\pi}{\lambda}\right] + 30\frac{\Delta}{\sqrt{\Delta^2 - m_\pi^2}}\log R(m_\pi)\right\}$$

$$r_2^2 = \frac{g_A^2 m_N}{8f_\pi^2\kappa\pi m_\pi} + \frac{c_A^2 m_N}{9f_\pi^2\kappa\pi^2\sqrt{\Delta^2 - m_\pi^2}}\log R(m_\pi) + \frac{24m_N}{\kappa}B_{c2}$$

$$R(m) = \frac{\Delta}{m} + \sqrt{\frac{\Delta^2}{m^2} - 1}\,, \quad \Delta = m_\Delta - m_N$$

Unpolarized Parton Distributions

$$\langle x \rangle = \int dx\, x \left[u(x, Q^2) - d(x, Q^2) \right]$$

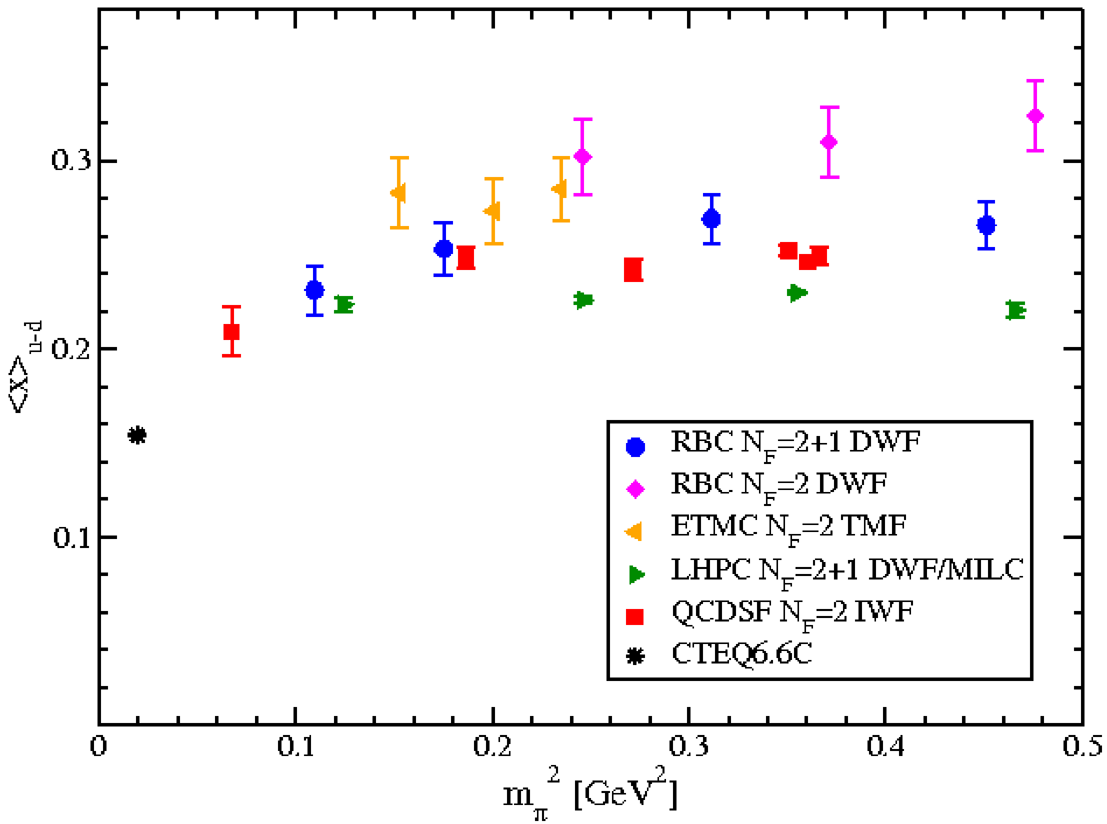


Fit to largest $m_\pi L$

$$
\begin{aligned}
\langle x \rangle_{u-d} &\equiv v_2 \\
&= v_2^0 + \frac{v_2^0 m_\pi^2}{(4\pi f_\pi)^2}\Bigg\{ -(3g_A^2+1)\ln\frac{m_\pi^2}{\lambda^2} - 2g_A^2 + g_A^2 \frac{m_\pi^2}{m_N^2}\left(1+3\ln\frac{m_\pi^2}{m_N^2}\right) \\
&- \frac{1}{2} g_A^2 \frac{m_\pi^4}{m_N^4}\ln\frac{m_\pi^2}{m_N^2} + g_A^2 \frac{m_\pi}{\sqrt{4m_N^2 - m_\pi^2}}\left(14 - 8\frac{m_\pi^2}{m_N^2} + \frac{m_\pi^4}{m_N^4}\right) \\
&\times \arccos\left(\frac{m_\pi}{2m_N}\right)\Bigg\} + \frac{\Delta v_2^0 g_A^0 m_\pi^2}{3(4\pi f_\pi)^2}\Bigg\{ 2\frac{m_\pi^2}{m_N^2}\left(1+3\ln\frac{m_\pi^2}{m_N^2}\right) - \frac{m_\pi^4}{m_N^4}\ln\frac{m_\pi^2}{m_N^2} \\
&+ \frac{2m_\pi(4m_N^2 - m_\pi^2)^{\frac{3}{2}}}{m_N^4}\arccos\left(\frac{m_\pi}{2m_N}\right)\Bigg\} + 4m_\pi^2 \frac{c_8^{(r)}(\lambda)}{M_0^2} + \mathcal{O}(p^3)
\end{aligned}
$$

Dorati, Gail & Hemmert

Finite size corrections not known so far

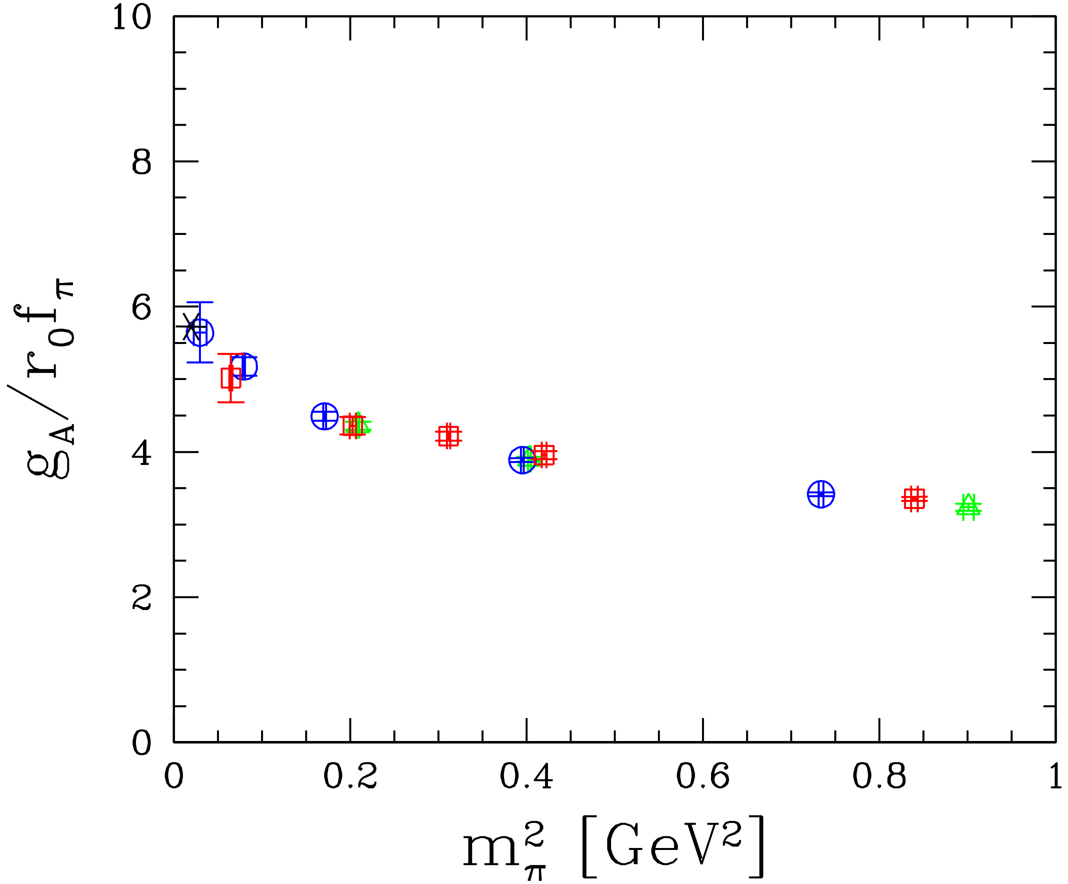


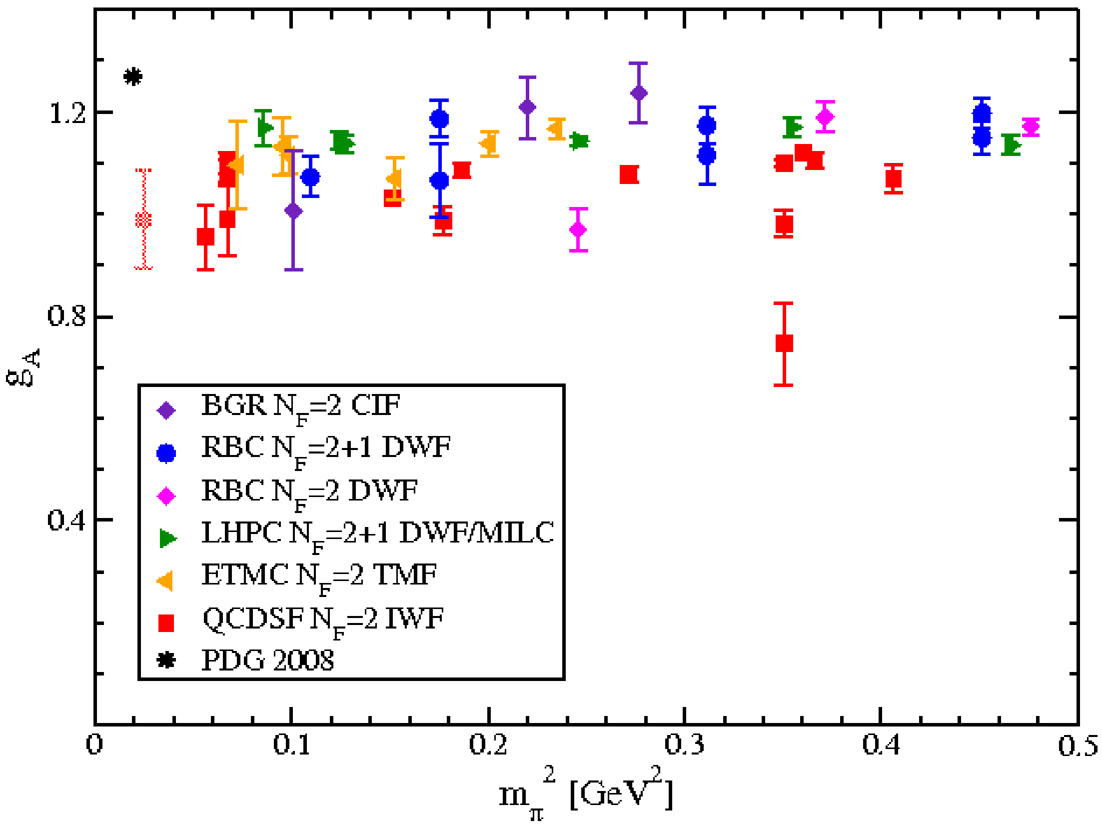
World data

Renner, Lat09

Polarized Parton Distributions

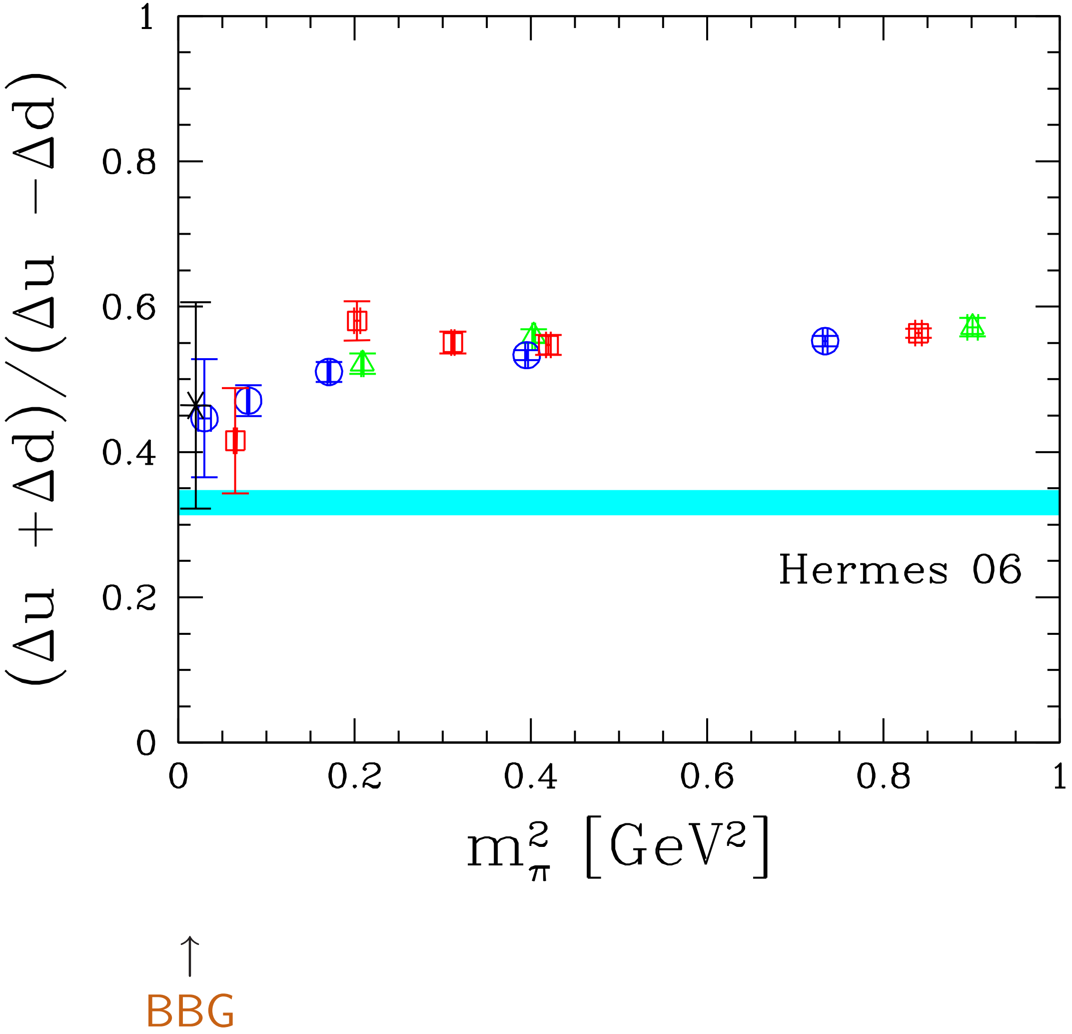
$$g_A \equiv \Delta u - \Delta d = \int dx\, [\Delta u(x, Q^2) - \Delta d(x, Q^2)]$$





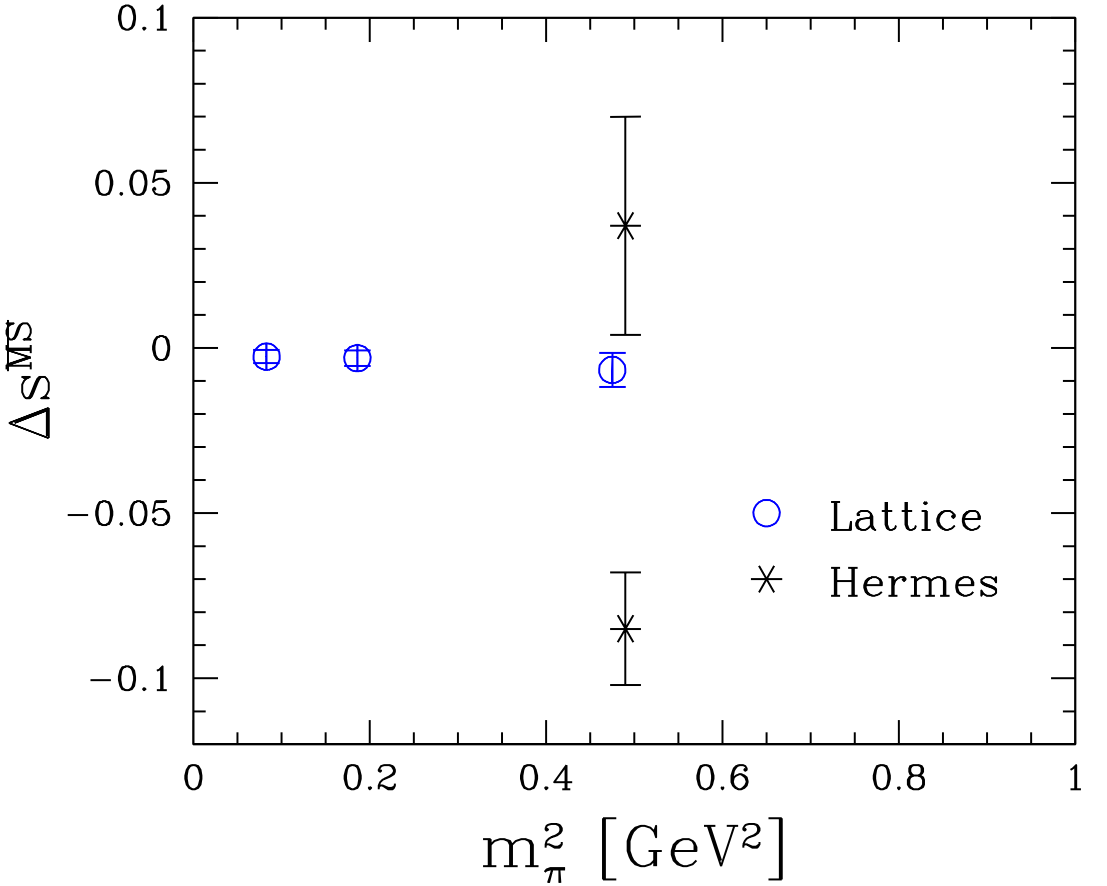
World data

Renner, Lat09



(Δu +Δd)/(Δu −Δd)
m²π [GeV²]
0
0.2
0.4
0.6
0.8
1
Hermes 06
↑
BBG

Disconnected



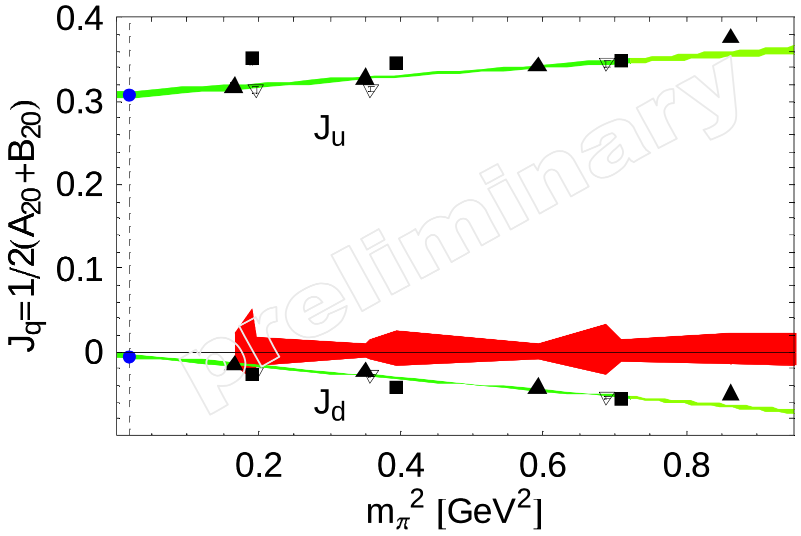
Bali, Collins & Schäfer

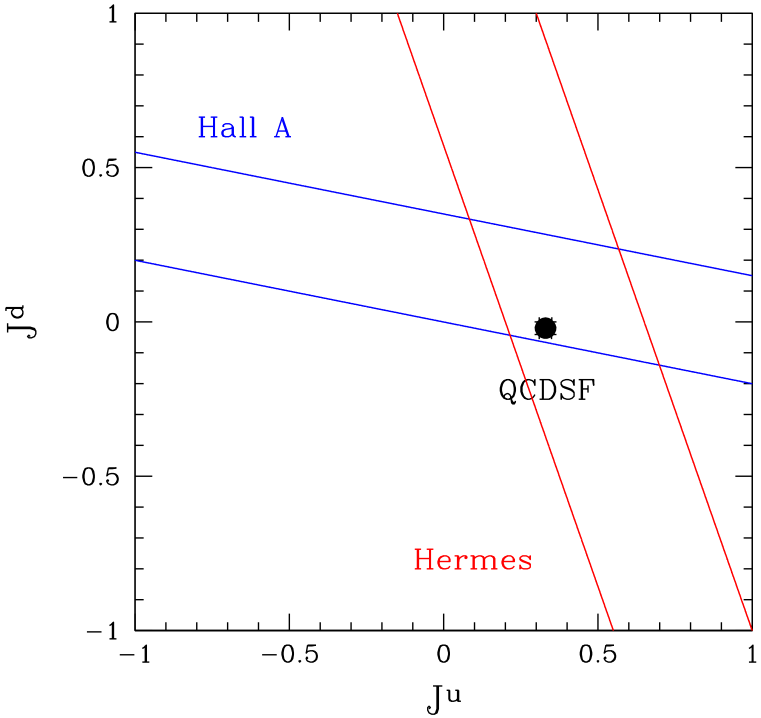
(Orbital) Angular Momentum

Valence

$$J^q = \frac{1}{2}\big(A_2^q(0) + B_2^q(0)\big) \equiv \frac{1}{2}\Delta\Sigma^q + L^q$$

$$\Delta\Sigma^q = \Delta u + \Delta d$$





$J^u = 0.33(2)$ $J^d = -0.02(2)$ $L^u = -0.08(4)$ $L^d = 0.10(4)$

Spin & Flavor Density

$\lambda_\perp$ quark spin
$s_\perp$ nucleon spin

Transverse spin density

$\Downarrow$

$$\langle p_+, s_\perp | \bar{q}(\mathbf{b}_\perp) \left[\gamma_+ - \lambda_{\perp i}\, \sigma_{+j}\, \gamma_5 \right] q(\mathbf{b}_\perp) | p_+, s_\perp \rangle = \Big\{ A_1^q(\mathbf{b}_\perp^2) + \lambda_{\perp i}\, s_{\perp i} \Big[A_1^{T\,q}(\mathbf{b}_\perp^2)$$

$$- \frac{1}{4m_N^2} \Delta_{b_\perp} \tilde{A}_1^{T\,q}(\mathbf{b}_\perp^2) \Big] - \frac{1}{m_N} \epsilon_{ij} b_{\perp j} \Big[s_{\perp i} B_1^q(\mathbf{b}_\perp^2)' + \lambda_{\perp i}\, \bar{B}_1^{T\,q}(\mathbf{b}_\perp^2)' \Big]$$

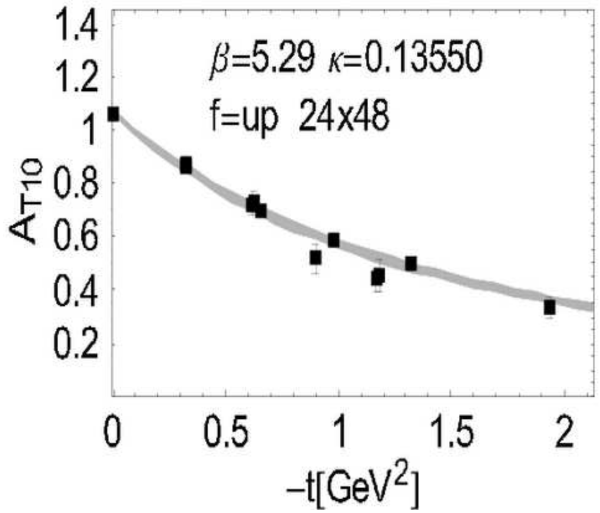
$$+ \frac{1}{m_N^2} \lambda_{\perp i}\, (2 b_{\perp i}\, b_{\perp j} - \mathbf{b}_\perp^2 \delta_{ij}) s_{\perp j}\, \tilde{A}_1^{T\,q}(\mathbf{b}_\perp^2)'' \Big\}$$

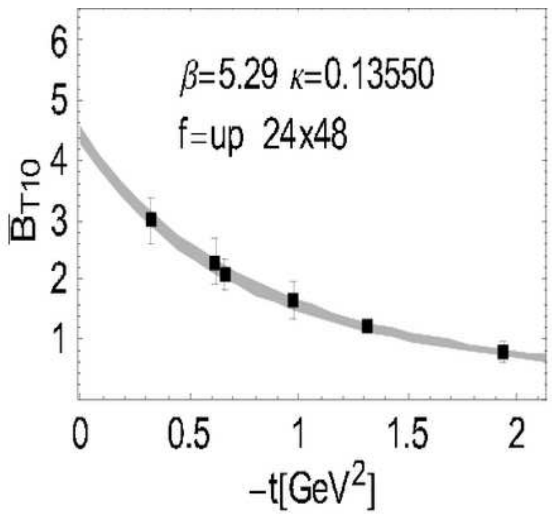
$\Uparrow$

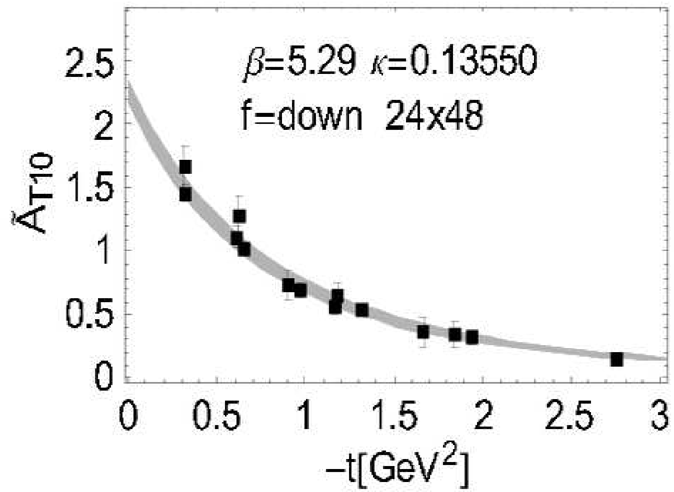
Quadrupole

Diehl & Hägler

So far

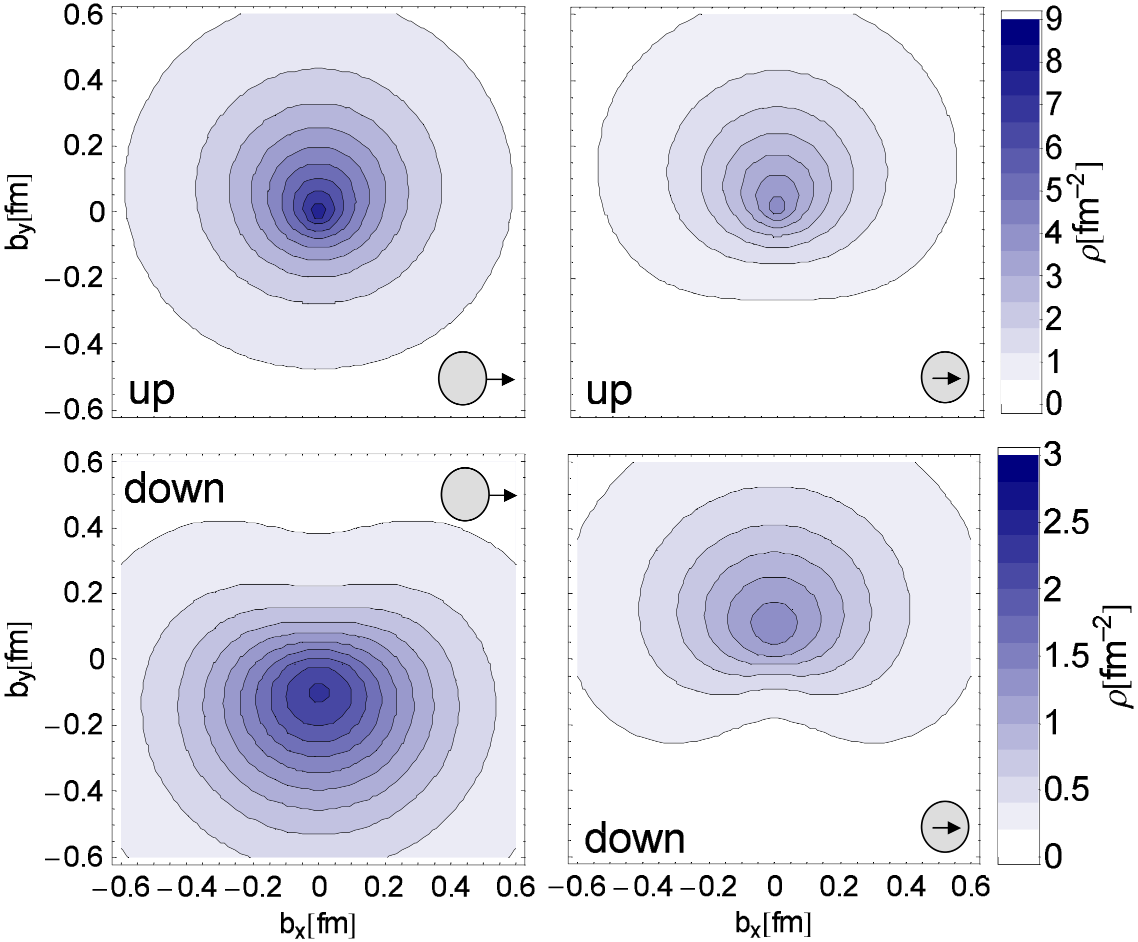




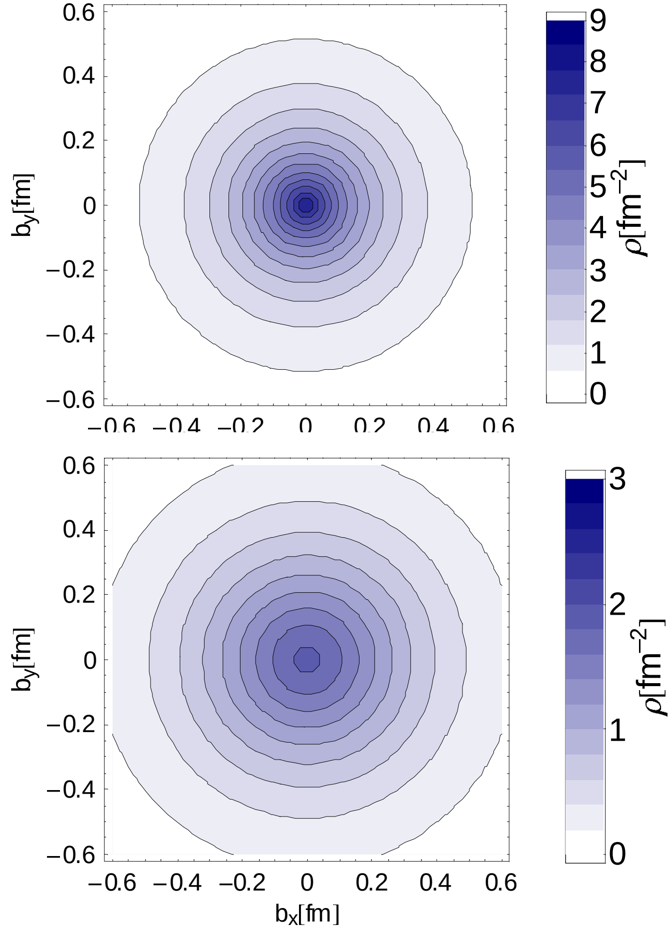


Dipole fit

To be extrapolated to chiral limit

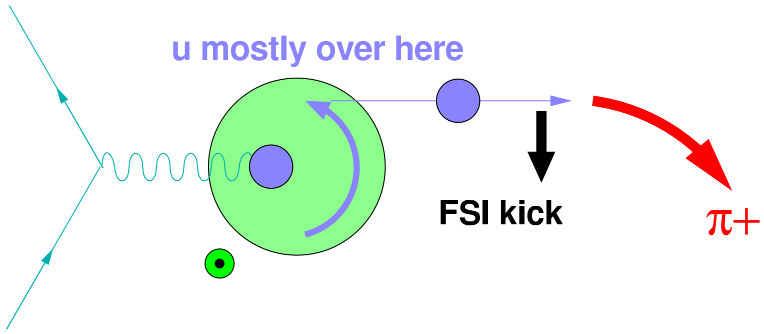


Sivers effect   Boer-Mulders effect



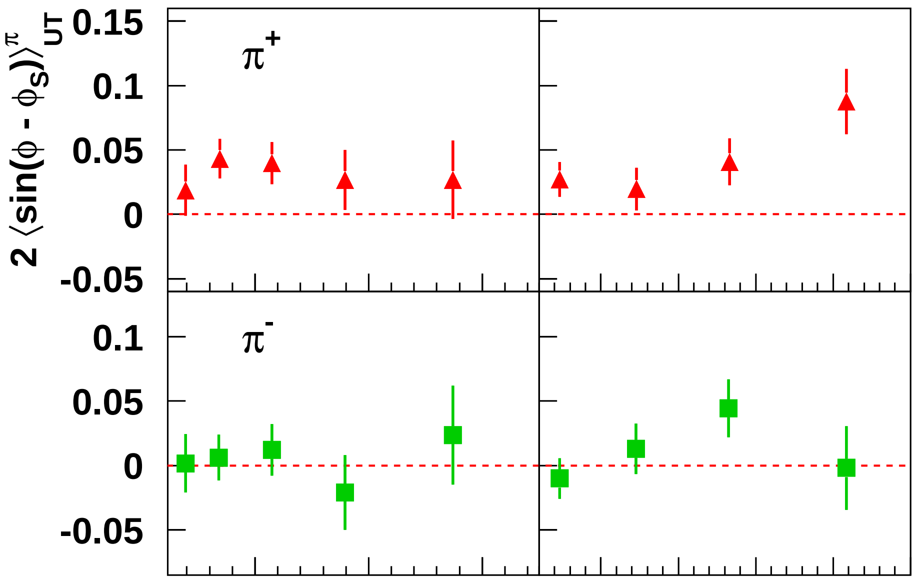
Unpolarized

$$r_T^d > r_T^u$$



Courtesy of G. Schnell

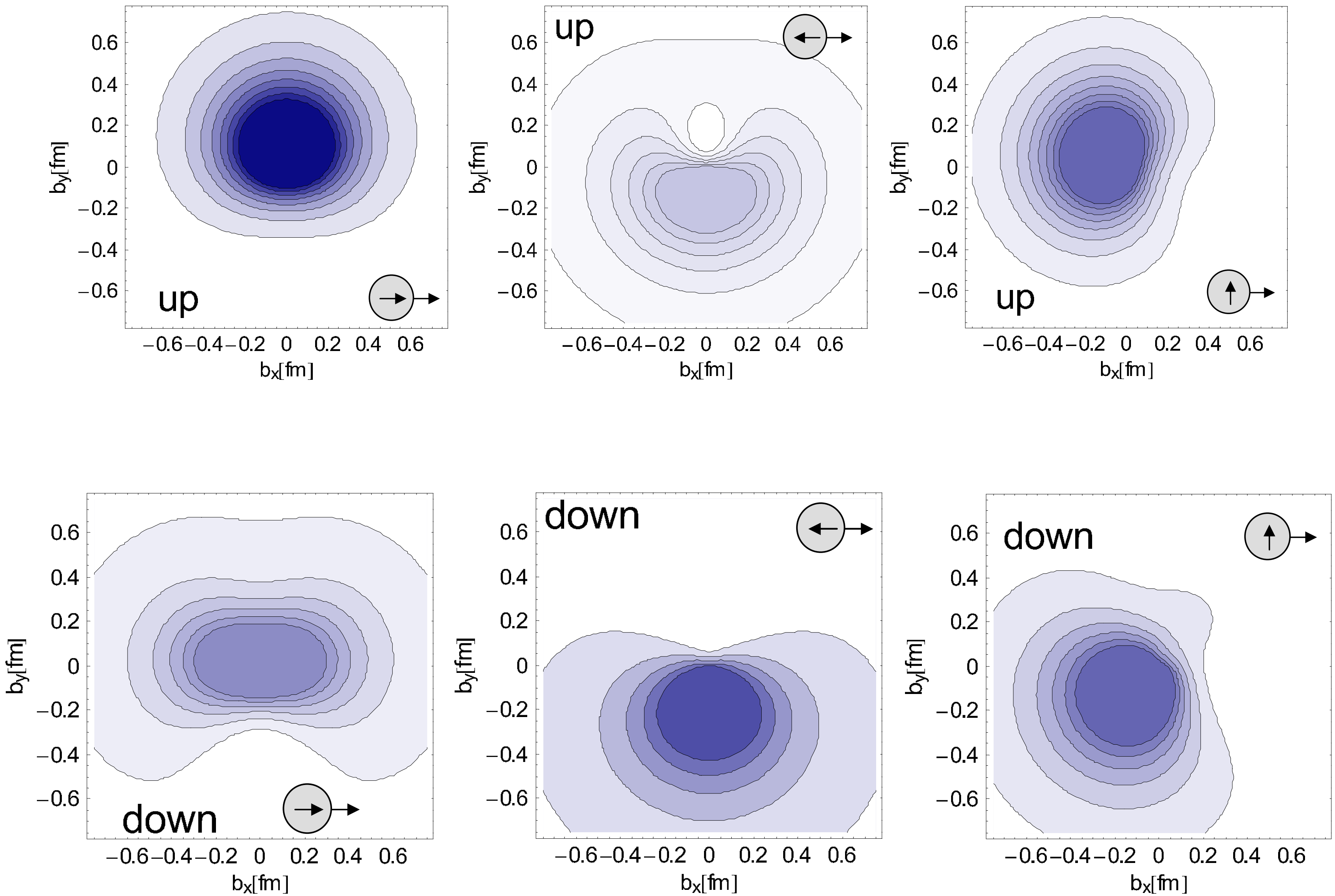
Sivers effect



Hermes

Nucleon and quarks both polarized

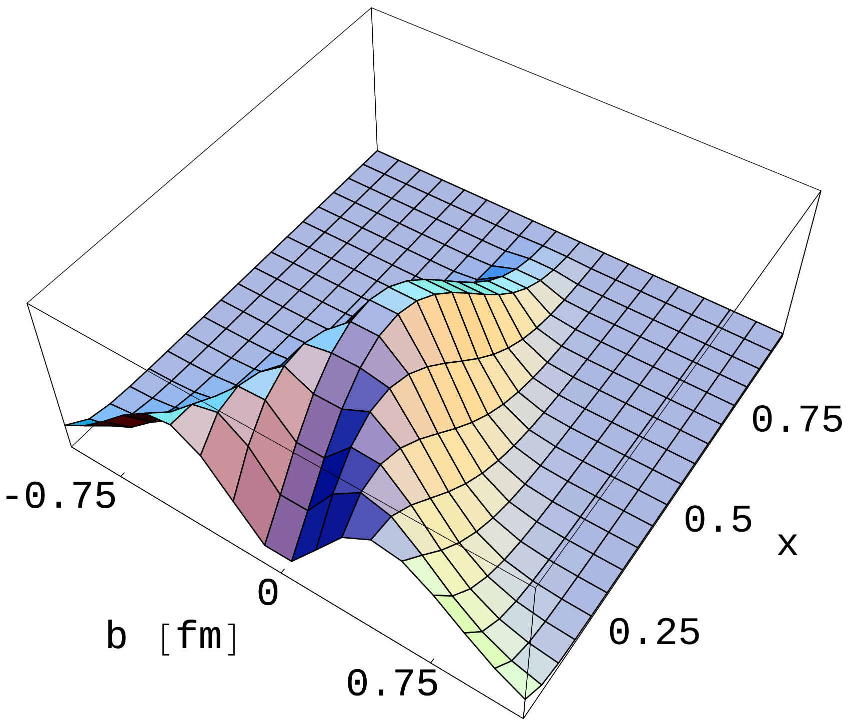
Spin-orbit coupling

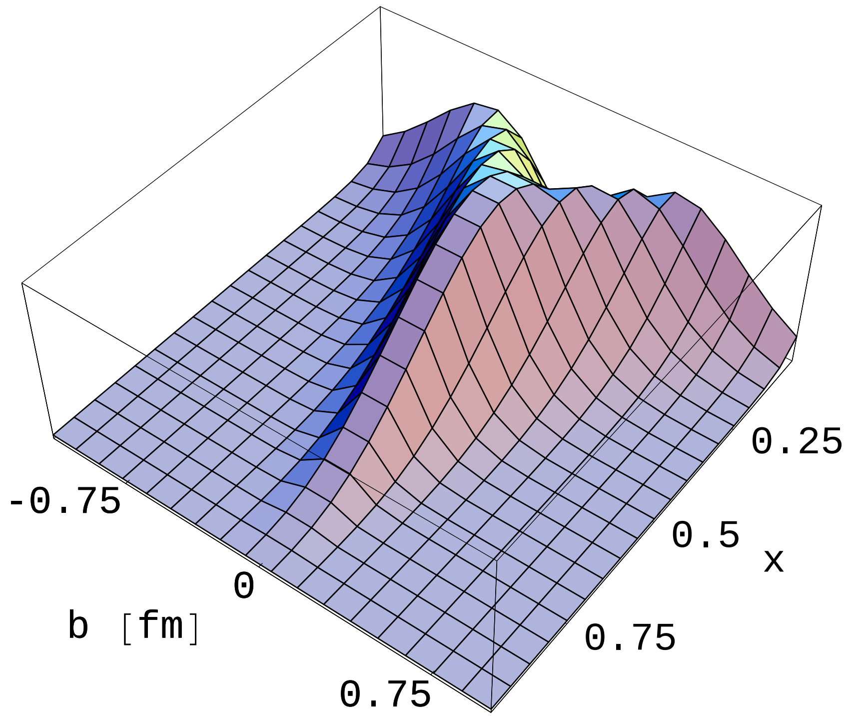


Generalized Parton Distribution

$$H^u(x, \mathbf{b}_\perp^2) = \int_x^1 \frac{dy}{y} C\left(\frac{x}{y}, \Delta^2\right) q(y)$$

$$\int_0^1 dx\, x^n C(x, \Delta^2) = \frac{A_{n+1}(\Delta^2)}{A_{n+1}(0)}$$





$$\langle b^2 \rangle = \frac{7}{2} \alpha^{\vee\, 2} (1-x)^2 + \mathcal{O}\left((1-x)^3\right)$$

$$\langle r^2 \rangle = \frac{7}{2} \alpha^{\vee\, 2} + \mathcal{O}\left(1-x\right)$$

Higher Twist

OPE without OPE

Unpolarized

$$\mathcal{M}_2(Q^2) = \int_0^1 dx\, F_2(x, Q^2) \, + \, \cdots$$

$$= c_2^{(2)}(Q^2/\mu^2, g(\mu^2))\, A_2^{(2)}(\mu) + c_2^{(4)}(Q^2/\mu^2, g(\mu^2))\, \frac{A_2^{(4)}(\mu)}{Q^2} \, + \, \cdots$$

IR UV

Renormalon ambiguity

$\hookrightarrow$ Nonperturbative solution

- Evaluate

$$W_{\mu\nu} \equiv \langle p|J_\mu(q)J_\nu(-q)|p\rangle + \cdots = \sum_{m,n} c^m_{\mu\nu\mu_1\cdots\mu_n}(aq)\,\langle p|\mathcal{O}^m_{\mu_1\cdots\mu_n}|p\rangle$$

input   input

between off-shell quark states $|p\rangle$ and solve for Wilson coefficients $c^m_{\mu\nu\mu_1\cdots\mu_n}(aq)$ by SV decomposition

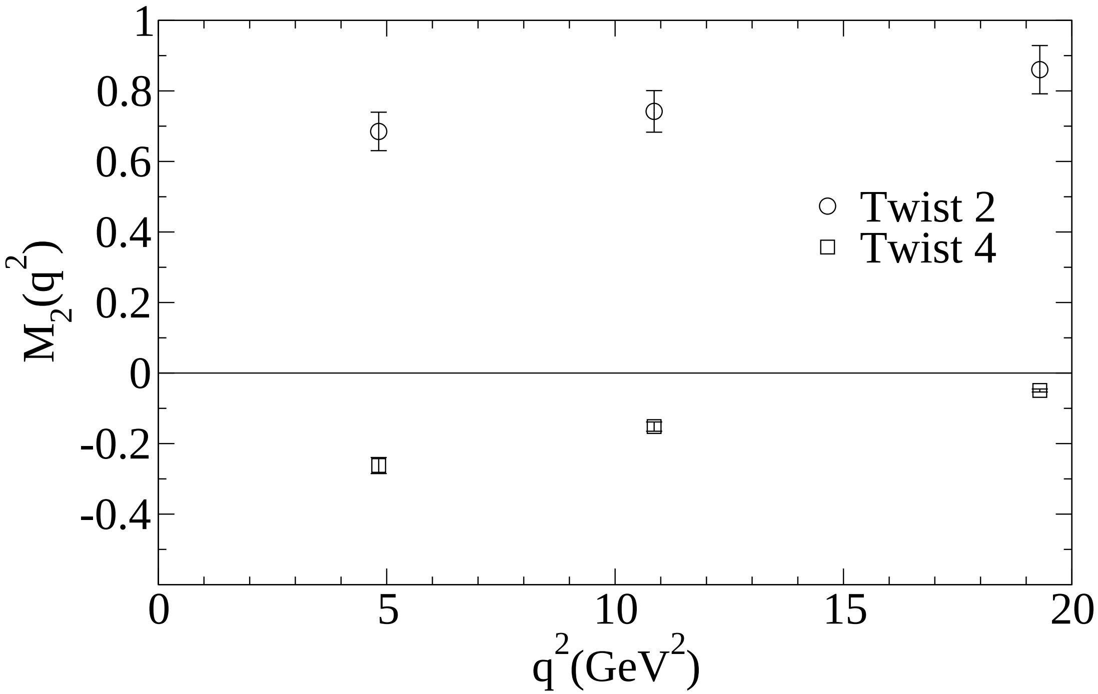
- Replace $\langle p|\mathcal{O}^m_{\mu_1\cdots\mu_n}|p\rangle$ by matrix element $\langle p_N, s|\mathcal{O}^m_{\mu_1\cdots\mu_n}|p_N, s\rangle$ between nucleon states and compute $W_{\mu\nu}$

Nachtmann moment

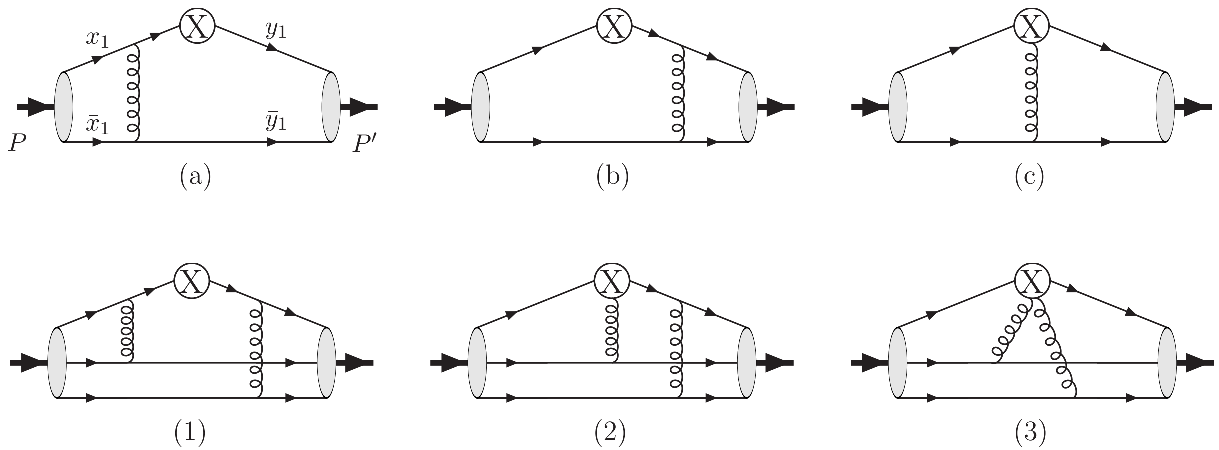
$$\mathcal{M}_2(Q^2) = \frac{3}{4}q^2 \int \frac{d\Omega_q}{4\pi^2}\left(n_\mu W_{\mu\nu} n_\nu - \frac{1}{4}W_{\mu\mu}\right) = \int_0^1 dx\left(F_2(x,Q^2) + \frac{1}{6}F_L(x,Q^2)\right)$$

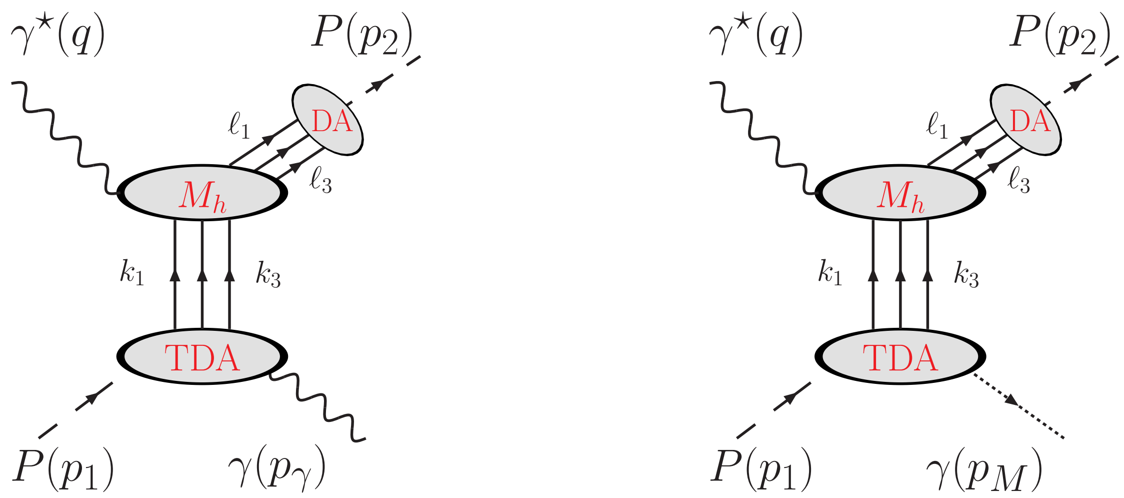
Bjorken limit

Quenched overlap fermions $m_\pi \approx 390\,\mathrm{MeV}$



Exclusive


X
x_1
y_1
$\bar{x}_1$
$\bar{y}_1$
P
P'
(a)
(b)
(c)
(1)
(2)
(3)


$\gamma^\star(q)$
$P(p_2)$
DA
ℓ_1
ℓ_3
M_h
k_1
k_3
TDA
$P(p_1)$
$\gamma(p_\gamma)$
$\gamma^\star(q)$
$P(p_2)$
DA
ℓ_1
ℓ_3
M_h
k_1
k_3
TDA
$P(p_1)$
$\gamma(p_M)$

Distribution Amplitudes

RG expansion Peskin, Brodsky et al.

$$\phi_N(x_1,x_2,x_3,\mu^2) = 120\, x_1x_2x_3 \sum_{n=0}^{\infty}\sum_{l=0}^{n} c_{nl}(\mu_0)\, P_{nl}(x_1,x_2,x_3)\, L^{\gamma_{nl}/\beta_0} \qquad \text{LLog}$$

$$= \int_{|k_\perp|<\mu} [d^2k_\perp]\, \phi_{BS}(x_1,x_2,x_3,[k_\perp])$$

$$L = \frac{\alpha_s(\mu)}{\alpha_s(\mu_0)}$$

$$c_{10} = \frac{7}{2}\left[3\left(\phi_N^{100}(\mu_0^2) + \phi_N^{001}(\mu_0^2)\right) - 2\right]$$

$$c_{11} = \frac{63}{2}\left(\phi_N^{100}(\mu_0^2) + \phi_N^{001}(\mu_0^2)\right)$$

$$c_{21} = -\frac{126}{5}\left(\phi_N^{200}(\mu_0^2) + \phi_N^{002}(\mu_0^2) + 3\phi_N^{101}(\mu_0^2)\right) + \frac{18}{5}(4 + c_{10})$$

$$\vdots$$

Moments

$$\langle 0|\mathcal{O}_{\rho\,\lambda_1\cdots\lambda_l\,\mu_1\cdots\mu_m\,\nu_1\cdots\nu_n\;\alpha}(0)|p\rangle \;=\; \phi_N^{lmn}(\mu^2)\,p_\rho\,p_{\lambda_1}\cdots p_{\lambda_l}\,p_{\mu_1}\cdots p_{\mu_m}\,p_{\nu_1}\cdots p_{\nu_n}\,N_\alpha^\uparrow(p)$$

$$\phi_N^{lmn}(\mu^2) = \int [dx]\; x_1^l\, x_2^m\, x_3^n\; \phi_N(x_1, x_2, x_3, \mu^2)$$

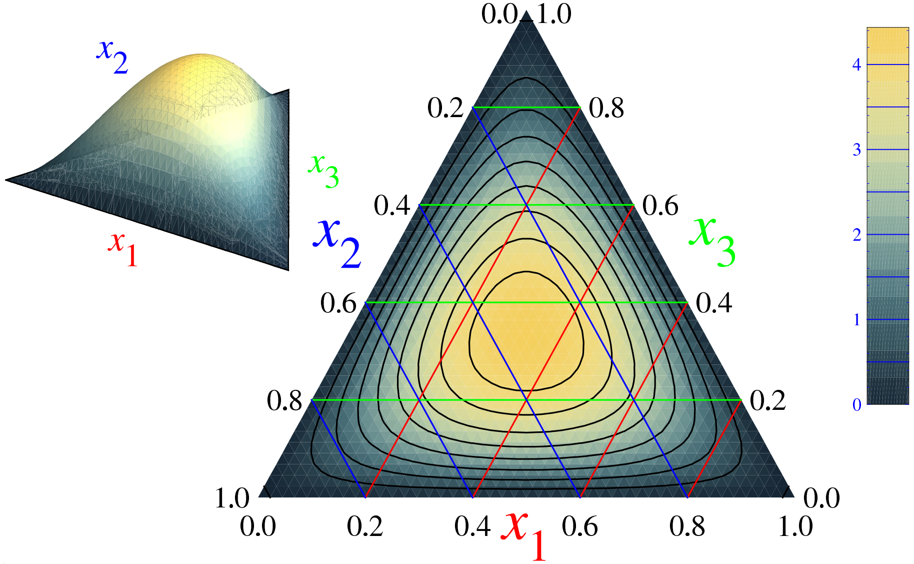
$$\mathcal{O}_{\rho\,\lambda_1\cdots\lambda_l\,\mu_1\cdots\mu_m\,\nu_1\cdots\nu_n\;\alpha}(0) \;=\; \big([i^l D_{\lambda_1}\cdots D_{\lambda_l}\,u^\uparrow(0)]_a\;C\gamma_\rho\;[i^m D_{\mu_1}\cdots D_{\mu_m}\,u^\downarrow(0)]_b\big)$$
$$\times\;\;[i^n D_{\nu_1}\cdots D_{\nu_n}\,d^\uparrow(0)]_{c\;\alpha}\;\epsilon_{abc}$$

↑

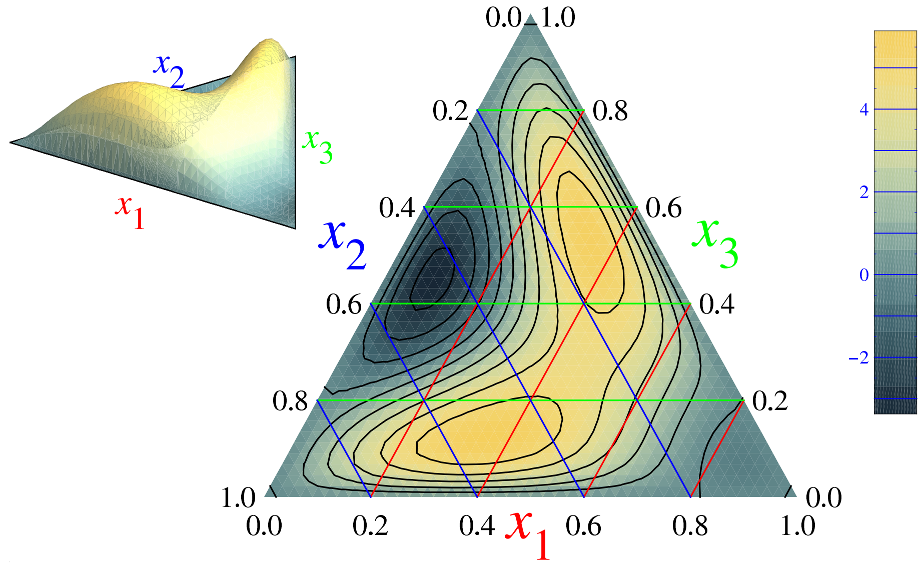
Need to be renormalized nonperturbatively

$\phi_N(x_1, x_2, x_3, \mu^2)$

Barycentric contour plot



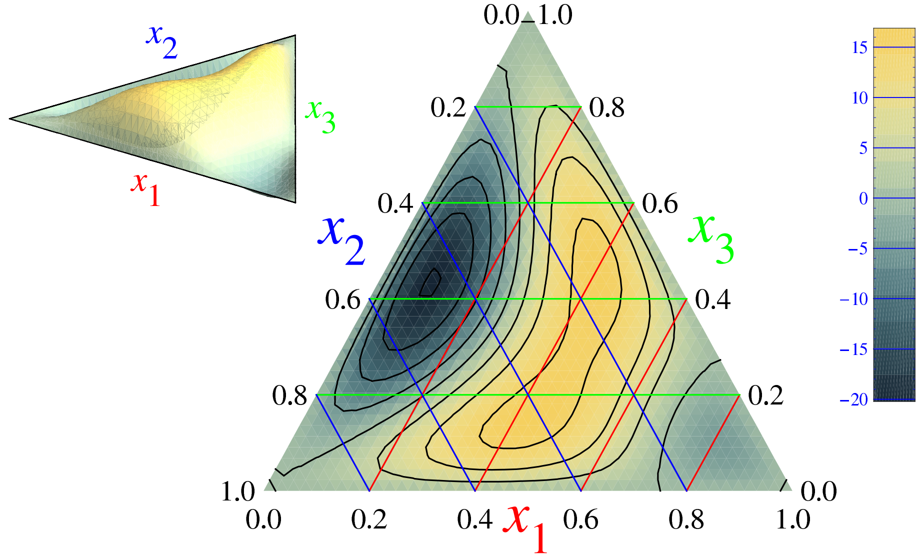
$\mu^2 \to \infty$



$\mu^2 = 4\,\text{GeV}^2$

Strong correlation of $(u^{\downarrow} d^{\uparrow})$ diquark

$N^*(1535)$



x2
x3
x1
0.0
1.0
0.2
0.8
0.4
0.6
0.6
0.4
0.8
0.2
1.0
0.0
0.0
0.2
0.4
0.6
0.8
1.0
x2
x3
x1
15
10
5
0
−5
−10
−15
−20

Applications

Form factor

$$G_M(\Delta^2) = \int [dy][dz]\, \phi_N^*(y_1, y_2, y_3, \mu^2)\, T_M([y], [z], \Delta^2)\, \phi_N(z_1, z_2, z_3, \mu^2) + \text{HT}$$

Δ^2 large

$\uparrow$

Perturbative above scale Δ^2

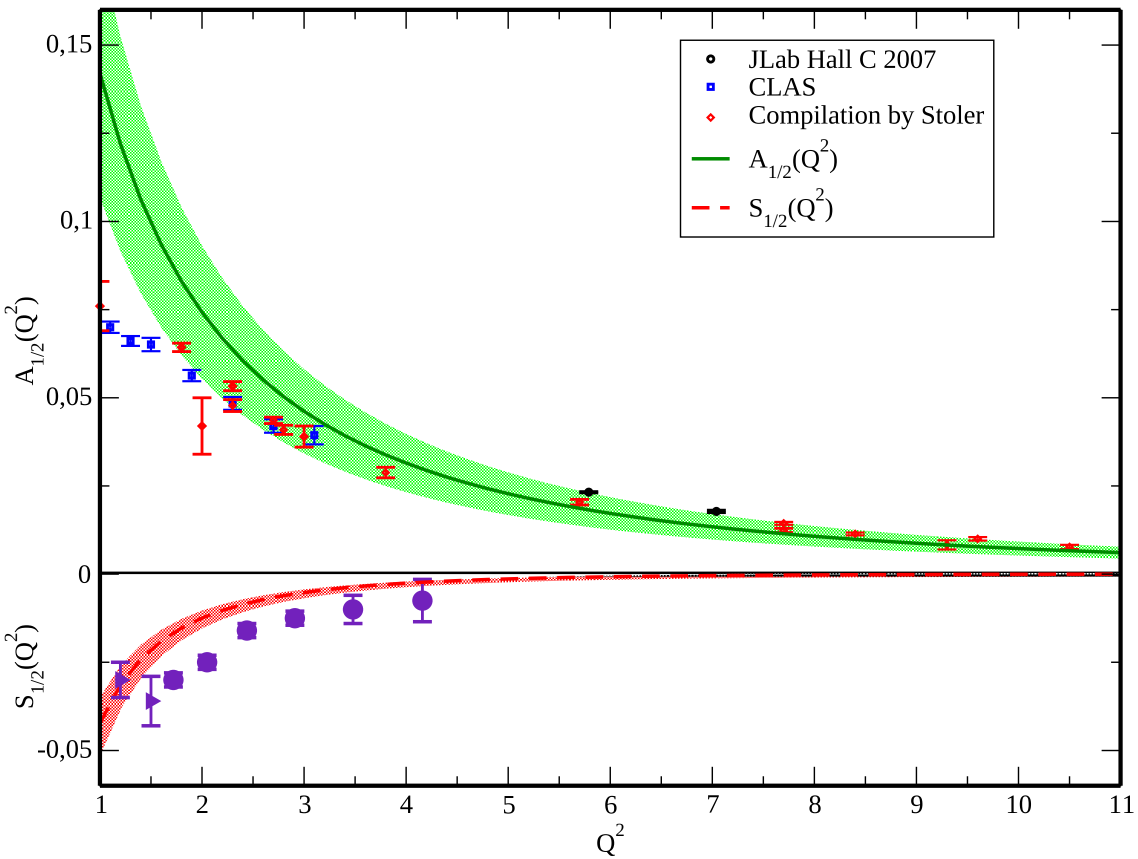
DVCS

$$H^q(x, \Delta^2, \xi) = \int [dy][dz]\, \phi_N^*(y_1, y_2, y_3, \mu^2)\, T_H^q([y], [z], x, \Delta^2, \xi)\, \phi_N(z_1, z_2, z_3, \mu^2) + \text{HT}$$

Δ^2 large

Hoodbhoy, Ji & Yuan

$\gamma^* N \to N^*(1535)$



Conclusions

- Simulations at the physical pion mass (with Wilson-type fermions) are progressing
  - Improvement of algorithms
  - Increase of computing power

- Structure of the nucleon changes significantly between $m_\pi \approx 300$ MeV and the physical pion mass

  Entering logarithmic mass dependence reflecting the pion cloud — ChPT

- Lattice calculations provide insight into nucleon structure not accessible by experiment

- Our final aim is to be better than experiment. This is still a long way to go. Costs [Pflops×y]:

a [fm]	L [fm]			
	3	4	5	6
0.08	0.02	0.07	0.21	0.53
0.05	0.28	1.18	3.60	8.96
0.03	6	25	80	190

↑ today