Geometry and causal flux in multi-loop Feynman diagrams.

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1. **Motivation**

2. **Loop-Tree Duality**
   A. Nested residues
   B. Causality at integrand level
   C. Geometry and causality
   D. Quantum algorithms for causal reconstruction

3. **Conclusions**
Motivation

- Experiments are collecting more and more data
- **More data available** → **More accurate results!!**
- In 2012, LHC discovered a «SM-Higgs»-compatible resonance in the 125 GeV region
- A full determination of its properties requires precision measurements
- Precision measurements ↔ **Accurate theoretical results!!**

Small effects can be discovered only if theoretical predictions match experimental accuracy…
Motivation

• What we need to calculate? Cross-sections and production/decay rates at colliders

• How to calculate? Use the parton model and SM (or other QFT…)

\[
\frac{d\sigma}{d^2 q_T \, dM^2 \, d\Omega \, dy} = \sum_{a,b} \int \, dx_1 \, dx_2 \, f_a^{h_1}(x_1) \, f_b^{h_2}(x_2) \, \frac{d\hat{\sigma}_{ab} \to V + X}{d^2 q_T \, dM^2 \, d\Omega \, dy}
\]

PDFs (non-perturbative) Partonic cross-section (perturbative)

• Intermediate steps contain mathematical issues

• Need for regularization \( \rightarrow \) DREG

• It changes the number of space-time dimensions in order to achieve integrability

\[
\mathcal{O}_d[F] = \int d^d x \, F(x) \quad d = 4 - 2\varepsilon
\]
Motivation

- **Parton Distribution Functions:**
  - Extracted from data (fits, neural networks, etc)
  - Scale dependence determined by DGLAP equations (perturbative kernels)
  - Several PDFs sets available in the market (different datasets, models, approximations, etc)

- **Partonic Cross Sections:**
  - Directly obtained from QFT (applying perturbative methods)
  - Several ingredients required (for higher-orders)

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**Loop contributions** (quantum fluctuations of vacuum)

**Real corrections** (additional particles)

**Counter-terms** (fix the problems of the other two)

**Appears after integration**

**FINITE NUMBER** (compare to experiments)

**CANCELLATION AFTER INTEGRATION**

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LTD features

• Loop amplitudes are a bottleneck in current high-precision computations
• Presence of **singularities and thresholds** prevents direct numerical implementations
• Well-known theorems (KLN) guarantee the **cancellation of singularities for physical observables**
• Real-radiation contributions are defined in **Euclidean space** (i.e. phase-space integrals)

Graphical representation of one-loop opening into trees (original idea by Catani et al '08)

**LOOP AMPLITUDES**
- Virtual internal momenta
- Defined in Minkowski space-time

**DUAL AMPLITUDES**
- On-shell cut momenta
- Defined in Euclidean space-time

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LTD roadmap

**Virtual Amplitudes**

**Real Amplitudes**

**Mappings**

**Combination at integrand level**

**Tree-level objects in Euclidean space**

- JHEP 09 (2008) 065
- JHEP 10 (2010) 073

**Localization of IR/thresholds**

- JHEP 11 (2014) 014

**UV counter-terms**

**Local 4-dimensional representation of cross-sections**

**FDU framework!**

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Nested residues: Details

- **Starting point:** multiloop Feynman integrals and scattering amplitudes
- **Iterated** application of the Cauchy residue theorem to remove one DOF for each loop momenta

**Notation setup**

- Sets of momenta: \( i_s \in s \)
- Combination of external momenta: \( q_{i_s} = \ell_s + k_{i_s} \)
- Loop momentum (integration): \( q_{i_s,0} = \sqrt{q_{i_s}^2 + m_{i_s}^2 - i\epsilon} \)

**Multiloop diagram**

- Using this notation, we write *any* L-loop N-particle scattering amplitude:

\[
A_N^{(L)}(1,\ldots,n) = \int_{\ell_1,\ldots,\ell_L} \mathcal{N} (\{\ell_i\}_L, \{p_j\}_N) G_F(1,\ldots,n) \quad \text{with} \quad G_F(1,\ldots,n) = \prod_{i=1}^{\text{un}} (G_F(q_i))^{a_i}
\]

D-dimensional loop momenta

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Nested residues: Details

- **Starting point:** multiloop Feynman integrals and scattering amplitudes
- **Iterated** application of the Cauchy residue theorem to remove one DOF for each loop momenta

**Iterated application of Cauchy’s theorem**

\[
G_D(1, \ldots, r; \eta, \mathbf{u}) = -2\pi i \sum_{i_r \in r} \text{Res}(G_D(1, \ldots, r-1, r, n), \text{Im}(\eta \cdot q_{i_r}) < 0)
\]

**Multiloop diagram**

- Dual representation for L-loop amplitudes is obtained after the L\(^{th}\) residue evaluation
- **Equivalent to:** “**Number of cuts equal number of loops**”
- **Sum over all possible poles is implicit:** some contributions vanish inside each iteration

**Iterated residues** (all the poles)
- **Nested residues** (only physical ones)
- “Displaced poles”

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Nested residues: Compact representations

- Cancellation of displaced poles leads to very compact formulae for the dual representation:

\[ A_{\text{MLT}}^{(L)}(1, 2, \ldots, L+1) = \sum_{i=1}^{L+1} A_D(1, \ldots, i-1, i+1, \ldots, L+1; i) \]

\[ = \sum_{i=1}^{L+1} \int_{i_1, \ldots, i_L} A_D(1, \ldots, i-1, i+1, \ldots, L+1; i) \]

- We define the Maximal Loop Topology (MLT) as a building block to describe multi-loop amplitudes
- **Important:** "Any one and two-loop amplitude can be described by MLT topologies"

**Remark:** External particles can be attached to each momenta set

**Lines = sets of propagators**

**Defined in Minkowski space**

**Defined in Euclidean space**

**Inductive proofs of these formulae to all-loop orders available in JHEP 02 (2021) 112**
Nested residues: Compact representations

- More complicated topologies can be described by convolutions with MLT-like diagrams

\[ A^{(L)}_{NMLT}(1, \ldots, n, 12) = A^{(2)}_{MLT}(1, 2, 12) \otimes A^{(L-2)}_{MLT}(3, \ldots, n) \]
\[ + A^{(1)}_{MLT}(1, 2) \otimes A^{(0)}(12) \]
\[ \otimes A^{(L-1)}_{MLT}(3, \ldots, \bar{n}) \]

**IMPORTANT FACTORIZATION FORMULAE**
Singular and causal structure is determined by the corresponding sub-topologies

\[ A^{(L)}_{NNMLT}(1, \ldots, n, 12, 23) = A^{(3)}_{NMLT}(1, 2, 3, 12, 23) \otimes A^{(L-3)}_{MLT}(4, \ldots, n) \]
\[ + A^{(2)}_{MLT}(1 \cup 23, 2, 3 \cup 12) \otimes A^{(L-2)}_{MLT}(4, \ldots, \bar{n}) \]

Inductive proofs of these formulae to all-loop orders available in JHEP 02 (2021) 112
Nested residues: Compact representations

- It works also for (much) more complicated topologies!!!

NNNN
Maximal Loop Topologies (6 vertices, L+5 lines)

Thanks to factorization properties, the singular and causal structure is given in terms of simpler objects

Lines = sets of propagators

\[ A_{N^4MLT}^{(L)}(1, \ldots, L+1, 12, 123, 234, J) = A_{N^4MLT}^{(4)}(1, 2, 3, 4, 12, 123, 234, J) \]
\[ \otimes A_{MLT}^{(L-4)}(5, \ldots, L+1) \]
\[ + A_{N^2MLT}^{(3)}(1 \cup 234, 2, 3, 4 \cup 123, 12, J) \]
\[ \otimes A_{MLT}^{(L-3)}(5, \ldots, L+1) \]
Causality at integrand level

- The cancellation of displaced poles implies un-physical terms vanish in the final representation.
- Moreover, there is a strict connection between **aligned contributions** and **causal terms**!!
- **MLT example**: If we sum over all the possible cuts, we get this extremely compact result:

\[
A^{(L)}_{MLT} (1, 2, \ldots, (L + 1)_{-p_1}) = - \int_{\ell_1, \ldots, \ell_L} \frac{1}{x_{L+1}} \left( \frac{1}{\lambda_1^-} + \frac{1}{\lambda_1^+} \right)
\]

with

\[
\lambda_1^\pm = \sum_{i=1}^{L+1} q_{i,0}^{(\pm)} \pm p_{1,0}
\]

and

\[
x_{L+k} = 2^{L+k} \prod_{i=1}^{L+k} q_{i,0}^{(+)}
\]

**CAUSAL PROPAGATORS**
• Similar formulae can be found for NMLT and NNMLT to all loop orders!

\[ A_{\text{NMLT}}^{(L)} (1, 2, \ldots, L + 2) = \int_{\vec{r}_1, \ldots, \vec{r}_L} \frac{2}{x_{L+2}} \left( \frac{1}{\lambda_1 \lambda_2} + \frac{1}{\lambda_2 \lambda_3} + \frac{1}{\lambda_3 \lambda_1} \right) \]

with \[ \lambda_1 = \sum_{i=1}^{L+1} q_{i,0}^{(+)} \]
\[ \lambda_2 = q_{1,0}^{(+)} + q_{2,0}^{(+)} + q_{L+2,0}^{(+)} \]
\[ \lambda_3 = \sum_{i=3}^{L+2} q_{i,0}^{(+)} \]

\[ A_{\text{N^2MLT}}^{(L)} (1, 2, \ldots, L + 3) = -\int_{\vec{r}_1, \ldots, \vec{r}_L} \frac{2}{x_{L+3}} \left[ \frac{1}{\lambda_1} \left( \frac{1}{\lambda_2} + \frac{1}{\lambda_3} \right) \left( \frac{1}{\lambda_4} + \frac{1}{\lambda_5} \right) \right. \]
\[ + \frac{1}{\lambda_6} \left( \frac{1}{\lambda_2} + \frac{1}{\lambda_4} \right) \left( \frac{1}{\lambda_3} + \frac{1}{\lambda_5} \right) + \frac{1}{\lambda_7} \left( \frac{1}{\lambda_2} + \frac{1}{\lambda_5} \right) \left( \frac{1}{\lambda_3} + \frac{1}{\lambda_4} \right) \]

with \[ \lambda_4 = q_{2,0}^{(+)} + q_{3,0}^{(+)} + q_{L+3,0}^{(+)} \]
\[ \lambda_5 = q_{1,0}^{(+)} + q_{L+3,0}^{(+)} + \sum_{i=4}^{L+1} q_{i,0}^{(+)} \]
\[ \lambda_6 = q_{1,0}^{(+)} + q_{3,0}^{(+)} + q_{L+2,0}^{(+)} + q_{L+3,0}^{(+)} \]
\[ \lambda_7 = q_{2,0}^{(+)} + \sum_{i=4}^{L+3} q_{i,0}^{(+)} \]
Causality at integrand level

- This is a Causal Representation and exists for any QFT amplitude!

- Advantages
  1. Causal denominators have **same-sign combinations of on-shell energies** (positive numbers), thus are **more stable numerically**!
  2. **Only physical thresholds remain**; spurious un-physical instabilities are removed!

Without causal representation

With causal representation

White lines = Numerical instabilities
Causality at integrand level: Implementation

- Numerical results in D=4:

**NMLT**
- 3-loop
- 4-loop

**NNMLT**
- 3-loop
- 4-loop

\[ A_{N^{k-1}}^{(L)}(1^2, 2^2, \ldots, L^2; L + 1, \ldots, L + k) \]

\[ = \prod_{i=1}^{L} \frac{\partial}{\partial (q_i^{(+)} q_i^{(-)})} \frac{A_{N^{k-1}}^{(L)}}{A_{N^{k-1}}^{(L)}}(1^2, 2^2, \ldots, L^2; L + 1, \ldots, L + k) \]

Is also causal by construction!
(derivatives preserve denominators)

**Solid lines: LTD**
**Dots: FIESTA**

Setup:

\[ A_{N^{k-1}}^{(L)}(1^2, 2^2, \ldots, L^2; L + 1, \ldots, L + k) \]

Mases:

\[ \{1, 2, \ldots, L\} \leftrightarrow m_4^2 \]

\[ \{L + 1, \ldots, L + k\} \leftrightarrow m_5^2 \]
Further studies were performed with several topological families


Graphical interpretation in terms of entangled thresholds

1. Each causal propagator represents a threshold of the diagram
2. Each diagram contains several thresholds
3. The causal representation involves products of (compatible) thresholds

\[
A_{\text{NMLT}}^{(L)} (1, 2, \ldots, L + 2) = \int \frac{2}{x_{L+2}} \left( \frac{1}{\lambda_1 \lambda_2} + \frac{1}{\lambda_2 \lambda_3} + \frac{1}{\lambda_3 \lambda_1} \right)
\]
**Geometric Algorithm for Causal Reconstruction**

- **Causal representation** obtained directly after summing over all the nested residues

\[
A_N^{(L)}(1, \ldots, L+k) = \sum_{\sigma \in \Sigma} \int \frac{N_{\sigma}(\{q_r(+)\}, \{p_j,0\})}{x_{L+k}} \prod_{i=1}^{k} \frac{1}{-\lambda_{\sigma(i)}} + (\sigma \leftrightarrow \bar{\sigma})
\]

- **Is it possible to do it in other way?**
  - **Geometrical reconstruction**
  - **Algebraic reconstruction** (Lotty)

- **Previous concepts**
  1. **Diagrams** are made of **vertices** and **multi-edges** (bunches of propagators, connecting two given vertices)
  2. **Multi-edges** define a **basis of momenta**, that lead to the "vertex matrix" Defines the casual structure!
  3. **Binary partitions** are given by **subsets of vertices** that splits in two the original diagram **Connected partitions!**
1. **Generate causal propagators**
   - Causal propagators are associated to **binary connected partitions** of the diagram, namely “connected sub-blocks of the diagram”
   - They encode the possible **physical thresholds**
   - Involve a **consistent (aligned) energy flow** through the cut lines

2. **Order of a diagram**: it quantifies the complexity of a given topology
   - $k=1$ for MLT, $k=2$ for NMLT and so on
   - $k = \text{vertices} - 1$
   - A diagram of order $k$ involves **products of $k$ causal propagators**

3. **Geometric compatibility rules**: determine the entangled thresholds
   a) All the multi-edges are cut at least once
   b) Causal propagators do no intersect; i.e. they are associated to disjoint or extended partitions of the diagram
   c) All the multi-edges involved in a causal threshold must carry momenta flowing in the same direction
      - Distinction $\lambda^+ / \lambda^-$

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More detailed explanation
-accepted in PRD-

Presence of intersections

Incompatible causal flux
Example: 1-loop hexagon (6 vertices, 1 external leg per vertex)

Input: vertex definition, i.e. labelling & momentum conservation

Vertex matrix: Basic object to generate the causal representation

Causal representation

Geometric Algorithm for Causal Reconstruction

Input:

- vertex definition, i.e. labelling
- momentum conservation

Generate causal propagators

Generate entangled thresholds (using selection rules)

Causal representation

(+ similar terms ...)

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Geometric Algorithm for Causal Reconstruction

SALIDA3a ([5])

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Quantum Algorithm for Causal Reconstruction

- New technology based on Grover’s algorithm to identify causal flux!
- We assign 1 qubit to each edge, and impose logical conditions to select configurations without closed cycles. **Non-cyclical configurations = Causal flux**
- Important: “loop” refers to integration variables; “eloop” to loops in the graph

\[
N = 2^n \quad \Rightarrow \quad |q\rangle = \frac{1}{\sqrt{N}} \sum_{x=0}^{N-1} |x\rangle
\]

Total number of orderings \((n = n^2\) of edges) \quad Quantum superposition of \(N\) flux configurations

\[
|w\rangle = \frac{1}{\sqrt{r}} \sum_{x \in w} |x\rangle \quad \Rightarrow \quad |\omega\rangle = \frac{1}{\sqrt{N - r}} \sum_{x \notin w} |x\rangle \quad \text{States with non-causal flow}
\]

\[
|q\rangle = \cos \theta |q_\perp\rangle + \sin \theta |w\rangle
\]

“Winning state” (causal flow)

- Grover’s algorithm **enhances** the probability of the winning state by using two operators:

\[
U_w = I - 2|w\rangle\langle w| \quad U_q = 2|q\rangle\langle q| - I \quad \Rightarrow \quad (U_q U_w)^t |q\rangle = \cos \theta_t |q_\perp\rangle + \sin \theta_t |w\rangle
\]

Oracle operator (changes sign of winning states) \quad Diffusion operator (reflects with respect to initial state)

\[
\sin^2 \theta_t \sim 1
\]

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Quantum Algorithm for Causal Reconstruction

- Implemented with Qiskit and run in IBM Q (simulator & real QC)
- Several topologies studied!! **Enhanced performance** with extra-qubits

The selected configurations are exactly $|001>$, $|011>$, $|101>$

*The algorithm identifies the causal flux, relying on geometrical concepts!*
Quantum Algorithm for Causal Reconstruction

- Optimized algorithm based on properties of the adjacency matrix
- Reduced number of qubits (allows to implement more complicated topologies in current devices)
- Successful identification of causal flux!!

Preliminary results!!
To be published soon!!

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Conclusions

- Use LTD to cleverly rewrite Feynman integrals: **Minkowski to Euclidean**
- Achieve **local integrand representations free of IR/UV singularities** for physical observables
- **Novel LTD approach** based on nested residues leads to **manifestly causal representations** of multiloop scattering amplitudes!
- Very compact formulae with **strong physical/conceptual** motivation

**Geometrical rules** select entangled thresholds. **Complete reconstruction** of multiloop amplitudes!

**Quantum algorithms** to speed-up causal flux selection. Exploring new disruptive tools for breaking the precision frontier!!
THANKS!
BACKUP SLIDES.
Nested residues: Displaced poles

- Practical (mathematical) example:
  \[
  f(x) = \frac{1}{(x_1^2 - y_1^2) \cdots (x_L^2 - y_L^2) (z_{L+1}^2 - y_{L+1}^2)}
  \]

  \[
  z_{L+1} = - \sum_{j=1}^{L} x_j + k_{L+1}
  \]

  to calculate
  \[
  I = \left( \prod_{i=1}^{L} \int \frac{dx_i}{2\pi i} \right) f(x)
  \]

- \textbf{1st step:} Apply C.R.T. in \(x_1\), by promoting \(x_1 \in \mathbb{R} \rightarrow \mathbb{C}\) (the other \(x\)'s remain real)

\[
I = - \left( \prod_{i=2}^{L} \int \frac{dx_i}{2\pi i} \right) \sum_{x_{1,j} \in \text{Poles}[f,x_1]} \text{Res} \left( f(x), \{x_1, x_{1,j}\} \right) \theta(-\text{Im}(x_{1,j}))
\]

\[
\sum_{x_{1,j} \in \text{Poles}^{(+)}[f,x_1]} \text{Res} \left( f(x), \{x_1, x_{1,j}\} \right)
\]

\[
Poles^{(+)}[f,x_1] = \{y_1, y_{L+1} - k_{L+1} - x_2 - \ldots - x_L\}
\]

\text{Theta functions removed}

\text{IMPORTANT!} \ x\text{'s are real, } y\text{'s are complex}
**Nested residues: Displaced poles**

- **Practical (mathematical) example:**

  \[
  I = - \left( \prod_{i=2}^{L} \int \frac{dx_i}{2\pi i} \right) \sum_{x_{i,j} \in \text{Poles}^+[f,x_1]} \text{Res}(f(x), \{x_1, x_{1,j}\})
  \]

  \[
  \text{Poles}^+\{f, x_1\} = \{y_1, y_{L+1} - k_{L+1}, -x_2 - \ldots - x_L\}
  \]

- **2\textsuperscript{nd} step:** Apply C.R.T. in \(x_2\), by promoting \(x_2 \in \mathbb{R} \rightarrow \mathbb{C}\) (the other \(x\)'s remain real)

  \[
  \text{Res}(f, \{x_1, \text{Im}(x_1) < 0\}, \{x_2, \text{Im}(x_2) < 0\})
  = \sum_{x_{2,l} \in \text{Poles}\{f,x_1,x_2\}} \text{Res}(\{f, \{x_1, \text{Im}(x_1) < 0\}, \{x_2, x_{2,l}\}\}) \theta(-\text{Im}(x_{2,l}))
  \]

  \[
  \text{Poles}(f, x_1; x_2) = \{\pm y_2, \pm y_1 + y_{L+1} - x_3 - \ldots - x_L + k_{L+1}, \pm y_{L+1} - y_1 - x_3 - \ldots - x_L + k_{L+1}\}
  \]

  **Sum of the residues in \(x_1\) (negative imaginary part)**

\[
\begin{align*}
\text{Res}(f, \{x_1, \text{Im}(x_1) < 0\}) &= \frac{1}{2y_1 (x_2^2 - y_2^2) \ldots (x_L^2 - y_L^2) ((y_1 + x_2 + \ldots + x_L - k_{L+1})^2 - y_{L+1}^2)} \\
&\quad + \frac{1}{2y_{L+1} ((y_{L+1} + k_{L+1} - x_2 - \ldots - x_L)^2 - y_1^2)(x_2^2 - y_2^2) \ldots (x_L^2 - y_L^2)}
\end{align*}
\]

**Theta functions remain!**

**All the possible poles:**

**SIGN OF IMAGINARY PART + or - !!!**
Nested residues: Displaced poles

- Practical (mathematical) example:

\[
\text{Res} \left( \text{Res}(f, \{x_1, \text{Im}(x_1) < 0\}), \{x_2, \text{Im}(x_2) < 0\} \right) = \sum_{x_2, i \in \text{Poles}[f, x_1, x_2]} \text{Res} \left( \text{Res}(f, \{x_1, \text{Im}(x_1) < 0\}), \{x_2, x_{2,i}\} \right) \theta(-\text{Im}(x_{2,i}))
\]

- 3rd step: Collect the different contributions according to \(\theta(-\text{Im}(x_{2,i}))\):

\[
\text{Res} \left( \text{Res}(f, \{x_1, \text{Im}(x_1) < 0\}), \{x_2, y_2\} \right) = \frac{1}{4y_1y_2(x_3^2 - y_3^2)\ldots(x_L^2 - y_L^2)((y_1 + y_2 + x_3 + \ldots + x_L - k_{L+1})^2 - y_{L+1}^2)}
\]

\[
+ \frac{1}{4y_{L+1}y_2((y_{L+1} - y_2 - x_3 - \ldots - x_L + k_{L+1})^2 - y_1^2)\ldots(x_L^2 - y_L^2)}
\]

\[
\text{Res} \left( \text{Res}(f, \{x_1, \text{Im}(x_1) < 0\}), \{x_2, y_1 + y_{L+1} - x_3 - \ldots - x_L + k_{L+1}\} \right) = \frac{1}{4y_1y_3((y_1 + y_L + y_{L+1} - x_3 - \ldots - x_L + k_{L+1})^2 - y_2^2)(x_3^2 - y_3^2)\ldots(x_L^2 - y_L^2)}
\]

\[
[\text{Res} \left( \text{Res}(f, \{x_1, y_1\}), \{x_2, y_{L+1} - y_1 - x_3 - \ldots - x_L + k_{L+1}\} \right) + \text{Res} \left( \text{Res}(f, \{x_1, y_{L+1} - x_2 - \ldots - x_L + k_{L+1}\}), \{x_2, y_{L+1} - y_1 - x_3 - \ldots - x_L + k_{L+1}\} \right) \theta(\text{Im}(y_1 - y_{L+1}))]
\]

Theta functions are trivially 1: y’s have negative imaginary part, x’s are real

Only sums of y’s!! ALIGNED CONTRIBUTIONS

Different-sign combinations of y’s: NON-TRIVIAL THETA!

DISPLACED POLES: VANISH!!

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Nested residues: Displaced poles

**Theorem:** Given a generic* rational function

\[ F(x_i, x_j) = \frac{P(x_i, x_j)}{(x_i - a_i)^2 - y_i^2)^\gamma_i(1 + x_j - a_{ij})^2 - y_k^2)^\gamma_k} \]

then:

\[
\text{Res}(\text{Res}(F(x_i, x_j), \{x_i, y_i + a_i\}), \{x_j, y_k - y_i + a_{ij} - a_i\})
\]

\[
= -\text{Res}(\text{Res}(F(x_i, x_j), \{x_i, y_k - x_j + a_{ij}\}), \{x_j, y_k - y_i + a_{ij} - a_i\})
\]

- **Physical consequences:**

  1. **Displaced poles** are associated to **un-physical** contributions:

     "they can not be mapped into cuts"

  2. After applying C.R.T. to all the loop momenta and **summing over the physical poles**:

     "only same-sign combinations of \( q_{k,0}^{(+)} \) remain"

---

Geometry and causal flux in multi-loop Feynman diagrams - G. Sborlini (DESY)
• **Theorem**: Given a generic* rational function $F(x_i, x_j) = \frac{P(x_i, x_j)}{(x_i - a_i)^2 - y_i^2)^\gamma_i ((x_i + x_j - a_{ij})^2 - y_k^2)^\gamma_k}$

then: \[ \text{Res}(\text{Res}(F(x_i, x_j), \{x_i, y_i + a_i\}), \{x_j, y_k - y_i + a_{ij} - a_i\}) \]
\[ = -\text{Res}(\text{Res}(F(x_i, x_j), \{x_i, y_k - x_j + a_{ij}\}), \{x_j, y_k - y_i + a_{ij} - a_i\}) \]

• **Mathematical consequences:**

1. In each iteration of C.R.T., contributions with **different sign combinations of** $y$’s **vanish**
2. Thus, after iterating over all integration variables, **only same-sign combinations of** $y$’s **remain**

**Example:**
$L = 2$

\[
\text{Res}(\text{Res}(f, \{x_1, \text{Im}(x_1) < 0\}), \{x_2, \text{Im}(x_2) < 0\})
= \frac{1}{4y_1y_2((y_1 + y_2 - k_3)^2 - y_3^2)} + \frac{1}{4y_2y_3((y_3 + y_1 + k_3)^2 - y_2^2)}
+ \frac{1}{4y_1y_3((y_3 - y_2 + k_3)^2 - y_2^2)}
= -\frac{1}{8y_1y_2y_3} \left( \frac{1}{y_1 + y_2 + y_3 - k_3} + \frac{1}{y_1 + y_2 + y_3 + k_3} \right)
\]

**Connection to QFT**

\[
y_i \leftrightarrow q_{i,0}^{(+)} = \sqrt{q_i^2 + m_i^2 - i0}
x_i \leftrightarrow q_{i,0}
a_i \leftrightarrow \{k_{m,0}\}
\]
Brief history of LTD-based methods

- Foundational paper: a new way to decompose loop amplitudes

- Application of Cauchy theorem taking care of Feynman prescription: leads to a new prescription!

Feynman integral

\[ L^{(1)}(p_1, \ldots, p_N) = \int_\ell \prod_{i=1}^N G_F(q_i) = \int_\ell \prod_{i=1}^N \frac{1}{q_i^2 - m_i^2 + i0} \]

Dual integral

\[ L^{(1)}(p_1, \ldots, p_N) = -\sum_{i=1}^N \int_\ell \delta(q_i) \prod_{j=1, j\neq i}^N G_D(q_i; q_j) \]
Brief history of LTD-based methods

- Extension to more general amplitudes, including possible local UV counter-terms
- Two-loop formula (2010)

$$L_{(2)}(p_1, p_2, \ldots, p_N) = \int \int_{\ell_1 \ell_2} \{-G_D(\alpha_1) G_F(\alpha_2) G_D(\alpha_3) + G_D(\alpha_1) G_D(\alpha_2 \cup \alpha_3) + G_D(\alpha_3) G_D(-\alpha_1 \cup \alpha_2)\}$$

- Formalism for dealing with higher-order poles (2012)

A tree-loop duality relation at two loops and beyond

Geometry and causal flux in multi-loop Feynman diagrams - G. Sborlini (DESY)
• Analysis of singular structures of loop amplitudes in LTD representation

• First clues for real-dual integrand level combination

• Forward (backward) on-shell hyperboloids associated with positive (negative) energy solutions

• Forward-backward intersections are physical divergences; FF cancel among them
Brief history of LTD-based methods

- Towards the computation of physical observables in four space-time dimensions
- Tested on toy scalar model; local cancellation of IR divergences

Towards gauge theories in four dimensions

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Abstract: The abundance of infrared singularities in gauge theories due to unresolved emission of massless particles (soft and collinear) represents the main difficulty in perturbative calculations. They are typically regularized in dimensional regularization, and their subtraction is usually achieved independently for virtual and real corrections. In this paper, we introduce a new method based on the loop-tree duality (LTD) theorem to accomplish the summation over degenerate infrared states directly at the integrand level such that the cancellation of the infrared divergences is achieved simultaneously, and apply it to reference examples as a proof of concept. Ultraviolet divergences, which are the consequence of the point-like nature of the theory, are also reinterpreted physically in this framework. The proposed method opens the intriguing possibility of carrying out purely four-dimensional implementations of higher-order perturbative calculations at next-to-leading order (NLO) and beyond free of soft and final-state collinear subtractions.

Keywords: NLO Computations

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Introduction of real-dual mappings, to achieve a local cancellation of IR singularities!

\[ p_1^{\mu} = q_1^{\mu}, \]
\[ p_1^{\mu} = q_3^{\mu} + \alpha_1 p_2^{\mu} = p_1^{\mu} - q_1^{\mu} + \alpha_1 p_2^{\mu}, \]
\[ p_2^{\mu} = (1 - \alpha_1) p_2^{\mu}, \quad \alpha_1 = \frac{q_3^2}{2q_3 \cdot p_2}, \]

- Purely four-dimensional representation of cross-sections
- First study of dual UV local counter-terms:

\[ F_{\text{UV}} = \int \frac{1}{(q_{\text{UV}}^2 - \mu_{\text{UV}}^2 + i\epsilon)^2} \]
Brief history of LTD-based methods

- Towards the computation of physical observables in four space-time dimensions
- **Tested on toy scalar model; local cancellation of IR divergences**

Partonic cross sections are obtained from QFT (applying perturbative methods)

- **Integrand-level** cancellation of IR and UV singularities!
- **No need of integrated counter-terms**
- Purely four-dimensional integration (**no DREG!**)

**FIRST APPROACH TO LOCAL REPRESENTATIONS!!**
• Development of the **Four Dimensional Unsubtraction (FDU)** framework @ NLO
• **Ingredients for local cancellation of IR singularities**
• Smooth numerical implementation (**massive to massless transition**)

- **Integrand-level cancellation of IR and UV singularities, for physical processes!**
- **No need of integrated counter-terms (up to NLO)**
- Purely four-dimensional integration (**no DREG!**)
Brief history of LTD-based methods

- Full analysis of Higgs decays at two-loop (inclusion of EW effects)
- First realization of local UV counter-terms at two-loop level

Locality explored at two-loops… what’s next?

- New singular structures arise beyond one-loop
- More diagrams, more variables… starts to be cumbersome!
- Explore novel representations of the integrands
- Point towards fully local cancellations of IR/UV singularities

UNDERSTANDING SINGULARITIES IS CRUCIAL!! EXPLORE THEM!!
Brief history of LTD-based methods

2008
2010-2012
2014
2015
2016
2017
2018-2019
2020-2021

Jan. ‘20

Jun. ‘20

Oct. ‘20

Geometry and causal flux in multi-loop Feynman diagrams - G. Sborlini (DESY)