Three-particle scattering amplitudes from lattice QCD

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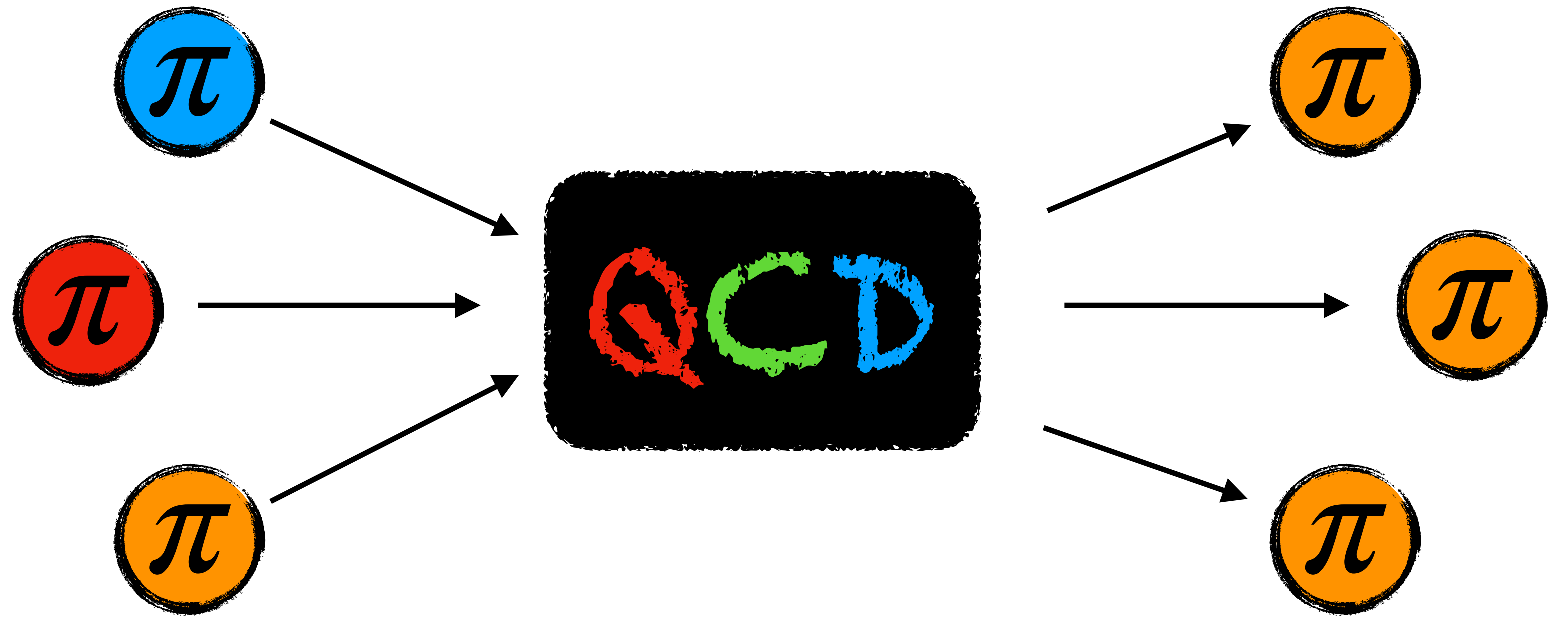

Outline

1. Introduction
2. Finite-Volume Spectrum
3. Three particles in finite volume
4. Applications to lattice QCD
5. Summary and Outlook

Introduction

Quantum Chromodynamics

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$$\mathscr{L}_{QCD} = \sum_i^{N_f} \bar{q}_i \left(D_\mu \gamma^\mu + m_i \right) q_i + \frac{1}{4} G^a_{\mu\nu} G^{\mu\nu}_a$$

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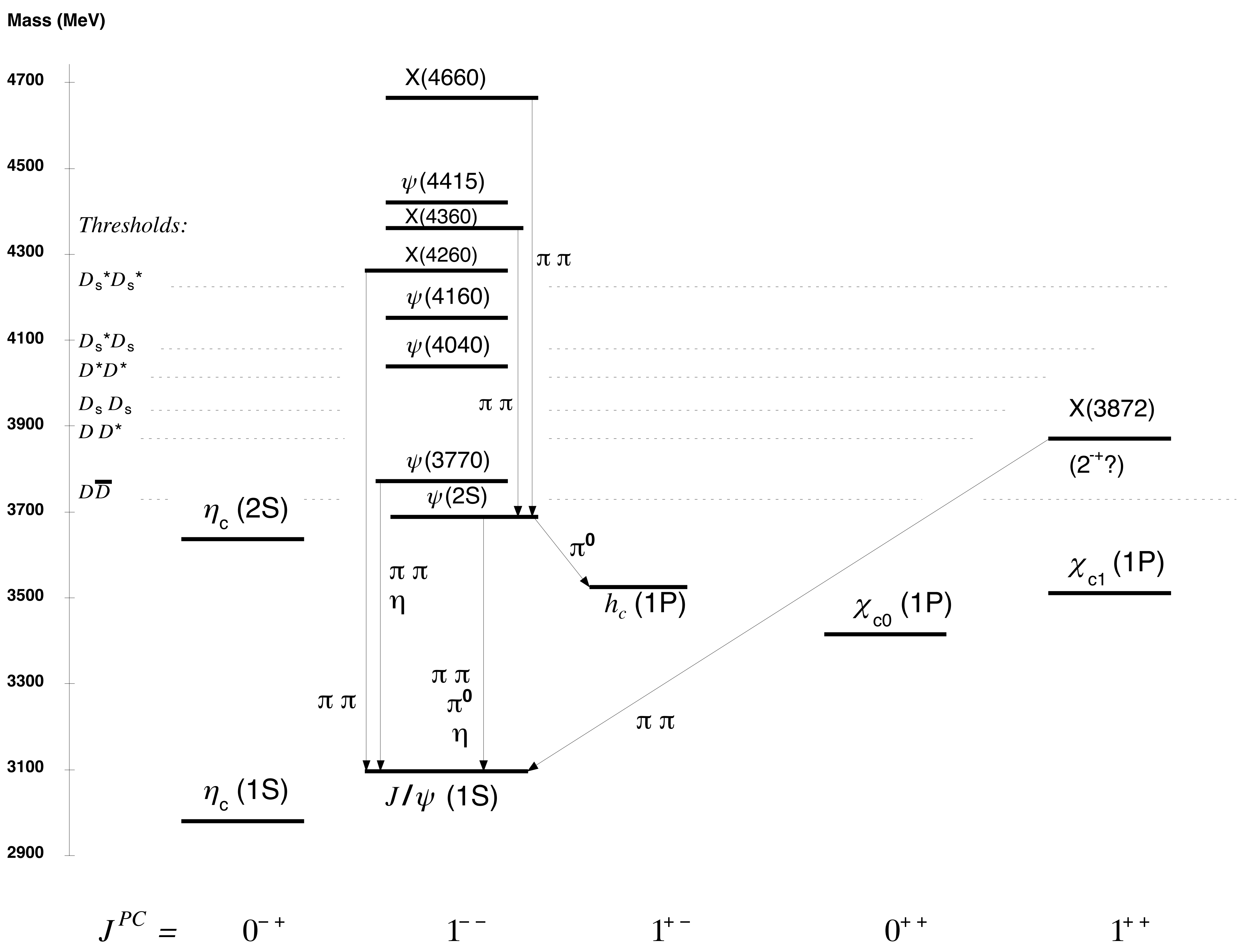
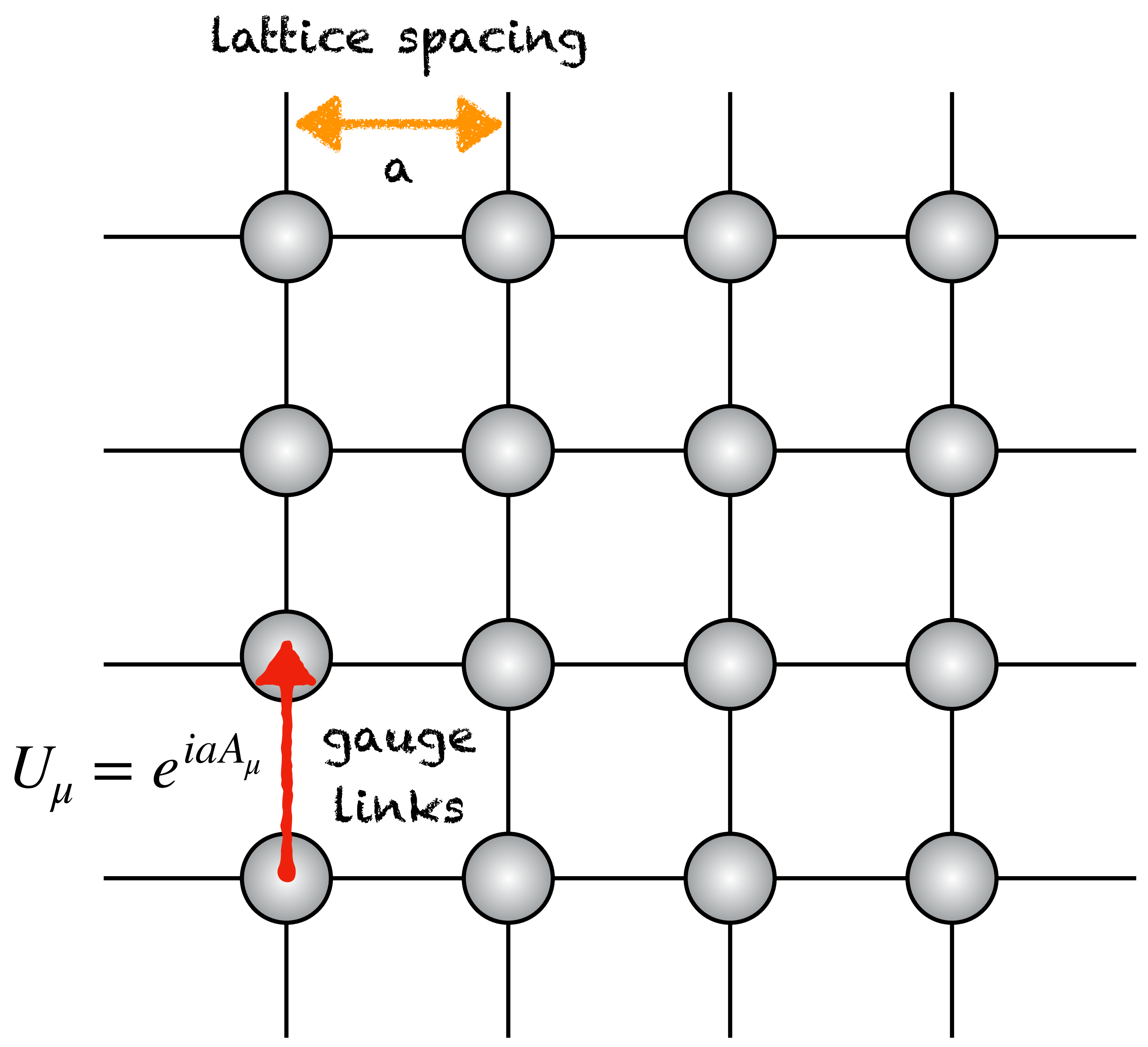
Charmonium Spectrum (PDG)



QCD on the lattice

- **Lattice QCD is a first-principles numerical approach to the strong interaction**



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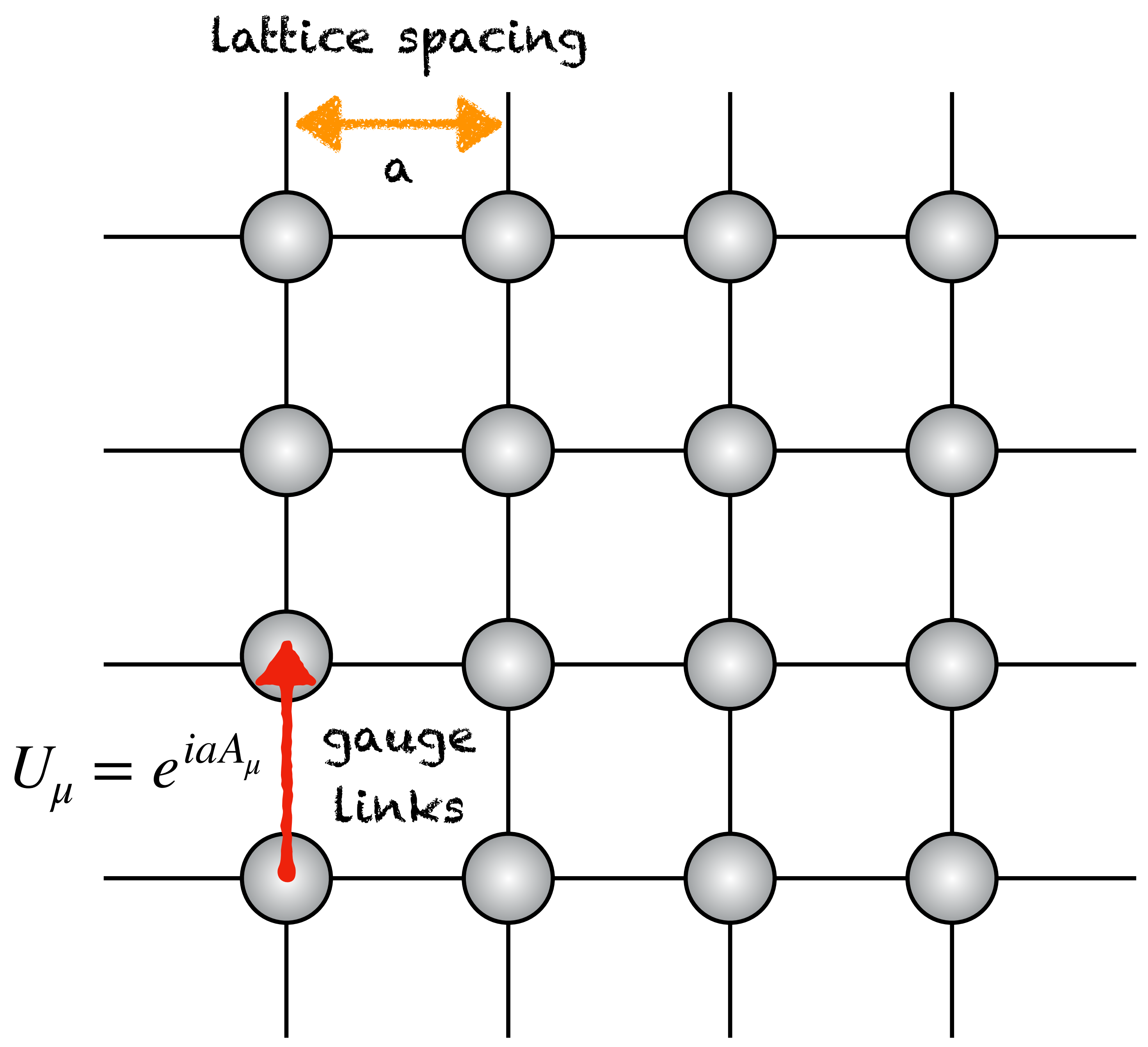
$$\langle \mathcal{O}(t)\mathcal{O}(0)\rangle = \frac{1}{\mathcal{Z}}\int D\psi D\bar{\psi} DA\, \mathcal{O}(t)\mathcal{O}(0) e^{-S_E(\psi,\bar{\psi},A_\mu)}$$



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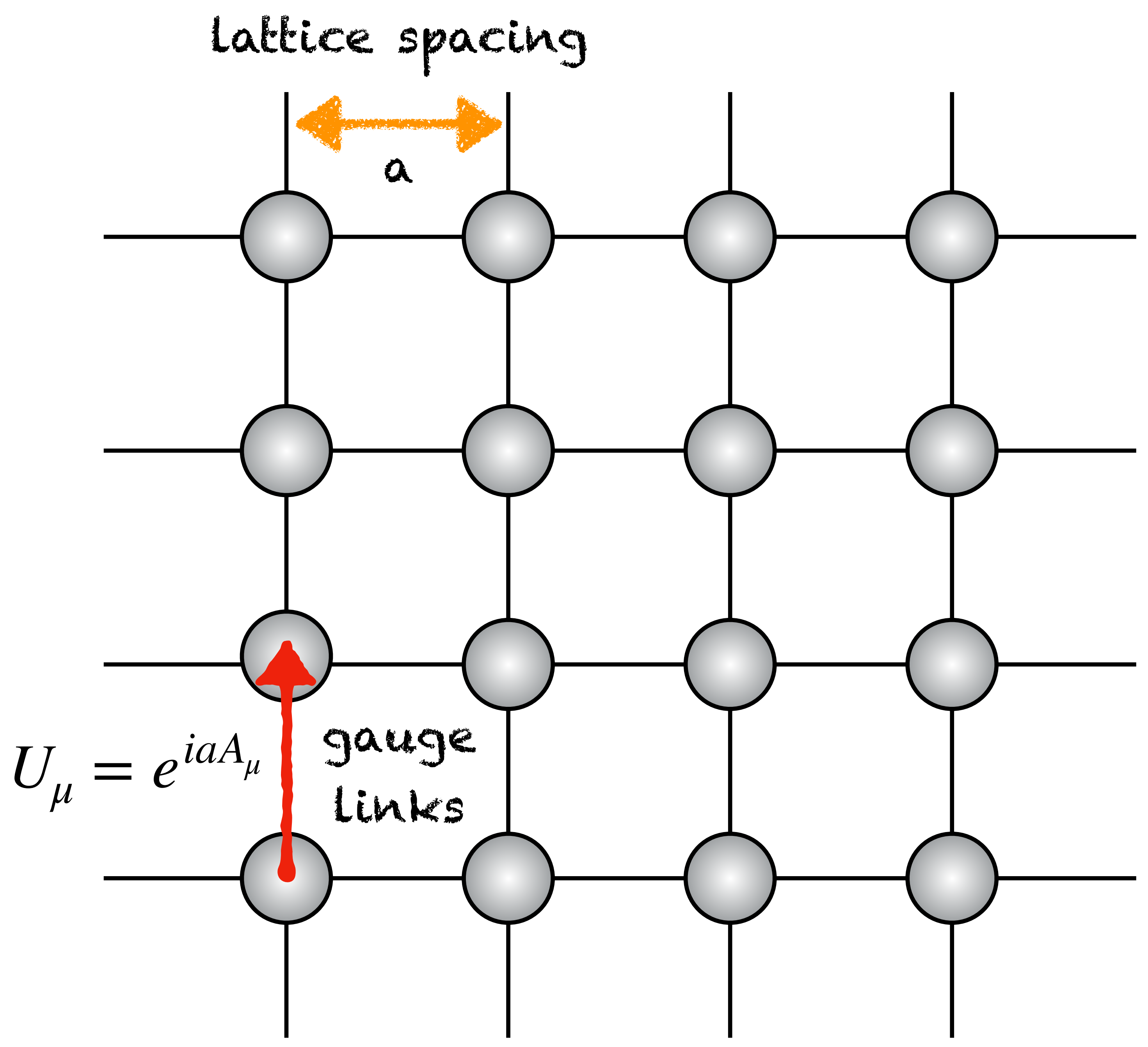

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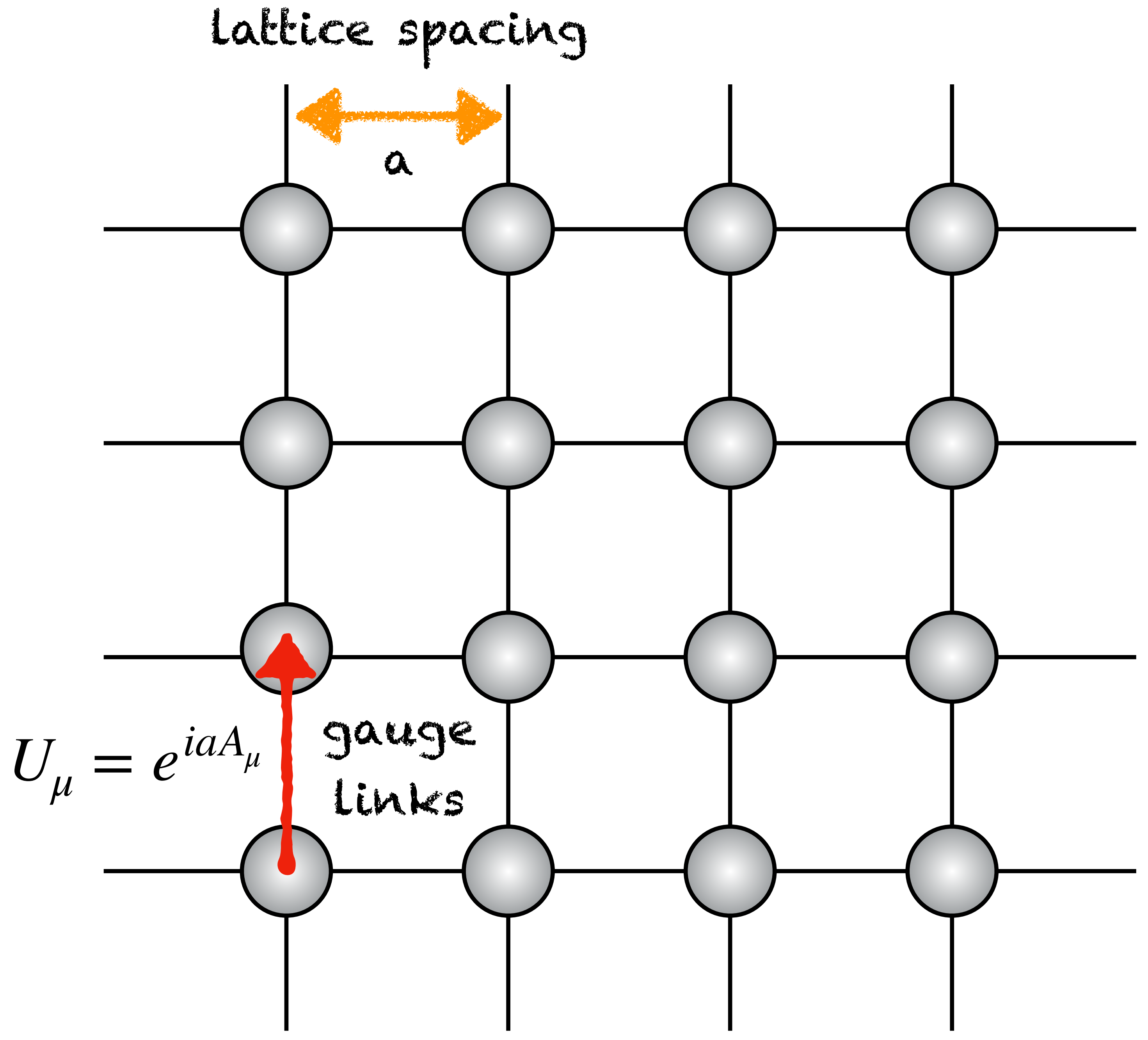

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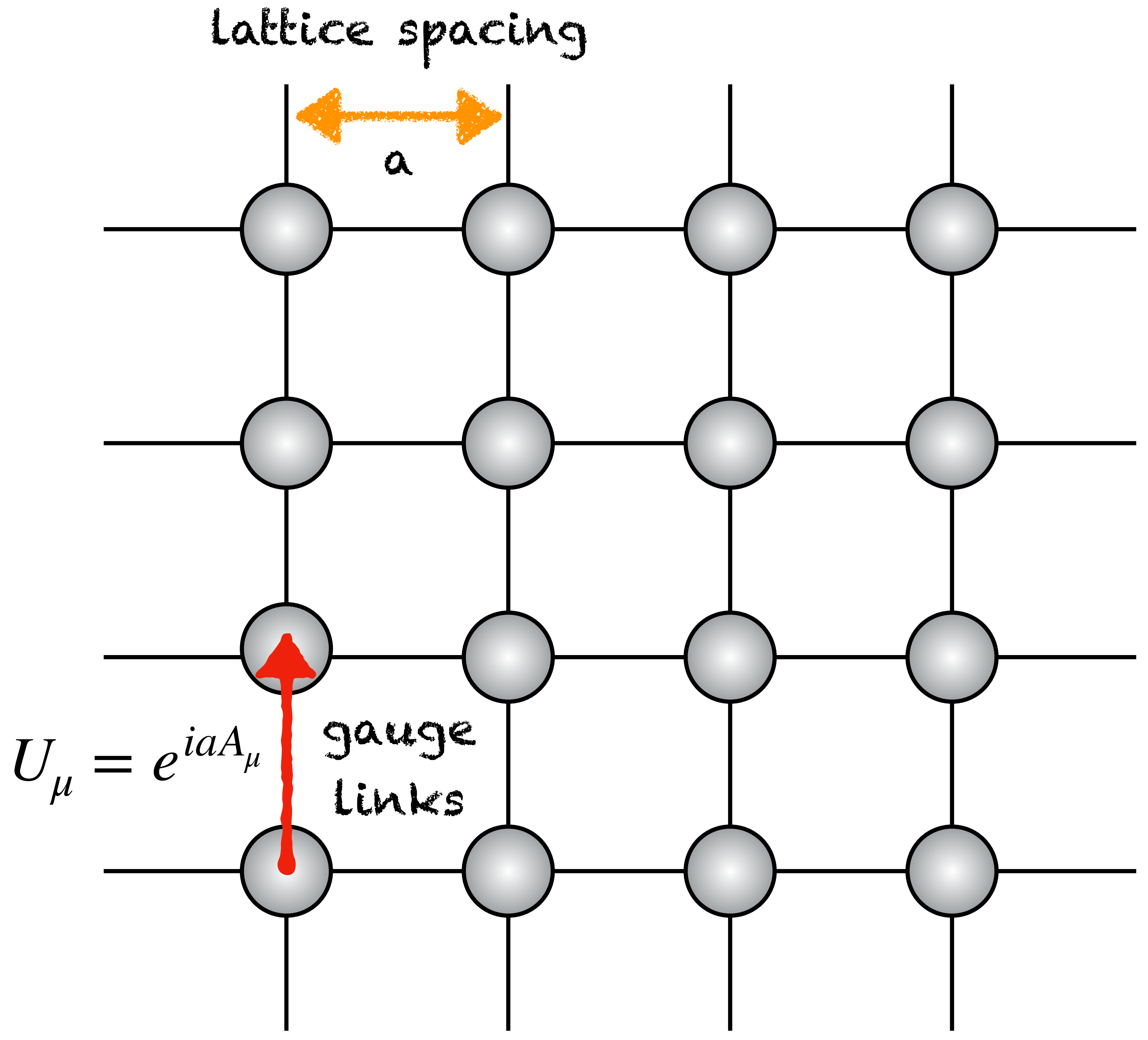

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Can we obtain scattering amplitudes from Euclidean correlation functions?

Scattering on the lattice

- **The Lüscher method is a well-established approach for two-particle scattering on the lattice.**

  See plenary by D. Wilson, Today

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Volume Dependence of the Energy Spectrum in Massive Quantum Field Theories

II. Scattering States

M. Lüscher
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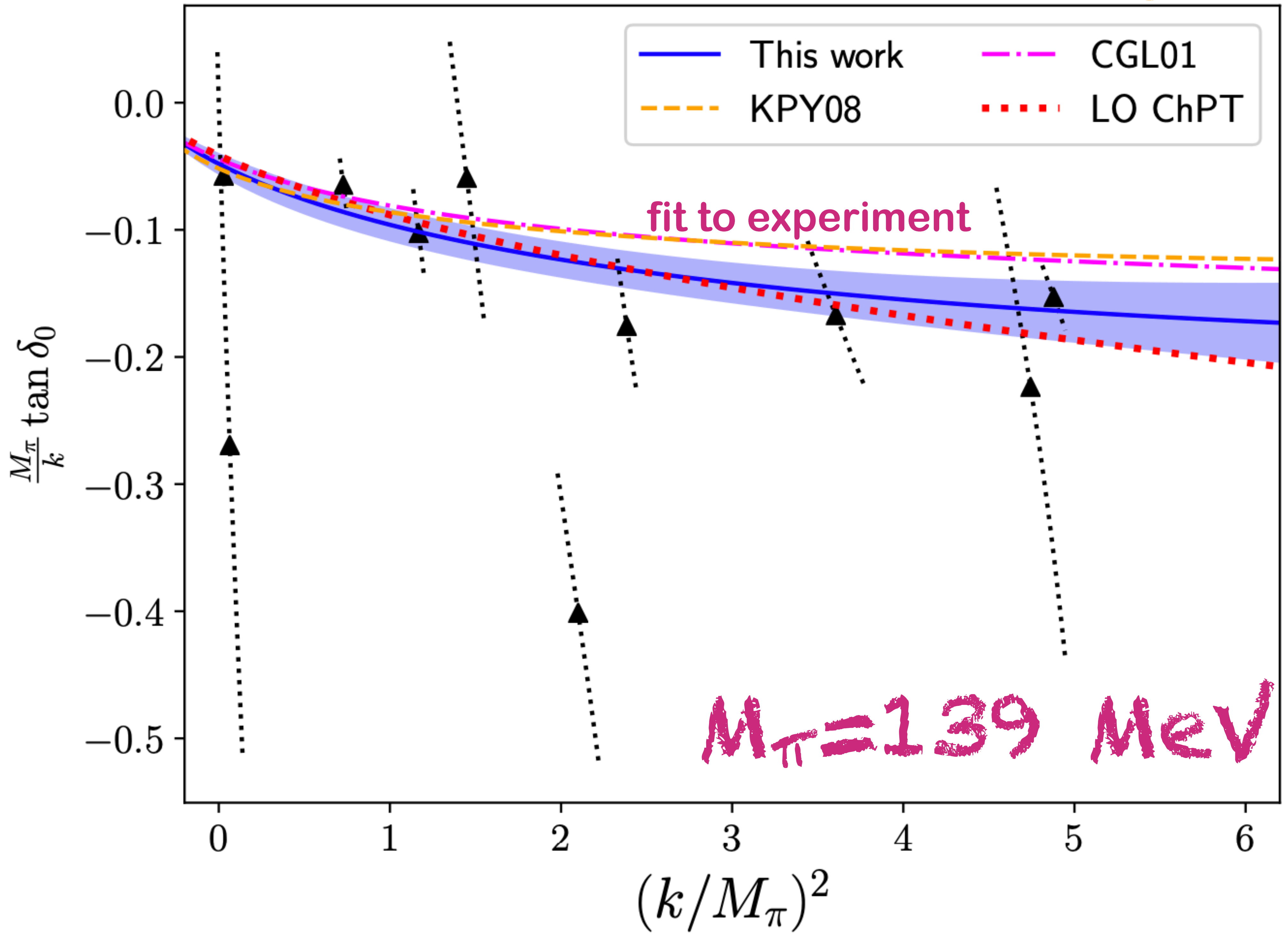

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However, most resonances decay to more than two hadrons

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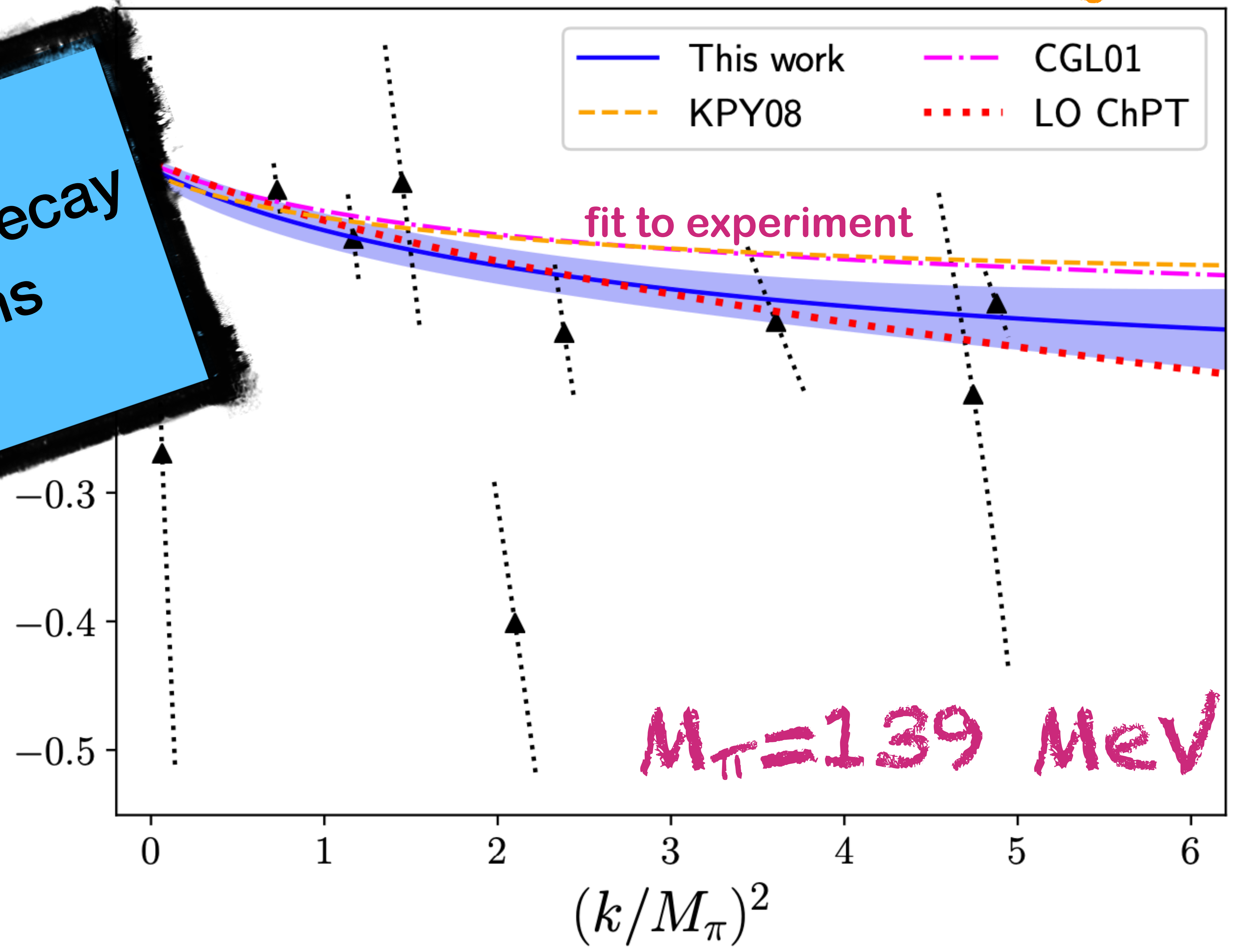

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Resonance	$I_{\pi\pi\pi}$	J^P
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$\omega_3(1670)$	0	3^-
$\pi(1300)$	1	0^-
$a_1(1260)$	1	1^+
$\pi_1(1400)$	1	1^-
$\pi_2(1670)$	1	2^-
$a_2(1320)$	1	2^+
$a_4(1970)$	1	4^+

(with $\geq 3\pi$ decay modes)

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- CP violation in D and K decays

$$K \to 3\pi, \quad D \to 4\pi$$

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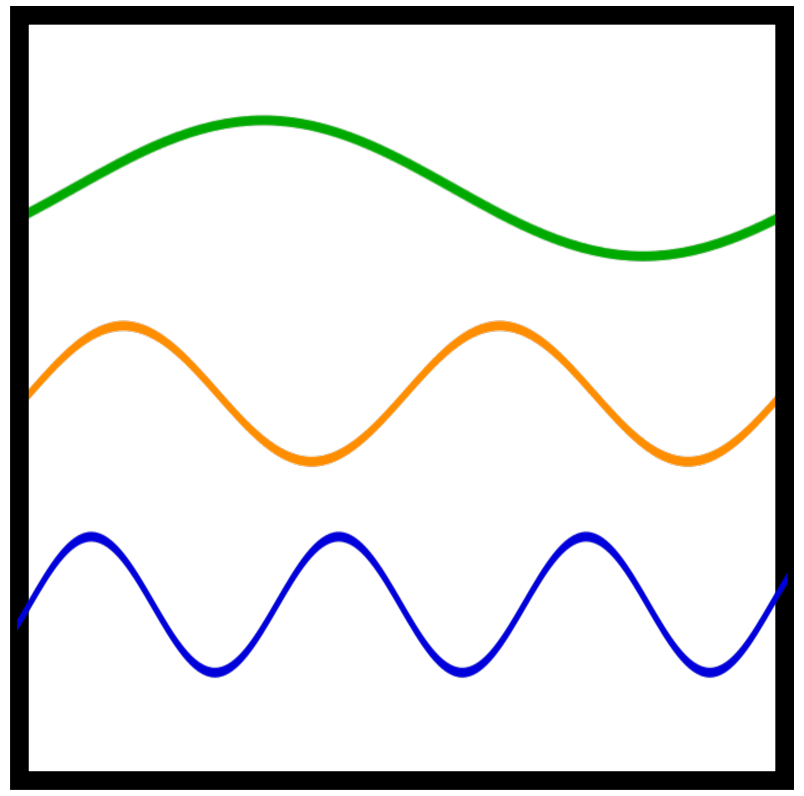
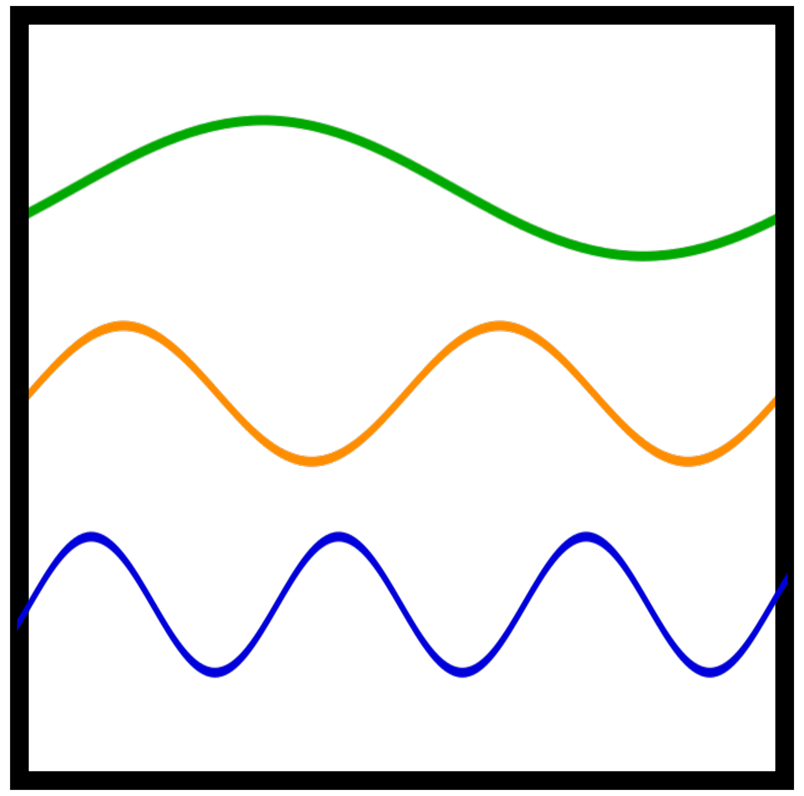
Finite-volume spectrum

Finite-Volume Spectrum



Finite-Volume Spectrum

Free scalar particles in finite volume with periodic BC

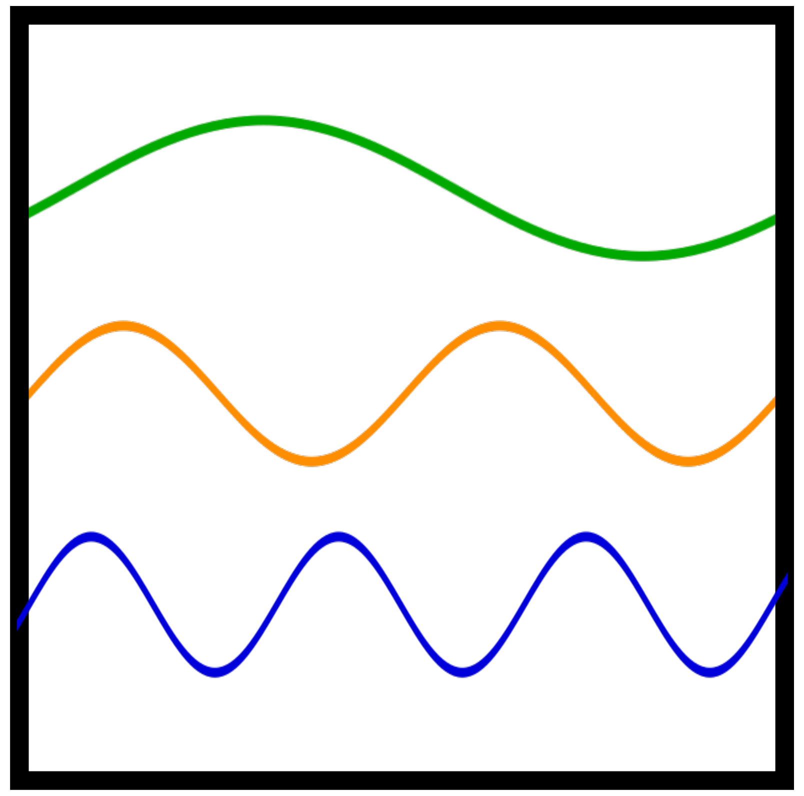


$$\vec{p} = \frac{2\pi}{L}(n_x, n_y, n_z)$$

Two particles: $E = 2\sqrt{m^2 + \frac{4\pi^2}{L^2}\vec{n}^2}$

Finite-Volume Spectrum

Free scalar particles in finite volume with periodic BC → **Interactions change the spectrum!**



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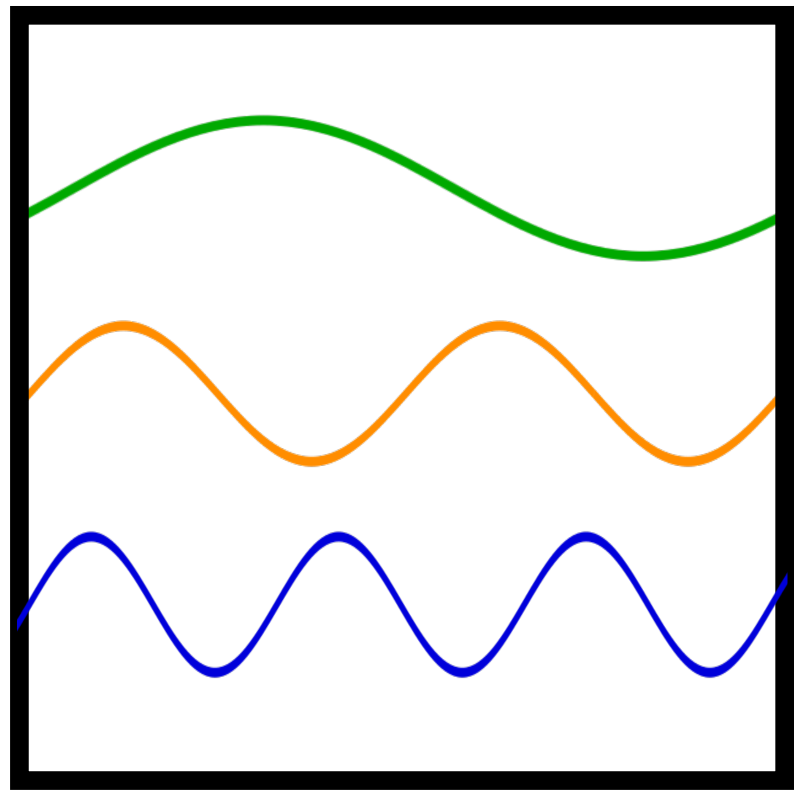

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Ground state to leading order

$$E_2 - 2m = \langle \phi(\vec{0})\phi(\vec{0}) \,|\, \mathbf{H_I} \,|\, \phi(\vec{0})\phi(\vec{0}) \rangle$$

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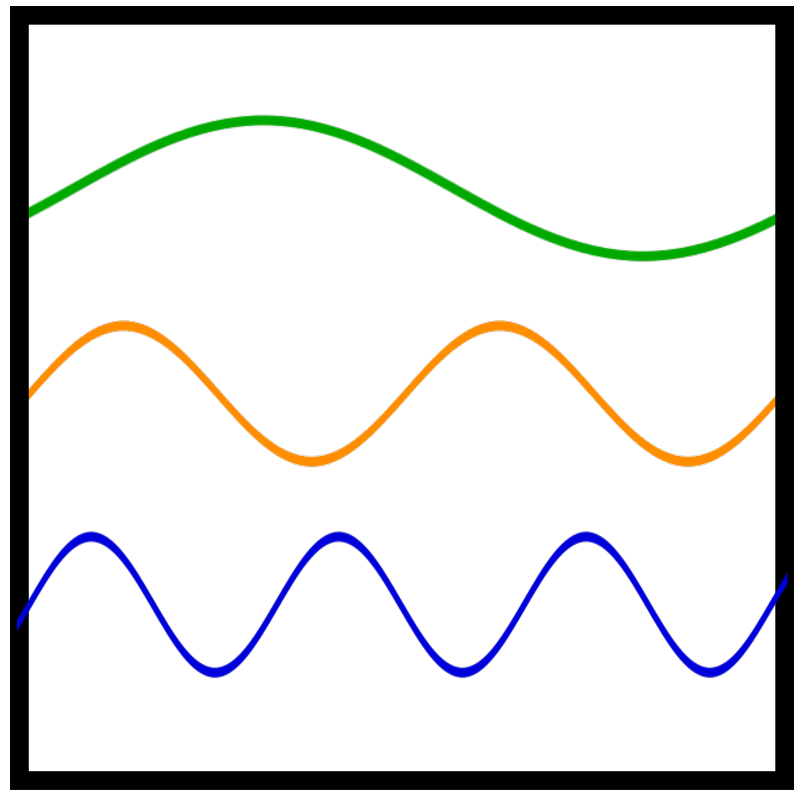

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[Huang, Yang, 1958]

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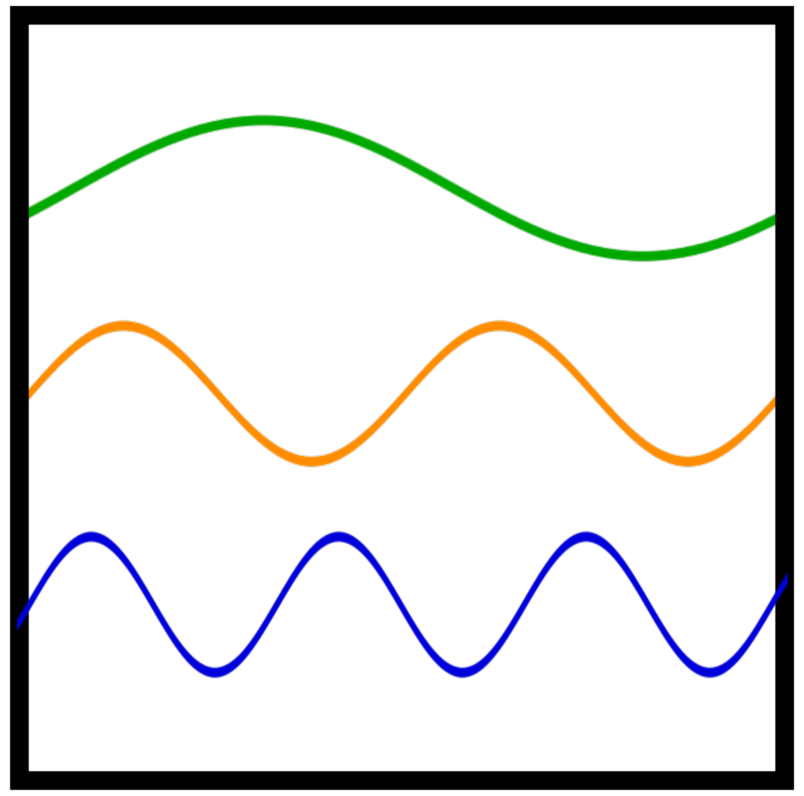

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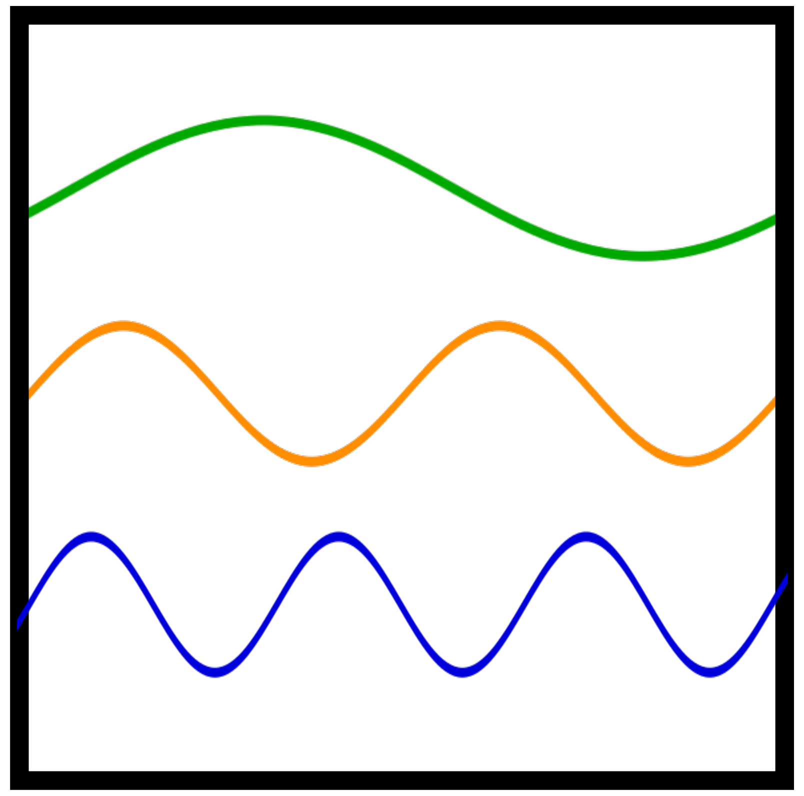

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Interactions change the spectrum!

In general a problem of Quantum Field Theory in finite volume

state to leading order

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Energy levels

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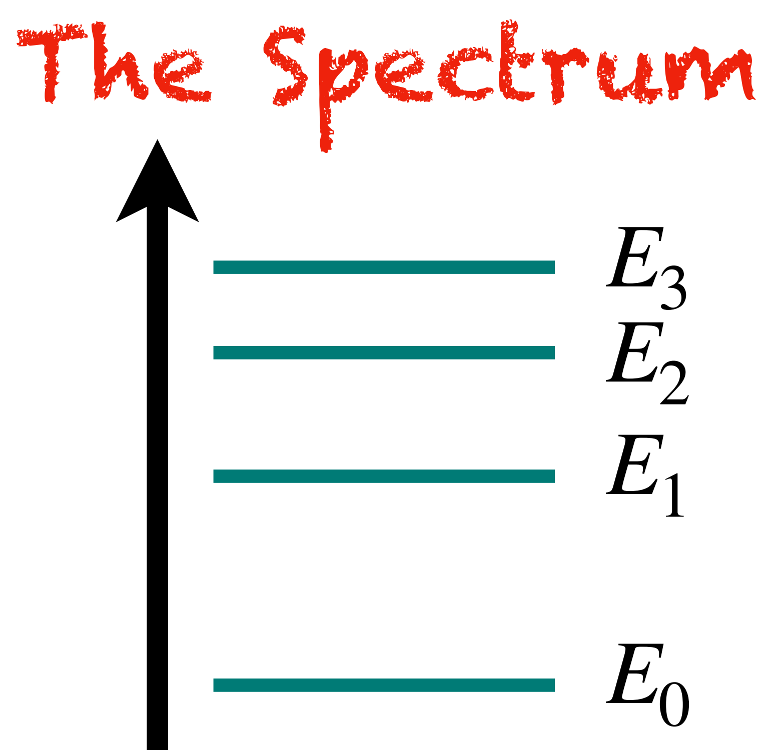

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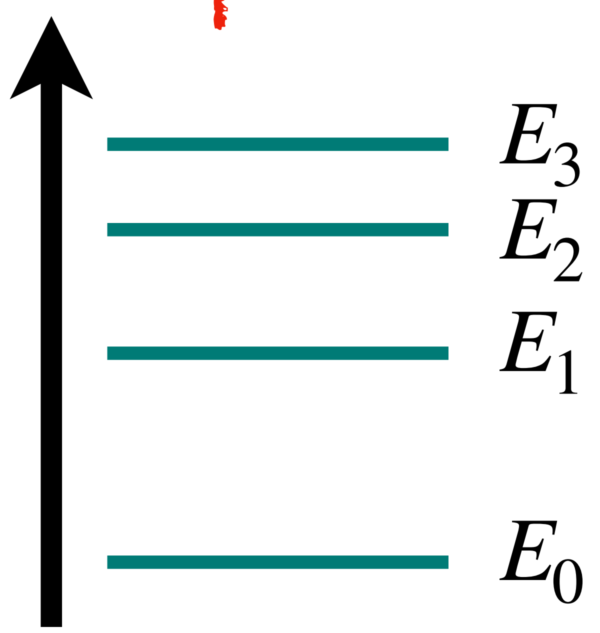
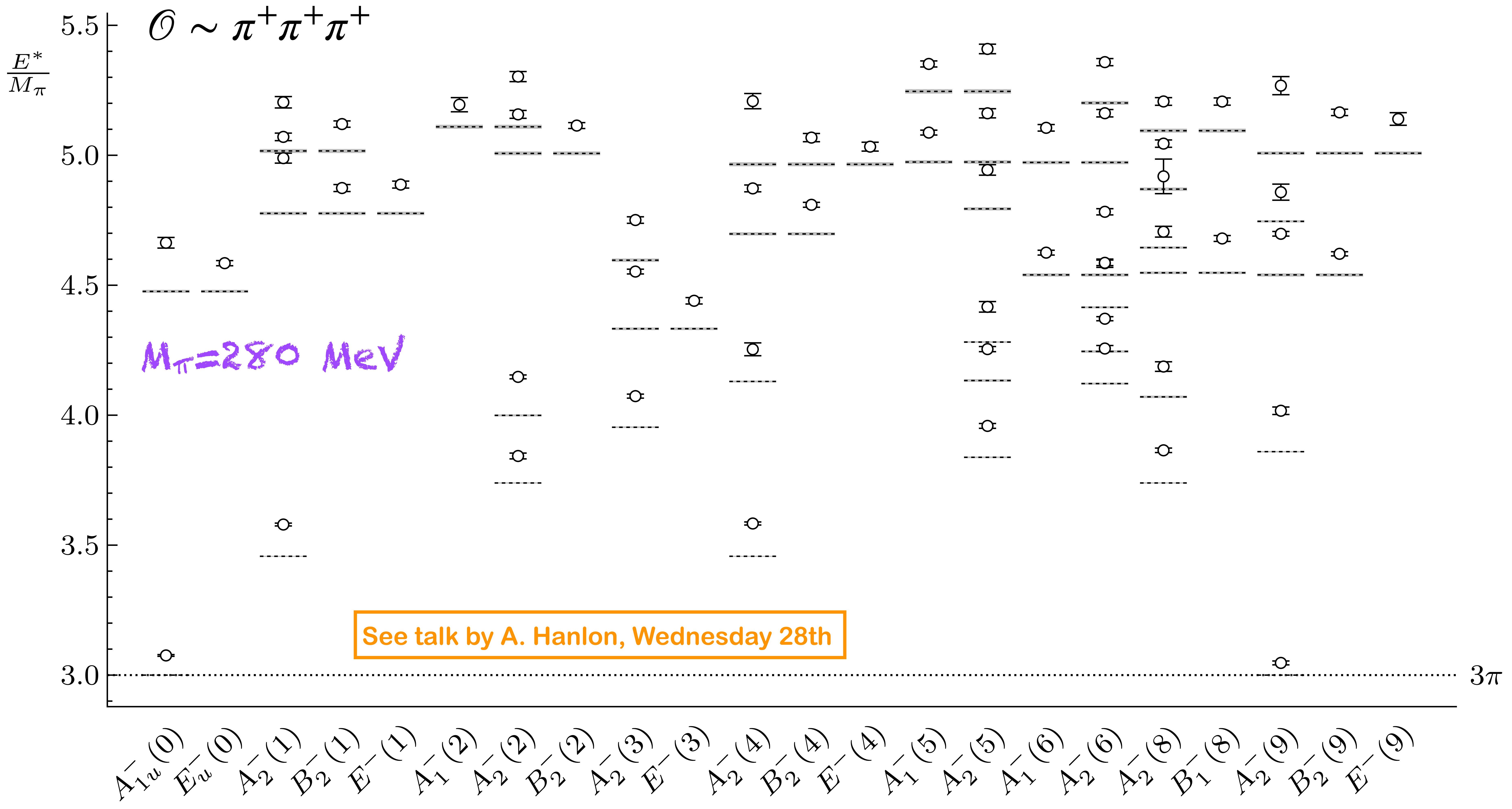
The Spectrum



Current techniques allow the determination of many energy levels!

3π+ energy levels



[Blanton, Hanlon, Hörz, Morningstar, FRL, Sharpe]

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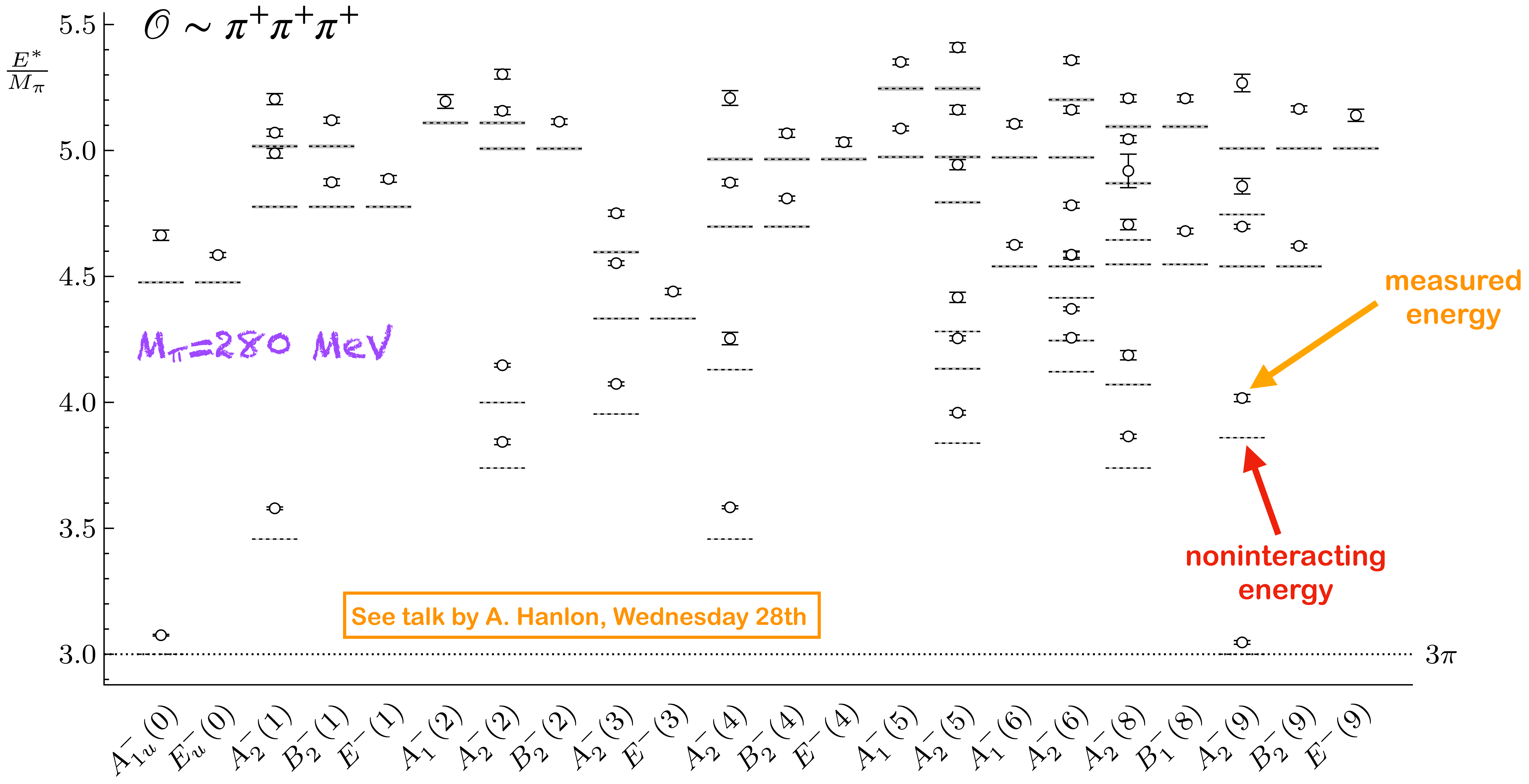

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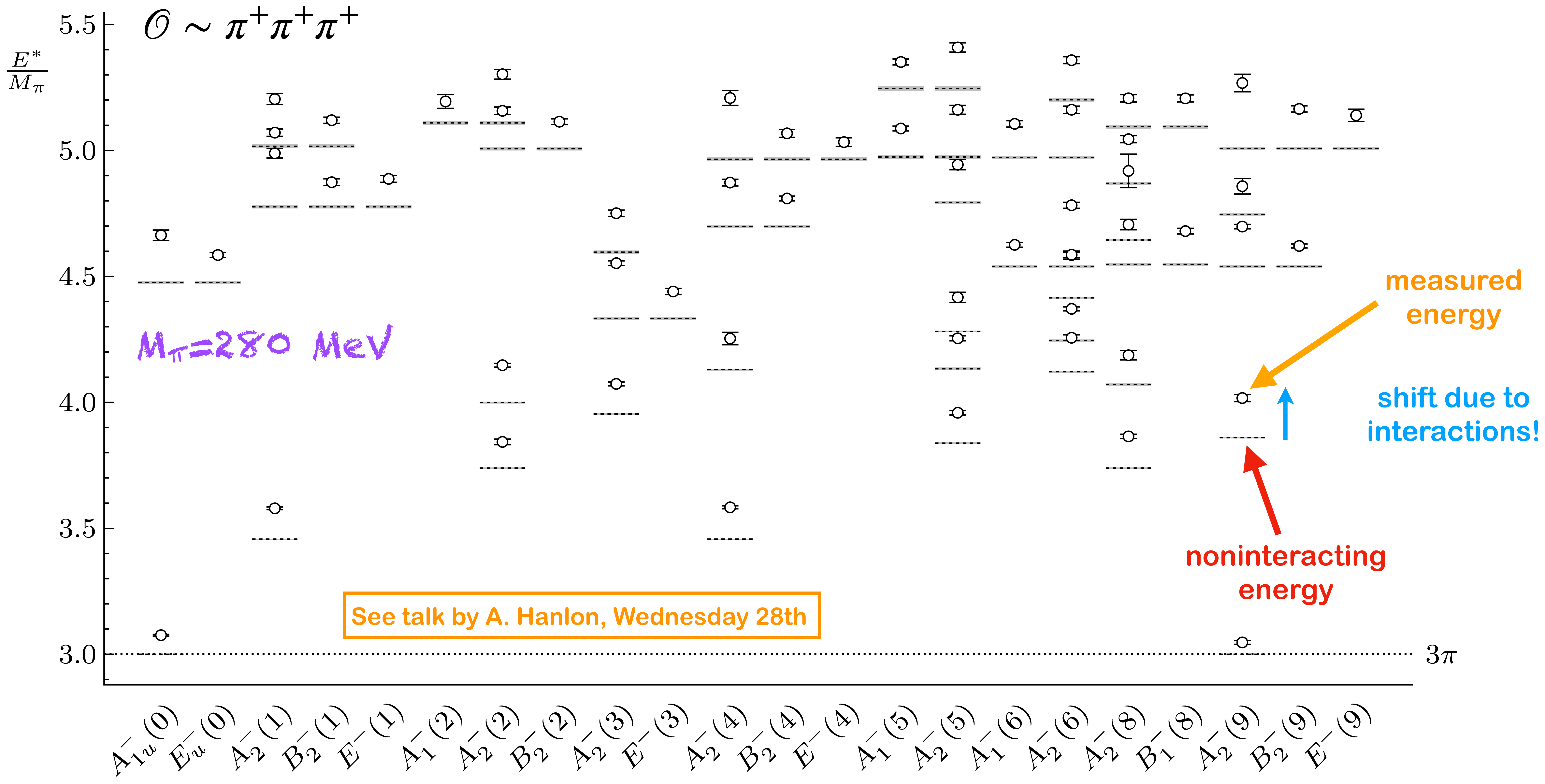

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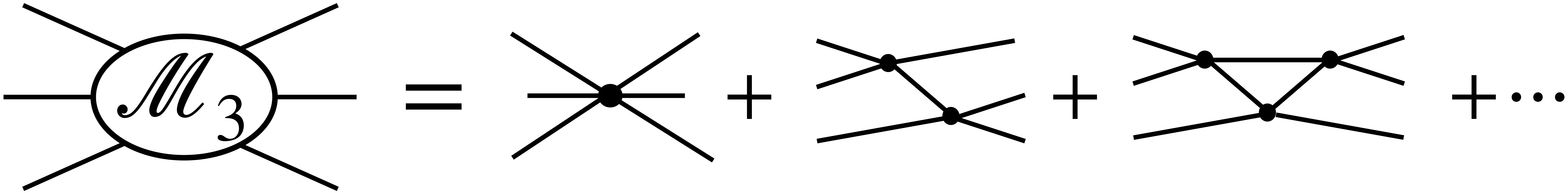
Three particles in finite volume

Three-particle amplitudes

Qualitatively more complicated than the two-particle case!



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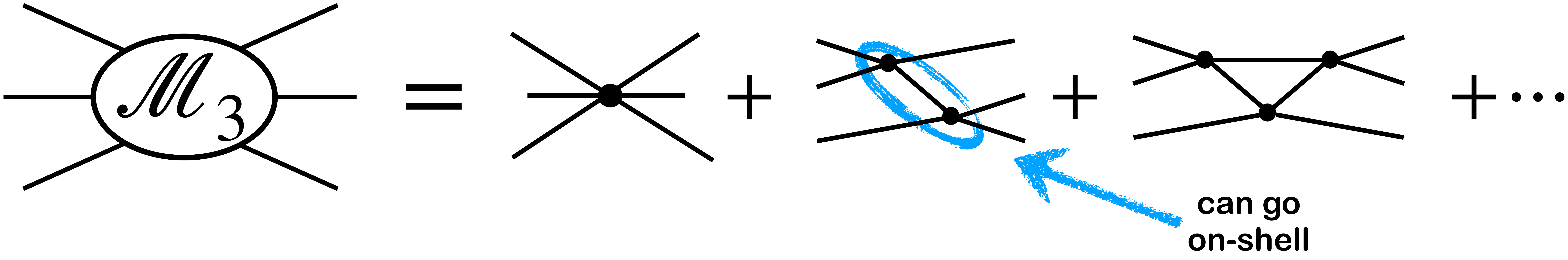

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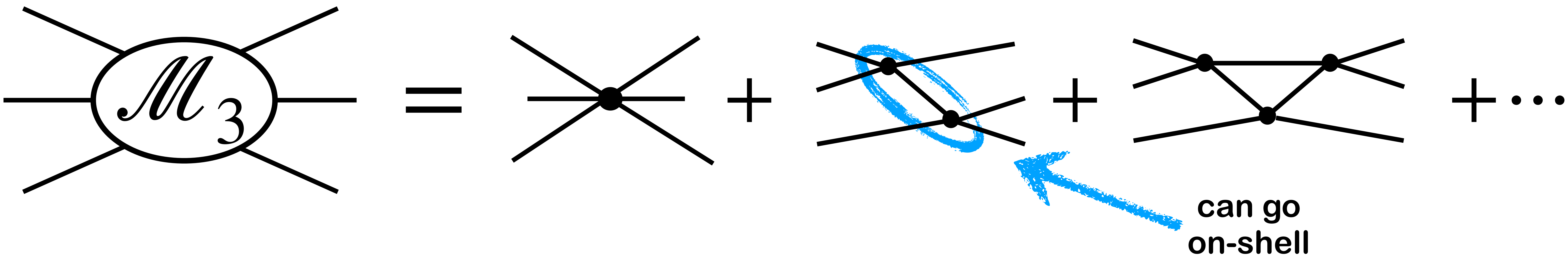

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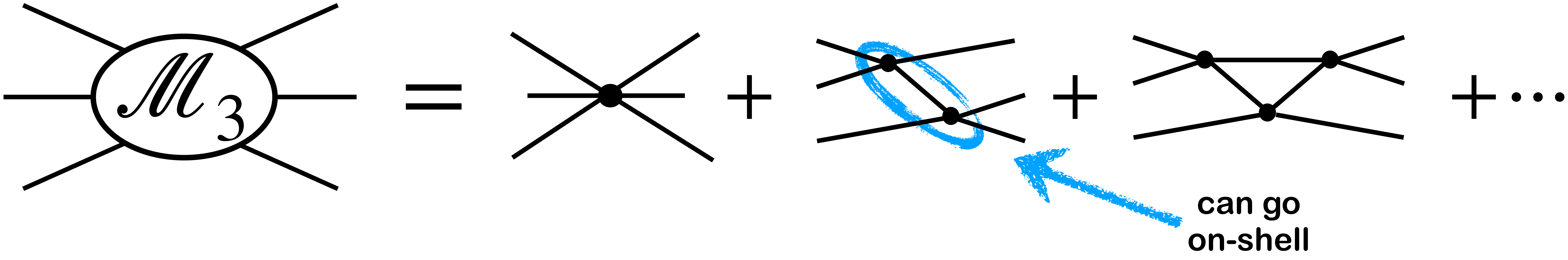

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- They depend also on two-to-two interactions.
  - But any separation between “two-particle” and “three-particle” effects is not well-defined
- However, the three-particle spectrum depends on S-matrix elements! [Polejaeva, Rusetsky]

Three-particle formalism(s)

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- **Generic Relativistic Field Theory (RFT)**

Relativistic, model-independent, three-particle quantization condition

Maxwell T. Hansen[1,*] and Stephen R. Sharpe[1,†]

Also [Blanton, Briceño, Hansen, Jackura, FRL, Szczepaniak, Sharpe]

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Three-particle quantization condition in a finite volume: 1. The role of the three-particle force

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- **Finite-Volume Unitarity (FVU)**

Three-body Unitarity in the Finite Volume

M. Mai[1,*] and M. Döring[1,2,†]

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  - Higher partial waves
  - Nondegenerate and nonidentical scalars
  - Two-to-three transitions
  - Three-particle decays
  - Analysis of lattice QCD data

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- **Non-Relativistic EFT (NREFT)**
  - Nondegenerate (DDK systems)
  - Perturbative expansions for three pions and excited states
  - Three-particle decays
  - Relativistic kinematics can be included

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Three-body Unitarity in the Finite Volume

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All three formalisms should be equivalent.

Explicitly shown for FVU and RFT!

[Blanton, Sharpe]

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- Finite-volume spectrum as solutions of the "quantization condition".

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- The two- and three-particle spectra are used to determine an intermediate quantity

$$\mathcal{K}_{df,3}\,, \quad H_0\,, \quad C_0$$

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Step 2:

- The scheme dependence is removed by solving integral equations.

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1. $\mathscr{K}_2$ and $\mathscr{K}_{df,3}$ parametrize interactions. They can be obtained from the spectrum

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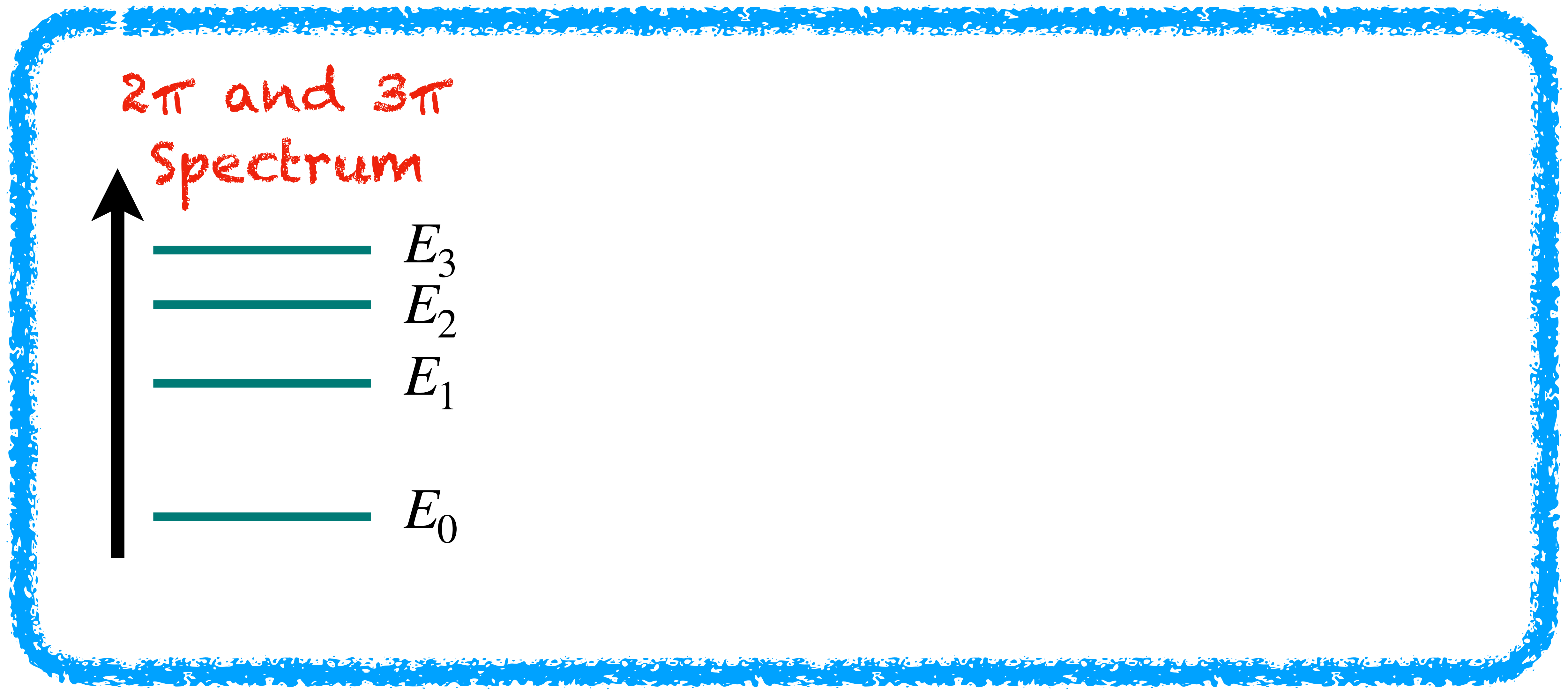

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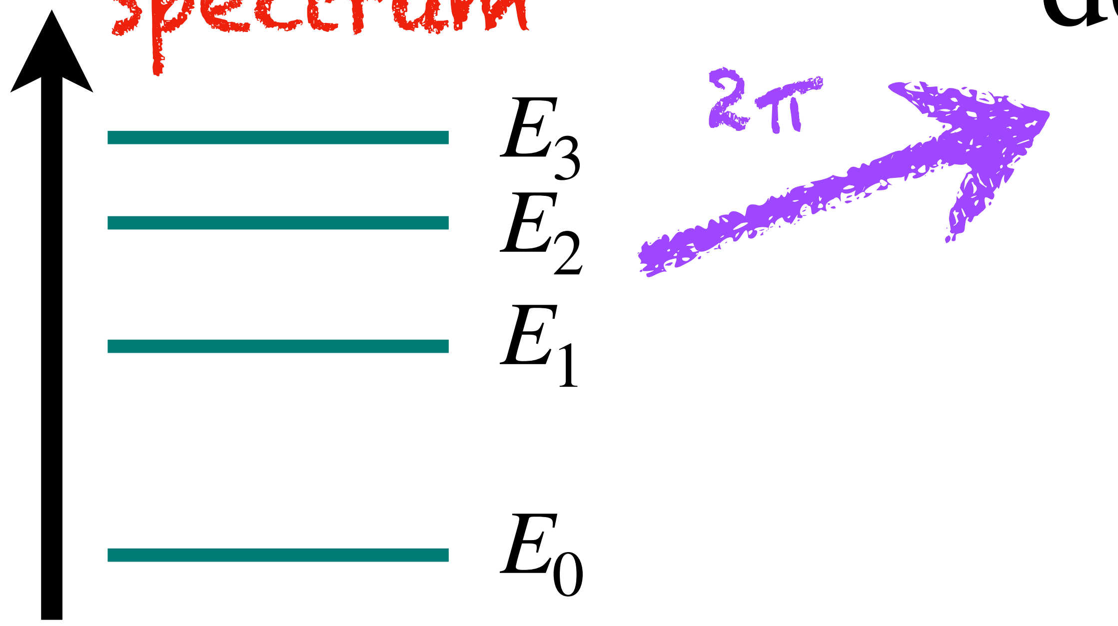
2π and 3π Spectrum



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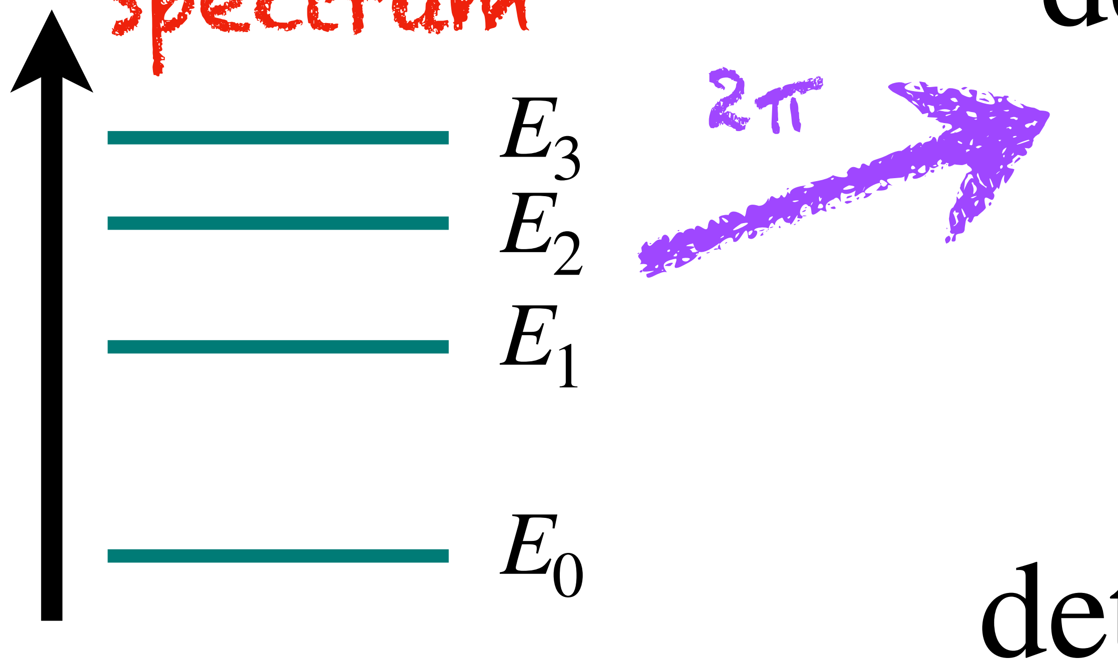
1. $\mathcal{K}_2$ and $\mathcal{K}_{df,3}$ parametrize interactions.
   They can be obtained from the spectrum

The RFT Formalism

2π and 3π Spectrum



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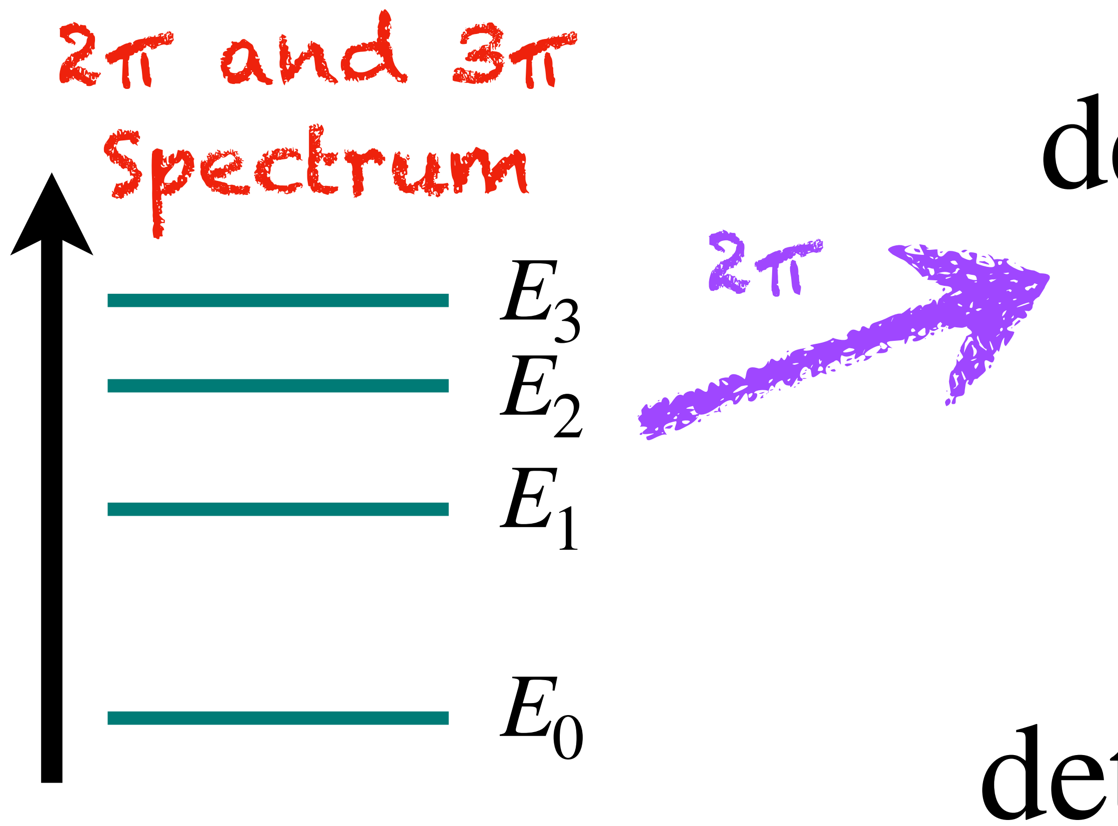

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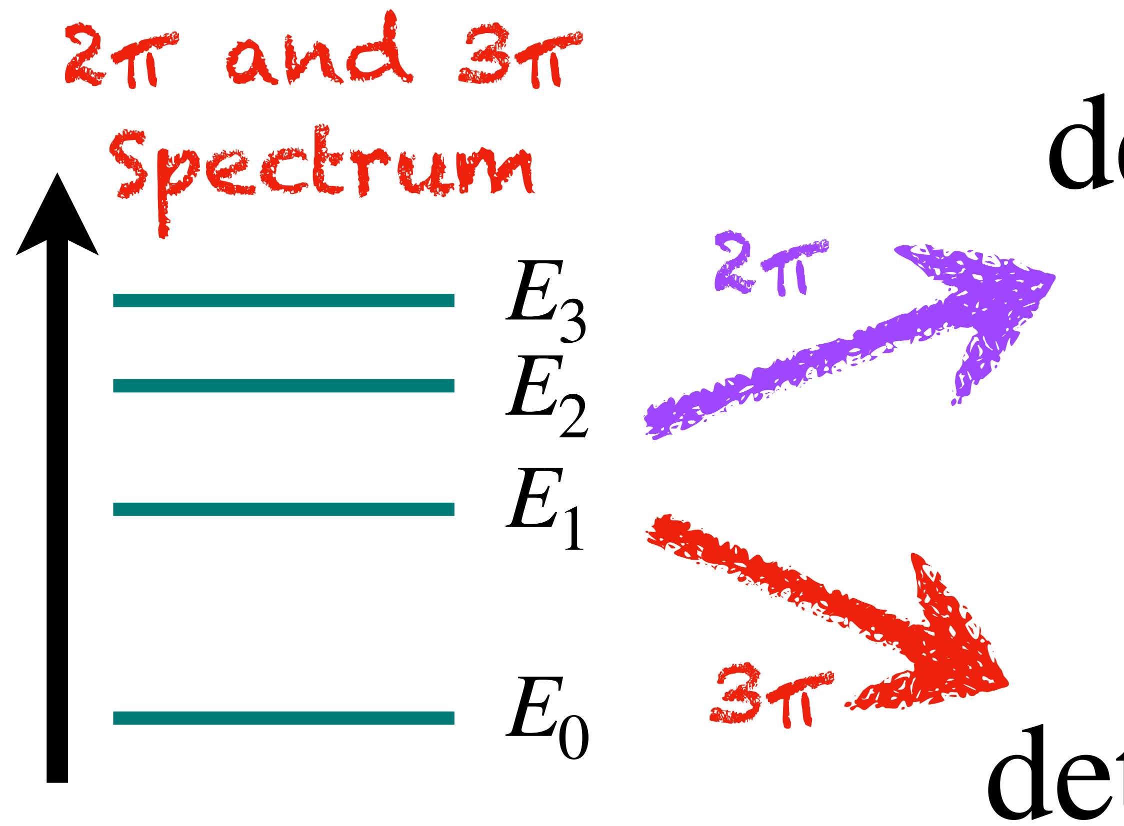

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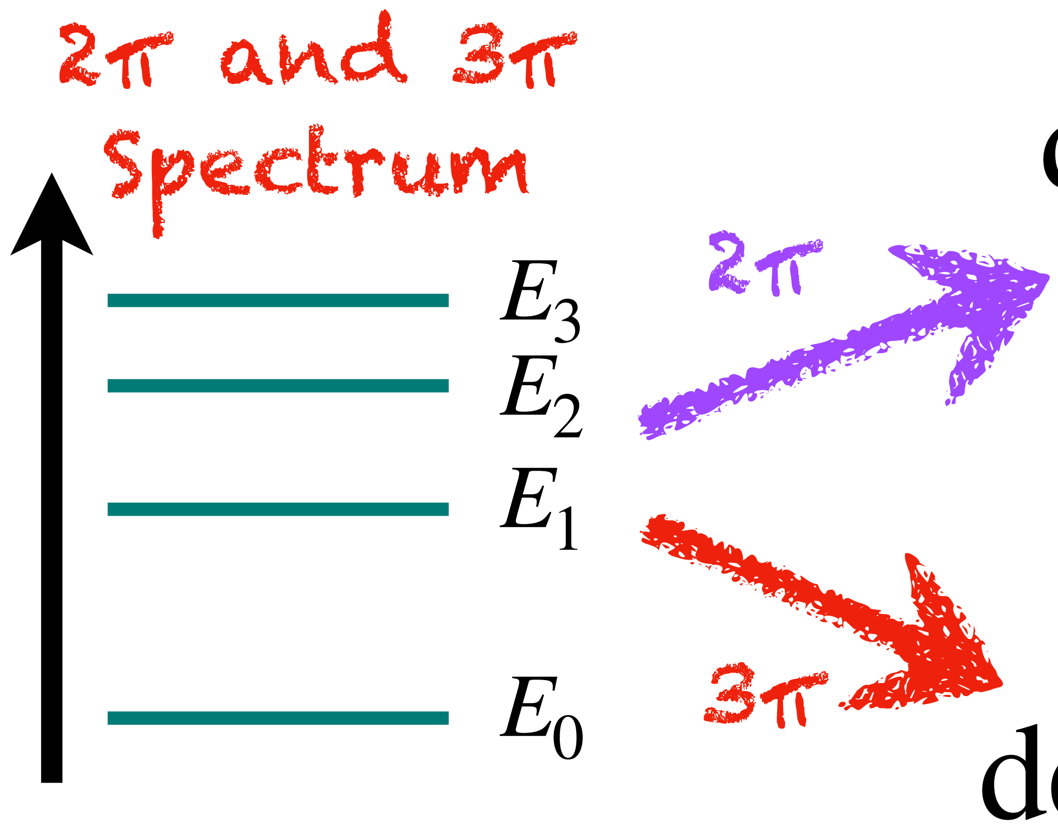

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2. Solve integral equations to obtain the physical three-to-three amplitude

Solved in [Briceño et al], [Hansen et al.], [Jackura et al.]

Physical 3->3 amplitude

$$\mathcal{K}_2, \mathcal{K}_{df,3} \xrightarrow{\text{Integral equations}} \mathcal{M}_3$$

Numerical implementations

Choose "toy" interactions:
$a_0\,, \mathscr{K}_{df,3}\ldots$

$\det\left[M(E,L)\right] = 0$
→

Generate spectrum from quantization condition

Numerical implementations

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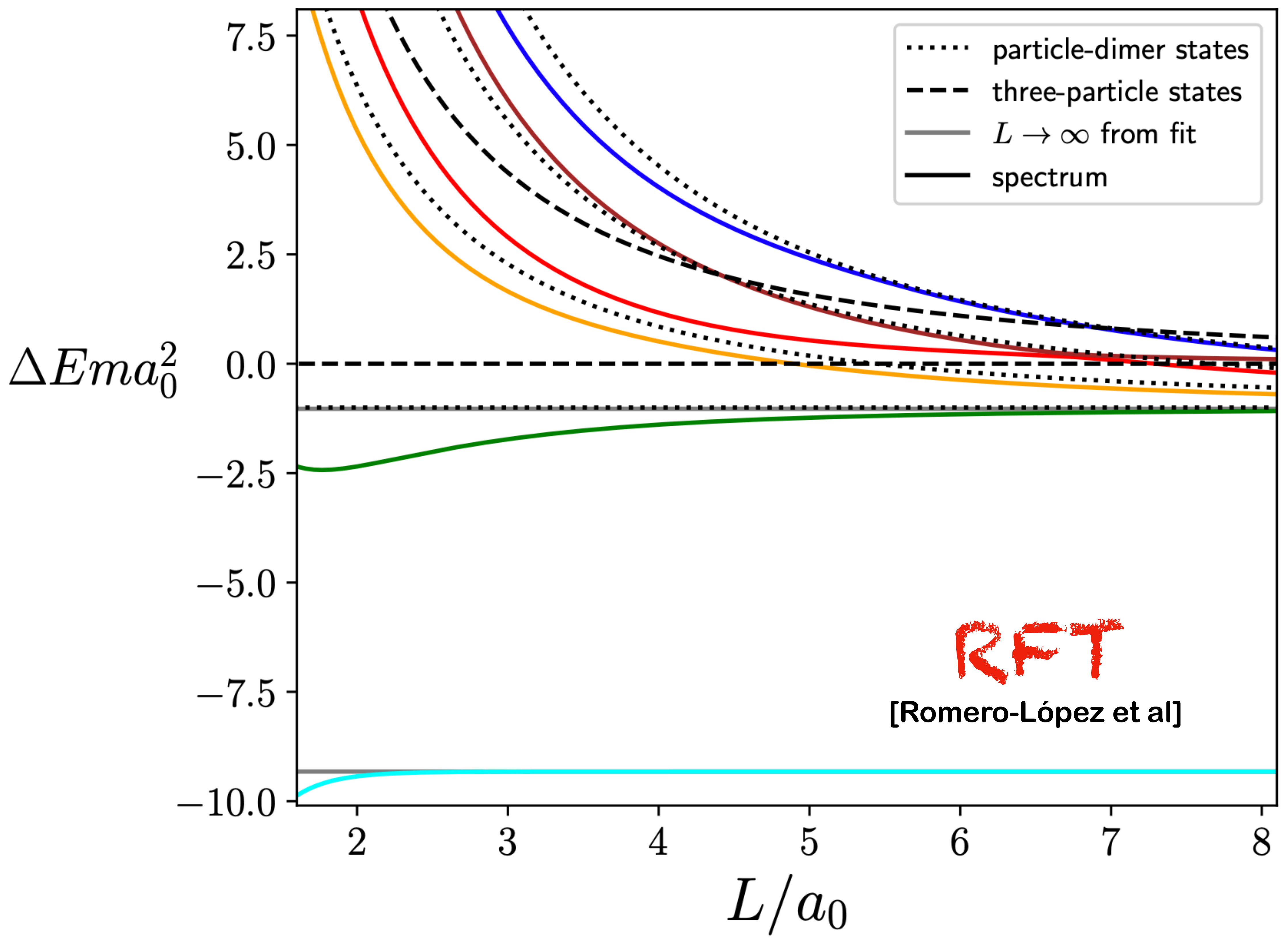

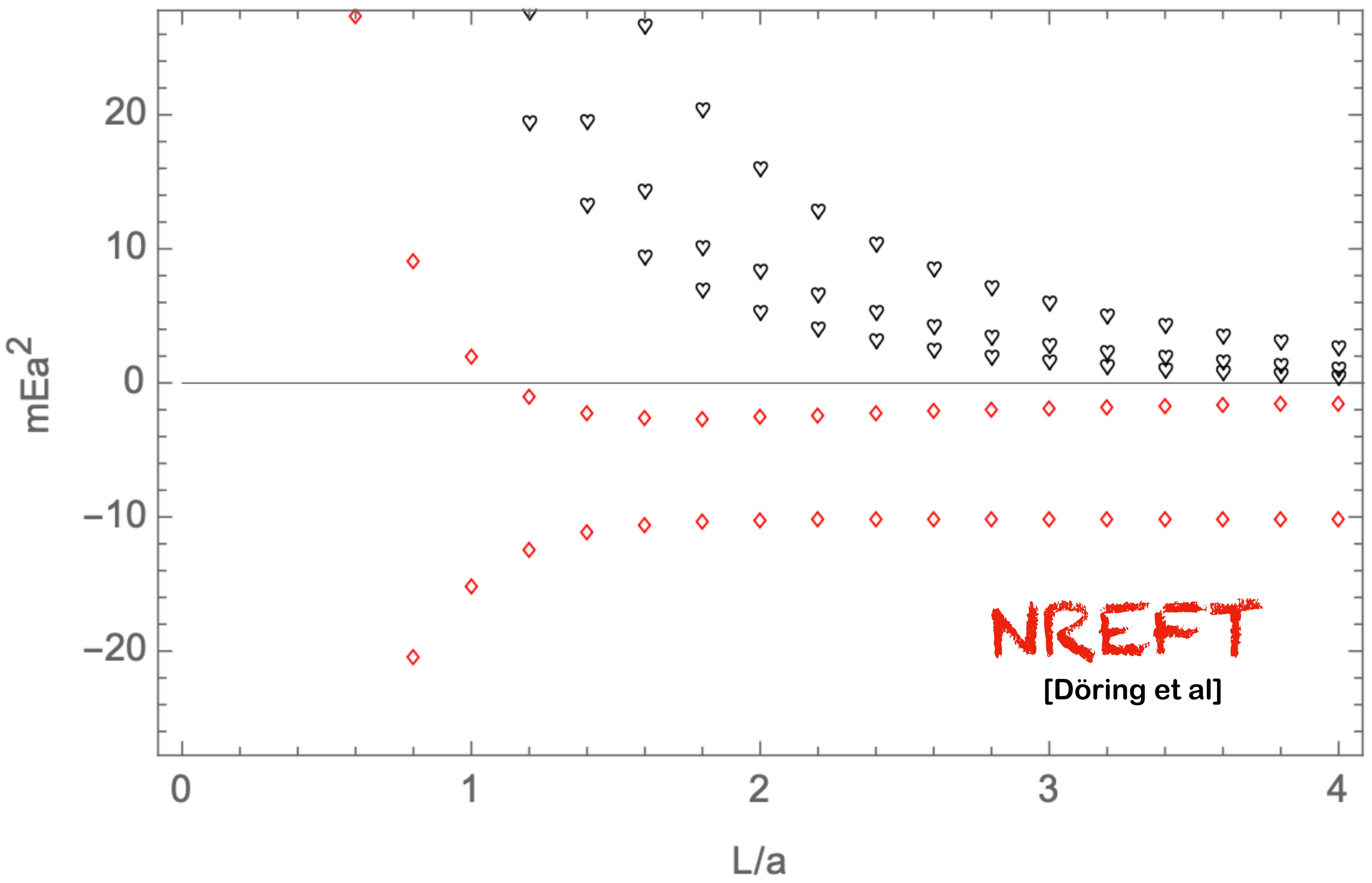

Three-particle decays ($K \to 3\pi$)

- **Decay processes get distorted in finite volume** [Lellouch,Lüscher]

Three-particle decays (K->3π)

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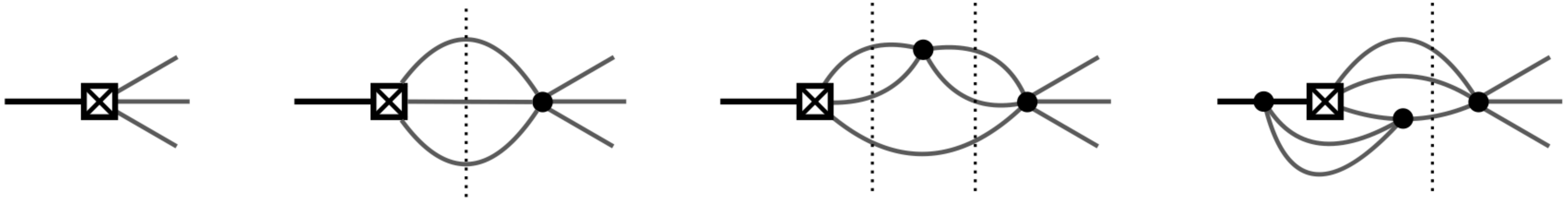

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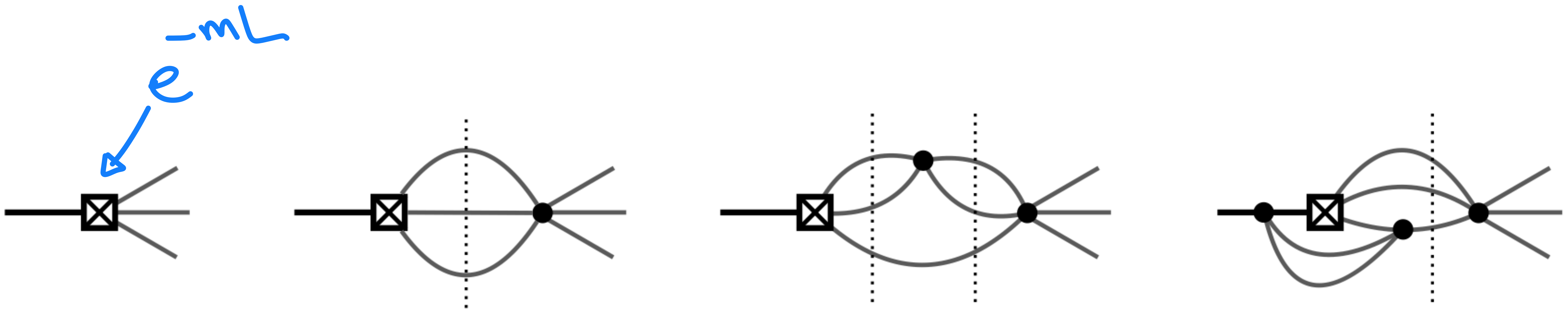

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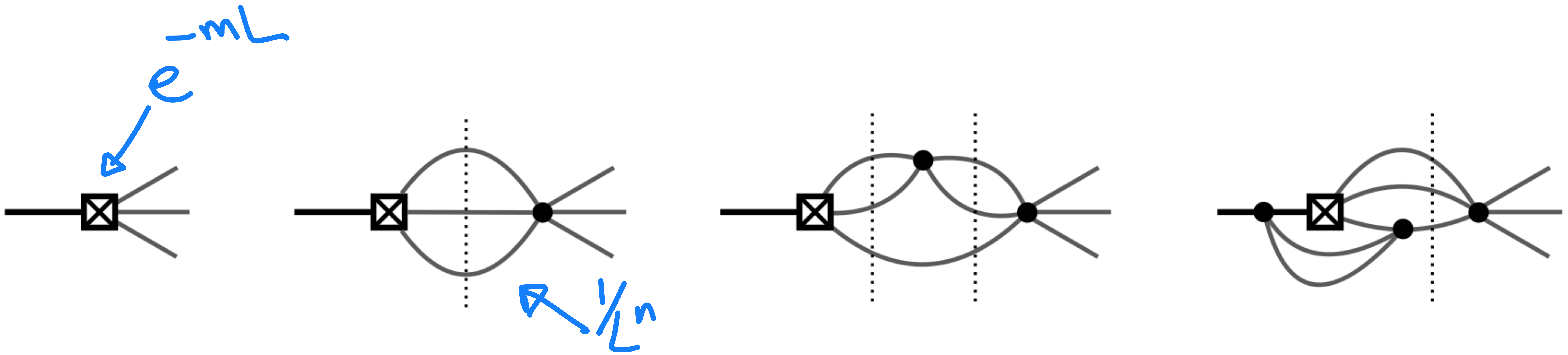

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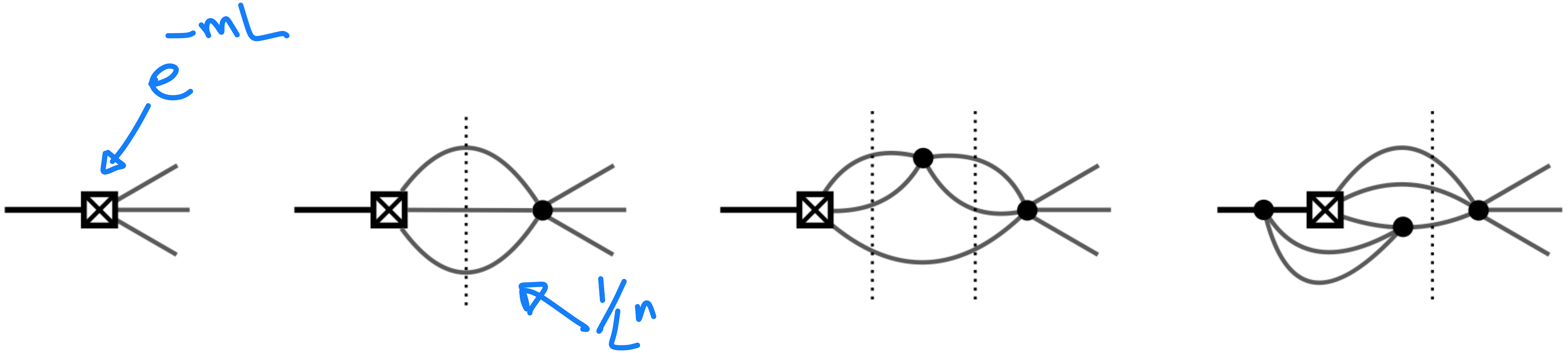

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$$\langle E_n, \boldsymbol{P}, \Lambda\mu, L|\mathcal{H}_W(0)|K, \boldsymbol{P}, L\rangle\,.$$

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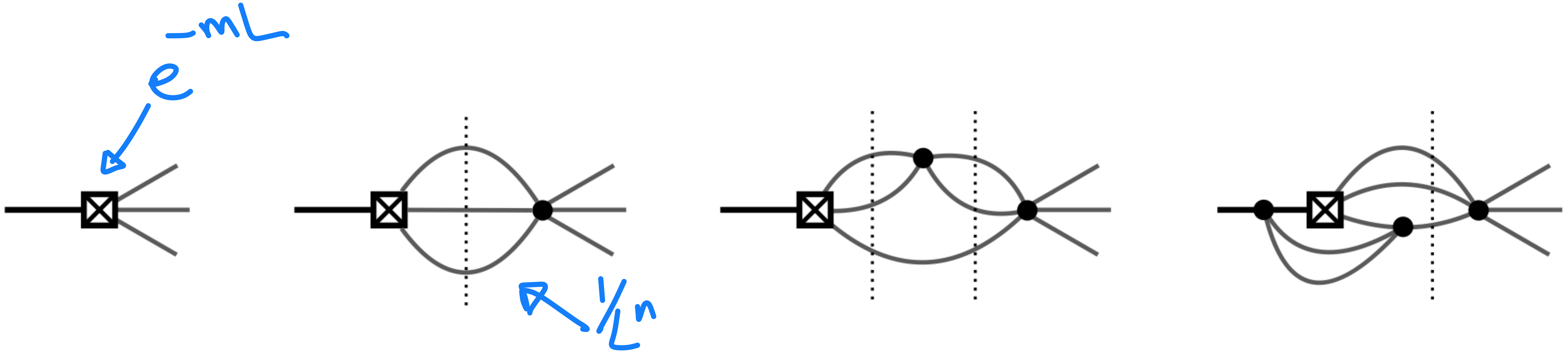

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$$T_{K3\pi} = \langle 3\pi, \text{out} | \, \mathcal{H}_W(0) \, | K, \boldsymbol{P} \rangle \;,$$

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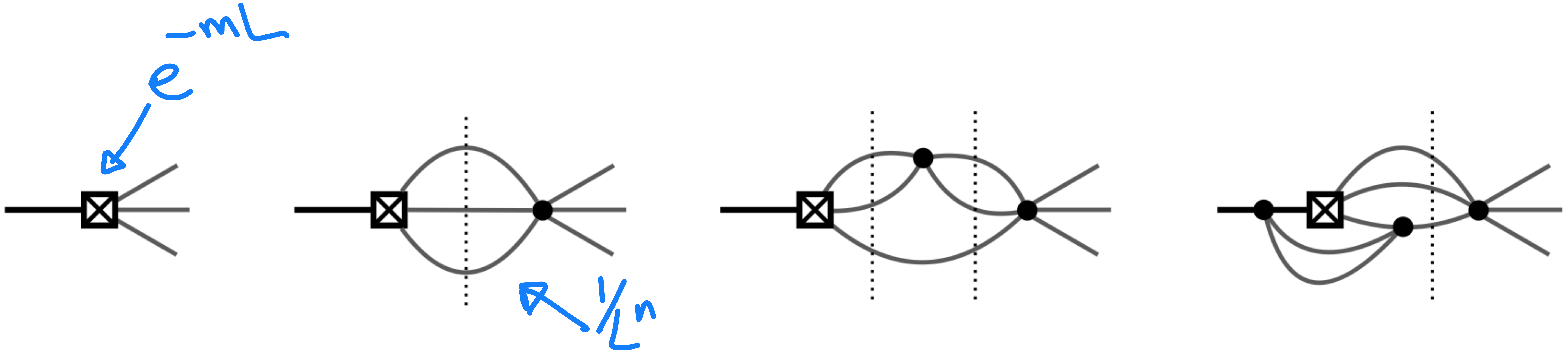

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RFT in [Hansen, FRL, Sharpe]

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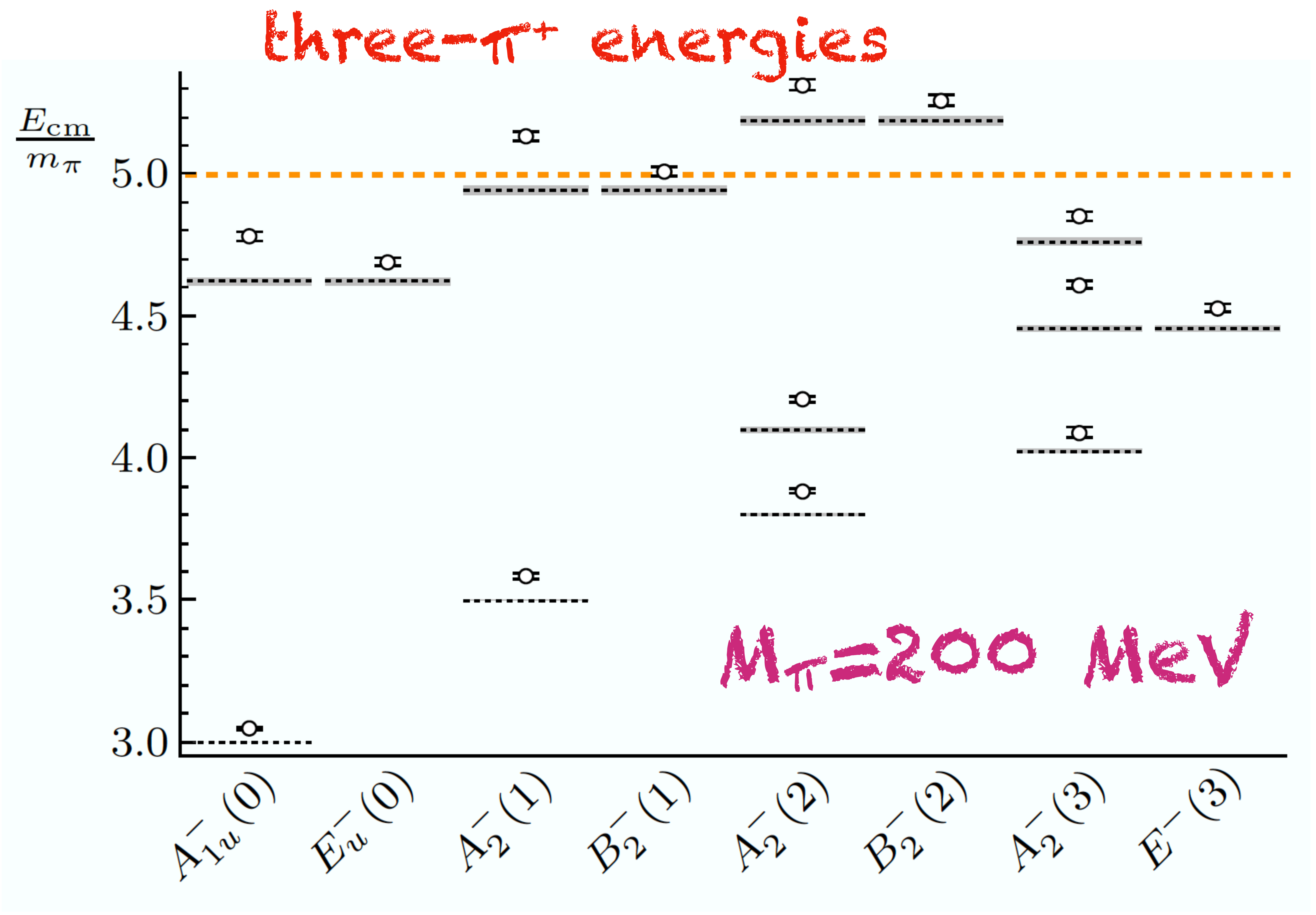
See talk by S. Sharpe, Wednesday 28th

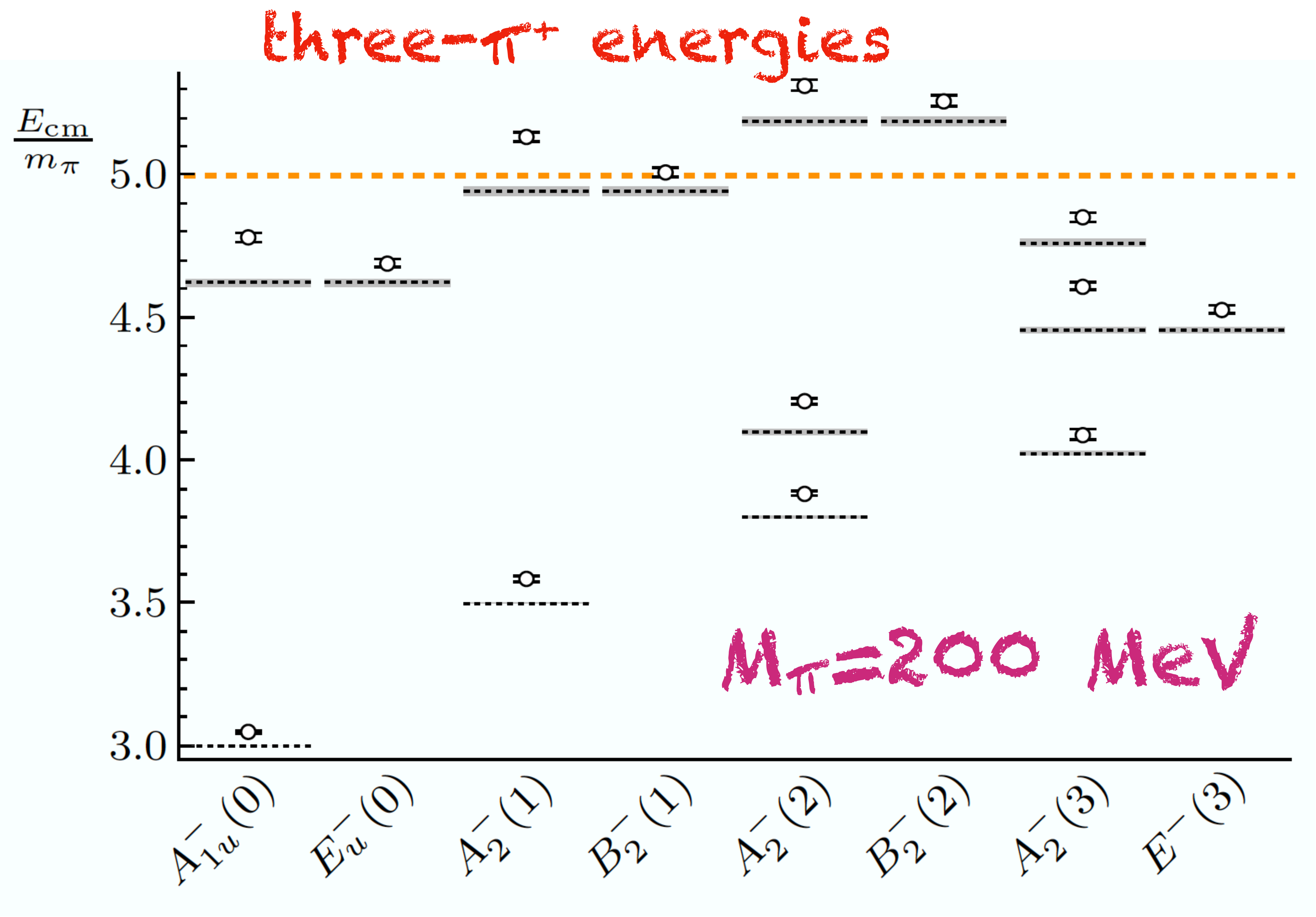
Results for three-meson amplitudes

Analyzing the spectrum



[Hörz,Hanlon (2019, PRL)]

Analyzing the spectrum



[Hörz,Hanlon (2019, PRL)]

$I = 3$ **three-pion scattering amplitude from lattice QCD**

Tyler D. Blanton,[1,*] Fernando Romero-López,[2,†] and Stephen R. Sharpe[1,‡]

[1] *Physics Department, University of Washington, Seattle, WA 98195-1560, USA*

[2] *Instituto de Física Corpuscular, Universitat de València and CSIC, 46980 Paterna, Spain*

(Dated: February 4, 2020)

First analysis of the full finite-volume spectrum of $2\pi^+$ and $3\pi^+$!

[see also earlier work using the ground state by Mai et al., and Beane et al.]

Applying the RFT approach

- **Parametrize K-matrices with only s-wave interactions:**

$$\frac{q}{M}\cot\delta_0 = \frac{\sqrt{s}M}{s - z_2^2}\left(B_0 + B_1 q^2 + \cdots\right)$$

$$\mathcal{K}_{df,3} = \mathcal{K}_{df,3}^{iso,0} + \mathcal{K}_{df,3}^{iso,1}\left(\frac{s - 9M^2}{9M^2}\right)$$

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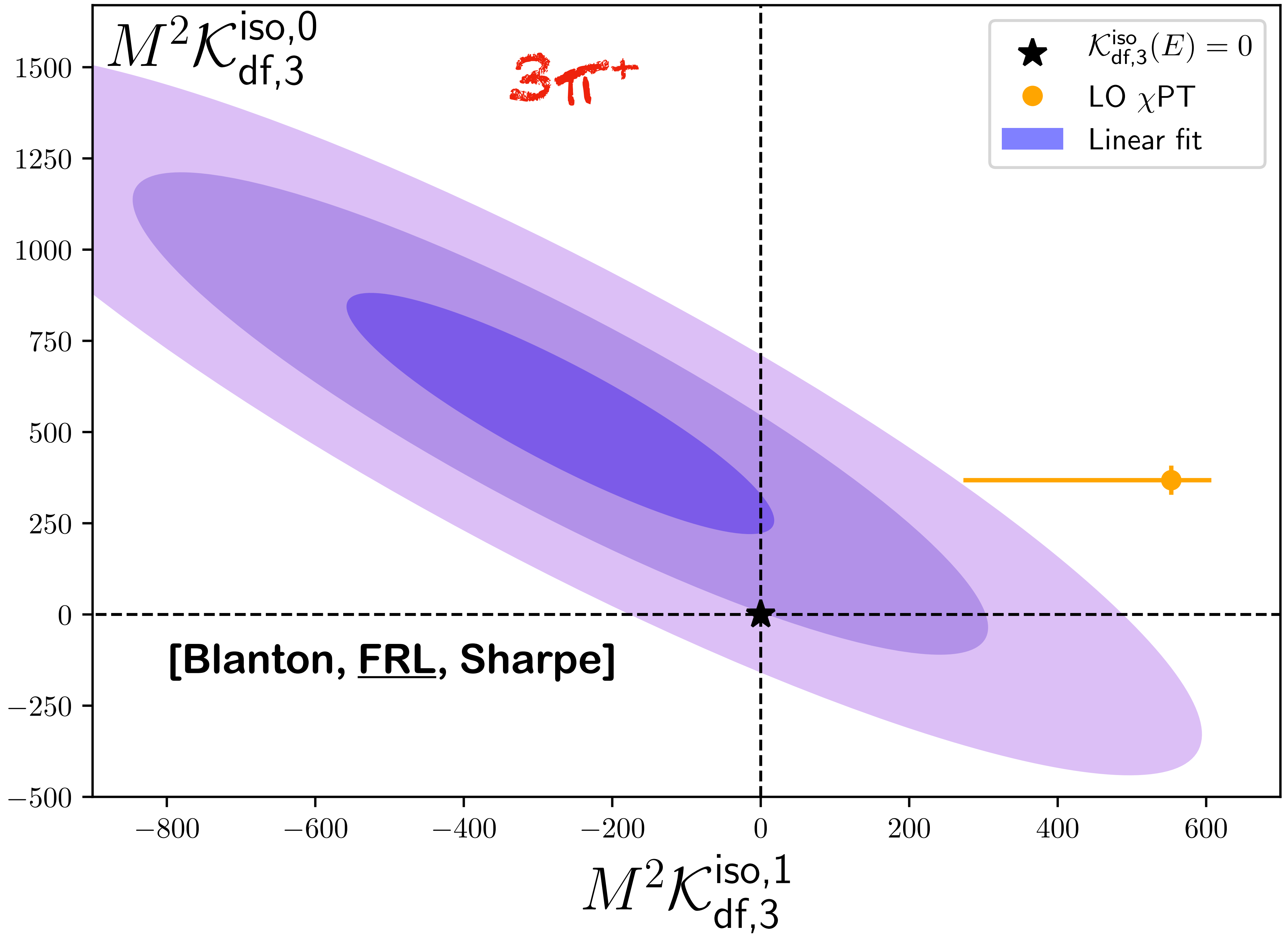
Fit	B_0	B_1	z_2^2/M^2	$M^2\mathcal{K}^{\rm iso,0}_{\rm df,3}$	$M^2\mathcal{K}^{\rm iso,1}_{\rm df,3}$	χ^2/dof	Ma_0	$M^2 r a_0$
5	-11.1(7)	-2.4(3)	1 (fixed)	550(330)	-280(290)	26.04/(22-4)	0.090(5)	2.57(8)



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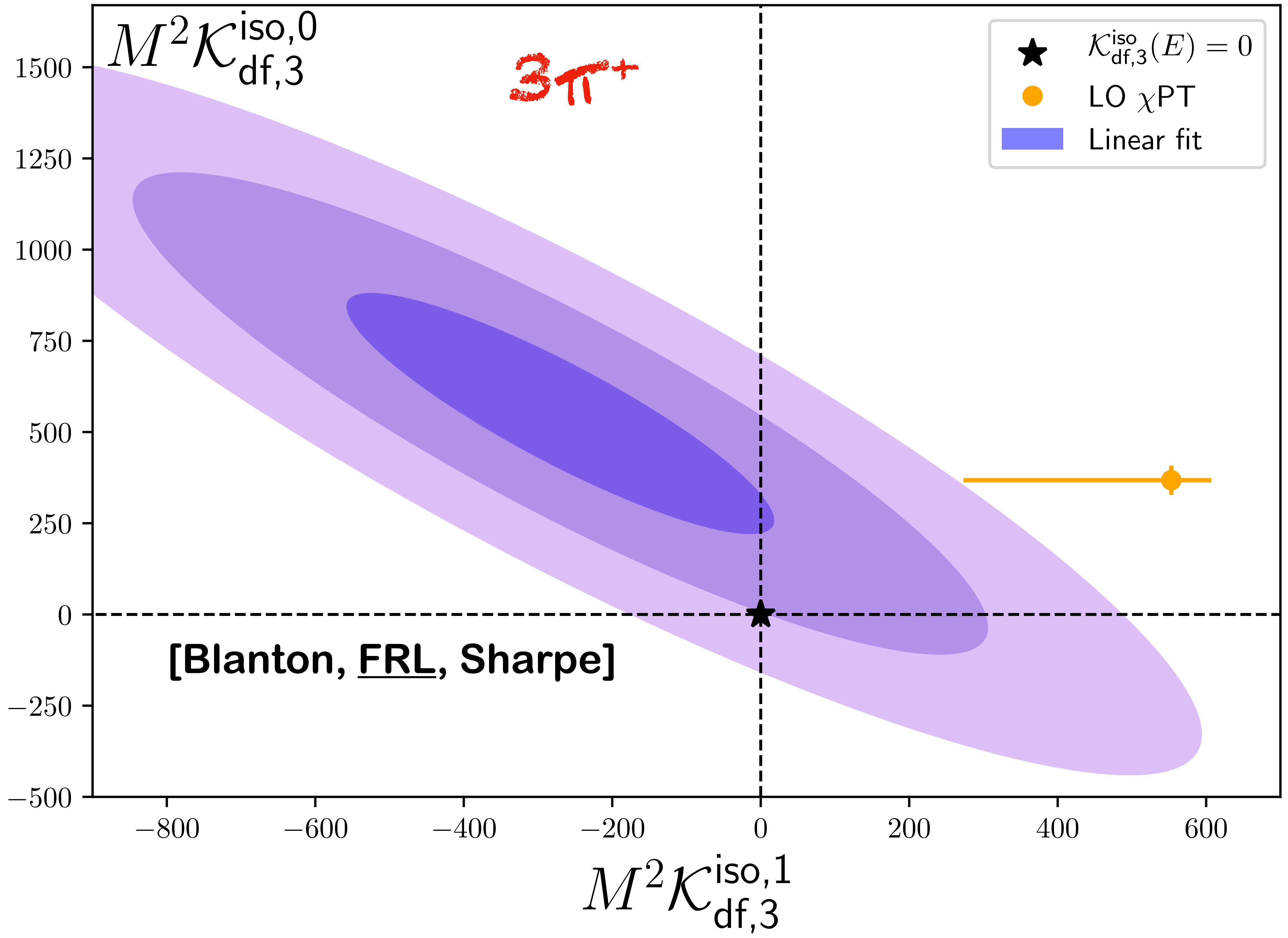

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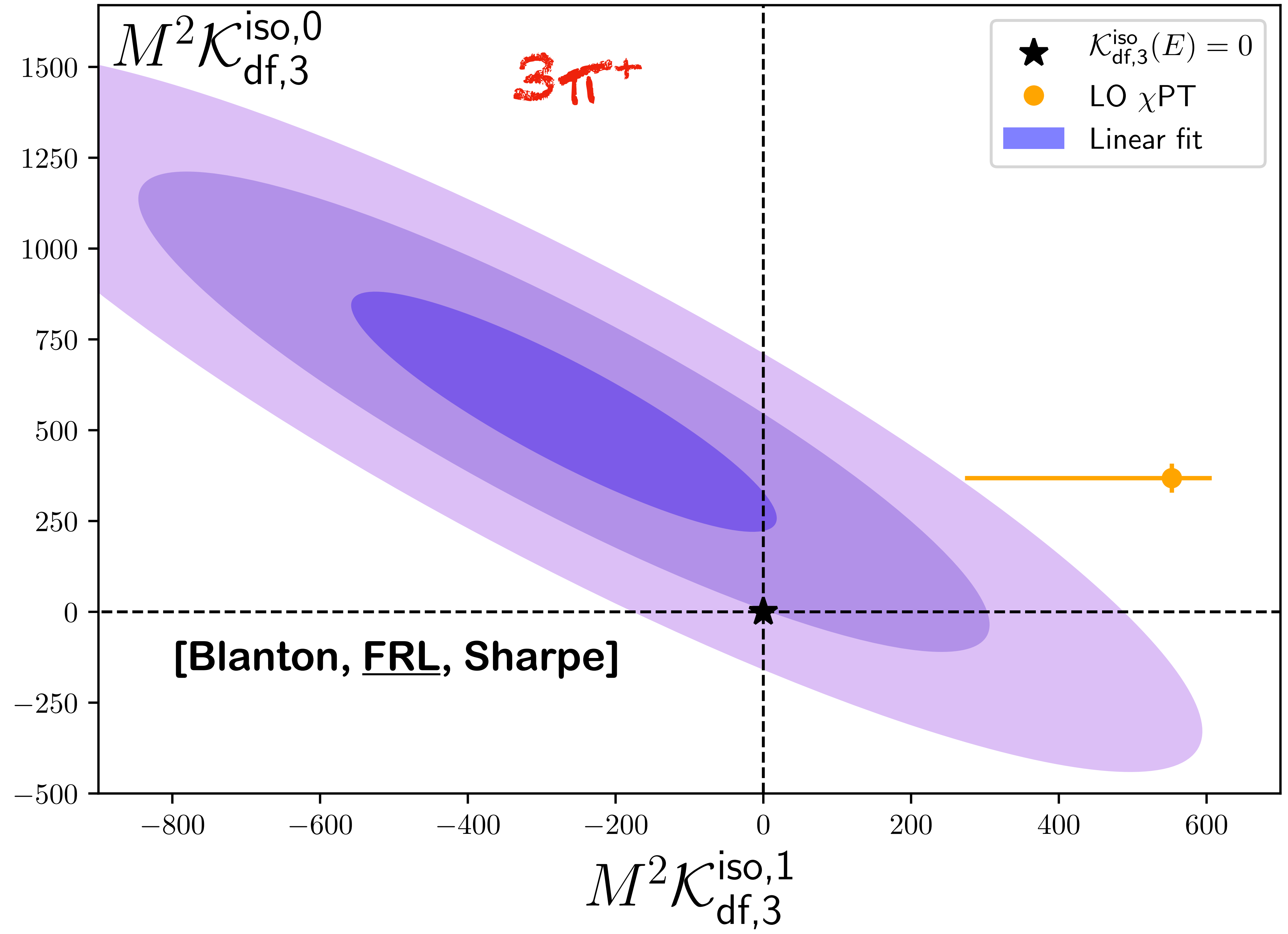

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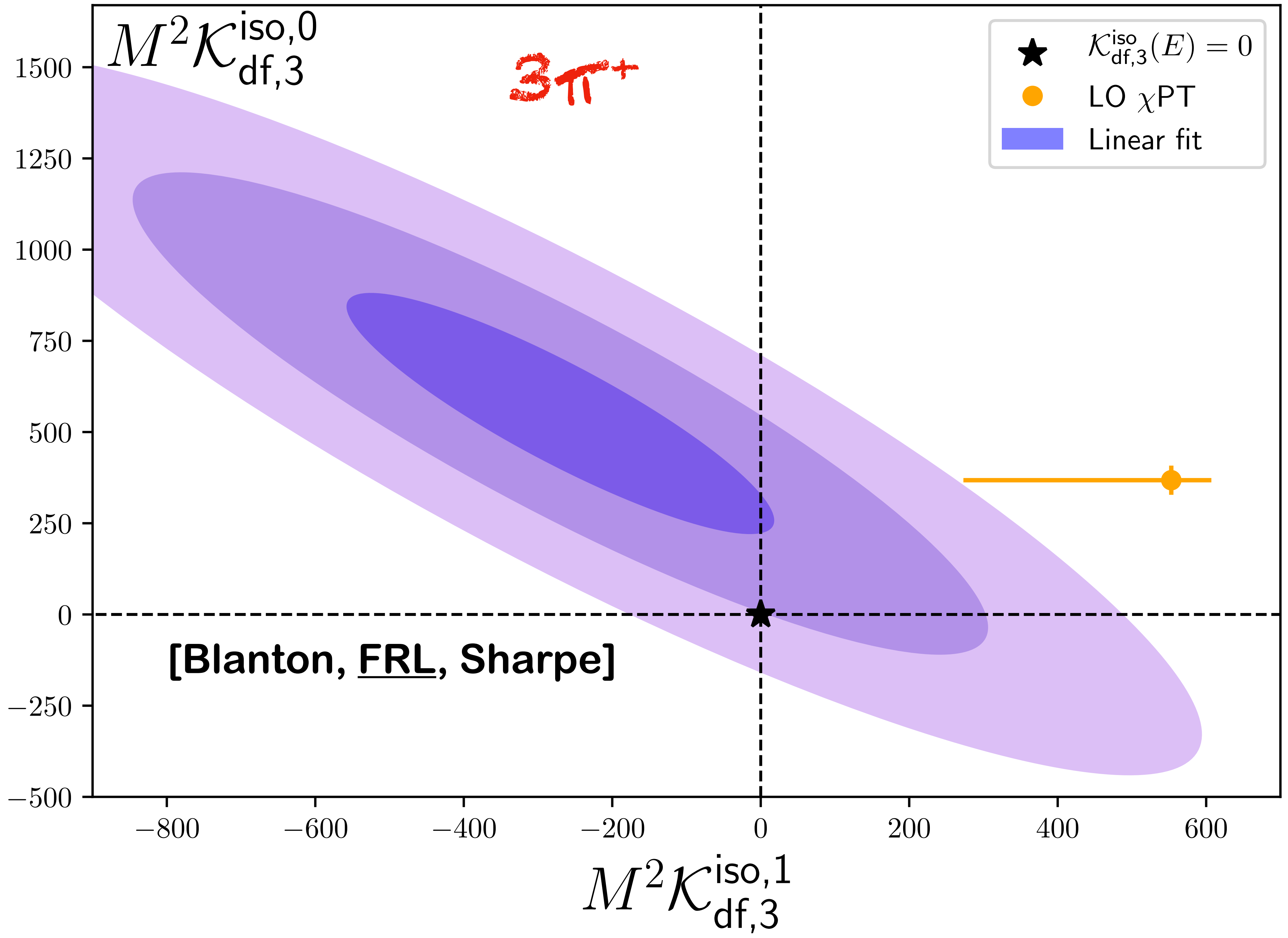

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Same spectrum has been analyzed by [Mai, Döring, Culver, Alexandru]

Chiral dependence ($3\pi^+$)

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  [Fischer, Kostrzewa, Liu, FRL, Ueding, Urbach (ETMC)]

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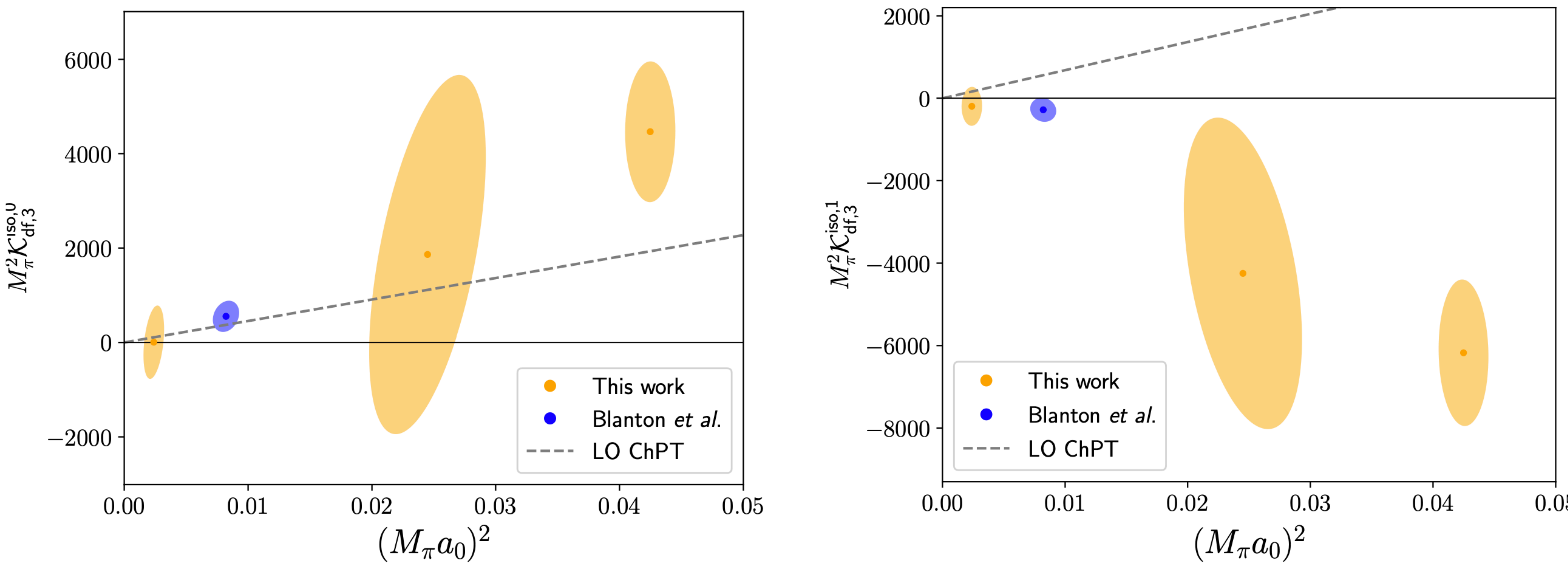

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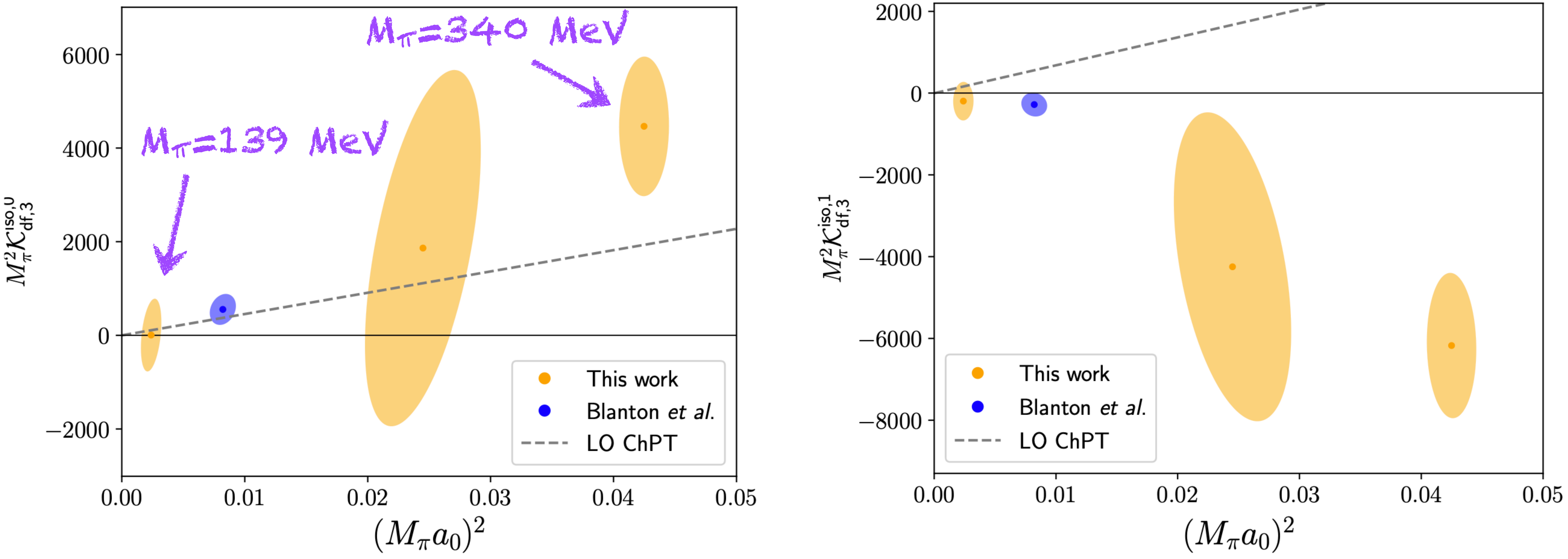

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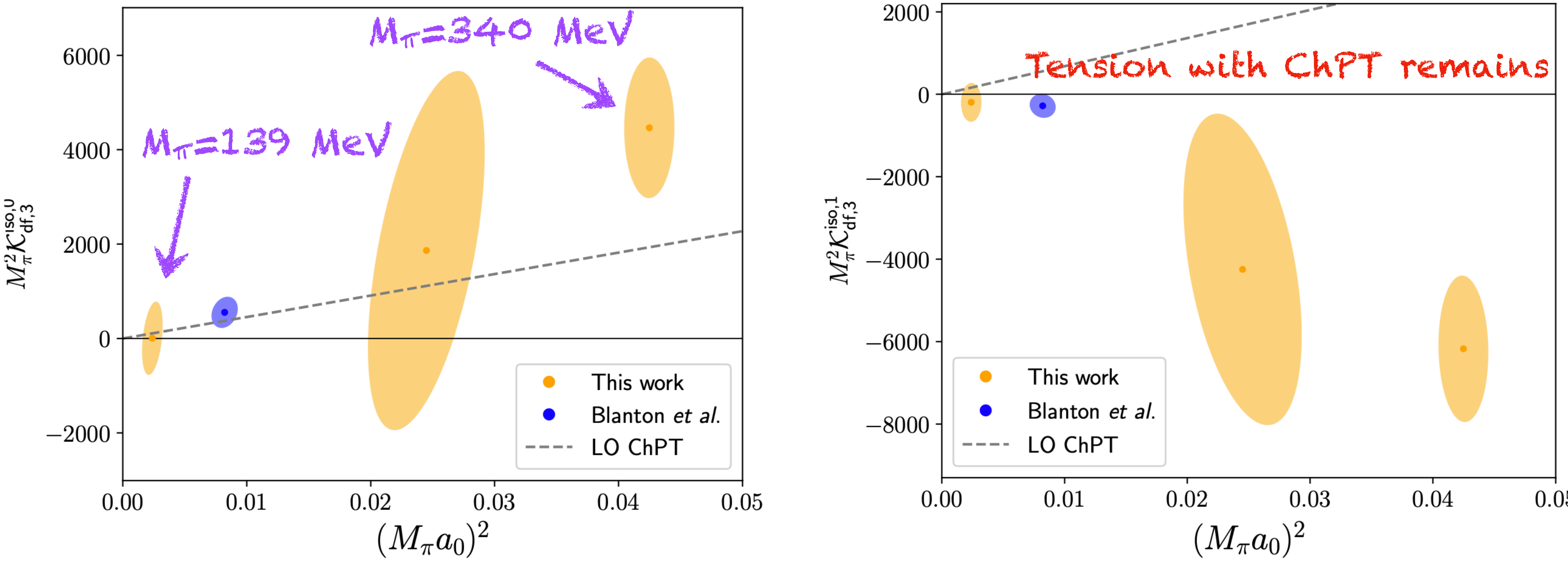

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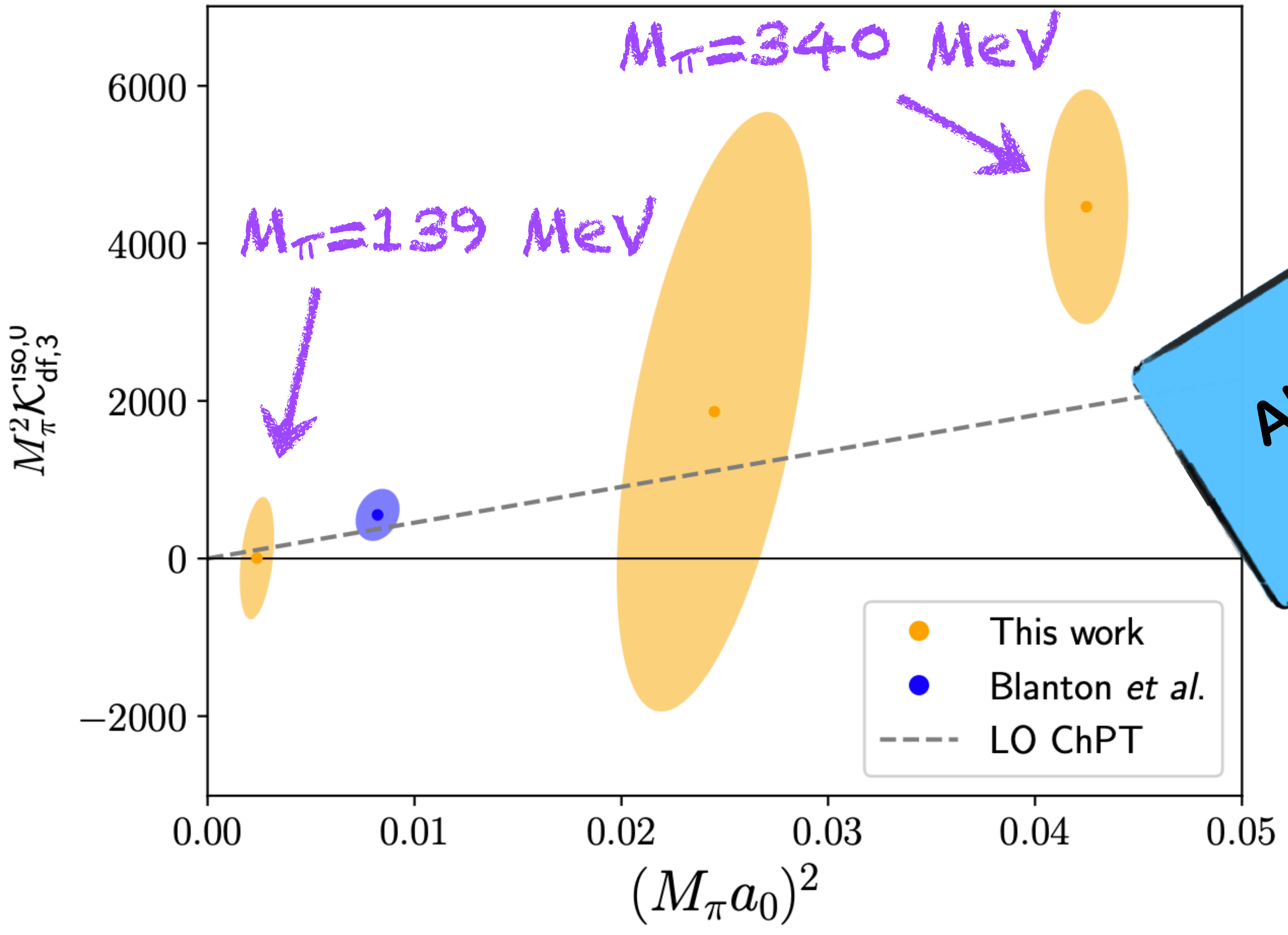
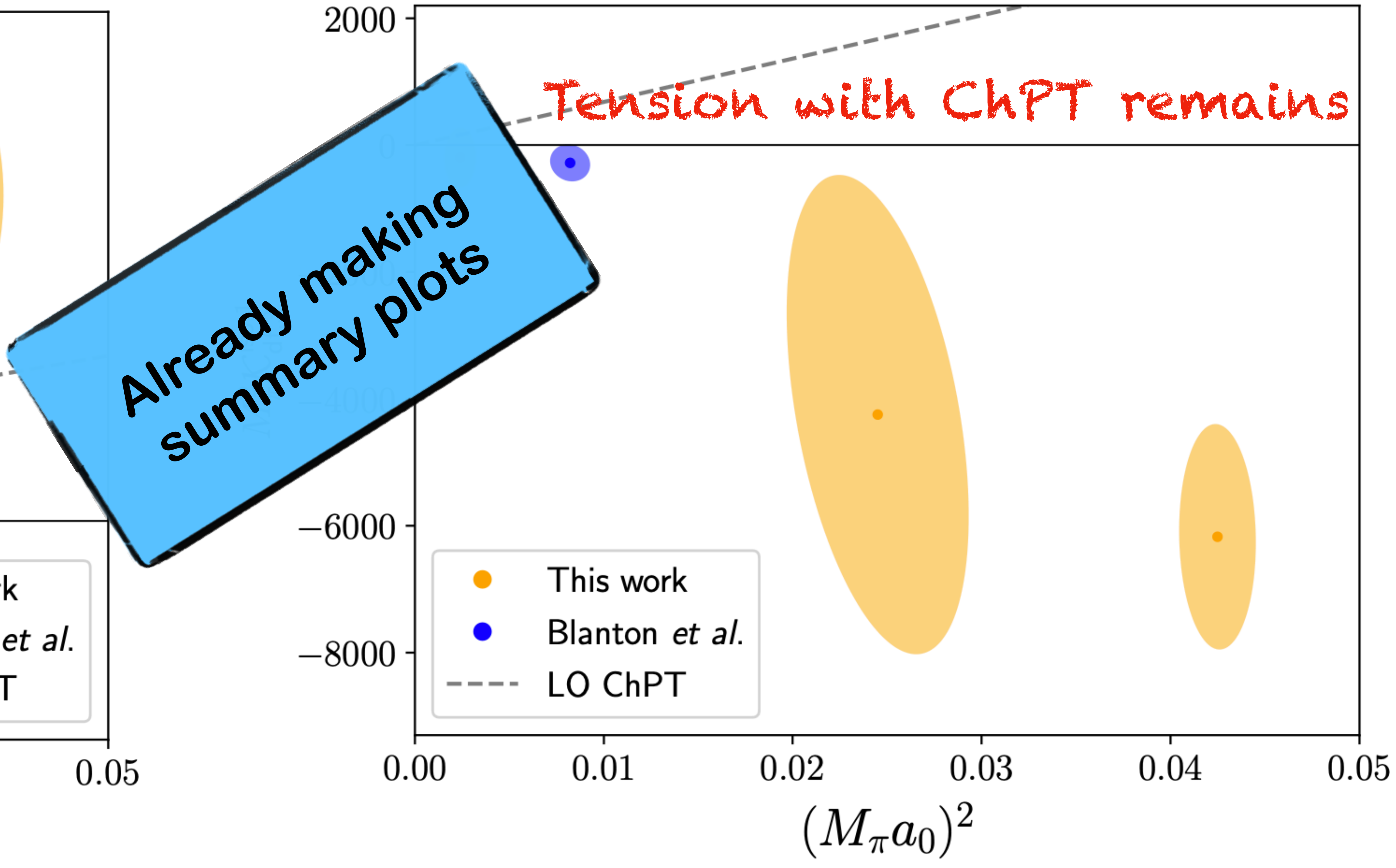


(a) $\mathcal{K}_{df,3}^{iso,0}$

Tension with ChPT remains

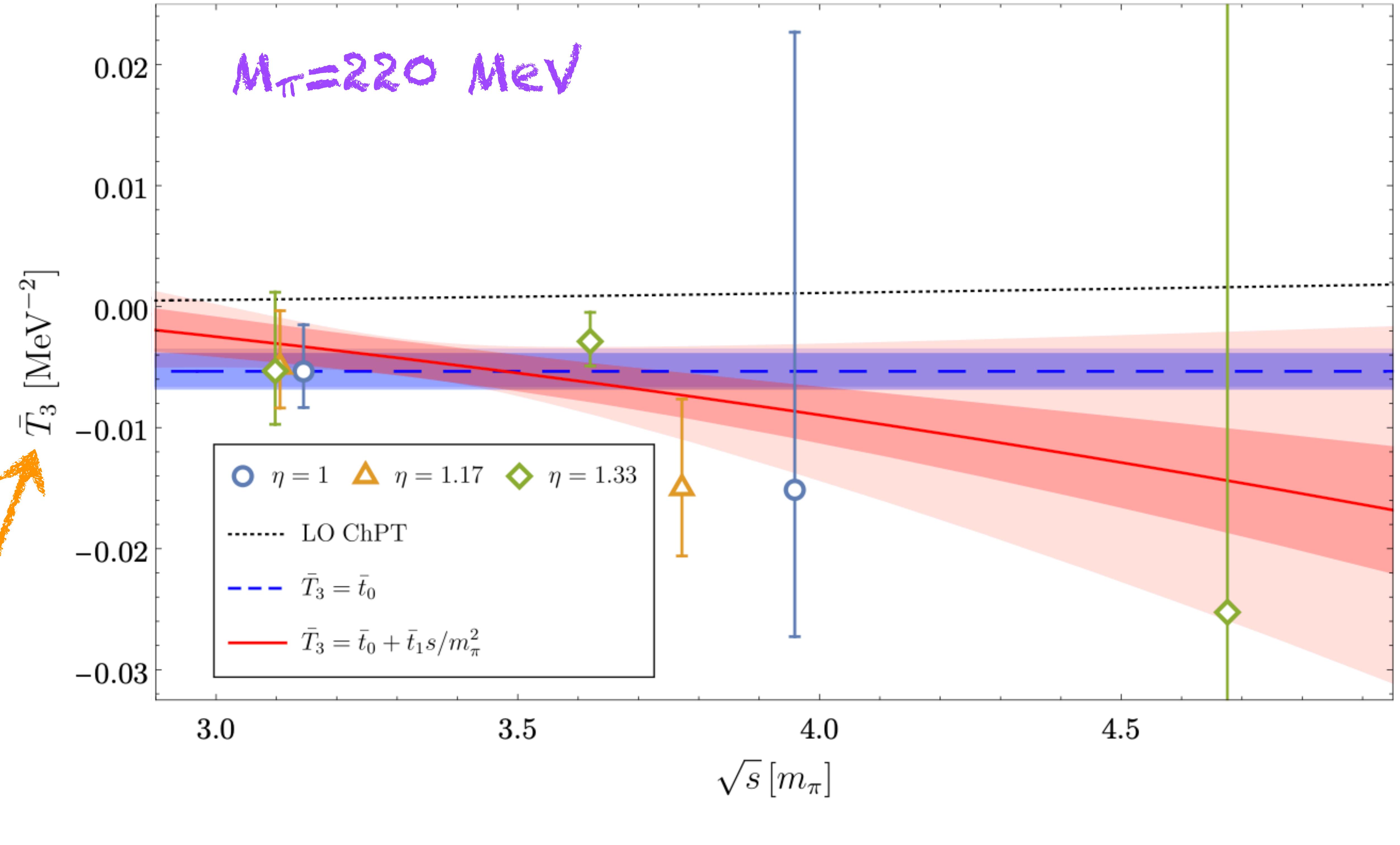


(b) $\mathcal{K}_{df,3}^{iso,1}$

Applying the FVU approach ($3\pi^+$)

- **The FVU formalism has also been applied to three-pion systems**
  [Brett et al.]

See talk by A. Alexandru, Wednesday 28th



three-particle “contact” term

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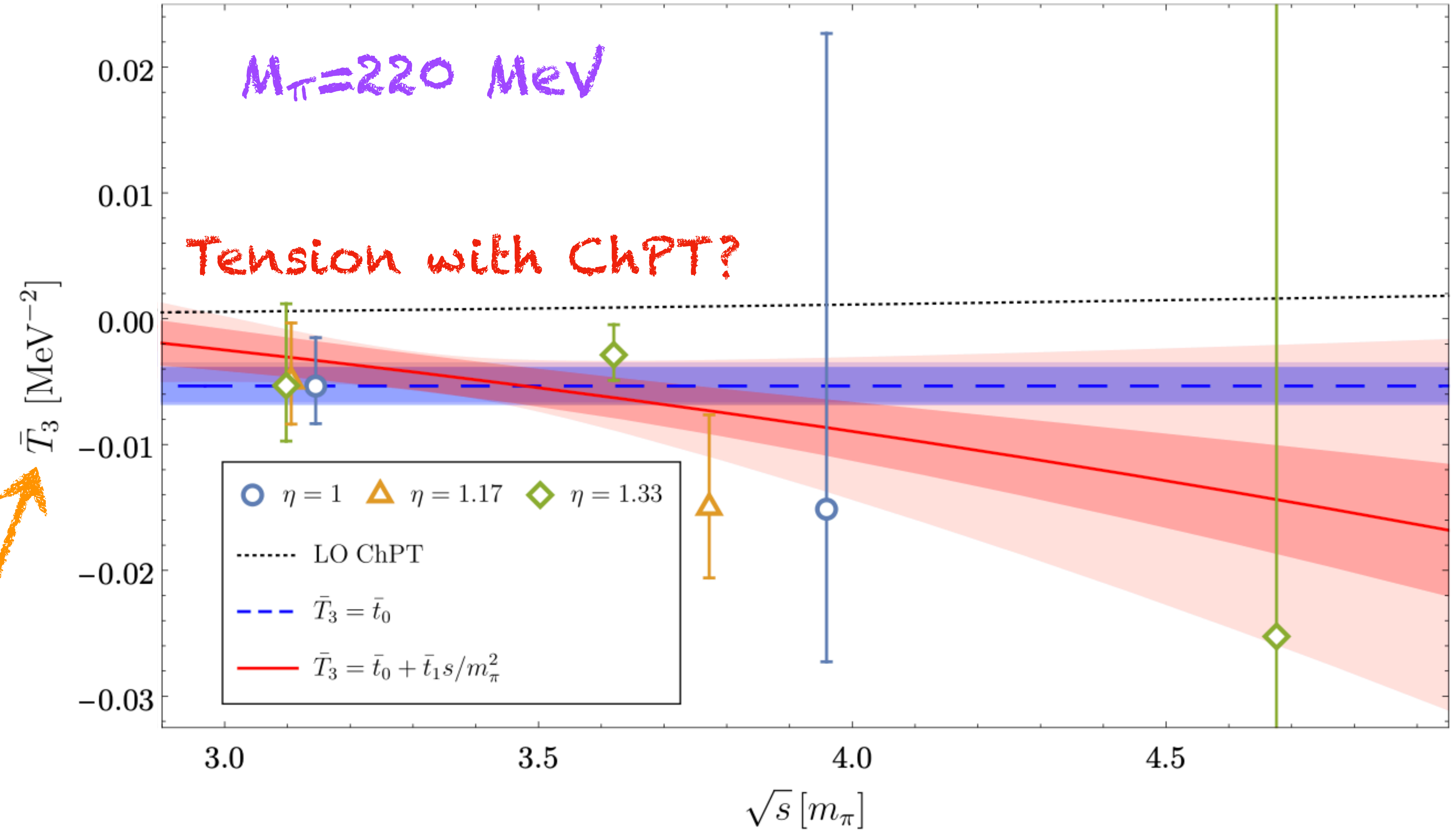

three-particle "contact" term

Higher partial waves

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Threshold expansion in the three-particle sector:

[Blanton, FRL, Sharpe]

$$\mathcal{K}_{\mathrm{df},3} = \mathcal{K}_{\mathrm{df},3}^{\mathrm{iso},0} + \mathcal{K}_{\mathrm{df},3}^{\mathrm{iso},1}\Delta + \mathcal{K}_{\mathrm{df},3}^{\mathrm{iso},2}\Delta^2 + \mathcal{K}_A\Delta_A + \mathcal{K}_B\Delta_B\,,$$

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s-wave

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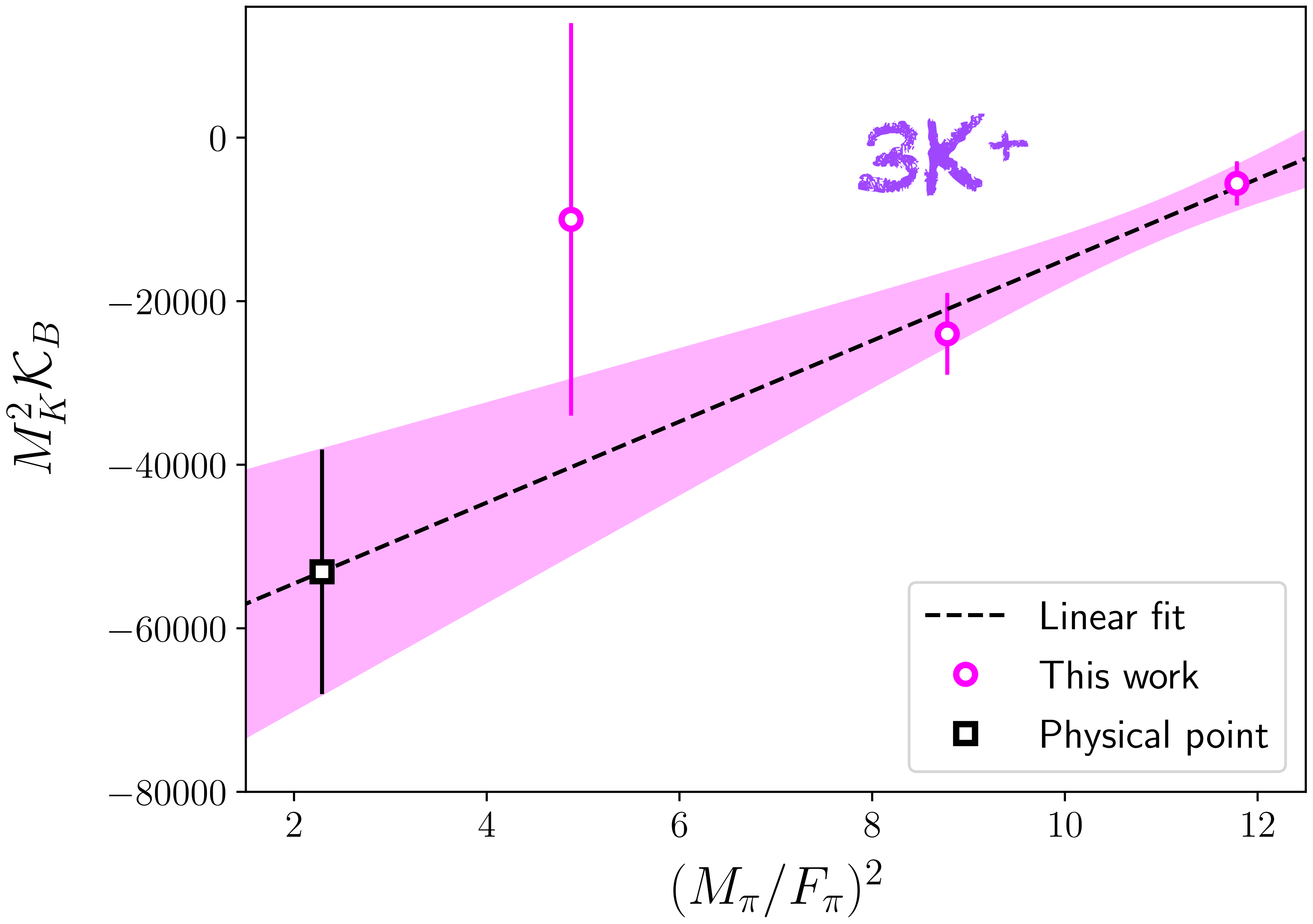
See talk by A. Hanlon, Wednesday 28th



[Blanton, Hanlon, Hörz, Morningstar, FRL, Sharpe]

Integral equations (RFT)

Final step

Physical 3->3 amplitude

$$\mathcal{K}_2, \mathcal{K}_{df,3} \longrightarrow \mathcal{M}_3$$

Integral equations

Integral equations (RFT)

Final step

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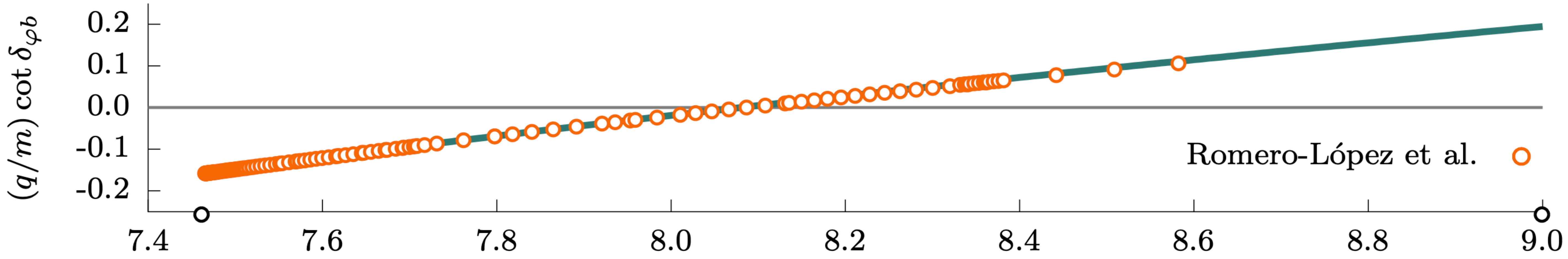
$$\mathcal{K}_2, \mathcal{K}_{df,3} \xrightarrow{\text{Integral equations}} \mathcal{M}_3$$

Particle-Dimer phase shift [Jackura et al.]



See talk by A. Jackura, Tuesday 27th

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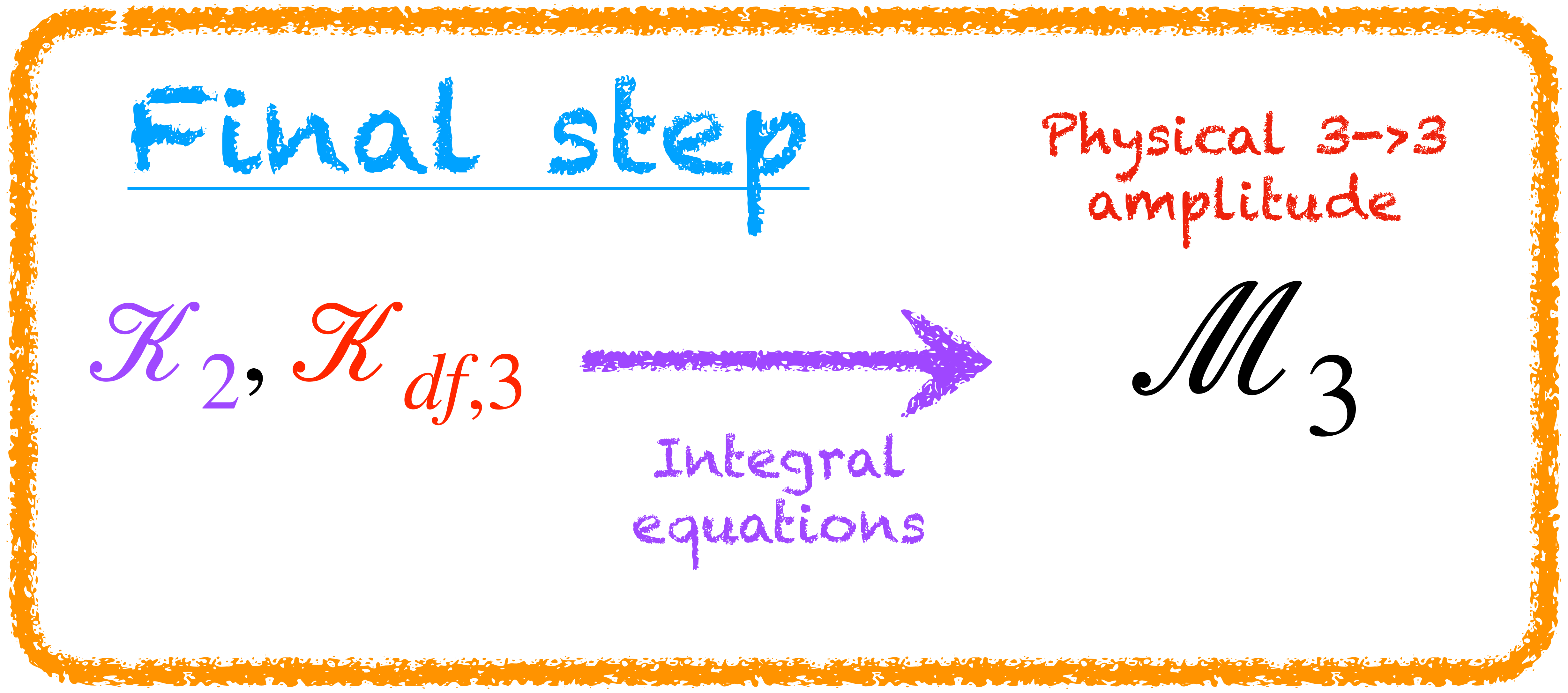
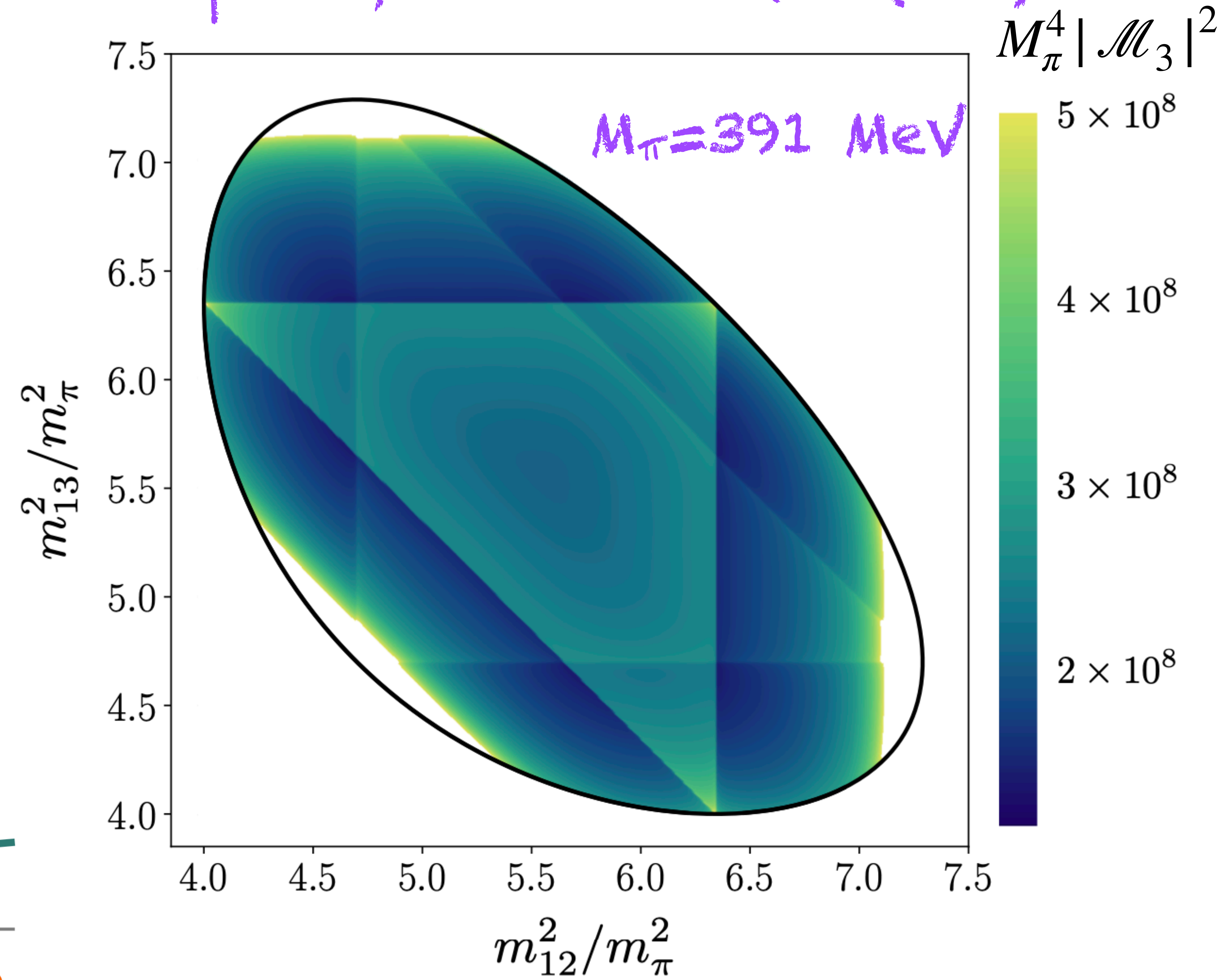


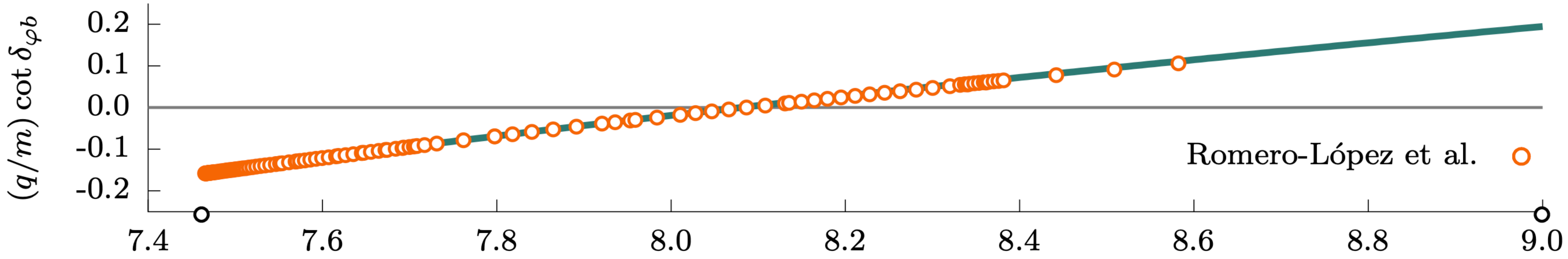
Dalitz plots from lattice QCD ($3\pi^+$)



[Hansen et al. (HadSpec)]

See talk by M. Hansen, Wednesday 28th

Particle-Dimer phase shift [Jackura et al.]



See talk by A. Jackura, Tuesday 27th

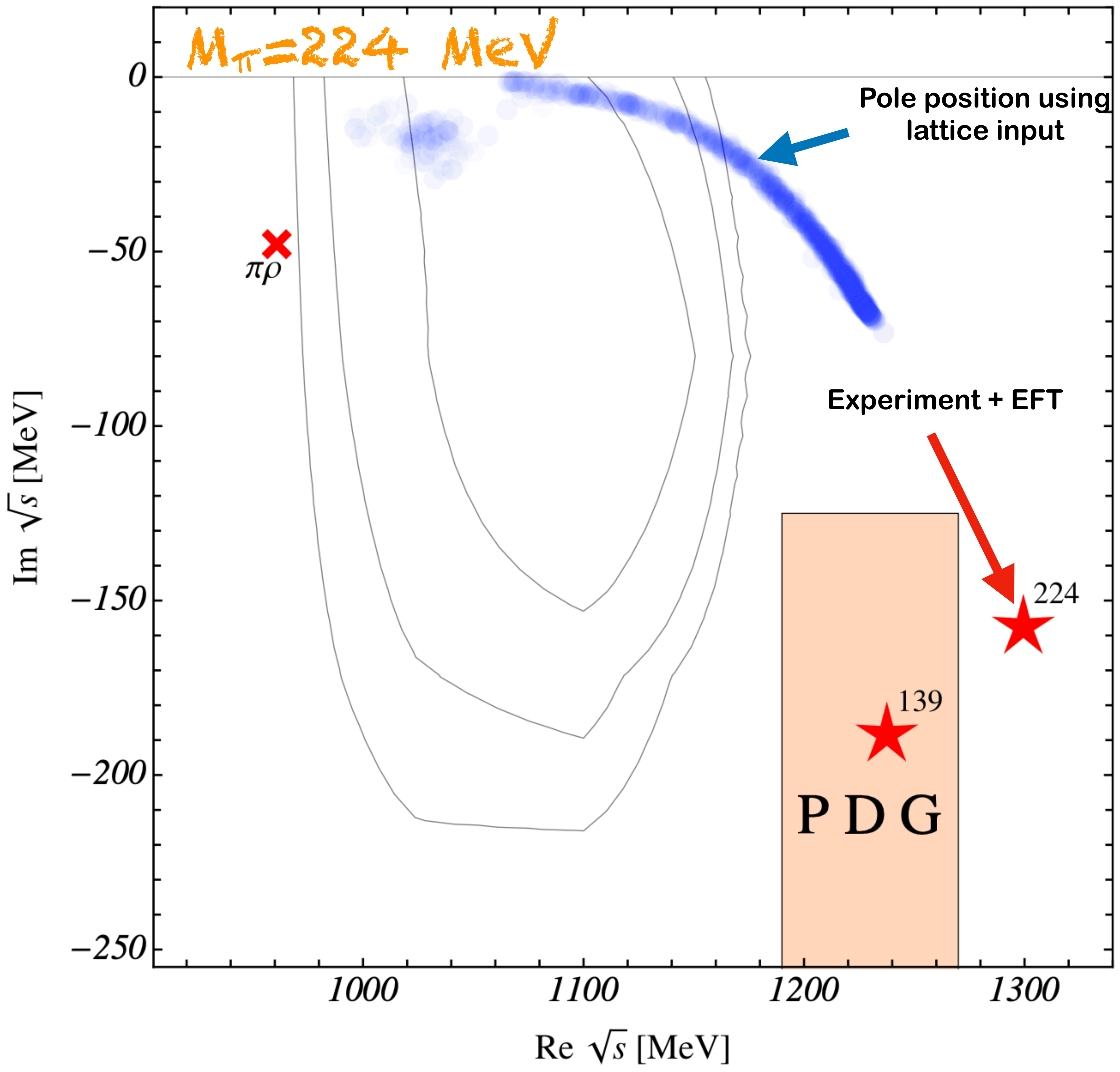
Integral equations (FVU)

Final step

Physical 3->3 amplitude

$$\mathcal{K}_2, C_{\ell\ell'} \xrightarrow{\text{Integral equations}} \mathcal{M}_3$$

Pole position of $a_1(1260)$



[Mai et al. (GWQCD)]

Summary & Outlook

Summary

The field is evolving very rapidly!

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- Many recent formal developments for three-particle systems.
  - Three versions of finite-volume formalism for identical particles.
  - Various generalizations for nonidentical scalar particles.
  - Formalism for three-body decays.

Summary

The field is evolving very rapidly!

- Many recent formal developments for three-particle systems.
  - Three versions of finite-volume formalism for identical particles.
  - Various generalizations for nonidentical scalar particles.
  - Formalism for three-body decays.
- Applications to simple systems successfully undertaken
  - Some lattice studies of three charged pions and kaons
  - It is possible to even study d-wave interactions [Blanton et al.]
  - Recent study in FVU formalism of the $a_1(1260)$ [Mai et al.]

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	Progress in relativistic three-hadron scattering from lattice QCD	*Dr. Andrew JACKURA*
	Mexico City	18:40 - 19:05
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	The six-pion amplitude	*Tomas HUSEK*
	Mexico City	19:30 - 19:55

Wednesday 28th

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Back-up

Higher partial waves

Higher partial waves

Threshold expansion:
[Blanton, FRL, Sharpe]

$$\mathcal{K}_{\text{df},3} = \mathcal{K}_{\text{df},3}^{\text{iso},0} + \mathcal{K}_{\text{df},3}^{\text{iso},1}\Delta + \mathcal{K}_{\text{df},3}^{\text{iso},2}\Delta^2 + \mathcal{K}_A\Delta_A + \mathcal{K}_B\Delta_B \,,$$

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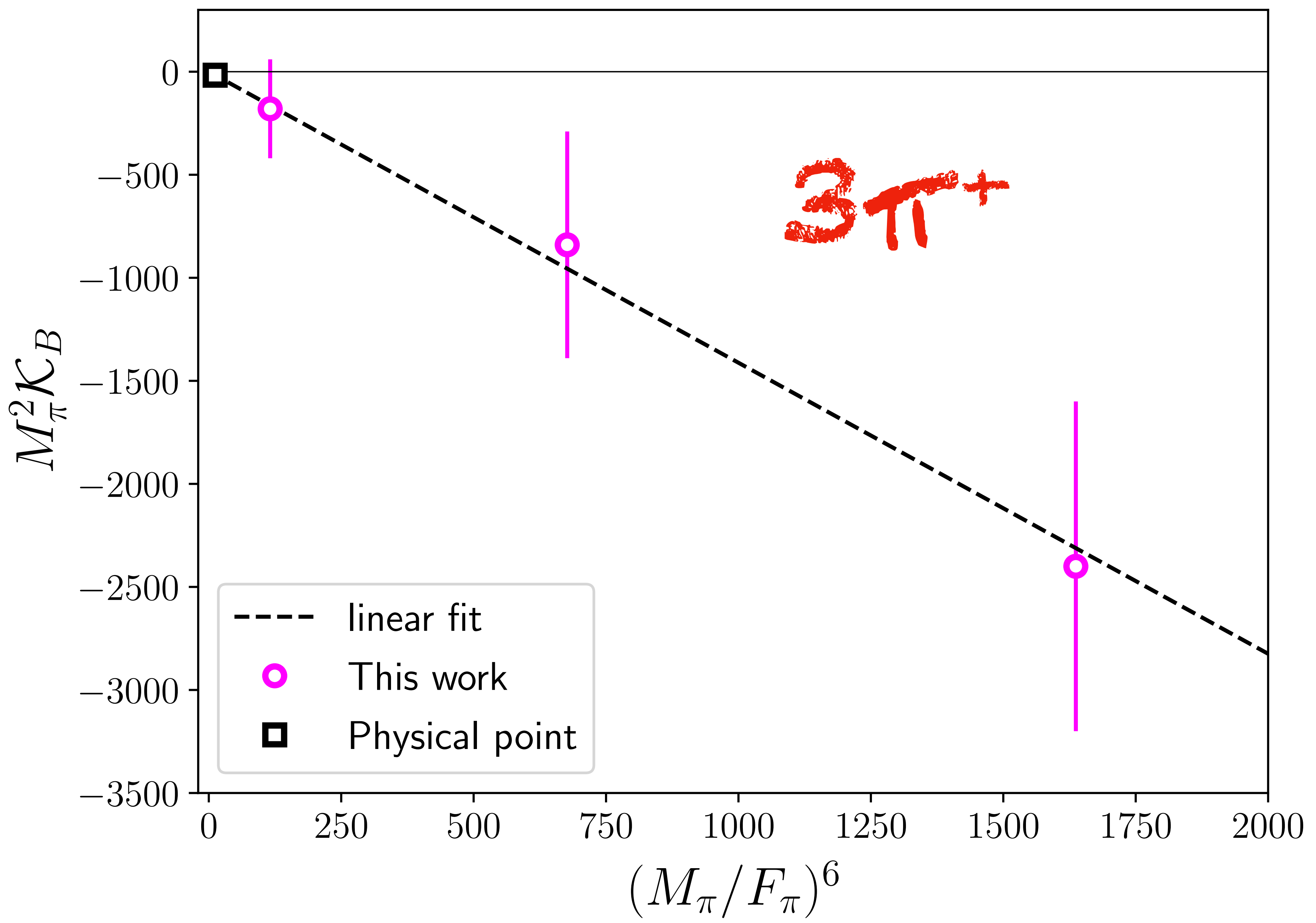

[Blanton, Hanlon, Hörz, Morningstar, FRL, Sharpe]

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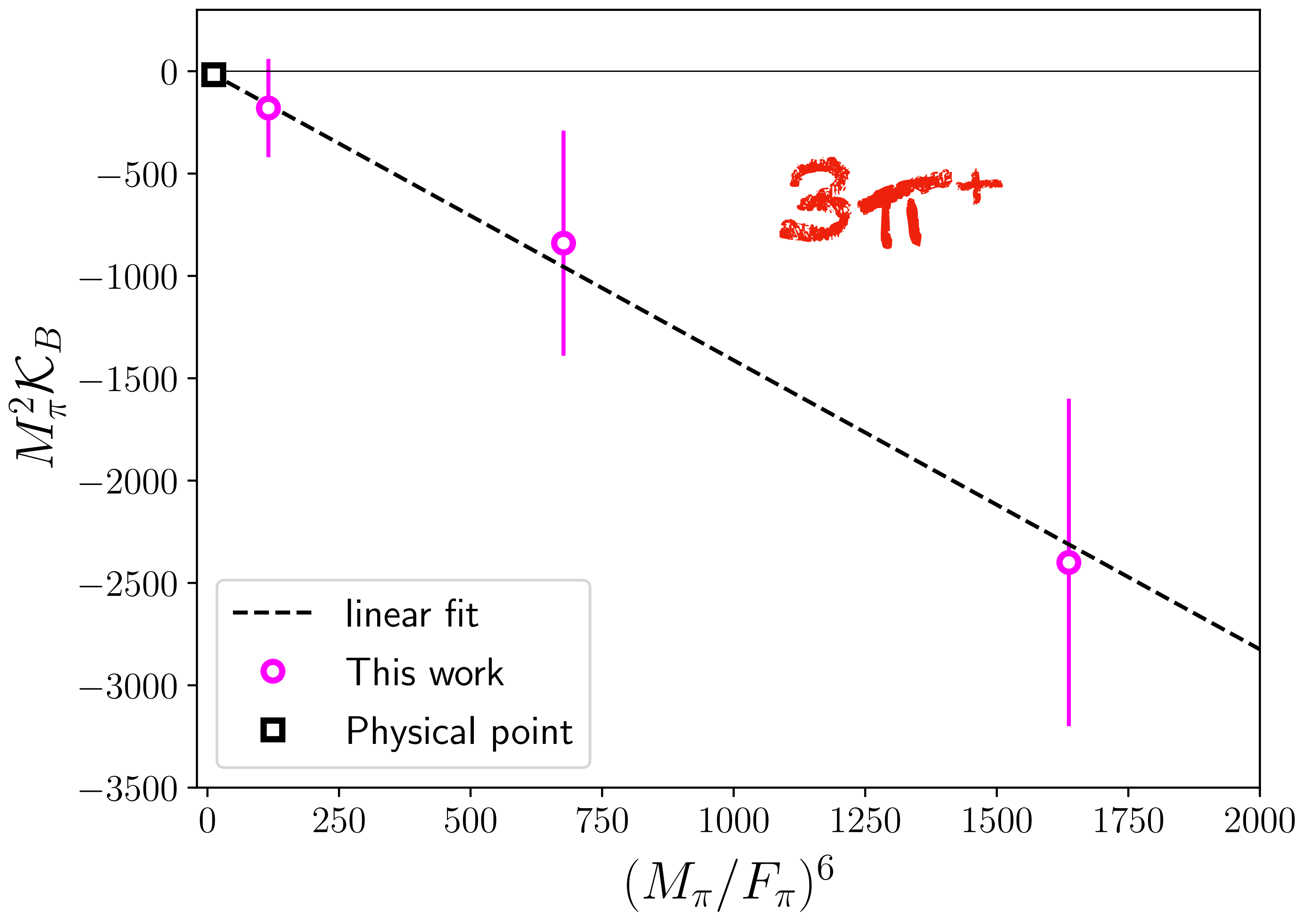

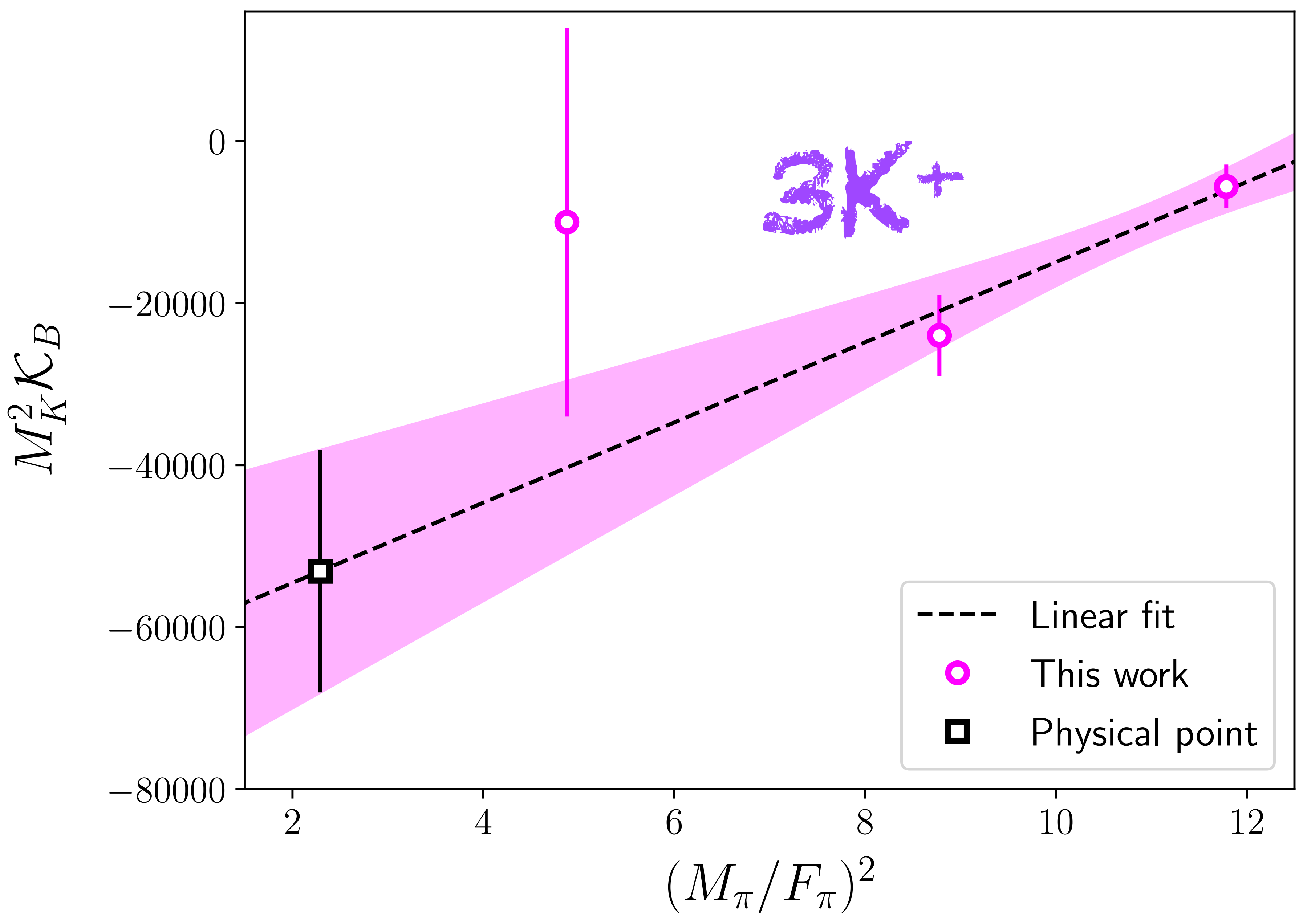

[Blanton, Hanlon, Hörz, Morningstar, FRL, Sharpe]

See talk by A. Hanlon, Wednesday 28th