Pion model with the Nakanishi Integral Representation for the Bethe-Salpeter amplitudes

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2. Pion quark-antiquark vertex
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Main aspects

- **Non-perturbative hadronic physics**
- **Quark model propagador (with self energy)**
- **Running mass**
- **Fit lattice data**
- **Reproduce experimental pion data**

⇒ **Main references:**

- **E. Rojas, J. de Melo, B. El-Bennich, O. Oliveira and T. Frederico, JHEP 1310, 193 (2013)**
- **Clayton S. Mello, J.de Melo, T. Frederico, Physics Letters B 766 (2017) 86.**
- **M. Parappilly, P. Bowman, U. Heller, D. Leinweber, A. Williams and J. Zhang, Phys. Rev. D 73, 054504 (2006)**

- **Pion quark-antiquark vertex**

$$\Gamma_\pi(k,p) = \gamma_5[\imath E_\pi(k,p) + \not{p}F_\pi(k,p) + k^\mu p_\mu \not{k}\, G_\pi(k,p) + \sigma_{\mu\nu}k^\mu p^\nu H_\pi(k,p)]\,.$$

- $\Longrightarrow$ **Four scalar amplitudes:** $E_\pi(k,p)$, $F_\pi(k;p)$, $G_\pi(k;p)$ **and** $H_\pi(k,p)$
- **First approximation, in the chiral limit** $\longrightarrow$ $m_\pi = 0$

$$E_\pi(k;p) = \imath\, B(k^2)/f_\pi^0$$

- **The quark propagator**

$$S_F(k) = \imath\, S_v(k^2)\not{k} + \imath\, S_s(k^2) = i\left(A(k^2)\not{k} - B(k^2)\right)^{-1}$$

- **Self-energy:** Scalar functions

$$A(k^2) = \frac{S_v(k^2)}{k^2 S_v^2(k^2) - S_s^2(k^2)}$$

$$B(k^2) = \frac{S_s(k^2)}{k^2 S_v^2(k^2) - S_s^2(k^2)}$$

- **Källen-Lehmann spectral decomposition:** $S_v(k)$ and $S_s(k)$

$$S_v(k^2) = \int_0^\infty d\mu^2 \frac{\rho_v(\mu^2)}{k^2 - \mu^2 + \imath\varepsilon}$$

$$S_s(k^2) = \int_0^\infty d\mu^2 \frac{\rho_s(\mu^2)}{k^2 - \mu^2 + \imath\varepsilon}$$

- **Integral representation of the scalar self-energy**

$$A(k^2) = \int_0^\infty d\mu^2 \frac{\rho_A(\mu^2)}{k^2 - \mu^2 + \imath\varepsilon}$$

$$B(k^2) = \int_0^\infty d\mu^2 \frac{\rho_B(\mu^2)}{k^2 - \mu^2 + \imath\varepsilon}$$

- **Spectral densitiues**

$$\rho_A(k^2) = -\frac{1}{\pi} Im[A(\mu^2)$$
$$\rho_B(k^2) = -\frac{1}{\pi} Im[B(\mu^2)$$

- **Khällen-Lehmann: Positivity Constraints**
- $\mathcal{P}_a = \rho_A(\mu^2) \geq 0$ and $\mathcal{P}_b = \mu\rho_A(\mu^2) - \rho_b(mu^2) \geq 0$

- **Bethe-Salpeter amplitude (only the dominant vertex function γ_5)**

$$
\begin{aligned}
\Psi_\pi(k,p) &= \gamma_5\ \chi_1(k,p) + \not{k}_q \gamma_5\ \chi_2(k,p) \\
&+ \gamma_5\ \not{k}_{\overline{q}}\ \chi_3(k,p) + \not{k}_q\ \gamma_5\ \not{k}_{\overline{q}}\ \chi_4(k,p)
\end{aligned}
$$

- $\Longrightarrow\ \chi_i(k,p)$: **Function of** $S_v(k^2)$, $S_s(k^2)$, $A(k^2)$ **and** $B(k^2)$.

- **Basis**
- **Non-orthogonal basis decomposition**

Auxiliary functional of the spectral densities

$$\chi_1(k,p) = \mathcal{F}(k,p;[\rho_B,\rho_s,\rho_s])\,,$$
$$\chi_2(k,p) = \mathcal{F}(k,p;[\rho_B,\rho_v,\rho_s])\,,$$
$$\chi_3(k,p) = \mathcal{F}(k,p;[\rho_B,\rho_s,\rho_v])\,,$$
$$\chi_4(k,p) = \mathcal{F}(k,p;[\rho_B,\rho_v,\rho_v])\ .$$

- Functional

$$\begin{aligned}\mathcal{F}(k,p;[o,f,h]) &= -i\int_0^\infty d\mu''^2 \frac{o(\mu''^2)}{k^2-\mu''^2+\imath\varepsilon} \\ &\times \int_0^\infty d\mu'^2 \frac{f(\mu'^2)}{k_q^2-\mu'^2+\imath\epsilon} \\ &\times \int_0^\infty d\mu^2 \frac{h(\mu^2)}{k_{\frac{q}{}}^2-\mu^2+\imath\epsilon}\end{aligned}$$

- **Symmetric properties for the scalar functions**

$$\chi_1(k,p) = \chi_1(-k,p)\,,$$
$$\chi_2(k,p) = \chi_3(-k,p)\,,$$
$$\chi_4(k,p) = \chi_4(-k,p)\,,$$

- **Bethe-Salpeter amplitude with orthogonal basis**

$$\Psi_\pi(k,p) = \sum_{i=1}^{4} S_i(k,p)\phi_i(k,p)$$

- **Orthogonal basis for the pion**

$$\begin{aligned}
S_1(k,p) &= \gamma^5, \\
S_2(k,p) &= \frac{\not p}{M_\pi}\gamma^5, \\
S_3(k,p) &= \frac{k\cdot p}{M_\pi^3}\,\not p\gamma^5 - \frac{1}{M_\pi}\not k\gamma^5, \\
S_4(k,p) &= \frac{i}{M_\pi^2}\sigma^{\mu\nu}p_\mu k_\nu\gamma^5
\end{aligned}$$

- $\sigma^{\mu\nu} = \frac{i}{2}(\gamma^\mu\gamma^\nu - \gamma^\nu\gamma^\mu)$

- **Scalar functions** $\phi_i(k,p)$ from orthogonality conditions

$$\phi_i(k,p) = \mathbf{Tr}[S_i(k,p)\Psi_\pi(k,p)]$$

- **The scalar amplitudes $\phi_i(k,p)$ are:**

$$\begin{aligned}
\phi_1(k,p) &= \chi_1(k,p) - \left(k^2 - \frac{p^2}{4}\right)\chi_4(k,p)\,, \\
\phi_2(k,p) &= \left(\frac{M_\pi}{2} + \frac{p\cdot k}{M_\pi}\right)\chi_2(k,p) \\
&+ \left(\frac{M_\pi}{2} - \frac{p\cdot k}{M_\pi}\right)\chi_3(k,p)\,, \\
\phi_3(k,p) &= M_\pi\,(\chi_3(k,p) - \chi_2(k,p))\,, \\
\phi_4(k,p) &= M_\pi^2\,\chi_4(k,p)\,.
\end{aligned}$$

- **Symmetry properties of $\phi_i(k,p)$ under the transformation $k \to -k$**
- **Consistent with the ones fulfilled by $\chi_i(k,p)$**

Nakanishi Integral Representation (NIR)

- **Feynman diagram amplitudes $\longrightarrow$ Nakanishi Representation**

$$\chi_i(k;p) = \int_{-1}^{1} dz \int_{-\infty}^{\infty} d\gamma \frac{G_i(\gamma, z)}{[k^2 + z\ k \cdot p - \gamma + i\epsilon]^3}$$

- Scalar functions $\phi_i(k,p)$ // NIR

$$\phi_i(k,p) = \int_{-1}^{1} dz' \int_{-\infty}^{\infty} d\gamma' \frac{g_i(\gamma', z')}{[k^2 + z'\ k \cdot p - \gamma' + i\epsilon]^3}$$

- **Naganishi references (some)**

Ref. **N. Nakanishi, Phy.Rev. 138 (1965) B1182, Pys.Rev.139 (1965) B1401.**
Graph Theory and Feynman Integrals
(Gordon and Breach, NY, 1971)

- Auxiliary functional

$$\int_{-1}^{1} dz \int_{-\infty}^{\infty} d\gamma \, \frac{i\, F(\gamma, z; \mu'', \mu', \mu)}{[k^2 + z\, k \cdot p - \gamma + i\epsilon]^3} \quad = \\ \frac{1}{[(k+p/2)^2 - \mu'^2 + \imath\epsilon][k^2 - \mu''^2 + \imath\epsilon]} \times \quad \frac{1}{[(k-p/2)^2 - \mu^2 + \imath\epsilon]},$$

- **Weight function in this case**

$$F(\gamma, z;\, \mu'', \mu', \mu) = -\frac{2\,\theta(z - 2\;\alpha + 1)\;\theta(\alpha - z)\;\theta(1-\alpha)\;\theta(\alpha)}{|\mu'^2 + \mu^2 - \frac{M_\pi^2}{2} - 2\mu''^2|}$$

and, $\alpha = \frac{\gamma - z(\mu''^2 - \mu^2 + \frac{M_\pi^2}{4}) - \mu''^2}{\mu'^2 + \mu^2 - \frac{M_\pi^2}{2} - 2\mu''^2}$

- **C. S. Mello, J. P. B. C. de Melo, and T. Frederico Phys. Lett. B, 766:86–93, 2017**

- **Function F has the symmetry property**

$$F(\gamma, z;\, \mu'', \mu', \mu) = F(\gamma, -z;\, \mu'', \mu, \mu')$$

- **Corresponds to the invariance of the product of the three denominators in the equation above**
- **$\mathcal{F}$ is written as NIR form**

$$\mathcal{F}(k, p; [o, f, h]) = \int_{-1}^{1} dz \int_{-\infty}^{\infty} d\gamma \frac{H(\gamma, z; [o, f, h])}{[k^2 + z'\ k \cdot p - \gamma + i\epsilon]^3}$$

- **Weight functional**

$$H(\gamma, z; [o, f, h]) = \int_0^{\infty} d\mu''^2 d\mu'^2 d\mu^2\, F(\gamma, z;\, \mu'', \mu', \mu) \times o(\mu''^2)\, f(\mu'^2)\, h(\mu^2)$$

Weight functions: non-orthogonal basis

- Weight functions of the scalar amplitudes χ_i

$$
\begin{aligned}
G_1(\gamma, z) &= H(\gamma, z; [\rho_B, \rho_s, \rho_s]) \\
G_2(\gamma, z) &= H(\gamma, z; [\rho_B, \rho_v, \rho_s] \\
G_3(\gamma, z) &= H(\gamma, z; [\rho_B, \rho_s, \rho_v]) \\
G_4(\gamma, z) &= H(\gamma, z; [\rho_B, \rho_s, \rho_s])
\end{aligned}
$$

- **Symmetry properties for the weight functions**

$$
\begin{aligned}
G_1(\gamma, z) &= G_1(\gamma, -z) \\
G_2(\gamma, z) &= G_3(\gamma, -z) \\
G_4(\gamma, z) &= G_4(\gamma, -z)
\end{aligned}
$$

Weight functions: orthogonal basis

- **Functions $\phi_1(k, P)$ written in terms of the χ's (NIR given above)** :

$$\phi_1(k,p) = \int_{-1}^{1} dz \int_0^\infty d\gamma \left\{ \frac{G_1(\gamma,z)}{[k^2 + z\,k\cdot p - \gamma' + i\epsilon]^3} + \frac{\left(\frac{M_\pi^2}{4} - k^2\right) G_4(\gamma,z)}{[k^2 + z\,k\cdot p - \gamma + i\epsilon]^3} \right\}$$

- Final expression for the weight function (NIR):

$$g_1(\gamma,z) = G_1(\gamma,z) + (M_\pi^2/4 - \gamma) G_4(\gamma,z) + \int_{-\infty}^{\gamma} d\gamma' \Big(G_4(\gamma',z) - z\,\partial_z [G_4(\gamma',z)] \Big)$$

- **Scalar amplitude $\phi_2(k,p)$ in terms of χ's**

$$\phi_2(k,p) = \int_{-\infty}^{\infty} d\gamma \int_{-1}^{1} dz \left\{ \frac{\left(\frac{M_\pi}{2} + \frac{k\cdot p}{M_\pi}\right) G_2(\gamma,z)}{[k^2 + z\,k\cdot p - \gamma + i\epsilon]^3} + \frac{\left(\frac{M_\pi}{2} - \frac{k\cdot p}{M_\pi}\right) G_3(\gamma,z)}{[k^2 + z\,k\cdot p - \gamma + i\epsilon]^3} \right\}$$

- **Final expression for the function $g_2(\gamma,z)$**

$$g_2(\gamma,z) = \frac{M_\pi}{2}\,[G_2(\gamma,z) + G_3(\gamma,z)] + \frac{1}{M_\pi}\int_{-\infty}^{\gamma} d\gamma'\;\partial_z\left[z\left(G_2(\gamma',z) - G_3(\gamma',z)\right)\right]$$

- **The third scalar amplitude ϕ_3**

$$\phi_3(k,p) = -\int_{-1}^{1} dz \int_0^{\infty} d\gamma \frac{M_\pi \; G_2(\gamma,z)}{[k^2 + z\,k\cdot p - \gamma + i\epsilon]^3} + \int_{-1}^{1} dz \int_0^{\infty} d\gamma \frac{M_\pi \; G_3(\gamma,z)}{[k^2 + z\,k\cdot p - \gamma + i\epsilon]^3}$$

- **Uniqueness of NIR**

$$g_3(\gamma,z) = M_\pi \; [G_3(\gamma,z) - G_2(\gamma,z)]$$

- **The fourth scalar amplitude $\phi_4(k,p)$**

$$\phi_4(k,p) = \int_{-1}^{1} dz \int_{-\infty}^{\infty} d\gamma \; \frac{M_\pi^2 \, G_4(\gamma,z)}{(k^2 + z\, k\cdot p - \gamma + i\epsilon)^3}$$

- **Uniqueness of the Nakanishi weight function**

$$g_4(\gamma,z) \;=\; M_\pi^2 \; G_4(\gamma,z)$$

The symmetry properties of the g_i's follows from their representation in terms of the G_i's, and are given by,

$$g_i(\gamma,z) = g_i(\gamma,-z)\,, \quad (i = 1,2,4)$$
$$g_3(\gamma,z) = -g_3(\gamma,-z)\,.$$

- Normalization

In order to calculate hadronic observables: Form Factors / Valence probability / momentum distributions

$$Tr\left[\int \frac{d^4k}{(2\pi)^4} \quad \frac{\partial}{\partial p'^{\mu}} \quad \{S^{-1}(k - p'/2)\bar{\Phi}(k,p) \right.$$
$$\left. \times \quad S^{-1}(k + p'/2)\Phi(k,p)|_{p=p'}\}\right] = -i\ 2p^{\mu}$$

Ref. **Claude Itzykson and Jean-Bernard Zuber Quantum field theory, 2012, Courier Corporation**

Model

- **Dressed fermion propagator**

$$S_F(k) = \imath\, Z(k^2) \left[\not{k} - M(k^2) + \imath\epsilon\right]^{-1}$$

- **Simplification with the function,** $Z(k^2) = 1$

$$S_F(k) = \imath \frac{\not{k} + M(k^2)}{(k^2 - M^2(k^2) + \imath\epsilon)}$$

- **Lattice QCD parametrization, mass running function**

$$M(k^2) = m_0 - m^3 \left[k^2 - \lambda^2 + i\epsilon\right]^{-1}$$

Parameters (IP), from the references [Rojas et al.],
$m_0 = 0.014$ GeV, $m = 0.574$ GeV and $\lambda = 0.846$ GeV.

- **Introduction mass running**

$$S_F(k) = \imath \, \frac{\not{k} + M(k^2)}{k^2 - \left[m_0 - \frac{m^3}{k^2 - \lambda^2 + i\epsilon}\right]^2 + i\epsilon}$$

$$= \frac{\imath \left(k^2 - \lambda^2 + i\epsilon\right)^2 \left(\not{k} + M(k^2)\right)}{k^2 \left(k^2 - \lambda^2 + i\epsilon\right)^2 - \left[m_0 \left(k^2 - \lambda^2 + i\epsilon\right)\right]^2 + i\epsilon}$$

The expression for the quark propagator, in the present model, has quark the poles, given by,

$$k^2 \left(k^2 - \lambda^2\right)^2 - \left[m_0 \left(k^2 - \lambda^2\right) - m^3\right]^2 \;\; = 0$$

- $\longrightarrow$ **Real Poles:** ($m_1 = 0.383$ GeV, $m_2 = 0.644$ GeV, and $m_3 = 0.954$ GeV)

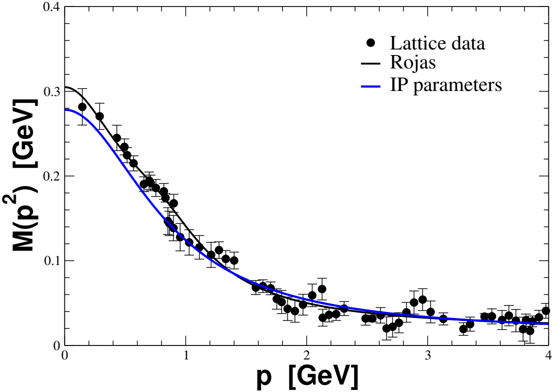


Figure: The running quark mass, as function of the momentum p, with the parameters utilized in the present work (IP), and, also, compared with the lattice, and the calculations from Rojas et al.

• Ref. **E. Rojas, J.P. de Melo, B. El-Bennich, O. Oliveira, and T. Frederico. JHEP, 10:193, 2013.**

- **factorized form of the quark propagator,**

$$S_F(k) = \imath \, \frac{\left(k^2 - \lambda^2\right)^2 \, (\not{k} + m_0) - \left(k^2 - \lambda^2\right) \, m^3}{\prod_{i=1,3}(k^2 - m_i^2 + \imath\epsilon)}$$

And,

$$A(k^2) = \frac{\left(k^2 - \lambda^2\right)^2}{\prod_{i=1,3}(k^2 - m_i^2 + \imath\epsilon)} \,,$$

$$B(k^2) = \frac{(\lambda^2 - k^2)\, m^3}{\prod_{i=1,3}(k^2 - m_i^2 + \imath\epsilon)} + A(k^2) m_0$$

- Solution of the previous equation

$$D_i = \frac{(\lambda^2 - m_i^2)^2}{(m_i^2 - m_j^2)(m_i^2 - m_k^2)}$$

- **With parameters in the present work provides the value**

$D_1 = 1.487$, $D_2 = -0.582$, and, $D_3 = -0.095$

- **Spectral representation of the function** $A(k^2)$

$$\int_0^\infty d\mu^2 \frac{\rho_A(\mu^2)}{k^2 - \mu^2} = \sum_{i=1}^{3} \frac{D_i}{k^2 - m_i^2}$$

- With,

$$\rho_A(\mu^2) = \sum_{i=1}^{3} D_i \delta(\mu^2 - m_i^2)$$

- **Decompose the second member**

$$\frac{(k^2-\lambda^2)\,m^3}{\prod_{i=1,3}(k^2-m_i^2)} = -\sum_{i=1}^{3}\frac{E_i}{(k^2-m_i^2)}$$

- **Solution**

$$E_i = -\frac{(m_i^2-\lambda^2)\,m^3}{(m_i^2-m_j^2)(m_i^2-m_k^2)}$$

- With model parameters: $E_1 = 0.480\,\mathrm{GeV}$, $E_2 = -0.375\,\mathrm{GeV}$ and $E_3 = -0.090\,\mathrm{GeV}$.
- Spectral density

$$\int_0^\infty d\mu^2 \frac{\rho_B(\mu^2)}{k^2-\mu^2} = \sum_{i=1}^{3}\frac{E_i}{k^2-m_i^2}$$

- **Spectral density** $\rho_B(\mu^2)$

$$\rho_B(\mu^2) = \sum_{i=1}^{3} E_i\delta(\mu^2-m_i^2) + m_0\rho_A(\mu^2)$$

BSA

$$\Psi_\pi(k;p) = -\left[A(k_q^2)\,\not{k}_q + B(k_q^2)\right] \times \frac{\mathcal{N}\,\gamma_5\;m^3}{k^2-\lambda^2+\imath\epsilon}\left[A(k_{\bar{q}}^2)\,\not{k}_{\bar{q}} + B(k_{\bar{q}}^2)\right]$$

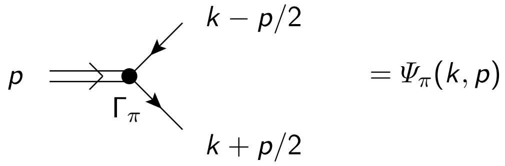


Figure: Vertex for the Bethe-Salpeter amplitude for pion

- Terms of scalar functions $\chi_i(k;p)$:

$$\chi_1(k;p) = -\mathcal{N}\ B(k_q^2)\ \frac{m^3}{k^2-\lambda^2+\imath\epsilon}\ B(k_{\bar{q}}^2)$$

$$\chi_2(k;p) = -\mathcal{N}\ A(k_q^2)\frac{m^3}{k^2-\lambda^2+\imath\epsilon}\ B(k_{\bar{q}}^2)$$

$$\chi_3(k;p) = -\mathcal{N}\ B(k_q^2)\ \frac{m^3}{k^2-\lambda^2+\imath\epsilon}\ A(k_{\bar{q}}^2)$$

$$\chi_4(k;p) = -\mathcal{N}\ A(k_q^2)\ \frac{m^3}{k^2-\lambda^2+\imath\epsilon}\ A(k_{\bar{q}}^2)$$

μ'	μ	γ_{min} [GeV]	γ_{max} $[GeV]$
m_1	m_1	-1.1253	1.3294
	m_2	-0.50973	1.0216
	m_3	0.10202	0.90521
m_2	m_1	-0.81753	1.3294
	m_2	-0.201924	1.0216
	m_3	0.409836	0.90521
m_3	m_1	-0.32215	1.32940
	m_2	0.29345	1.0216
	m_3	0.52621	1.28422

Table: **Values of γ_{min} and γ_{max} for each pair of pole masses**

- **NIR for χ_1**

$$\chi_1(k,p) = -\sum_{1\leq i,j\leq 3} \frac{E_i + m_0 D_i}{(k+\frac{P}{2})^2 - m_i^2 + i\epsilon} \times \frac{\mathcal{N} m^3}{k^2 - \lambda^2 + i\epsilon} \frac{E_j + m_0 D_j}{(k+\frac{P}{2})^2 - m_j^2 + i\epsilon}$$

- with $1 \leq i,j \leq 3$ denotes i and j running from 1 to 3.
- Define,

$$G_1(\gamma,z) = -\sum_{1\leq i,j\leq 3} (E_i + m_0 D_i)(E_j + m_0 D_j) \times \mathcal{N} m^3 F(\gamma, z, m_i, m_j)$$

- χ_2

$$\chi_2(k,P) = -A(k_q^2)\frac{\mathcal{N} m^3}{k^2-\lambda^2+i\epsilon}\; B(k_{\bar{q}}^2)$$

$$\chi_2(k,P) = -\sum_{1\leq i,j\leq 3} \frac{D_i}{(k+\frac{P}{2})^2 - m_i^2 + i\epsilon} \times \frac{\mathcal{N} m^3}{k^2-\lambda^2+i\epsilon} \frac{E_j + m_0 D_j}{(k-\frac{P}{2})^2 - m_j^2 + i\epsilon}$$

$$G_2(\gamma,z) = -\sum_{1\leq i,j\leq 3} (E_j + m_0 D_j) \times \mathcal{N} m^3\, F(\gamma,z;\; m_i, m_j)$$

- χ_3

$$\chi_3(k,P) = -B(k_q^2)\,\frac{\mathcal{N} m^3}{k^2-\lambda^2+i\epsilon}\,A(k_q^2),$$

$$\chi_3(k;P) = -\sum_{1\leq i,j\leq 3}\frac{1}{[(k+\frac{P}{2})^2-m_i^2+i\epsilon]} \times \frac{\mathcal{N} m^3\,(E_i+m_0 D_i)\,D_j}{[k^2-\lambda^2+i\epsilon][(k-\frac{P}{2})^2-m_j^2+i\epsilon]}$$

$$G_3(\gamma,z) = -\sum_{1\leq i,j\leq 3}(E_i+m_0 D_i)\,D_j\times\mathcal{N} m^3\,F(\gamma,z;\;m_i,m_j)$$

- χ_4

$$\chi_4(k,P) = -A(k_q^2)\,\frac{\mathcal{N} m^3}{k^2-\lambda^2+\imath\epsilon}\,A(k_{\bar{q}}^2)$$

$$\chi_4(k,P) = -\sum_{1\le i,j\le 3} \frac{D_i}{(k+\frac{P}{2})^2 - m_i^2 + i\epsilon} \times \frac{\mathcal{N} m^3}{k^2-\lambda^2+i\epsilon}\,\frac{D_j}{(k-\frac{P}{2})^2 - m_j^2 + i\epsilon}$$

The fourth weight function of Nakanish representation

$$G_4(\gamma,z) = -\mathcal{N} m^3 \sum_{i,j} D_i D_j F(\gamma,z;\; m_i,m_j)$$

Orthogonal Basis

- Light-front projection of the Bethe-Salpeter amplitude

$$\psi_i(\gamma, z) = \int \frac{dk^-}{2\pi}\ \phi_i(k, p)$$

- Notation $|\vec{k}_\perp|^2 = \gamma$,

$$\phi_i(k, p) = \int_{-1}^{1} dz' \int_0^{\gamma'_f} d\gamma' \frac{g_i(\gamma', z')}{[k^2 + z'\ k \cdot p - \gamma' + i\epsilon]^3}$$

$$\psi_i(\gamma, z) = -\frac{i}{M_\pi} \int_0^{\infty} d\gamma'\ \frac{g_i(\gamma', z)}{[\frac{z^2 M_\pi^2}{4} + \gamma + \gamma' - i\epsilon]^2}$$

- Applying the principle of uniqueness

$$g_1(\gamma', z') = G_1(\gamma', z') + (M_\pi^2/4\gamma')\ G_4(\gamma', z') + \int_0^{\gamma'} d\Gamma\left(G_4(\Gamma, z') - z'\partial_{z'} G_4(\Gamma, z')\right)$$

$$g_2(\gamma', z') = \frac{M_\pi}{2}\left[G_2(\gamma', z') + G_3(\gamma', z')\right] + \frac{2}{M_\pi}\int_0^{\gamma'} d\Gamma \partial_{z'}\left[G_2(\Gamma, z') - G_3(\Gamma, z')\right]$$

$$g_3(\gamma', z') = M_\pi\left[-G_2(\gamma', z') + G_3(\gamma', z')\right]$$

$$g_4\gamma', z') = M_\pi^2 G_4(\gamma', z') + G_3(\gamma', z')$$

- Wave function projected on the front of light

$$\psi_i(\gamma, z) = -\frac{\imath}{M_\pi} \int_0^{\gamma_f'} d\gamma' \; \frac{g_i(\gamma', z)}{[\frac{z^2 M_\pi^2}{4} + \gamma + \gamma' - i\epsilon]^2}$$

with, $z = 1 - 2\xi$ (ξ internal momentum)

$$\psi_3(\gamma, z) = -\frac{\imath}{M_\pi} \int_0^{\gamma_f'} d\gamma' \frac{g_3(\gamma', z)}{[\frac{z^2 M_\pi^2}{4} + \gamma + \gamma' - i\epsilon]^2}$$

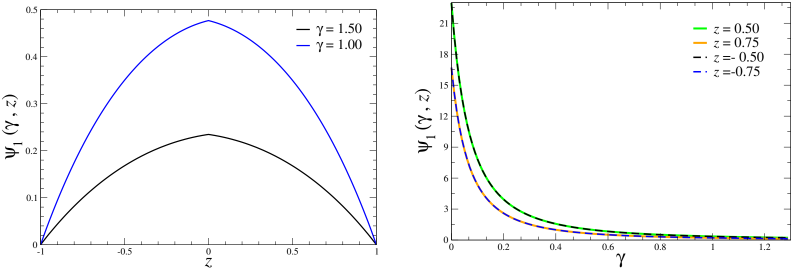


Figure: Right panel: Figure for the function $\psi_1(\gamma, z)$ with fixed γ and varying $z(z = x)$ from -1 to 1. Left panel: Figure for the function $\psi_1(\gamma, z)$, with fixed $z(z = x)$ and varying γ $0 < 3$

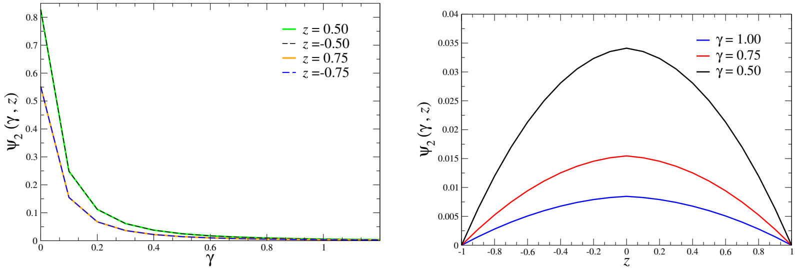


Figure: Left panel: Figure for the function $\psi_2(\gamma, z)$ with fixed z, and varying γ. Right panel: Figure for plot of $\psi_2(\gamma, z)$ with fixed γ, and varying z.

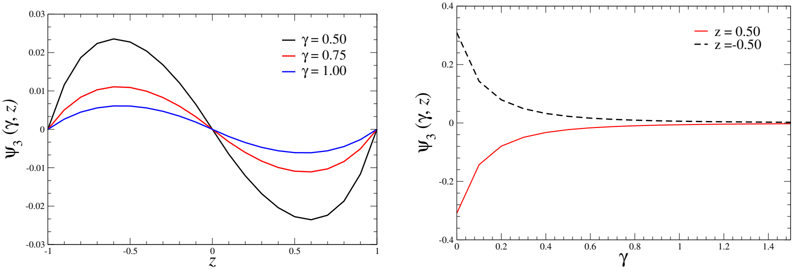


Figure: Left panel: Plot of $\psi_3(\gamma, z)$ with fixed γ and varying $x = z$ from -1 to 1. Right panel: Plot of $\psi_3(\gamma, z)$ with fixed z and varying γ from 0 to 3

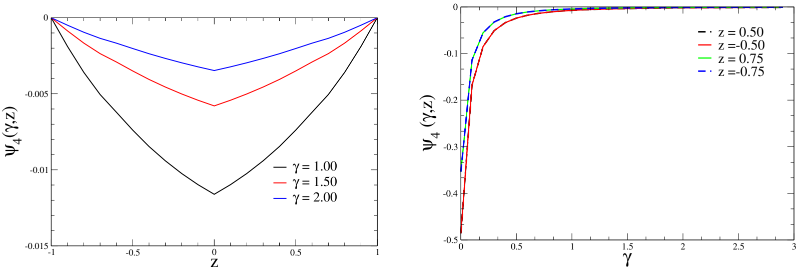


Figure: Left panel: Figure for the function $\psi_4(\gamma, z)$ with fixed γ and varying $z = x$ from -1 to 1.
Right panel: Plot of $\psi_4(\gamma, z)$ with fixed $x = z$ and varying γ from 0 to 3

.

Observables

- **Pion Eletroweak decay constant: BSA**

$$\imath p^\mu f_\pi = N_c \int \frac{d^4k}{(2\pi)^4} \, \mathbf{Tr}[\,\gamma^\mu \gamma^5 \, \Phi(p,k)\,]$$

- $N_c = 3$ **is the number of color**
- **Only** $\phi_2(k,p)$

$$f_\pi = \frac{N_c}{8\pi^2 M_\pi} \int_{-1}^{1} dz' \int_0^{\gamma'_f} d\gamma' \frac{g_2(\gamma', z')}{\gamma' + z'^2 M_\pi^2/4}$$

- **Ref. W. de Paula, E. Ydrefors, J. H. Alvarenga Nogueira, T. Frederico, and G. Salme, Phys. Rev. D, 103(1):014002, 2021**

- **Probability valence component // bound state**
- **Parell spin case**

$$\psi_{\uparrow\uparrow}(\gamma, z) = \frac{\sqrt{\gamma}}{M_\pi}\ \psi_4(\gamma, z)$$

- **Anti-symmetric spin case**

$$\psi_{\uparrow\downarrow}(\gamma, z) = \psi_2(\gamma, z) + \frac{z}{2}\psi_3(\gamma, z) + \frac{i}{M_\pi^3}\int_0^{\gamma_f'} d\gamma' \frac{\frac{\partial}{\partial z}[g_3(\gamma', z)]}{\left(\gamma + \gamma' + \frac{z^2 M_\pi^2}{4} - i\epsilon\right)}$$

- Ref. **W. de Paula, E. Ydrefors, J. Nogueira, T. Frederico, and G. Salme, Phys. Rev. D, 103(1):014002, 2021**

- **Final valence probability**

$$P_{\text{val}} = \int_{-1}^{1} dz \int_{0}^{\infty} d\gamma \;\rho(\gamma, z)$$

- $\Longrightarrow$ **Valence momentum distribution density**

$$\rho(\gamma, z) = \rho_{\uparrow\downarrow}(\gamma, z) + \rho_{\uparrow\uparrow}(\gamma, z)$$

- Density for the spin aligned configuration

$$\rho_{\uparrow\uparrow}(\gamma, z) = \frac{N_c}{16\pi^2} |\psi_{\uparrow\uparrow}(\gamma, z)|^2 = \frac{N_c}{16\pi^2} \frac{\gamma}{M_\pi^2} |\psi_4(\gamma, z)|^2$$

- **Anti-aligned quark spin probability density**

$$\rho_{\uparrow\downarrow}(\gamma, z) = \frac{N_c}{16\pi^2} |\psi_{\uparrow\downarrow}(\gamma, z)|^2$$

- $\Longrightarrow$ **Slide [41] for** $\psi_{\uparrow\downarrow}$

- **Numerical Results**

$$P_{\uparrow\uparrow} = \int_0^\infty d\gamma \int_{-1}^1 dz \; \rho_{\uparrow\uparrow}(\gamma, z)$$

$$P_{\uparrow\downarrow} = \int_0^\infty d\gamma \int_{-1}^1 dz \; \rho_{\uparrow\downarrow}(\gamma, z)$$

Set	parameters	P_{val}	$P_{\uparrow\downarrow}$	$P_{\uparrow\uparrow}$	f_π(MeV)
(I)	[**]	0 .70	0.58	0.12	130.1

Ref.[**] **C. S. Mello, J. P. B. C. de Melo, and T. Frederico Phys. Lett. B766 (2017) 86–93**

- **Longitudinal distribution:** $\phi(\xi)$

$$
\begin{aligned}
\phi(\xi) &= \int_0^\infty d\gamma\ \rho(\gamma, z) \\
&= \int_0^\infty d\gamma\ \rho_{\uparrow\downarrow}(\gamma, z) + \int_0^\infty d\gamma\ \rho_{\uparrow\uparrow}(\gamma, z) \\
&= \phi_{\uparrow\downarrow}(\xi) + \phi_{\uparrow\uparrow}(\xi)
\end{aligned}
$$

- with $z = 2\xi - 1$
- **Transverse distribution:** $p(\gamma)$

$$
\begin{aligned}
p(\gamma) &= \int_{-1}^1 dz\ \rho(\gamma, z) \\
&= \int_{-1}^1 dz\ \rho_{\uparrow\downarrow}(\gamma, z) + \int_{-1}^1 dz\ \rho_{\downarrow\downarrow}(\gamma, z) \\
&= p_{\uparrow\downarrow}(\gamma) + p_{\downarrow\downarrow}(\gamma)
\end{aligned}
$$

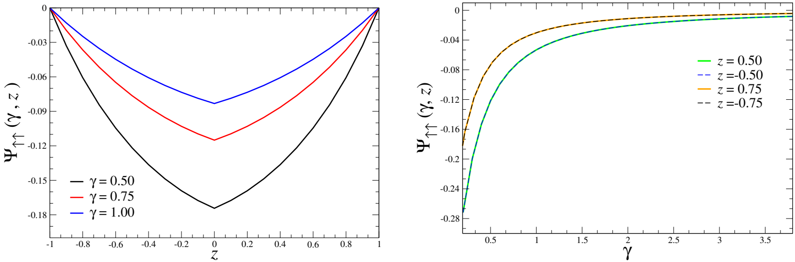
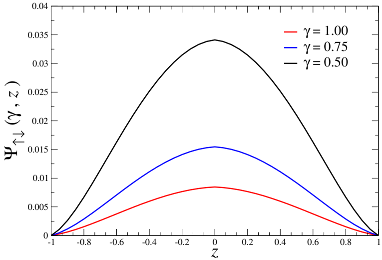


Figure: Left panel: Figure for the function $\psi_{\uparrow\uparrow}(\gamma, z)$ with fixed γ and varying z from -1 to 1. Right panel: Figure for $\psi_{\uparrow\uparrow}(\gamma, z)$ with γ variation, and fix z.



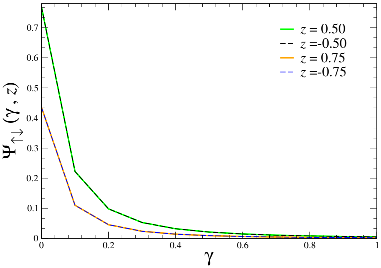


Figure: Left panel: Figure for the function plot of $\psi_{\uparrow\downarrow}(\gamma, z)$ with fixed γ and varying z from -1 to 1. Right panel: Figure for $\psi_{\uparrow\downarrow}(\gamma, z)$

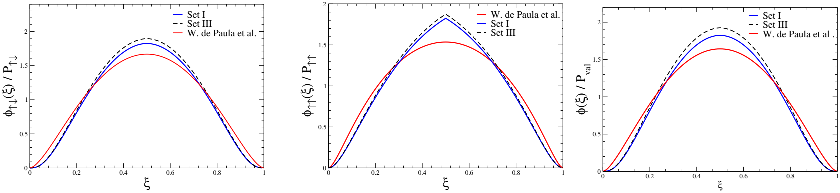


Figure: **Left panel: Antialigned pion valence longitudinal-momentum distributions, normalized to 1. Middle panel: Aligned pion valence longitudinal-momentum distributions, normalized to 1. Right Panel: Total pion valence longitudinal-momentum distributions, normalized to 1.**

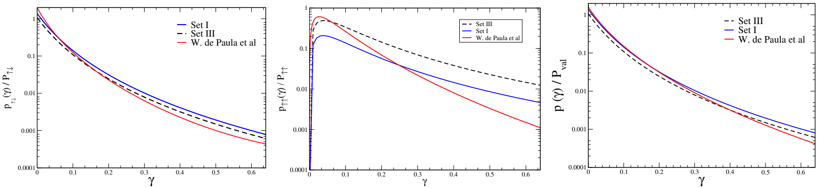


Figure: Left panel: Antialigned pion valence transverse-momentum distributions. Middle panel: Aligned pion valence transverse-momentum distributions. Right Panel: Total valence transverse momentum distributions for the pion.

Summary

• **With NIR plus Light-front projection is possible to described quite well pseudoscalar mesons**
• **Extraction of the Behe-Salpeter amplitudes and wave funcions**
• **Observables calculations**
• **Distribuition's amplitudes**
• **Compare with experimental data, and also, with Lattice results**

$\Longrightarrow$ **Next**

$\star$ **Pion Form Factors**
$\star$ **Constituent quark masses (constants) to see mass effects**
$\star$ **Partons distributions amplitudes, GPD's, TMD's ...**
$\star$ **Kaon (like pion)**
$\star$ **Vector mesons**

Thanks to the Organizers

Support LFTC and Brazilian Agencies

- **FAPESP** , **CNPq** and **CAPES**

Thanks (Obrigado)!!