THE 2-FLAVOR SCHWINGER MODEL AT FINITE TEMPERATURE AND IN THE δ-REGIME

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ABSTRACT

We present a study of the two flavor Schwinger model by means of lattice simulations, using Wilson fermions and the Hybrid Monte Carlo algorithm. At finite temperature, we measure the mass of the bosons, which are related to m_π and $m_{\eta'}$ as a function of the degenerate fermion mass m. We compare the results with the numerical solution of a set of equations obtained by Hosotani et al. based on bosonization, which predict these masses when $m \ll \sqrt{2g^2/\pi}$, where g is the gauge coupling. Furthermore, we measured the pion decay constant F_π, in the so-called δ-*regime*, where finite size effects of the pion mass lead to $F_\pi = 0.6688(5)$. We also computed F_π through an independent method, which consists of measuring the quenched topological susceptibility and applying a 2d version of the Witten-Veneziano formula. This yields a lower value of $F_\pi = 0.424(8)$. The discrepancy might be due to subtleties in the application of the Witten-Veneziano formula to the Schwinger model.

QCD

Quantum Chromodynamics is the theory for the strong interaction, its Lagrangian is given by

$$\mathcal{L} = \sum_f^{N_f} \overline{\psi}_f(\gamma^\mu D_\mu - m_f)\psi_f - \frac{1}{4}\mathrm{Tr}[G_{\mu\nu}G^{\mu\nu}], \quad G_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu + ig[A_\mu, A_\nu].$$

At low energy regime a non-perturbative approach is required, such as lattice simulations or an effective field theory. For low energy we consider those quarks whose masses satisfy $m_f \ll \Lambda_{\text{QCD}} \approx 300$ MeV. Hence, we work with two flavors u and d, since $m_u, m_d \lesssim 5$ MeV .

CHIRAL SYMMETRY

To build the effective field theory, we have to analyze the chiral symmetry by applying the chiral projection operators.
$P_R = \frac{1}{2}(\mathbb{I} + \gamma_5)$, $\quad P_L = \frac{1}{2}(\mathbb{I} - \gamma_5)$, $\quad \psi_{f_{L,R}} = P_{L,R}\psi_f$, $\quad \overline{\psi}_{f_{L,R}} = \overline{\psi}_f P_{R,L}$.

$$\mathcal{L} = \sum_f^{N_f} [\overline{\psi}_{f,L}\gamma_\mu D_\mu \psi_{f,L} + \overline{\psi}_{f,R}\gamma_\mu D_\mu \psi_{f,R} - m_f(\overline{\psi}_{f,R}\psi_{f,L} + \overline{\psi}_{f,L}\psi_{f,R})] - \frac{1}{4}\mathrm{Tr}[G_{\mu\nu}G^{\mu\nu}].$$

Writing

$$\psi_{R,L} = \begin{pmatrix} \psi_{u_{R,L}} \\ \psi_{d_{R,L}} \end{pmatrix}, \quad \overline{\psi}_{R,L} = \left(\overline{\psi}_{u_{R,L}}, \overline{\psi}_{d_{R,L}}\right),$$

we can transform

$$\psi_R \to \psi'_R = R\psi_R, \quad \overline{\psi}_R \to \overline{\psi}'_R = \overline{\psi}_R R^\dagger, \quad R \in \mathrm{SU}(N_f)_R,$$
$$\psi_L \to \psi'_L = L\psi_L, \quad \overline{\psi}_L \to \overline{\psi}'_L = \overline{\psi}_L L^\dagger, \quad L \in \mathrm{SU}(N_f)_L$$

and $\mathcal{L}$ remains invariant when $m_f = 0$, for that reason this is known as the chiral limit. However the chiral condensate

$$\Sigma = -\langle\overline{\psi}\psi\rangle \to -\langle\overline{\psi}'\psi'\rangle = -\langle\overline{\psi}_R R^\dagger L\psi_L + \overline{\psi}_L L^\dagger R\psi_R\rangle.$$

only remains invariant when $R = L$. Therefore, if $\langle\overline{\psi}\psi\rangle = 0$, the symmetry is spontaneously broken: $\mathrm{SU}(N_f)_L \otimes \mathrm{SU}(N_f)_R \to \mathrm{SU}(N_f)_{L=R}$. The Goldstone theorem implies that $N_f^2 - 1$ massless NGB appear. If $m_f \gtrsim 0$, there is explicit symmetry breaking and for $N_f = 2$ three quasi-NGB appear: π^+, π^-, π^0. We will consider $m_u \approx m_d \equiv m$.

CHIRAL PERTURBATION THEORY (χPT)

Chiral perturbation theory is the most successful effective field theory for QCD. To build the theory for $N_f = 2$ and $m = 0$ we introduce a field

$$U(x) = \exp\left(i\frac{\phi(x)}{F_\pi}\right), \quad \phi(x) = \begin{pmatrix} \pi^0(x) & \sqrt{2}\pi^+(x) \\ \sqrt{2}\pi^-(x) & -\pi^0(x) \end{pmatrix}$$

and we write an effective Lagrangian $\mathcal{L}_{\text{eff}}(U)$ with global $\mathrm{SU}(2)_L \otimes \mathrm{SU}(2)_R$ symmetry. All the symmetric terms must be included in $\mathcal{L}_{\text{eff}}(U)$. Nevertheless, this leads to an infinite number of terms, so one truncates in the number of derivatives

$$\mathcal{L}_{\text{eff}} = \frac{1}{4}F_\pi^2\mathrm{Tr}[\partial_\mu U^\dagger \partial_\mu U] - \frac{1}{4}l_1(\mathrm{Tr}[\partial_\mu U^\dagger \partial_\mu U])^2 + \cdots$$

By considering $m \neq 0$ and adding the term $\frac{1}{2}\Sigma m \mathrm{Tr}\left[U + U^\dagger\right]$, there is explicit symmetry breaking $\mathrm{SU}(2)_L \otimes \mathrm{SU}(2)_R \to \mathrm{SU}(2)_{L=R}$. For $m \ll \Lambda_{\text{QCD}}$ and infinite volume we have

$$\mathcal{L}_{\text{eff}} = \frac{F_\pi^2}{4}\mathrm{Tr}[\partial_\mu U^\dagger \partial^\mu U] + \frac{1}{2}\Sigma m \mathrm{Tr}\left[U + U^\dagger\right], \quad \Sigma = \frac{F_\pi^2 m_\pi^2}{m}.$$

The effective Lagrangian can be formulated in terms of a normalized field $\vec{S}(x) \in \mathrm{O}(4)$, $|\vec{S}(x)| = 1$ as well, since there is a local isomorphism between O(4) and $\mathrm{SU}(2)_L \otimes \mathrm{SU}(2)_R$. The symmetry breaking pattern takes the form

$$\mathrm{O}(4) \to \mathrm{O}(3) \quad \Longleftrightarrow \quad \mathrm{SU}(2)_L \otimes \mathrm{SU}(2)_R \to \mathrm{SU}(2).$$

In terms of the field $\vec{S}(x)$, the $\mathcal{L}_{\text{eff}}$ with the least number of derivatives reads

$$\mathcal{L}_{\text{eff}} = \frac{F_\pi^2}{2}\partial^\mu \vec{S} \cdot \partial_\mu \vec{S}.$$

Now, we introduce an external field $\vec{H}$ that explicitly breaks the symmetry by adding the term

$$\mathcal{L}_{\text{s.b.}} = -\Sigma \vec{H} \cdot \vec{S},$$

where $\vec{H}$ plays the same role as the quark mass. This is known as the non-linear σ model.

REGIMES OF CHIRAL PERTURBATION THEORY

For finite volume V, there are three regimes to study χPT, based on the dimensions of V

- p-regime: $V = L^4, L \gg \frac{1}{m_\pi}$.
- ϵ-regime: $V = L^4, L \lesssim \frac{1}{m_\pi}$.
- δ-regime: $V = L^3 \times L_t, L \lesssim \frac{1}{m_\pi} \ll L_t$ [1].

In the δ-regime, when we take the chiral limit $m \to 0$ there is a residual pion mass m_π^R. Also, the finite space volume enables us to treat the theory in this regime as a quasi-1D field theory. Based on this, the system can be modeled as an O(4) quantum rotor [2]. The energies are given by

$$E_j - E_0 = \frac{j(j+2)}{2\Theta_{\text{eff}}}, \quad \Theta_{\text{eff}} = F_\pi^2 L^{d-1}\left[1 + \frac{N-2}{4\pi F_\pi^2 L^{d-2}}\left(2\frac{d-1}{d-2}\right) + \cdots\right]$$

for $d \geq 3$. m_π^R is obtained with the energy gap

$$m_\pi^R = E_1 - E_0 = \frac{3}{2\Theta_{\text{eff}}} = \frac{3}{2F_\pi^2 L^3(1+\triangle)}, \quad \triangle = \frac{0.477...}{F_\pi^2 L^2} + ... \quad d = 4.$$

For two dimensions we can only consider the leading order term of Θ_{eff}, yielding

$$m_\pi^R \simeq \frac{3}{2F_\pi^2 L}.$$

By performing lattice simulations in 2D, one obtains m_π^R for different values of L and one can verify the prediction in the δ-regime. We did that by using the Schwinger model.

MASSIVE SCHWINGER MODEL

The Schwinger model 2D QED [3]. It is a toy model for QCD, since it is simpler and has common properties, such as confinement, chiral symmetry breaking and topology [4, 5]. If one performs a Wick rotation $\tau = it$, the Lagrangian takes the following form in Euclidean space

$$\mathcal{L} = \sum_{f=1}^{N_f} \bar{\psi}_f\left[\gamma_\mu^E(i\partial_\mu + gA_\mu) + m_f\right]\psi_f + \frac{1}{4}F_{\mu\nu}F_{\mu\nu}.$$

We consider degenerate masses $m \equiv m_f$. Schwinger proved that when $m = 0$ and $N_f = 1$, a boson of mass $\mu^2 = g^2/\pi$ appears. This result has been generalized for $N_f > 1$, where $\mu^2 = N_f g^2/\pi$ [6]. For 2 massive flavors, it is known that two bosons appear and their masses can be related to m_π and $m_{\eta'}$ from QCD. No general solution for the dependence of these masses on m exists, but there have been several analytic approaches. In particular, we analysed the approach made by Hosotani et al. [7, 8]. He maps the Schwinger model onto a circle by imposing the boundary conditions

$$\psi_f\left(\tau + \frac{1}{T}, x\right) = -e^{-i2\pi\alpha_f}\psi_f(\tau, x), \quad A_\mu\left(\tau + \frac{1}{T}, x\right) = A_\mu(\tau, x)$$

where T is temperature of the system, related to the Euclidean time extent by $T = 1/L_t$. Later he uses *bosonization*, which allows him to reduce the model to a quantum mechanical system, governed by the following set of equations, valid when $m \ll \mu$.

$$\left[\left(i\frac{d}{d\varphi} - \delta\alpha\right)^2 - \kappa\cos\varphi\right]f(\varphi) = \epsilon f(\varphi), \quad f(\varphi + 2\pi) = f(\varphi), \quad \delta\alpha = \alpha_2 - \alpha_1,$$
$$\kappa = \frac{4}{\pi}mL_t\left[B(m_{\eta'}L_t)B(m_\pi L_t)\right]^{1/2}e^{-\pi/(2\mu L_t)}, \quad \mu^2 = 2g^2/\pi,$$
$$B(z) = \frac{z}{4\pi}\exp\left[\gamma + \frac{\pi}{z} - 2\int_1^\infty \frac{du}{(e^{uz}-1)\sqrt{u^2-1}}\right], \quad \gamma = 0.5772...$$
$$m_\pi^2 = \frac{2\pi^2}{L_t^2}\kappa\int_{-\pi}^{\pi} d\varphi\, \cos\varphi|f_0(\varphi)|^2, \quad m_{\eta'}^2 = \mu^2 + m_\pi^2.$$

One can compute values for m_π and $m_{\eta'}$ by solving this set of equations in a self-consistent way. Hosotani's solution for m_π converges to

$$m_\pi = 2.1633...(m^2 g)^{1/3}$$

when one approximates $m_{\eta'} \approx \mu$ and takes $L_t \to \infty$. There is another prediction for small mass and large volume given by Smilga [9], which states

$$m_\pi = 2.008...(m^2 g)^{1/3}.$$

Finite temperature results

We computed m_π and $m_{\eta'}$ at finite temperature by using a Hybrid Monte Carlo algorithm and we compared them with Hosotani's solution. Taking 10^3 measurements we obtained the following for $L_t = 10$ and $\beta = 1/g^2 = 4$.

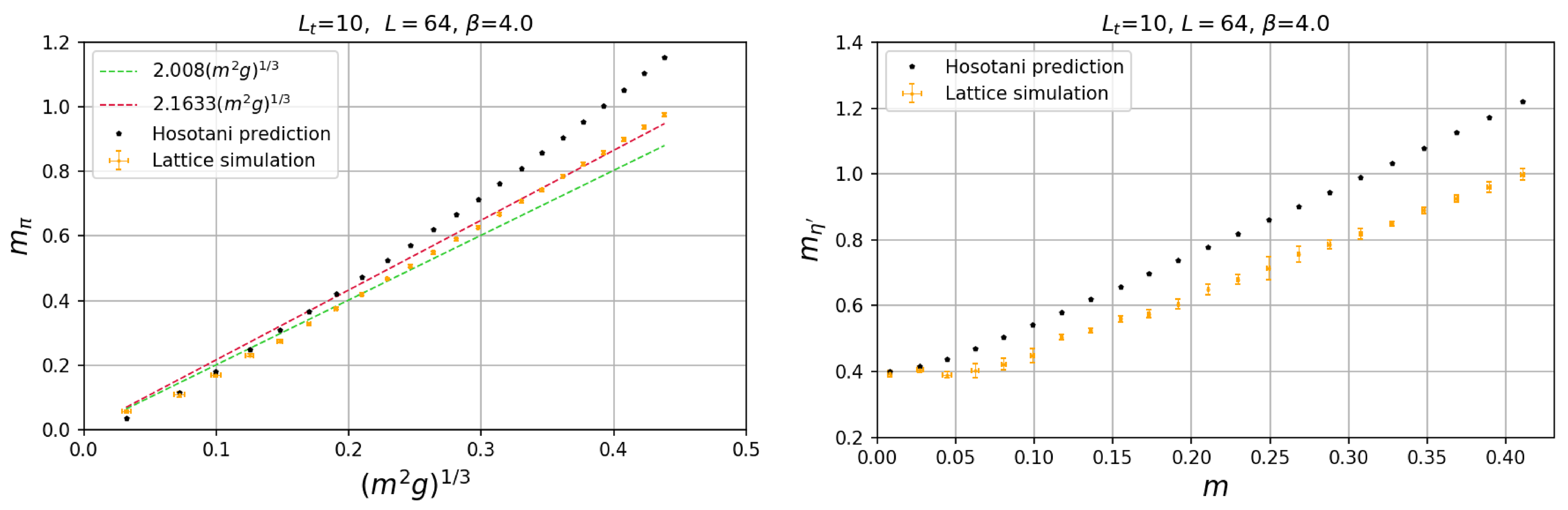

Figure 1

Only for $m \ll \mu$ Hosotani's solution agrees with the lattice results. We confirm that m_π vanishes for $m = 0$ and that $m_{\eta'}$ converges to $\mu = \sqrt{2/(\pi\beta)} \simeq 0.39$.

δ-regime results

In the δ-regime we obtained the pion mass for different values of the degenerate fermion mass m and for different spatial size L by taking 10^3 measurements. To extrapolate m_π^R, we fitted a function of the form $m_\pi = \sqrt{a + b\,m^2 g}$. In figure 2 we show m_π as function of $(m^2g)^{1/3}$ for $L = 10$ and $\beta = 4$.

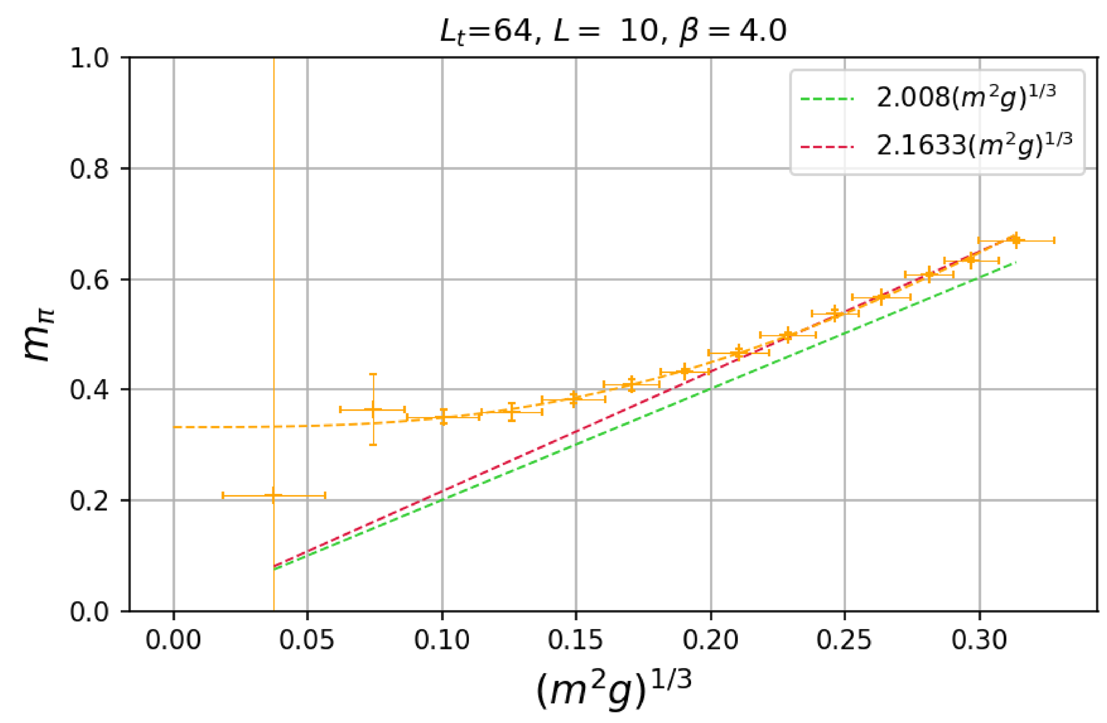

Figure 2: Pion mass in the δ-regime for $L = 10$ and $\beta = 4$. Close to the chiral limit the pion mass result is plagued by large errors, so one has to extrapolate m_π^R. This yields $m_\pi^R = 0.3323(22)$.

We measured m_π^R for several values of L. In figure 3 we show the residual pion mass as a function of L; we observe a $1/L$ behavior, as predicted in the δ-regime. We also computed F_π.

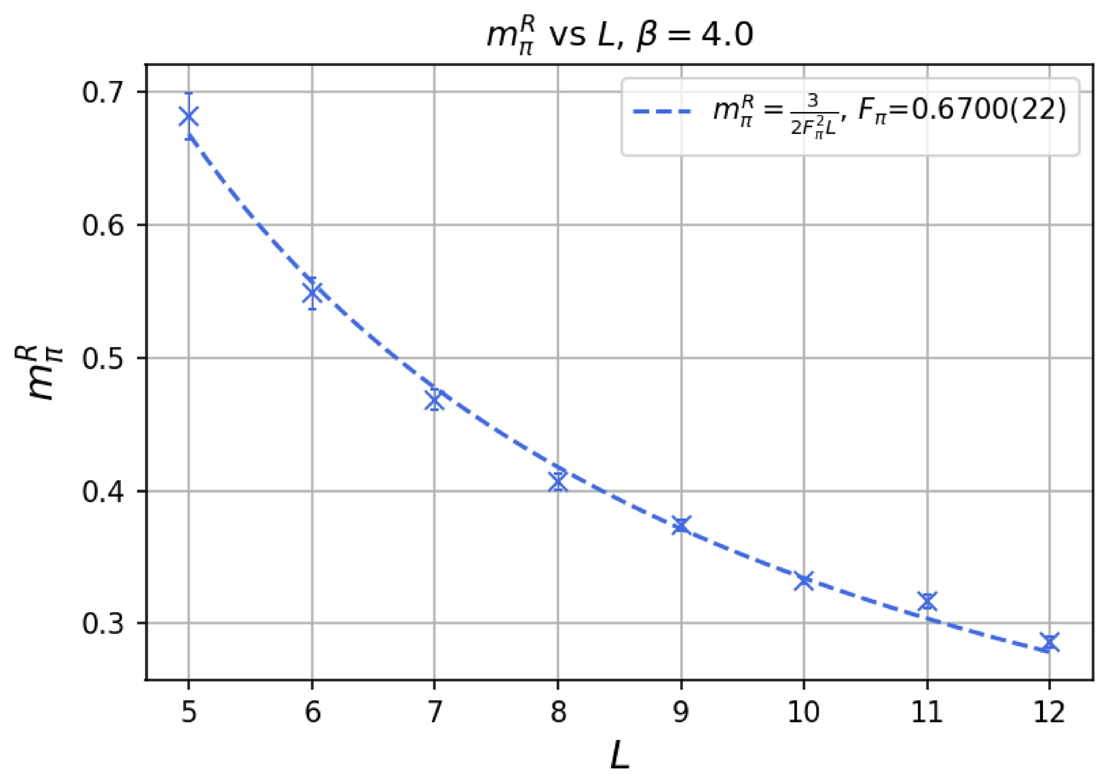

Figure 3: Residual pion mass as a function of the spatial size L for $\beta = 4$.

We repeated everything for $\beta = 2$ and 3 to verify that F_π is β independent. In figure 4 we show the result.

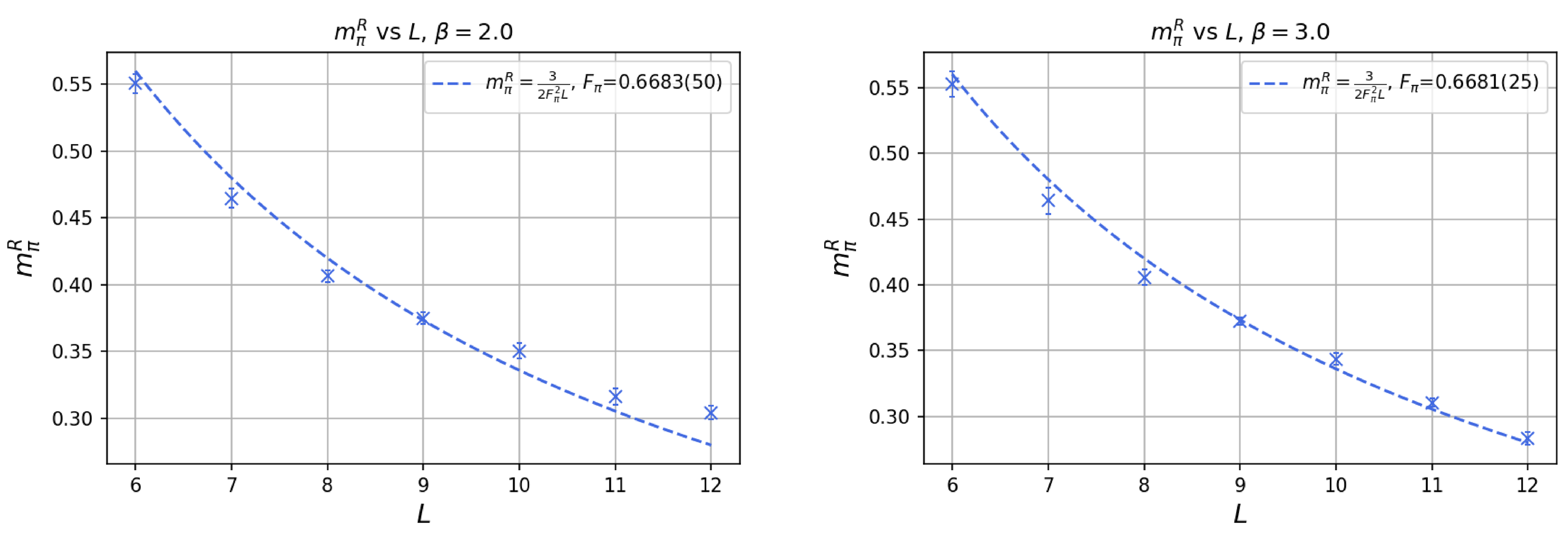

Figure 4: Residual pion mass as a function of the spatial size L for $\beta = 2$ and 3.

In table 1 we show the three results for F_π in two dimensions. An average of the values yields

$$F_\pi = 0.6688(5).$$

β	2	3	4
F_π	0.6683(50)	0.6681(25)	0.6700(22)

Table 1: Pion decay constant for different β.

Witten-Veneziano formula

The Witten-Veneziano formula relates the topological susceptibility χ_T with the masses of the η, η' and π mesons in QCD [10, 11]. It is obtained from the leading order term of a $1/N_c$ expansion, for large N_c (color number). For three flavors, the formula reads

$$m_{\eta'}^2 - \frac{1}{2}m_\eta^2 - \frac{1}{2}m_\pi^2 = \frac{6}{F_{\eta'}^2}\chi_T^{\text{que}},$$

where "que" stands for quenched, *i.e.* its value when the fermion mass $m \to \infty$. For large N_c, $F_{\eta'} = F_\pi$ is valid in QCD. In the Schwinger model for massless fermions, the formula takes the following form, for $N_f \geq 2$ [12]

$$m_{\eta'}^2 = \frac{2N_f}{F_{\eta'}^2}\chi_T^{\text{que}}, \quad m_{\eta'}^2 = \frac{N_f g^2}{\pi}.$$

However, in this case the literature is not clear whether $F_{\eta'} = F_\pi$ is valid. The topological susceptibility in the Schwinger model is defined as

$$\chi_T = \int d^2x\,\langle q(x)q(0)\rangle, \quad q(x) = \frac{g}{2\pi}\epsilon_{\mu\nu}F_{\mu\nu}(x),$$

where $q(x)$ is the topological charge density. We computed χ_T as a function of m by taking 10^4 measurements. In figure 5 we show $\chi_T\beta$ vs. m.

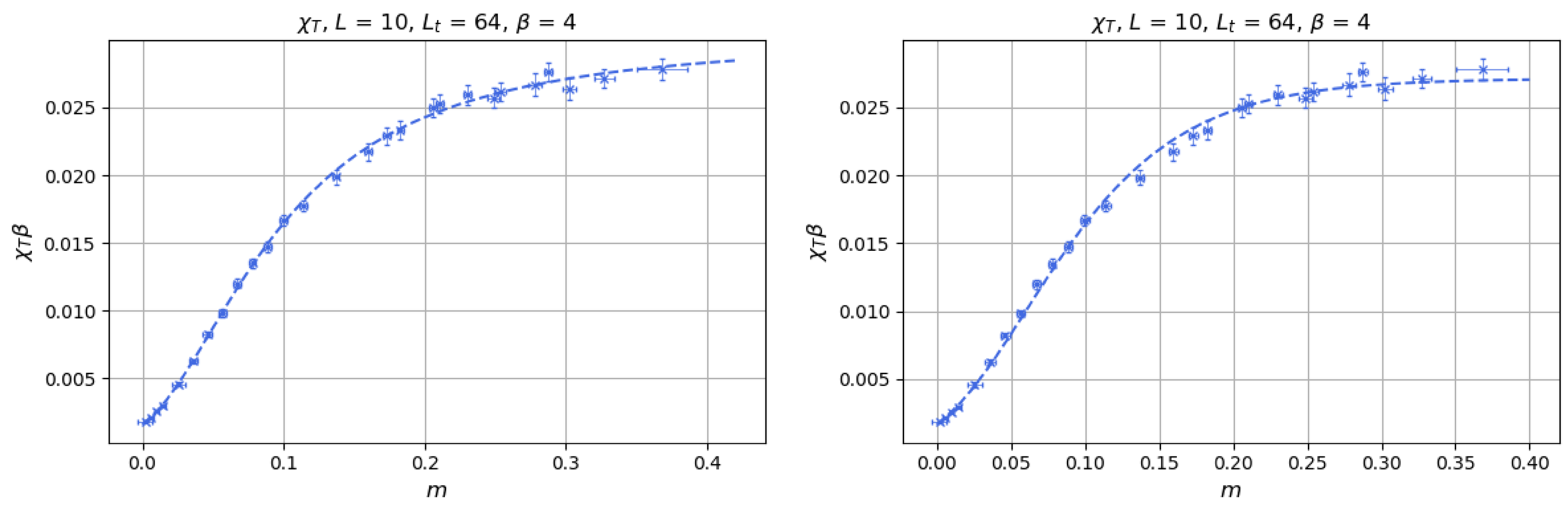

Figure 5: χ_T as a function of the fermion mass. To extrapolate to the quenched value we fitted two different functions and took the average as the final result. On the left-hand side plot we fitted a function of the form $y = \frac{a+bx+cx^2}{d+fx+gx^2}$, while on the right-hand side we fitted $y = ae^{-be^{-cx}}$.

We obtained $\chi_T^{\text{que}} = 0.029(1)g^2$. With the Witten-Veneziano formula we calculate

$$F_\pi = \sqrt{\frac{\chi_T^{\text{que}}2\pi}{g^2}} = 0.4243(76).$$

Conclusions

We performed an analysis of the Schwinger model at finite temperature, where we computed m_π and $m_{\eta'}$ for different values of the degenerate fermion mass by means of lattice simulations. We compared the results with a numerical solution to the equations proposed by Hosotani in his bosonization approach.

We verified the $1/L$ behavior predicted in the δ-regime for m_π^R in 2D, which allowed us to determine the pion decay constant F_π. In the δ-regime our final result is $F_\pi = 0.6688(5)$ and we checked that it is independent of the gauge coupling constant. Nevertheless, this result disagrees with the value computed by using the Witten-Veneziano formula: $F_\pi = 0.4243(76)$. This might be due to a wrong interpretation of F_π in the Witten-Venziano formula. Further investigation is needed in this direction.

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