Graviton scattering amplitudes in first quantisation

James P. Edwards
ifm.umich.mx/∼jedwards

XIX Mexican School of Particles and Fields

August 2021

In collaboration with **Christian Schubert** (UMSNH) and Memo Estrada Gonzalez (UMSNH)

Thanks to CONACyT for funding through a "Ciencias de Frontera" project

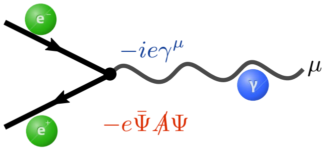
Outline

Field theory

Let's revise standard QED: a Dirac field coupled to a gauge potential $\mathrm{A}(x) = A_\mu(x)dx^\mu$ described by the familiar Dirac action with *"minimal coupling"*

$$S[\Psi, A] = \int d^D x \left[-\frac{1}{4}\mathrm{tr}F^2 + \bar{\Psi}\big(i\not{D} - m\big)\Psi \right]$$

where the covariant derivative is $D_\mu := \partial_\mu + ieA_\mu$, which couples the fields together and produces the interaction vertex of the quantum theory

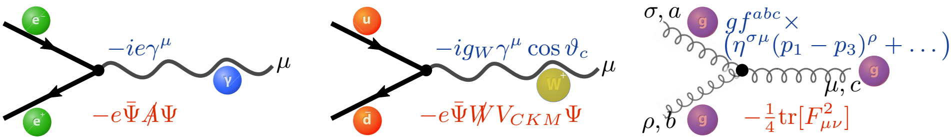


We can interpret "particles" as excitations of the quantum field about *"the vacuum,"* a state of minimal energy denoted by $|0\rangle$:

$$|p, \sigma\rangle = \widehat{a}^\dagger_{p,\sigma}|0\rangle \qquad\qquad |k, h\rangle = \widehat{\alpha}^\dagger_{k,h}|0\rangle$$

Interactions

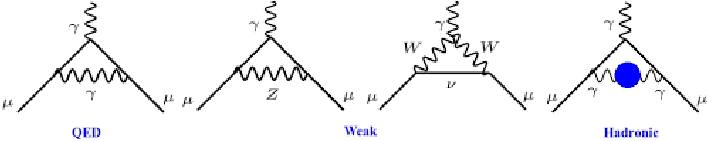
It can be useful to separate the action into a "free" part and an "interaction" part which couples the fields together – these indicate the interaction vertices of the theory:



Working with canonical quantisation in the interaction picture the fundamental objects of interest are **correlation functions**:

$$\langle\Omega|T\{\hat{\Psi}(x_1)\dots\hat{\bar{\Psi}}(x_n)\dots\hat{A}_{\mu_N}(x_N)\}|\Omega\rangle\,.$$

In perturbation theory we use the above vertices to form Feynman diagrams that contribute order by order in the coupling constants:



Hunting simplifications

Perturbative calculations of amplitudes often involve tedious manipulation of long expressions....

But lead to surprisingly simple final results:

> **Weisskopf** *was so unhappy with the conventional calculation of the Compton cross section that he asked me, and several other Ph.D. students, to find a better way to get the* **simple final result**. *(R. Stora).*

There is also a **factorial explosion** in the number of diagrams at a given loop order that makes organisation of the calculation progressively more difficult.

Example: electron $g-2$ (QED):

- **1** diagram at one-loop order: $\frac{g-2}{2}+=\frac{1}{2}\frac{\alpha}{\pi}$ (Schwinger[1])
- **7** diagrams at two-loops: $\frac{g-2}{2}+=-0.328\ldots\left(\frac{\alpha}{\pi}\right)^2$ (Petermann / Sommerfeld[2])
- **72** diagrams at three-loops: $\frac{g-2}{2}+=1.181\ldots\left(\frac{\alpha}{\pi}\right)^3$ (Laporta, Remiddi[3])
- **891** diagrams at 4-loops: $\frac{g-2}{2}+=-1.912\ldots\left(\frac{\alpha}{\pi}\right)^4$ (Laporta[4])
- **12672** diagrams at 5-loop order (some numerical evaluations)

[1] Schwinger, J.S. Phys. Rev. 73, (1948), 416–417.

[2] Petermann, Helv. Phys. Acta 30, (1957), 407–408 & Sommerfield, C.M. Ann. Phys. 5 (1958), 26–57

[3] Laporta, S., Remiddi, E. Phys. Lett. B 379, (1996), 283–291

[4] Laporta, S. Phys. Lett. B 772 (2017), 232–238

Miraculous cancellations

Natural question: *¿Why are these final results so small?*
Aside from this, for $g-2$ there is also a cancellation of spurious UV and IR divergences between diagrams[5] due to gauge symmetry.

[5]See G. V. Dunne et al., J. Phys. Conf. Ser.37, 59–72 (2006) for a discussion on the influence of gauge cancellations on divergence structure in QFT which is still not very well understood.

Miraculous cancellations

Natural question: *¿Why are these final results so small?*
Aside from this, for $g-2$ there is also a cancellation of spurious UV and IR divergences between diagrams[5] due to gauge symmetry.

Is this gauge invariance also responsible for the spectacular cancellations of the finite part of the amplitude?

Motivated P. Cvitanović to examine the contributions from individual "**gauge-sets**" – sets of gauge invariant diagrams. The contributions of different gauge sets are very close to being **integer multiples** of $\pm\frac{1}{2}\times(\frac{\alpha}{\pi})^n$!

Cvitanović also made the conjecture that this suggests that the perturbative series for $g-2$ may in fact converge, having asymptotic form

$$\frac{g-2}{2} = \sum_{n=1}^{\infty} c_n \Big(\frac{\alpha}{\pi}\Big)^{2n}\,, \qquad |c_n| \sim \frac{n}{2}\,, \tag{1}$$

rather than the usual factorial growth.

[5] See G. V. Dunne et al., J. Phys. Conf. Ser.37, 59–72 (2006) for a discussion on the influence of gauge cancellations on divergence structure in QFT which is still not very well understood.

Graviton amplitudes

In the case of gravity the explosion in Feynman diagrams is even more apparent.

We can see this by expanding the Einstein-Hilbert action about a flat space metric,

$$S_{\rm EH} = \frac{2}{\kappa^2}\int d^D x\,\sqrt{-g}R\,, \qquad g_{\mu\nu}(x) \to \eta_{\mu\nu} + \kappa h_{\mu\nu}(x)\,.$$

After fixing the diffeomorphism symmetry for the metric perturbation we get a propagator (de Donde gauge)

$$P_{\mu\nu,\alpha\beta}(k) = \frac{1}{2}\frac{i}{k^2+i\epsilon}\Big[\eta_{\mu\alpha}\eta_{\nu\beta} + \eta_{\mu\beta}\eta_{\nu\alpha} - \frac{2}{D-2}\eta_{\mu\nu}\eta_{\alpha\beta}\Big]\,.$$

For later: *The final "trace term" makes the organisation of the perturbative expansion very different to the string theory case.*

Graviton amplitudes

In the case of gravity the explosion in Feynman diagrams is even more apparent.

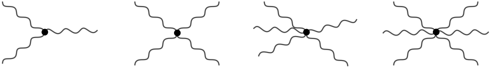
We can see this by expanding the Einstein-Hilbert action about a flat space metric,

$$S_{\rm EH} = \frac{2}{\kappa^2}\int d^Dx\,\sqrt{-g}R\,, \qquad g_{\mu\nu}(x) \to \eta_{\mu\nu} + \kappa h_{\mu\nu}(x)\,.$$

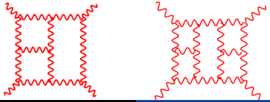
After fixing the diffeomorphism symmetry for the metric perturbation we get a propagator (de Donde gauge)

$$P_{\mu\nu,\alpha\beta}(k) = \frac{1}{2}\frac{i}{k^2+i\epsilon}\Big[\eta_{\mu\alpha}\eta_{\nu\beta} + \eta_{\mu\beta}\eta_{\nu\alpha} - \frac{2}{D-2}\eta_{\mu\nu}\eta_{\alpha\beta}\Big]\,.$$

For later: *The final "trace term" makes the organisation of the perturbative expansion very different to the string theory case.*



There are infinitely many vertices whose Feynman rules involve large ($\mathcal{O}(10^{2+})$) kinematic factors involving the participating momenta:



Strings and fields

During the development of string theory (around the 70s) it was discovered that its **infinite** tension limit is closely related to the quantum field theory of point particles. J. Scherk & H. H. Schwarz, T. Yoneya, J. L. Gervais, A. Neveu.

String theory is a first quantised theory – suggests we can do S-matrix calculations by analysing scattering amplitudes of a single string in the appropriate limit.

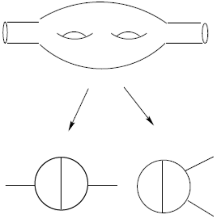


Figure: Amplitude in ϕ^3 theory reorganised into a single string process: there are in general fewer string diagrams corresponding to the field theory ones.

This relation was studied more systematically in the 90s by Bern and Kosower for QCD and Bern, Dunbar, Shimada and Norridge for gravity. Final products are so-called "**Master Formulae**" and auxiliary rules which can be used to generate field theory

String theory amplitudes

String theory amplitudes can be written in Polyakov's formulation as a path integral over string worldsheets Σ involving insertions of **vertex operators** for external states (here for bosonic strings):

$$\langle V_1(k_1,\varepsilon_1)\ldots V_N(k_1,\varepsilon_N)\rangle \sim \int \mathcal{D}h(\tau,\sigma)\int \mathcal{D}X(\tau,\sigma)V_1(k_1,\varepsilon_1)\ldots V_N(k_1,\varepsilon_N)\mathrm{e}^{-S[X,h]}$$

with worldsheet action

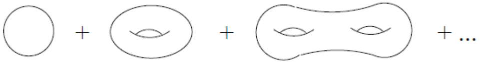
$$S[X,h] = \frac{1}{4\pi\alpha'}\int_\Sigma d^2\sigma\sqrt{h}h^{\alpha\beta}\partial_\alpha X\cdot\partial_\beta X$$

and where for scalar, photon and **graviton** states we have

$$V^\phi[k] = \int_{\partial\Sigma} d\tau\,\mathrm{e}^{ik\cdot X(\tau,\sigma^\pm)} \qquad k^2 = -\frac{1}{\alpha'}$$

$$V^\gamma[k,\varepsilon] = \int_{\partial\Sigma} d\tau\,\varepsilon\cdot\dot{X}(\tau)\mathrm{e}^{ik\cdot X(\tau,\sigma^\pm)} \qquad k^2 = 0 = k\cdot\varepsilon$$

$$\mathbf{V}^{\mathbf{g}}[\mathbf{k},\varepsilon] = \int_{\boldsymbol{\Sigma}} \mathbf{d}^{\mathbf{2}}\sigma\,\partial\mathbf{X}(\tau,\sigma)\cdot\varepsilon\cdot\bar{\partial}\mathbf{X}(\tau,\sigma)\mathrm{e}^{\mathbf{ik}\cdot\mathbf{X}(\tau,\sigma)} \qquad \mathbf{k}^{\mathbf{2}} = \mathbf{0} = \mathbf{k}\cdot\varepsilon = \varepsilon\cdot\mathbf{k}$$



Amplitudes on the annulus

On a given Riemann surface the reparameterisation symmetry allows the metric to be **gauge-fixed** to a conformally flat one, and eventually (in the critical dimension) $\int \mathcal{D}h$ reduces to a Riemann integral over a finite space of conformal equivalence classes.

The "matter" path integral is Gaussian and can be computed using Wick's theorem. On the annulus, the fundamental contraction evaluates to a simple function,

$$\begin{aligned}\langle X^\mu(\tau_1)X^\nu(\tau_2)\rangle &= -\eta^{\mu\nu}\Big[\log\big|2\sinh(\tau_1-\tau_2)\big| - \frac{(\tau_1-\tau_2)^2}{\tau} - 4\mathrm{e}^{-2\tau}\sinh^2\big(\tau_1-\tau_2\big)\Big]\\ &\equiv \eta^{\mu\nu}G(\tau_1-\tau_2;\tau)\end{aligned}$$

where $q=\mathrm{e}^{-2\tau}$ is the modulus parameterising the ratio of the annulus' radii.

Amplitudes on the annulus

On a given Riemann surface the reparameterisation symmetry allows the metric to be **gauge-fixed** to a conformally flat one, and eventually (in the critical dimension) $\int \mathcal{D}h$ reduces to a Riemann integral over a finite space of conformal equivalence classes.

The "matter" path integral is Gaussian and can be computed using Wick's theorem. On the annulus, the fundamental contraction evaluates to a simple function,

$$\begin{aligned}\langle X^\mu(\tau_1)X^\nu(\tau_2)\rangle &= -\eta^{\mu\nu}\Big[\log\big|2\sinh(\tau_1-\tau_2)\big| - \frac{(\tau_1-\tau_2)^2}{\tau} - 4\mathrm{e}^{-2\tau}\sinh^2\big(\tau_1-\tau_2\big)\Big] \\ &\equiv \eta^{\mu\nu}G(\tau_1-\tau_2;\tau)\end{aligned}$$

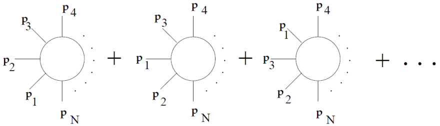
where $q=\mathrm{e}^{-2\tau}$ is the modulus parameterising the ratio of the annulus' radii.

The infinite tension limit corresponds to $\tau\to\infty$ and gives the leading contributions

$$G(\tau_1-\tau_2)\sim \mathrm{const}-\Big[|\tau_1-\tau_2|-\frac{(\tau_1-\tau_2)^2}{\tau}\Big], \quad \dot{G}(\tau_1-\tau_2)\sim -\Big[\sigma(\tau_1-\tau_2)-2\frac{\tau_1-\tau_2}{\tau}\Big]+\dots$$

Universal rules

This treatment is appropriate to produce the one-loop scattering amplitudes for external gauge bosons in **various** field theories:



The generation of field theory amplitudes is based upon the so-called kinematic factor

$$\mathcal{K}_N \sim \int \prod_{i=1}^{N} du_i \prod_{i<j} \exp\left[G_{ij} k_i \cdot k_j + i\dot{G}_{ij}(k_i \cdot \varepsilon_j - k_j \cdot \varepsilon_i) + \ddot{G}_{ij} \varepsilon_i \cdot \varepsilon_j\right]$$

which acts as a kind of *generating function* for amplitudes.

In fact it turns out that there are a set of replacement rules which provide a means to generate amplitudes for different theories.

Comparison with standard approach

It is worth pausing to consider the advantages these “Bern-Kosower Rules” for constructing amplitudes have over the conventional approach to perturbation theory:

- Better organisation of gauge invariance (but see worldline approach later).
- Loop momentum integrals already done – so fewer kinematic invariants that enter intermediate calculations.
- Universal basis of the rules – handling different field theories amounts to adapting the replacement rules for the internal loop.
- Combines nicely with space-time supersymmetry and other internal symmetries (spin, colour etc).

Later on we shall see how to adapt this method to describe graviton amplitudes based only on ϕ^3 vertices and a set of replacement rules for ensuring that the correct particle degrees of freedom circulate in the loop.

Comparison with standard approach

It is worth pausing to consider the advantages these "Bern-Kosower Rules" for constructing amplitudes have over the conventional approach to perturbation theory:

- Better organisation of gauge invariance (but see worldline approach later).
- Loop momentum integrals already done – so fewer kinematic invariants that enter intermediate calculations.
- Universal basis of the rules – handling different field theories amounts to adapting the replacement rules for the internal loop.
- Combines nicely with space-time supersymmetry and other internal symmetries (spin, colour etc).

Later on we shall see how to adapt this method to describe graviton amplitudes based only on ϕ^3 vertices and a set of replacement rules for ensuring that the correct particle degrees of freedom circulate in the loop.

First we look at how to derive this representation of scattering amplitudes **without** *the need to refer to string theory...*

The worldline representation

The **worldline formalism** is an alternative method for field quantisation. The **worldline** method is based on the *first quantisation* of relativistic particles.

In fact it was proposed by **Feynman**[6] and developed in pioneering work by **Strassler**[6]:

PHYSICAL REVIEW VOLUME 80, NUMBER 3 NOVEMBER 1, 1950

Mathematical Formulation of the Quantum Theory of Electromagnetic Interaction

R. P. FEYNMAN*
Department of Physics, Cornell University, Ithaca, New York
(Received June 8, 1950)

The validity of the rules given in previous papers for the solution of problems in quantum electrodynamics is established. Starting with Fermi's formulation of the field as a set of harmonic oscillators, the effect of the oscillators is integrated out in the Lagrangian form of quantum mechanics. There results an expression for the effect of all virtual photons valid to all orders in $e^2/\hbar c$. It is shown that evaluation of this expression as a power series in $e^2/\hbar c$ gives just the terms expected by the aforementioned rules.
In addition, a relation is established between the amplitude for a given process in an arbitrary unquantized potential and in a quantum electrodynamical field. This relation permits a simple general statement of the laws of quantum electrodynamics.
A description, in Lagrangian quantum-mechanical form, of particles satisfying the Klein-Gordon equation

I. INTRODUCTION

IN two previous papers[1] rules were given for the calculation of the matrix element for any process in electrodynamics, to each order in $e^2/\hbar c$. No complete

net effect of the field is a delayed interaction of the particles. It is possible to do this easily only if it is not necessary at the same time to analyze completely the motion of the particles. The only advantage in our problems of the form of quantum mechanics in C is to

APPENDIX A. THE KLEIN-GORDON EQUATION

In this Appendix we describe a formulation of the equations for a particle of spin zero which was first used to obtain the rules given in II for such particles. The complete physical significance of the equations has not been analyzed thoroughly so that it may be preferable to derive the rules directly from the second quantization formulation of Pauli and Weisskopf. This can be done in a manner analogous to the derivation of the rules for the Dirac equation given in I or from the Schwinger-Tomonaga formulation[2] in a manner described, for example, by Rohrlich.[18] The formulation given here is therefore not necessary for a description of spin zero particles but is given only for its own interest as an alternative to the formulation of second quantization.
We start with the Klein-Gordon equation

$$(i\partial/\partial x_\mu - A_\mu)^2\psi = m^2\psi \quad (1A)$$

for the wave function ψ of a particle of mass m in a given external potential A_μ. We shall try to represent this in a manner analogous

$$\varphi(x, u+\epsilon) = \int \exp i\epsilon\left[-\frac{(x_\mu - x_\mu')^2}{2\epsilon^2} - \frac{1}{2}\left(\frac{x_\mu - x_\mu'}{\epsilon}\right)(A_\mu(x) + A_\mu(x'))\right] \cdot \varphi(x', u)d^4\tau_{x'}(2\pi i\epsilon)^{-1}(-2\pi i\epsilon)^{-1} \quad (3A)$$

where $(x_\mu - x_\mu')^2$ means $(x_\mu - x_\mu')(x_\mu - x_\mu')$, $d^4\tau_{x'} = dx_1'dx_2'dx_3'dx_4'$ and the sign of the normalizing factor is changed for the x_4 component since the component has the reversed sign in its quadratic coefficient in the exponential, in accordance with our summation convention $a_\mu b_\mu = a_4b_4 - a_1b_1 - a_2b_2 - a_3b_3$. Equation (3A), as can be verified readily as described in C, Sec. 6, is equivalent to first order in ϵ, to Eq. (2A). Hence, by repeated use of this equation the wave function at $u_0 = n\epsilon$ can be represented in terms of that at $u = 0$ by:

$$\varphi(x_{\nu,n}, u_0) = \int \exp -\frac{i\epsilon}{2}\sum_{i=1}^{n}\left[\left(\frac{x_{\mu,i} - x_{\mu,i-1}}{\epsilon}\right)^2 + \epsilon^{-1}(x_{\mu,i} - x_{\mu,i-1})(A_\mu(x_i) + A_\mu(x_{i-1}))\right] \cdot \varphi(x_{\nu,0}, 0)\prod_{i=0}^{n-1}(d^4\tau_i/4\pi^2\epsilon^2 i). \quad (4A)$$

Nuclear Physics B 385 (1992) 145–184
North-Holland
NUCLEAR PHYSICS B

Field theory without Feynman diagrams: One-loop effective actions

Matthew J. Strassler *
Stanford Linear Accelerator Center, Stanford University, Stanford, CA 94309, USA

Received 20 March 1992
Accepted for publication 23 June 1992

In memory of Brian J. Warr

In this paper the connection between standard perturbation theory techniques and the new Bern–Kosower calculational rules for gauge theory is clarified. For one-loop effective actions of scalars, Dirac spinors, and vector bosons in a background gauge field, Bern–Kosower type rules are derived without the use of either string theory or Feynman diagrams. The effective action is written as a one-dimensional path integral, which can be calculated to any order in the gauge coupling; evaluation leads to Feynman parameter integrals directly, bypassing the usual algebra required from Feynman diagrams, and leading to compact and organized expressions. This formalism is valid off-shell, is explicitly gauge invariant, and can be extended to a number of other field theories.

1. Introduction

In the past year significant advances have been made in techniques for calculating one-loop scattering amplitudes in gauge theories. Following on the successes of

[6]Phys. Rev. **E80**, 3 (1950), 440

[7]Nucl. Phys. **B385**, (1992), 145

Fundamental idea

Here we illustrate the simplest construction: of the effective action in the QED of a scalar field coupled to an (Abelian) electromagnetic potential.
Integrating over the degrees of freedom of the scalar we define ($D \equiv \partial + ieA$)

$$\begin{aligned}\Gamma[A] &= \log\left[\int \mathscr{D}\left(\bar{\Phi}\Phi\right) \exp\left(-\int d^4x\, \bar{\Phi}\left(-D^2 + m^2\right)\Phi\right)\right] \\ &= -\log \mathrm{Det}\left(-D^2 + m^2\right)\end{aligned}$$

Fundamental idea

Here we illustrate the simplest construction: of the effective action in the QED of a scalar field coupled to an (Abelian) electromagnetic potential.
Integrating over the degrees of freedom of the scalar we define $(D \equiv \partial + ieA)$

$$\Gamma[A] = \log\left[\int \mathscr{D}\left(\bar{\Phi}\Phi\right) \exp\left(i\int d^4x\, \bar{\Phi}\left(-D^2+m^2\right)\Phi\right)\right]$$
$$= -\mathrm{Tr}\,\log\left(-D^2+m^2\right)$$
$$= \int_0^\infty \frac{dT}{T}\int d^Dx\,\langle x|\mathrm{e}^{iT\left(-D^2+m^2\right)}|x\rangle,$$

which is finally expressed as a transition amplitude for auxiliary particles traversing closed loops in Minkowski space:

$$\Gamma[A] = \int_0^\infty \frac{dT}{T} e^{-im^2T}\oint_{PBC}\mathscr{D}x\,\mathrm{e}^{iS[x]}$$

with the **worldline action** (scalar QED)

$$S[x] = \int_0^T d\tau\left[\frac{\dot{x}^2}{4} + eA(x)\cdot\dot{x}\right]$$

Worldline representation

The effective action contains the quantum modifications to the dynamics of the gauge field (here to one-loop order). Our worldline representation in spinor QED amounts to

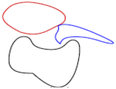
$$\Gamma\left[\mathrm{A}\right] = -\frac{1}{2}\int_0^\infty \frac{dT}{T} e^{-im^2 T} \oint_{PBC} \mathscr{D}x \oint_{ABC} \mathscr{D}\psi \, e^{iS[x,\psi]},$$

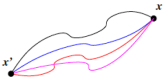
with

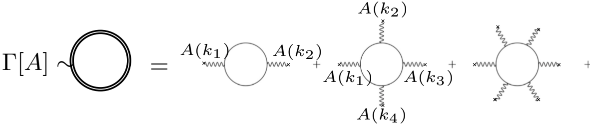
$$S\left[x,\psi\right] = \int_0^T d\tau \left[\frac{\dot{x}^2}{4} + \frac{i}{2}\psi\cdot\dot{\psi} + eA(x)\cdot\dot{x} + ie\psi^\mu F_{\mu\nu}(x)\psi^\nu\right].$$

that describes the propagation of an auxiliary particle whose *closed trajectory* is described by the bosonic variables $x^\mu(\tau)$ with its spin represented by the Grassmann fields $\psi^\mu(\tau)$ along the worldline coupled to an external electromagnetic field, $A_\mu(x)$.

There is also a diagrammatic representation, organised according to the number of interactions with the background field, here given in momentum space as:







1-loop amplitudes

1-loop photon scattering amplitudes in vacuum are extracted by specialising the gauge potential to a sum of asymptotic wavefunctions of fixed momenta and polarisations:

$$A_\mu(x) = \sum_{i=1}^{N} \varepsilon_{i\mu} \mathrm{e}^{ik_i \cdot x} \,.$$

We then expand the "interaction exponential" to multi-linear order in the ε_i. The path integral is conveniently computed in **Euclidean** space...

$$\Gamma\left[\{k_i, \varepsilon_i\}\right] = (-ie)^N \int_0^\infty \frac{dT}{T} e^{-m^2 T} \oint_{PBC} \mathscr{D}x \oint_{ABC} \mathscr{D}\psi \, e^{-\int_0^T d\tau \left[\frac{\dot{x}^2}{4} + \frac{1}{2}\psi \cdot \dot{\psi}\right]} \prod_{i=1}^{N} V[k_i, \varepsilon_i],$$

where the (Euclidean) free particle action is now **quadratic** in the fields and we have introduced a product of "vertex operators" familiar from string theory:

$$V[k, \varepsilon] = \int_0^T d\tau \big[\varepsilon \cdot \dot{x}(\tau) - i\psi(\tau) \cdot f \cdot \psi(\tau)\big] \mathrm{e}^{ik \cdot x(\tau)} \,.$$

Master formula

The path integral is now **Gaussian** and can be evaluated using Wick's theorem after separating off the *zero mode*, $x^\mu(\tau) \to x_0^\mu + q^\mu(\tau)$, based on the **Green functions**

$$\langle q^\mu(\tau_i) q^\nu(\tau_j)\rangle_\perp = -G_{Bij}\eta^{\mu\nu}\,, \quad G_{Bij} \equiv G_B(\tau_i,\tau_j) = |\tau_i - \tau_j| - \frac{(\tau_i-\tau_j)^2}{T}$$

$$\langle \psi^\mu(\tau_i)\psi^\nu(\tau_j)\rangle = \frac{1}{2}G_{Fij}\,, \quad G_{Fij} \equiv G_F(\tau_i,\tau_j) = \sigma(\tau_i - \tau_j)\,.$$

Thus, we recover the Bern-Kosower Master Formula (*scalar QED* for brevity):

$$\Gamma_{\rm scal}[\{k_i,\varepsilon_i\}] = (-ie)^N (2\pi)^D \delta^D\Big(\sum_i k_i\Big) \int_0^\infty \frac{dT}{T}(4\pi T)^{-\frac{D}{2}} \mathrm{e}^{-m^2 T} \prod_{i=1}^N \int_0^T d\tau_i$$
$$\times\, \mathrm{e}^{\frac{1}{2}\sum_{i,j=1}^N G_{Bij} k_i\cdot k_j - 2i\dot{G}_{Bij}\varepsilon_i\cdot k_j + \ddot{G}_{Bij}\varepsilon_i\cdot\varepsilon_j}\Big|_{\mathrm{lin}\,\varepsilon_1\ldots\varepsilon_N}$$

¡Eventual integration over proper time T gives Feynman parameterised denominator!

Master formula

The path integral is now **Gaussian** and can be evaluated using Wick's theorem after separating off the *zero mode*, $x^\mu(\tau) \to x_0^\mu + q^\mu(\tau)$, based on the **Green functions**

$$\langle q^\mu(\tau_i) q^\nu(\tau_j)\rangle_\perp = -G_{Bij}\eta^{\mu\nu}\,, \quad G_{Bij} \equiv G_B(\tau_i,\tau_j) = |\tau_i - \tau_j| - \frac{(\tau_i-\tau_j)^2}{T}$$
$$\langle \psi^\mu(\tau_i)\psi^\nu(\tau_j)\rangle = \frac{1}{2}G_{Fij}\,, \quad G_{Fij} \equiv G_F(\tau_i,\tau_j) = \sigma(\tau_i - \tau_j)\,.$$

Thus, we recover the Bern-Kosower Master Formula (*scalar QED* for brevity):

$$\Gamma_{\rm scal}[\{k_i,\varepsilon_i\}] = (-ie)^N (2\pi)^D \delta^D\Big(\sum_i k_i\Big) \int_0^\infty \frac{dT}{T}(4\pi T)^{-\frac{D}{2}} {\rm e}^{-m^2 T} \prod_{i=1}^N \int_0^T d\tau_i$$
$$\times\, {\rm e}^{\frac{1}{2}\sum_{i,j=1}^N G_{Bij} k_i\cdot k_j - 2i\dot{G}_{Bij}\varepsilon_i\cdot k_j + \ddot{G}_{Bij}\varepsilon_i\cdot\varepsilon_j}\Big|_{{\rm lin}\,\varepsilon_1\ldots\varepsilon_N}$$

(Momentum conservation came from the integral over x_0).

¡Eventual integration over proper time T gives Feynman parameterised denominator!

Gravity replacement rules

The rules for *on-shell* **graviton** scattering were derived from **closed string theory** by **B**ern, **D**unbar and **S**himada in 1993[7] and applied by Dunbar and Norridge to calculate the one-loop four-graviton amplitude in quantum gravity[8].

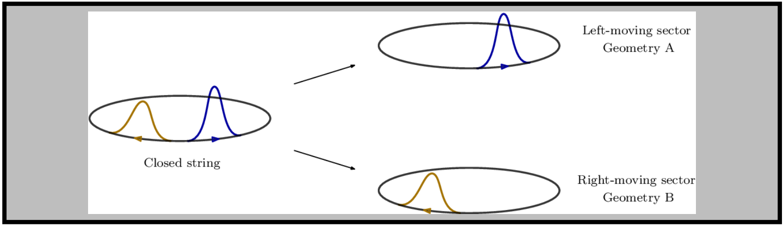
Here will recapitulate the rules rather than deriving them from string theory.

Gravity replacement rules

The rules for *on-shell* **graviton** scattering were derived from **closed string theory** by **B**ern, **D**unbar and **S**himada in 1993[7] and applied by Dunbar and Norridge to calculate the one-loop four-graviton amplitude in quantum gravity[8].

Here will recapitulate the rules rather than deriving them from string theory.

In this case we have two "sectors" that begin life as distinct contributions to the amplitude coming from left-moving and right-moving string modes:



Gravity replacement rules

The rules for *on-shell* **graviton** scattering were derived from **closed string theory** by **B**ern, **D**unbar and **S**himada in 1993[9] and applied by Dunbar and Norridge to calculate the one-loop four-graviton amplitude in quantum gravity[10].

Here will recapitulate the rules rather than deriving them from string theory.

In this case we have two "sectors" that begin life as distinct contributions to the amplitude coming from left-moving and right-moving string modes:

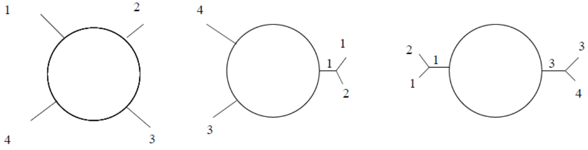
1. The worldsheet Green function, $G(\tau, \sigma)$, becomes a genuine function of two variables, which can be taken to be $\tau + i\sigma$ and $\tau - i\sigma$.
2. We use $\dot{G}$ and $\ddot{G}$ for the derivatives with respect to left-moving variables.
3. We use $\dot{\bar{G}}$ and $\ddot{\bar{G}}$ for the derivatives with respect to right-moving variables.
4. We use H as the derivative with respect to one left-mover and one right-mover.
5. Finally, we decompose the (one-shell) polarisation tensor into these sectors by setting $\varepsilon_{\mu\nu} \to \varepsilon_{\mu}\bar{\varepsilon}_{\nu}$ and reconstruct it by identification $\varepsilon_{\mu}\bar{\varepsilon}_{\nu} \equiv \varepsilon_{\mu\nu}$ at the end.

[9] Z. Bern, D.C. Dunbar, T. Shimada, Phys. Lett. B **312** (1993) 277 [arXiv:9307001 [hep-th]]
[10] D.C. Dunbar, P.S. Norridge, Nucl. Phys. B **433** (1995) 181 [arXiv:9408014 [hep-th]])

BDS Step 1

The N-graviton amplitudes are generated from some "primordial Feynman diagrams" according to

STEP 1: Draw all possible one-loop Φ^3 diagrams with N external legs with appropriate labels:



Notes:

- All **permutations** of external legs are to be included (and labelled as in ordinary Feynman diagrams) – no need to worry about colour ordering as in gauge theory.
- Internal legs attached to "*external trees*" are assigned a label equal to the *smallest* label of the external legs it opens up to.
- So called "*tapole*" diagrams are **ignored**, along with *loops* on external legs

BDS Step 2

We need to calculate the contribution from each diagram after a reduction according to **STEP 2:** Each diagram is associated with an integral (dimensionally regularised so $D = 4 - 2\epsilon$)

$$\mathscr{D} = i\frac{(-\kappa)^N}{(4\pi)^{2-\epsilon}}\Gamma\left[\ell-2+\epsilon\right]\int_0^1 du_{\ell-1}\int_0^{u_{\ell-1}} dx_{\ell-2}\cdots\int_0^{u_2} du_1 \, \frac{\mathcal{K}_{\rm red}}{\left[\sum_{i<j} K_i\cdot K_j G_{ij}\right]^{\ell-2+\frac{\epsilon}{2}}}$$

Notes:

1. The ordering of the parameter integrals (over the u_i) coincides with the ordering of the ℓ lines attached to the (*massless*) loop.
2. K_i is the momentum entering the loop at point i (sum of external momenta entering trees that join to the loop there).
3. $\mathcal{K}_{\rm red}$ is the reduced kinematic factor (arising in the limiting string theory) to be constructed from the kinematic factor – **double copy** –

$$\mathcal{K} = \int \prod_{i=1}^{N} du_i d\bar{u}_i \prod_{i<j}^{N} {\rm e}^{k_i\cdot k_j G_{ij}}\,{\rm e}^{(k_i\cdot\varepsilon_j - k_j\cdot\varepsilon_i)\dot{G}_{ij} - \varepsilon_i\cdot\varepsilon_j\ddot{G}_{ij}}\,{\rm e}^{(k_i\cdot\bar{\varepsilon}_j - k_j\cdot\bar{\varepsilon}_i)\dot{\bar{G}}_{ij} - \bar{\varepsilon}_i\cdot\bar{\varepsilon}_j\ddot{\bar{G}}_{ij}} \times {\rm e}^{-(\varepsilon_i\cdot\bar{\varepsilon}_j + \varepsilon_j\cdot\bar{\varepsilon}_i)H_{ij}}\Big|_{\rm Multi\text{-}linear}$$

that is expanded to multi-linear order in each ε_i and each $\bar{\varepsilon}_i$.

BDS Step 3

To determine the reduced kinematic factor we apply a set of rules to each diagram following

STEP 3: Integration by parts

- We integrate the kinematic expression by parts to remove all $\ddot{G}_{ij}$ and $\ddot{\bar{G}}_{ij}$
- The functions G_{ij} and all second derivatives are taken to be symmetric in their indices whilst first derivatives are anti-symmetric.
- Cross terms are handled according to the following relations:

$$\frac{\partial}{\partial u_k}\dot{\bar{G}}_{ij} = (\delta_{ki} - \delta_{kj})H_{ij}\,, \qquad \frac{\partial}{\partial \bar{u}_k}\dot{G}_{ij} = (\delta_{ki} - \delta_{kj})H_{ij}$$
$$\frac{\partial}{\partial u_k}\ddot{\bar{G}}_{ij} = 0\,, \qquad \frac{\partial}{\partial \bar{u}_k}\ddot{G}_{ij} = 0$$

- Once this has been achieved we can drop the leading exponential factor (with the G_{ij}) and parameter integrals which leaves behind $\mathcal{K}_{\rm red}$.

BDS Step 4

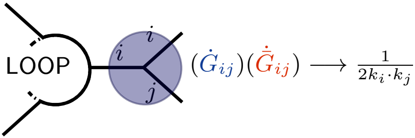
The reduced kinematic factor is now transformed according to two sets of **replacement rules:**

Step 4a: Tree Replacement rules:

- Working from the *outside* in, we **pinch** away trees attached to the loop by the replacement

$$(\dot{G}_{ij})(\ddot{G}_{ij}) \longrightarrow \frac{1}{2k_i \cdot k_j}\,,$$

  (setting all remaining powers to zero) and replacing $j \to i$ in other G_{jk}.



- Iterate until only the loop remains!

BDS Step 4

The reduced kinematic factor is now transformed according to two sets of **replacement rules:**

Step 4b: Loop Replacement rules:

- The loop replacement rules depend on the theory – ¿what particle(s) circulate in the loop? They are independent implementations of the original gauge theory replacement rules in the left and right moving sector.
- For the simplest case, a scalar running in the loop, we make the replacements:

$$G_{ij} \to G_{Bij} = |u_i - u_j| - (u_i - u_j)^2 = (u_i - u_j)(1 - (u_i - u_j))$$

$$\dot{G}_{ij} \to -\frac{1}{2}\dot{G}_{Bij} = -\frac{1}{2}(\sigma(u_i - u_j) - 2(u_i - u_j))$$

$$\dot{\bar{G}}_{ij} \to -\frac{1}{2}\dot{G}_{Bij} = -\frac{1}{2}(\sigma(u_i - u_j) - 2(u_i - u_j))$$

$$H_{ij} \to \frac{1}{2T}$$

- For a complex scalar, multiply the whole expression by 2.

At this stage we have reduced $\mathcal{K}_{\text{red}}$ to a function of external momenta and parameters u_i, so we can compute the integral in $\mathscr{D}$ for the diagram in question.

BDS Step 4

The loop replacement rules can be generalised to include other particles in the loop.

Substitution	Particle Content
$2[S,S]$	complex scalar
$-2[S,F]$	Weyl Fermion
$2[S,V]$	Vector
$-4[V,F]$	gravitino and Weyl Fermion
$4[V,V]$	graviton and complex scalar
$4[V,V]-2[S,S]$	graviton
$-4[V,F]+2[S,F]$	gravitino

Figure: A summary of the replacement rules – the notation $[A,B]$ indicates the substitution rules applied to $\dot{G}$ and to $\ddot{G}$ respectively.

The notation F and V refers to two types of contribution:

$$F = S + C_F\,, \qquad V = S + C_V \tag{2}$$

where S is the scalar loop replacement and C_V and C_F are **cycle replacement rules** that act on "*closed cycles*" like $\dot{G}_{ij}\dot{G}_{jk}\ldots\dot{G}_{si}$.

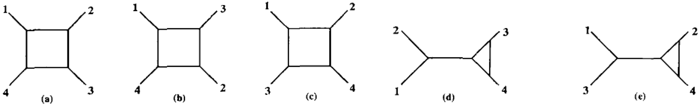
Application to 4-graviton amplitudes

The BDS rules combine perfectly with the **spinor-helicity** method for external gravitons, where we decompose $\varepsilon^{\pm\pm}_{\mu\nu} \to \varepsilon^{\pm}_{\mu}\bar{\varepsilon}^{\pm}_{\nu}$.

- Good choices of the references spinors $|q\rangle$ or $|q]$ can significantly reduce the number of terms in the exponent of $\mathcal{K}$ by nullifying products like $\varepsilon^+_i \cdot \varepsilon^+_j$ or $\varepsilon^+_i \cdot k_j$.

For the amplitude $\mathscr{A}(1^-, 2^+, 3^+, 4^+)$ the standard formalism involves **12** different types of diagram that lead to **54** diagrams with vertices containing $\mathcal{O}(100)$ terms.

- With the BDS formalism (and clever choice of reference vectors) this counting is reduced to just five Φ^3 diagrams that survive the *tree-replacement rules*!



- Their contributions are based on the kinematic factor (no second derivatives)

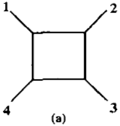
$$\mathcal{K}_{\rm red} = \mathscr{S}(\dot{G}_{13} - \dot{G}_{12})(\dot{G}_{24} - \dot{G}_{23})(\dot{G}_{34} + \dot{G}_{23})(\dot{G}_{34} - \dot{G}_{24})$$
$$\times(\dot{\bar{G}}_{13} - \dot{\bar{G}}_{12})(\dot{\bar{G}}_{24} - \dot{\bar{G}}_{23})(\dot{\bar{G}}_{34} + \dot{\bar{G}}_{23})(\dot{\bar{G}}_{34} - \dot{\bar{G}}_{24})$$

where $\mathscr{S} = \left(\frac{s^2 t}{4}\right)^2 \left(\frac{[24]^2}{[12]\langle 23\rangle\langle 34\rangle[41]}\right)^2$

4-graviton contributions

We look at two examples:

Diagram (a)



1. There are no trees so we only need the loop replacement rules and we find

$$\mathcal{K}_{\text{red}} = 2\mathscr{S} u_2^2 (1-u_3)^2 (u_3-u_2)^4$$

which is inserted into the $\mathscr{D}$ for this diagram (¡finite in $D=4$!):

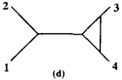
$$\mathscr{D}_a = \frac{2i\kappa^4}{(4\pi)^2}\mathscr{S}\int_0^1 du_3 \int_0^{u_3} du_2 \int_0^{u_2} du_1 \frac{u_2^2(1-u_3)^2(u_3-u_2)^4}{\left[su_1(u_3-u_2)+t(u_2-u_2)(1-u_1)\right]^2}$$

This integral evaluates easily to $\mathscr{D}_a = \frac{2i\kappa^4}{(4\pi)^2}\frac{\mathscr{S}}{840st}$.
Likewise we evaluate diagrams (b) and (c) as

$$\mathscr{D}_b = \frac{2i\kappa^4}{(4\pi)^2}\frac{\mathscr{S}}{840ut}\,, \qquad \mathscr{D}_c = \frac{2i\kappa^4}{(4\pi)^2}\frac{\mathscr{S}}{252su}$$

4-graviton contributions

Diagram (d)



(d)

1. There is a 1-2 tree attached to the loop so

$$\mathcal{K}_{\rm red} = \mathscr{S}(\dot G_{13} - \dot{\mathbf{G}}_{\mathbf{12}})(\dot G_{24} - \dot G_{23})(\dot G_{34} + \dot G_{23})(\dot G_{34} - \dot G_{24})$$
$$\times(\dot{\bar G}_{13} - \dot{\bar{\mathbf{G}}}_{\mathbf{12}})(\dot{\bar G}_{24} - \dot{\bar G}_{23})(\dot{\bar G}_{34} + \dot{\bar G}_{23})(\dot{\bar G}_{34} - \dot{\bar G}_{24})$$
$$\to -\frac{\mathscr{S}}{s}(\dot G_{24} - \dot G_{23})(\dot G_{34} + \dot G_{23})(\dot G_{34} - \dot G_{24})(\dot{\bar G}_{24} - \dot{\bar G}_{23})(\dot{\bar G}_{34} + \dot{\bar G}_{23})(\dot{\bar G}_{34} - \dot{\bar G}_{24})$$

2. The loop replacement rule turns this into a genuine function which yields

$$\mathscr{D}_d = -\frac{2i\kappa^4}{(4\pi)^2}\frac{\mathscr{S}}{s}\int_0^1 du_3 \int_0^{u_3} du_2 \frac{u_2^2(1-u_3)^2(u_3-u_2)^2}{s(u_3-u_2)} \tag{3}$$

which is again finite and easy to determine.

In this way we get

$$\mathscr{D}_d = \frac{2i\kappa^4}{(4\pi)^2}\frac{\mathscr{S}}{360s^2}, \qquad \mathscr{D}_e = \frac{2i\kappa^4}{(4\pi)^2}\frac{\mathscr{S}}{360u^2}$$

4-graviton amplitude

The final result is the sum of these contributions and can be written neatly as

$$\mathscr{A}(1^-, 2^+, 3^+, 4^+) = \frac{i\kappa^4}{(4\pi)^2} \frac{s^2 t^2}{2880 u^2} (u^2 - st) \left(\frac{[24]^2}{[12]\langle 23\rangle\langle 34\rangle[41]} \right)^2$$

which, as in the introduction is a simple, invariant function of the external momenta.

The calculations of the remaining helicity amplitudes are more or less comparable in difficulty and length.

The amplitudes can also be computed for more "exotic" theories and the results can be consistency checked against unitarity and other constraints. In general the method has been shown to provide a powerful, alternative calculational tool for studying graviton amplitudes in various field theories.

Worldline derivation

We would like to understand how to derive the gravitational Master Formula directly using worldline techniques and generalise it for *massive* particles running in the loop. This presents some difficulties and raises some questions:

Worldline derivation

We would like to understand how to derive the gravitational Master Formula directly using worldline techniques and generalise it for *massive* particles running in the loop. This presents some difficulties and raises some questions:

1. For photons there is no significant difference between Worldline and Bern-Kosower, nor between working on-shell or off-shell
2. For gravity, BDS requires **on-shell** gravitons – used early on in the construction of string theory amplitudes and seems to survive the infinite tension limit. But this is not necessary for the worldline construction!
3. We understand on-shell *irreducible* diagrams in the worldline formalism but producing the *reducible* ones still requires invoking the BDS procedure.
4. In the worldline approach there is no intrinsic separation into left- and right-moving modes that is an essential part of the string theory method.

The starting point is the worldline representation of the amplitude:

$$\frac{1}{2}\left(\frac{-\kappa}{4}\right)^{N}\int_{0}^{\infty}\frac{dT}{T}(4\pi T)^{-\frac{D}{2}}\,\mathrm{e}^{-\mathbf{m^2 T}}\big\langle V_g[k_1,\varepsilon_1]\dots V_g[k_N,\varepsilon_N]\big\rangle \tag{4}$$

with graviton vertex operator $V^g[k,\varepsilon]=\int_0^T d\tau\,\dot{x}\cdot\varepsilon\cdot\dot{x}\,\mathrm{e}^{ik\cdot x}$.

Conclusion

Alternative methods for calculating graviton scattering amplitudes are desirable (and probably necessary). **String inspired** techniques can be very useful.

For gauge theories (QED, QCD etc) we already understand the relationship between the string based "Bern-Kosower" method thanks to the **worldline formalism**. For gravity we are still working out the details of this correspondence.

1. These methods involve Master Formulas for entire classes of diagrams
2. Various theories (scalar / spinor / QCD) are unified by the first quantised methods which involve replacement rules to transform kinematic factors according to the particles running in the loop
3. It remains to extend the approach to **off-shell**, massive amplitudes and to produce the reducible contributions within a single framework.

Conclusion

Alternative methods for calculating graviton scattering amplitudes are desirable (and probably necessary). **String inspired** techniques can be very useful.

For gauge theories (QED, QCD etc) we already understand the relationship between the string based "Bern-Kosower" method thanks to the **worldline formalism**. For gravity we are still working out the details of this correspondence.

1. These methods involve Master Formulas for entire classes of diagrams
2. Various theories (scalar / spinor / QCD) are unified by the first quantised methods which involve replacement rules to transform kinematic factors according to the particles running in the loop
3. It remains to extend the approach to **off-shell**, massive amplitudes and to produce the reducible contributions within a single framework.

For more information on worldline techniques see

- Classic review: C. Schubert Phys. Rept. 355 (2001) 73 [arXiv:0101036 [hep-th]]
- More up-to-date report: **JPE** and C. Schubert [arXiv:1912.10004 [hep-th]]

and please get in touch if you're interested in joining the team!

Conclusion

Alternative methods for calculating graviton scattering amplitudes are desirable (and probably necessary). **String inspired** techniques can be very useful.

For gauge theories (QED, QCD etc) we already understand the relationship between the string based "Bern-Kosower" method thanks to the **worldline formalism**. For gravity we are still working out the details of this correspondence.

1. These methods involve Master Formulas for entire classes of diagrams
2. Various theories (scalar / spinor / QCD) are unified by the first quantised methods which involve replacement rules to transform kinematic factors according to the particles running in the loop
3. It remains to extend the approach to **off-shell**, massive amplitudes and to produce the reducible contributions within a single framework.

For more information on worldline techniques see

- Classic review: C. Schubert Phys. Rept. 355 (2001) 73 [arXiv:0101036 [hep-th]]
- More up-to-date report: **JPE** and C. Schubert [arXiv:1912.10004 [hep-th]]

and please get in touch if you're interested in joining the team!

¡Thank you for your attention!