High-precision determination of the electric and magnetic radius of the proton

Yonghui Lin

HISKP, Universität Bonn

In collaboration with Hans-Werner Hammer, Ulf-G. Meißner

Based on PLB816, 136254(2021) and arXiv2106.06357 (EPJA accepted)

19th International Conference on Hadron Spectroscopy and Structure in memoriam Simon Eidelman and Steven Weinberg (HADRON2021)

July 31, 2021

Deutsche Forschungsgemeinschaft

Outline

- Proton charge radius
- Dispersion theoretical determination of r_p^E
  - Parametrization of nucleon FFs
  - Application to data
  - Results and uncertainties
- Status of the proton radius puzzle
- Summary

Proton charge radius

- Definition $\langle r_p^2 \rangle_{\mathrm{E}} \equiv \int r^2 \rho_{\mathrm{E}}(\vec{r})\, d\vec{r},$

$$G_{\mathrm{E}}(Q^2) = 1 - \frac{r_p^2}{3!}Q^2 + \frac{\langle r^4 \rangle_{\mathrm{E}}}{5!}Q^4 - \frac{\langle r^6 \rangle_{\mathrm{E}}}{7!}Q^6 + \ldots$$

$$r_p^2 = -6\, \frac{dG_E(Q^2)}{d\,Q^2}\bigg|_{Q^2=0}$$

- Measurement
  - Leptonic hydrogen Lamb shift

$$(\Delta E_L)_{\mathrm{measured}} = E_1 + E_2 C(r_p^2) + \mathcal{O}(m_r \alpha^6), \quad C(r_p^2) = c_1 + c_2 r_p^2 + \mathcal{O}(\alpha^2)$$

  - Lepton-proton Scattering (XS & pol. transfer)

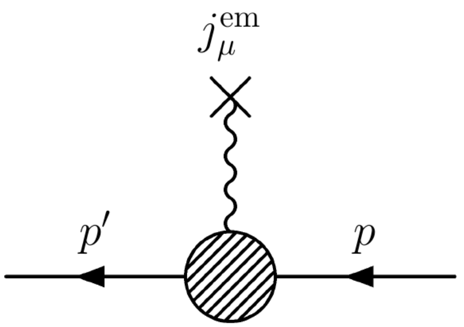
$$\left(\frac{d\sigma}{d\Omega}\right)_{\mathrm{measured}} = \frac{d\sigma_{\mathrm{Mott}}}{d\Omega}\frac{1}{1+\tau_p}\left(G_E^2 + \frac{\tau_p}{\varepsilon}G_M^2\right)(1+\delta_{\mathrm{TPE}}) + \mathcal{O}(\alpha^2)$$

C. Peset, *et al.* arXiv2106.00695

Nucleon Form Factors

- Definition

$$\langle p'|j_\mu^{\rm em}|p\rangle = \bar{u}(p')\left[F_1(t)\gamma_\mu + i\frac{F_2(t)}{2m}\sigma_{\mu\nu}q^\nu\right]u(p)\,,$$



$$t \equiv q^2 = -Q^2 = (p'-p)^2,\;\; t>0 \text{ for time-like, } t<0 \text{ for space-like}$$

- Normalization $F_1^p(0)=1\,, F_1^n(0)=0\,,\; F_2^p(0)=\kappa_p\,, F_2^n(0)=\kappa_n\,.$

- Isoscalar & isovector NFFs

$$F_i^s = \frac{1}{2}(F_i^p + F_i^n)\,, F_i^v = \frac{1}{2}(F_i^p - F_i^n)\,, i=1,2$$

- Sachs NFFs $\tau = -t/(4m_N^2)$

$$G_E(t) = F_1(t) - \tau F_2(t)\,, G_M(t) = F_1(t) + F_2(t)$$

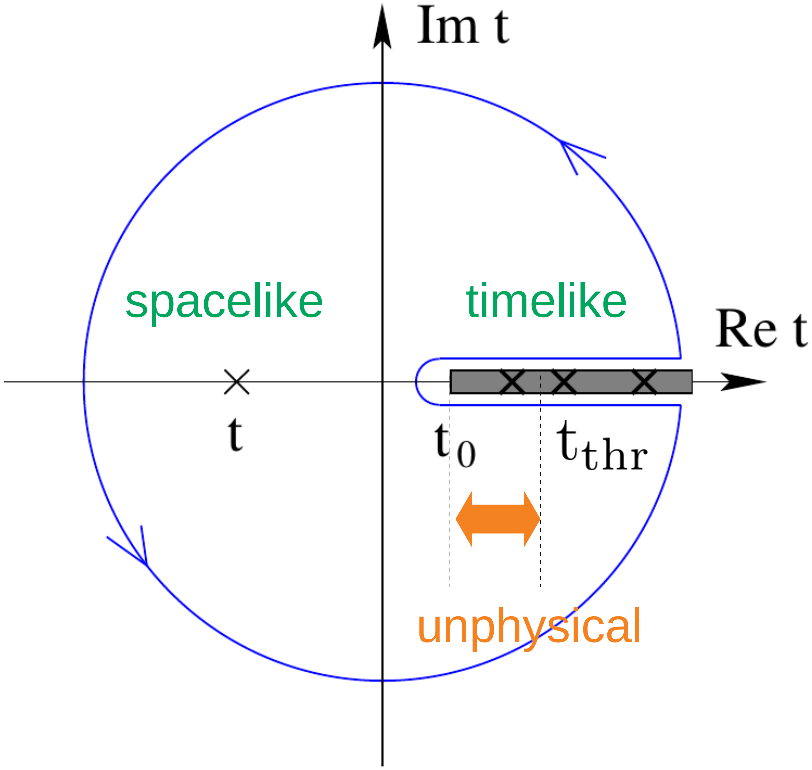
Why Dispersion Theory?

- Difficulties on NFFs
  - Unknown expression ⟶ parametrization-dependent
  - Data at $Q^2 = 0$ is unachievable ⟶ extrapolation needed
- Dispersion theoretical NFFs

$$F(t) = \frac{1}{\pi}\int_{t_0}^{\infty} \frac{\mathrm{Im}\, F(t')}{t' - t - i\epsilon}\, dt'$$

  - Unitarity and analyticity guaranteed,
  - Works well in the whole t-region, $(\sim 10^{-4}\text{-}10\mathrm{s}\ \mathrm{GeV}^2)$ experimentally
  - Theoretical constraints of asymptotic behavior of NFFs can be added easily,
  - Connects to data from different process. $(\pi N\text{-scattering}, \cdots)$

Dispersion Relations of NFFs



$$F(t) = \frac{1}{\pi}\int_{t_0}^{\infty} \frac{\mathrm{Im}\, F(t')}{t' - t - i\epsilon}\, dt'$$

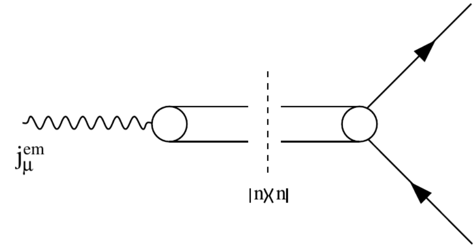
Ingredients: multiple cuts
(starting from $t_0 = (2M_\pi)^2$)
& vector meson poles

The spectral functions $\mathrm{Im} F(t)$ are central quantities, different from zero only over the cut $[t_0, +\infty)$.

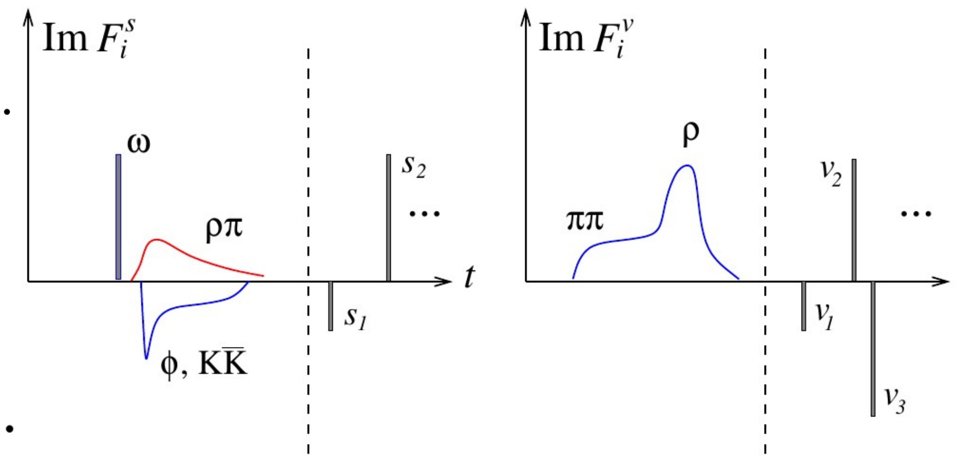
Spectral functions

- Spectral Decomposition (low-mass part)
  - Crossing symmetry $\langle N(p')|j_\mu^{\text{em}}|N(p)\rangle \longleftrightarrow \langle N(p)\bar{N}(\bar{p})|j_\mu^{\text{em}}|0\rangle$
  - Spectral decomposition G. F. Chew, *et al.* PhysRev110, 265(1958)

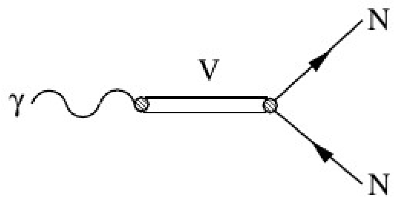
$$\text{Im}\langle N(p)\bar{N}(\bar{p})|j_\mu^{\text{em}}|0\rangle \sim \sum_n \langle N(p)\bar{N}(\bar{p})|n\rangle\langle n|j_\mu^{\text{em}}|0\rangle$$



  - Intermediate mass states $|n\rangle$
    - Isoscalar: $3\pi, 5\pi, \ldots, K\bar{K}, \pi\rho, \ldots$
    - Isovector: $2\pi, 4\pi, \ldots$



- Vector meson dominance

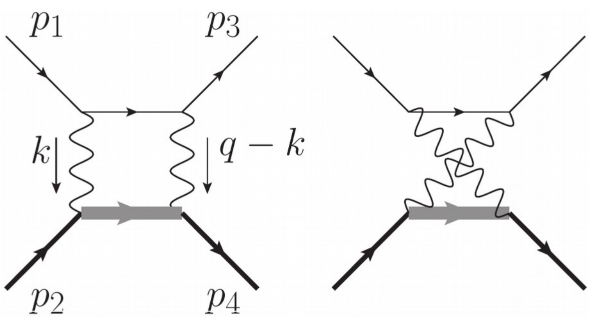


Higher mass states $s_1, s_2, \ldots$
(effective poles) $v_1, v_2, \ldots$

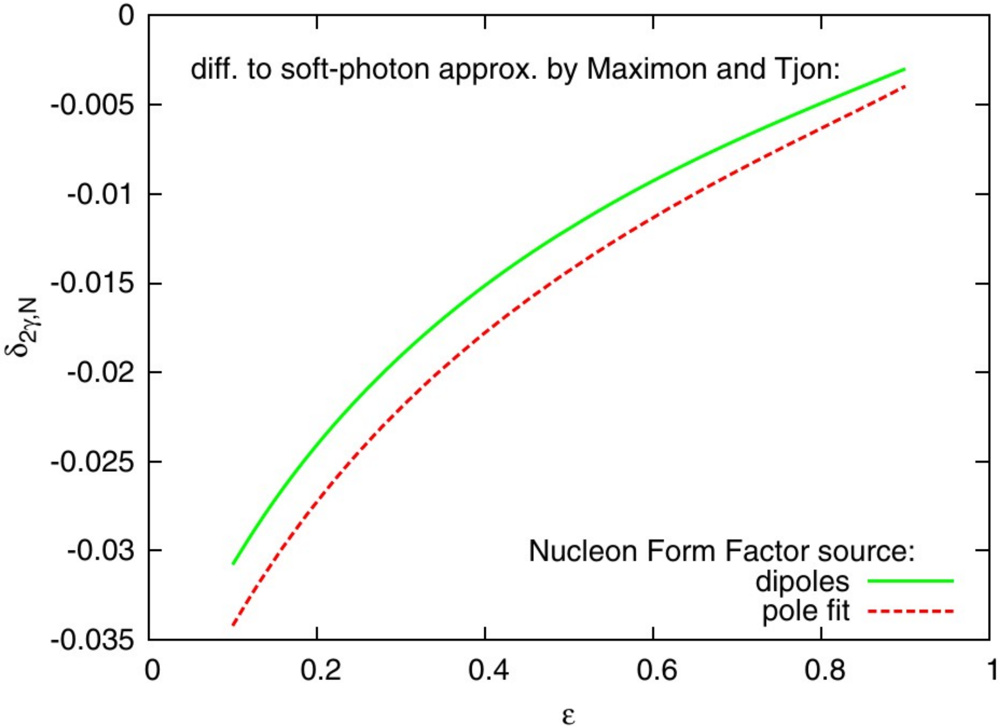
Two-photon effects

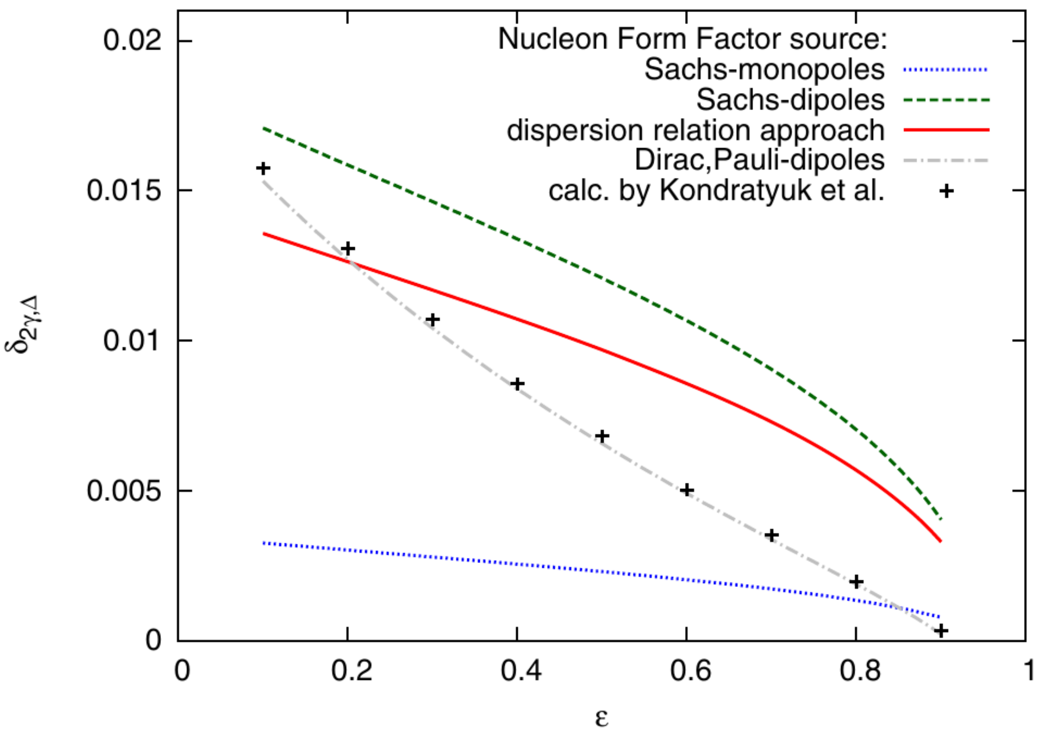
- Diagrams I. T. Lorenz, *et al.* PRD91, 014023(2015)

$$\frac{d\sigma_{\text{corr}}}{d\Omega} = \frac{d\sigma_{1\gamma}}{d\Omega}(1+\delta_{2\gamma}+\ldots)\ ,\ \delta_{2\gamma}\underbrace{\approx}_{\mathcal{O}(\alpha)}\frac{2\text{Re}(\mathcal{M}_{1\gamma}^{\dagger}\mathcal{M}_{2\gamma})}{|\mathcal{M}_{1\gamma}|^2}$$



- Numerical result for $\delta_{2\gamma,N}$, $\delta_{2\gamma,\Delta}$ with $Q^2 = 3\,\text{GeV}^2$





Two-photon effects

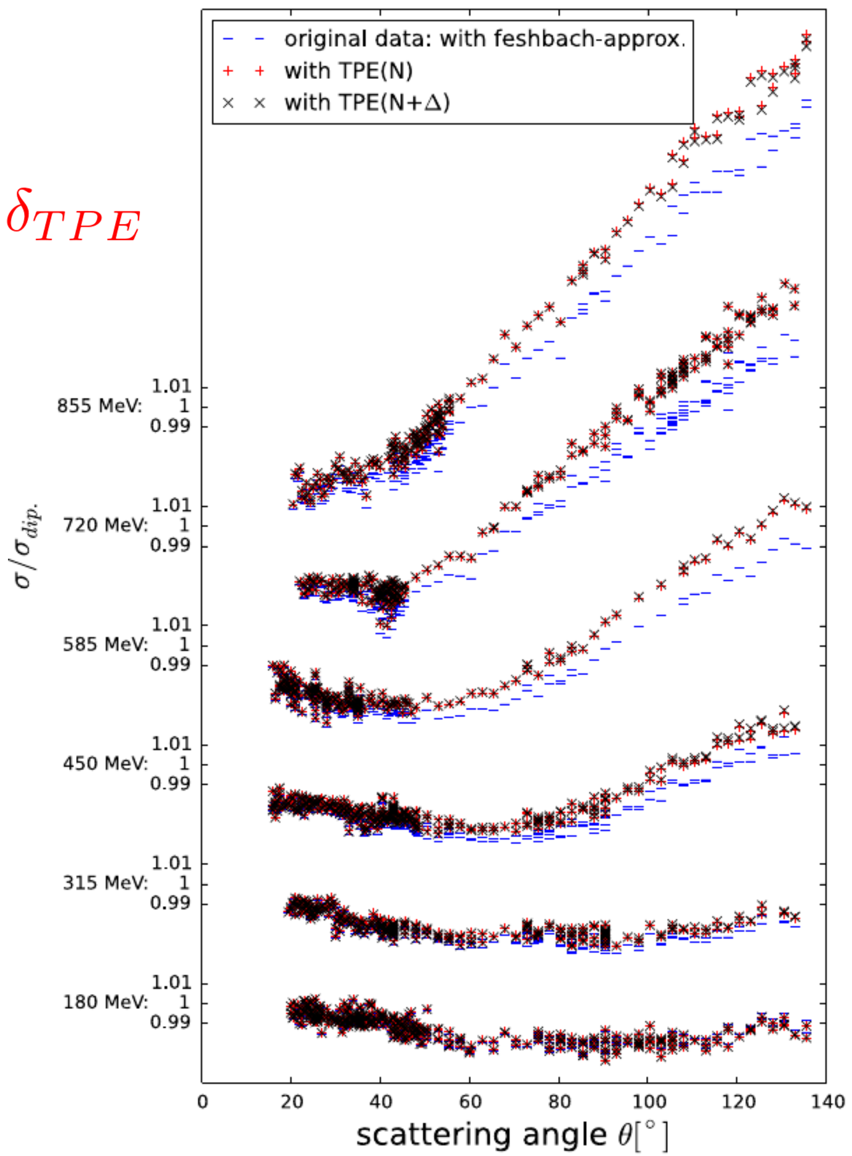
- Correction to XS

$\text{smaller}\ \theta \longrightarrow \text{lower}\ Q^2 \longrightarrow \text{smaller}\ \delta_{TPE}$

A simple form for $\delta_{2\gamma}$ that only works in the limit $Q^2 \to 0$ contained in MAMI data.

$$\delta_F = Z\alpha\pi \frac{\sin(\theta/2) - \sin^2(\theta/2)}{\cos^2(\theta/2)}$$

W. A. McKinley, *et al.* PhysRev74, 1759(1948)



Theoretical constraints

- Normalization (4)
- Neutron charge radius squared (1)

A. A. Filin, *et al.* PhysRevLett124, 082501(2020)

$$\langle r_n^2 \rangle = -0.105^{+0.005}_{-0.006}\ \mathrm{fm}^2$$

- Superconvergence relations from pQCD (6)

$$\int_{t_0}^{\infty} \mathrm{Im}\, F_i(t)\, t^n dt = 0\,, \quad i = 1, 2$$

with $n = 0$ for F_1, $n = 0, 1$ for F_2

Data base

- Differential cross section
  - MAMI (0.00384-0.977 GeV^2, 1422)
  - PRad (0.000215-0.058, 71)
- World data on Neutron form factor
  - G_E^n (0.14-1.47, 25)
  - G_M^n (0.071-10.0, 23)
- JLab data on Proton FFs ratio

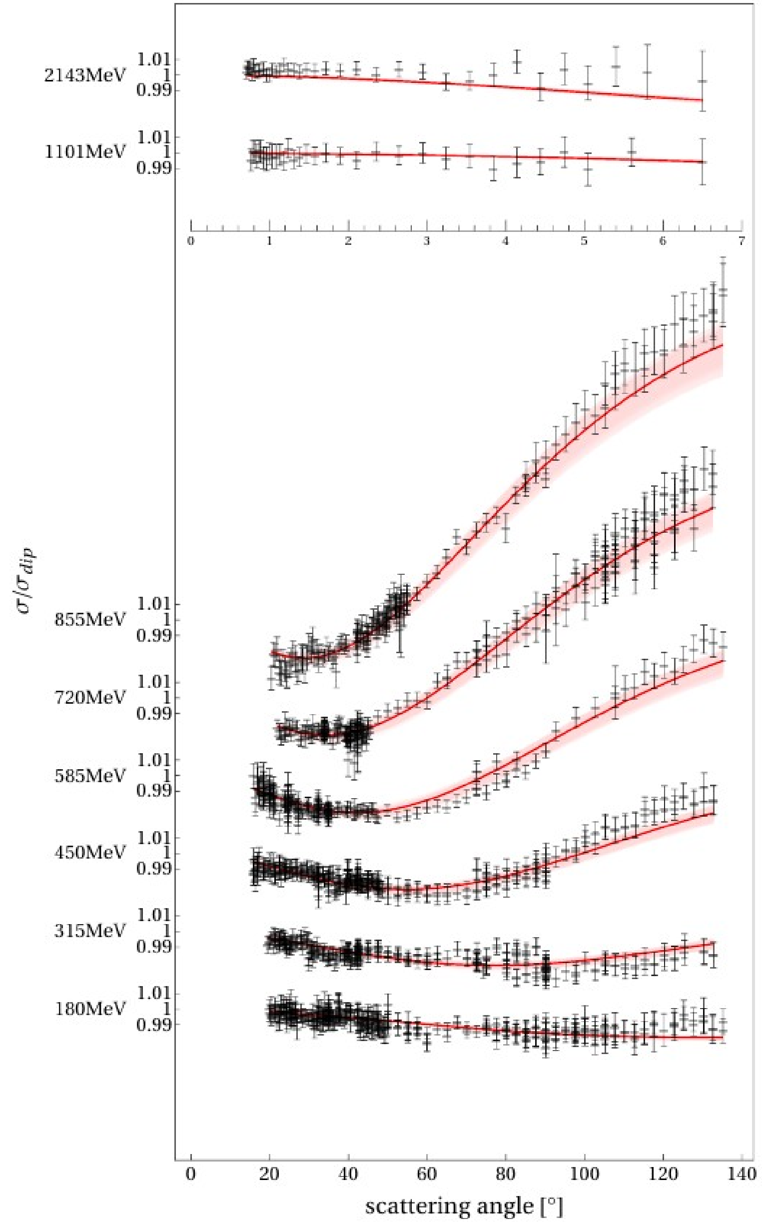
  $\mu_p G_E^p / G_M^p$ (1.18-8.49, 16)

- Number of free parameters

$$1557 \Longrightarrow 4 + 3(N_s + N_v) - 11 + 31 + 2$$

Results I: Differential cross section

- Best configuration '6s+4v'
  - 50 parameters
  - $\omega, \phi, s_1, s_2, s_3, s_4 + K\bar{K} + \rho\pi$
  - $v_1, v_2, v_3, v_4 + \pi\pi$
  - $\chi^2/\text{dof} = 1.927$

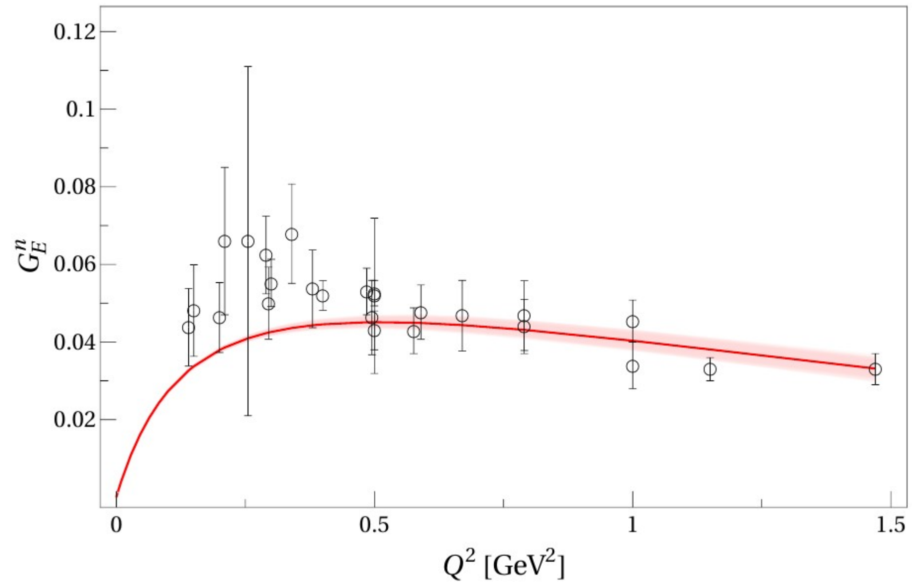


Line best fit

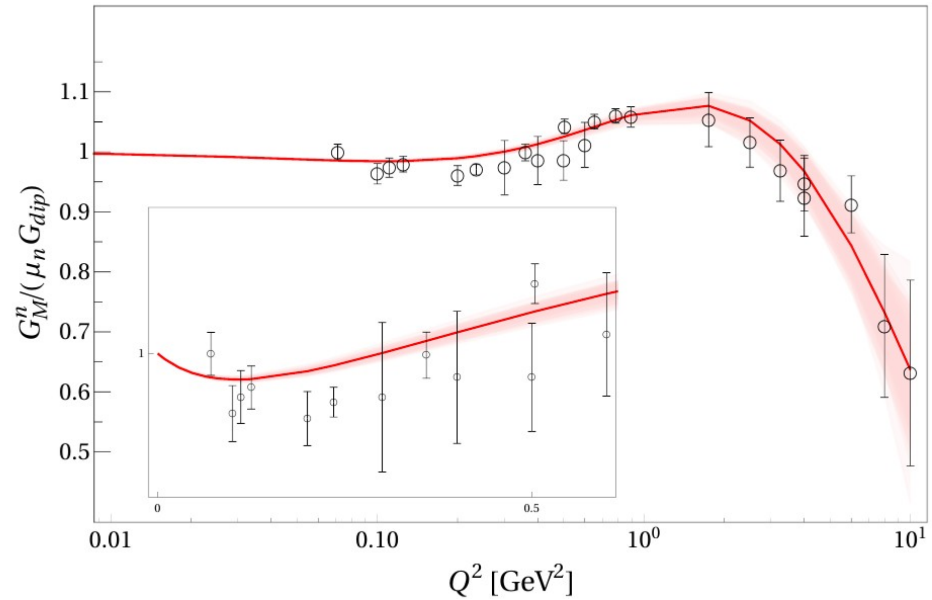
Band error from bootstrap sampling.

Results II: NFFs

G_E^n

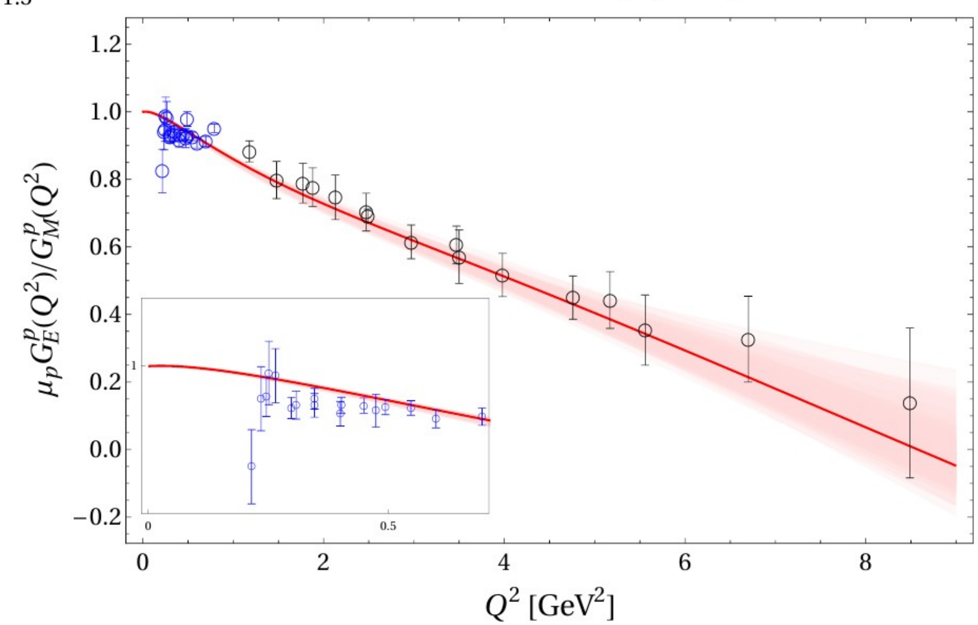


$G_M^n/(\mu_n G_{\text{dip}})$



$\mu_p G_E^p / G_M^p$

Blue not fitted



Line best fit

Band error from bootstrap sampling.

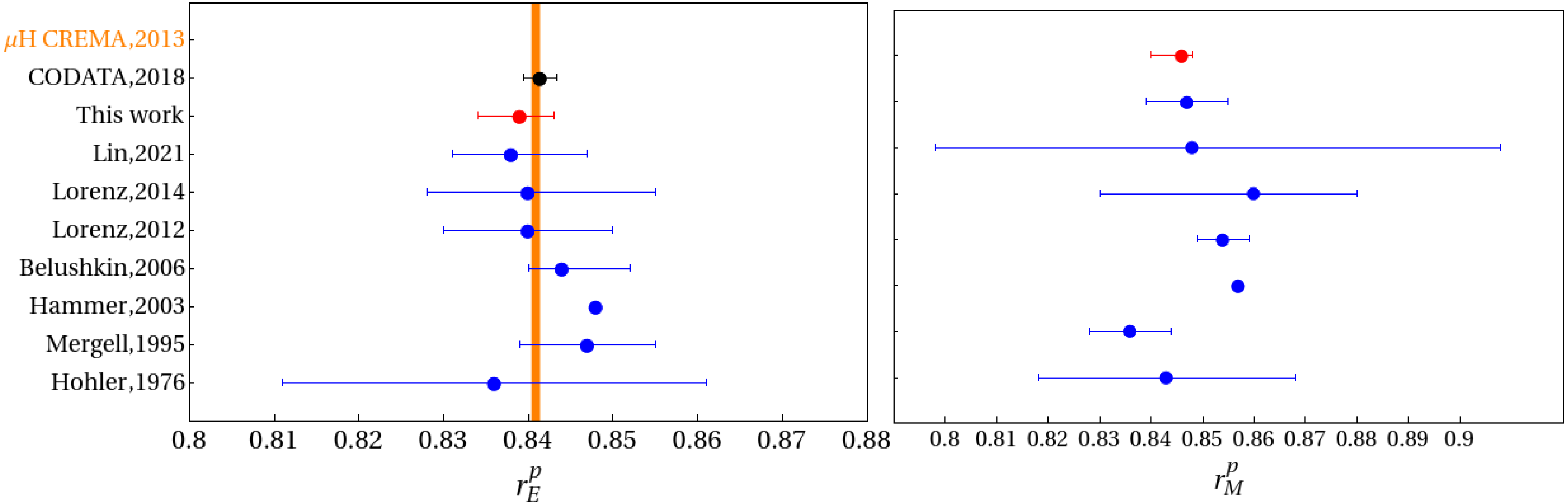
Focusing on proton charge radius

- Our determination

"Systematical" error from variation of the spectral functions
"Statistical" error form bootstrap

$$r_E^p = 0.839 \pm 0.002^{+0.002}_{-0.003}\,\text{fm},\; r_M^p = 0.846 \pm 0.001^{+0.001}_{-0.005}\ \text{fm}$$

- Comparing to existing DR determinations

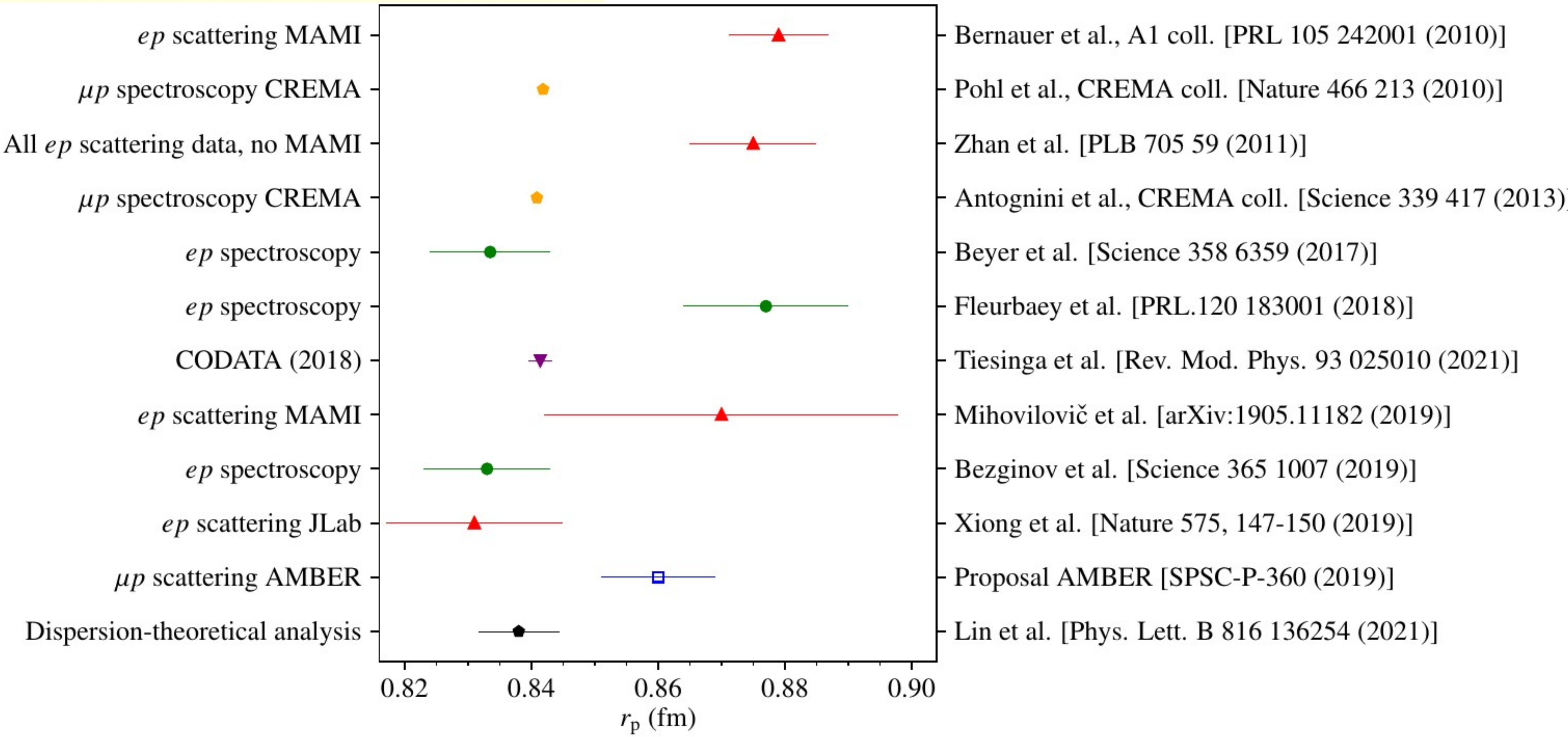


Comparing to recent measurements

Fundamental Physical Constants From CODATA website

proton rms charge radius r_p	
Numerical value	8.414 x 10^{-16} m
Standard uncertainty	0.019 x 10^{-16} m
Relative standard uncertainty	2.2 x 10^{-3}
Concise form	8.414(19) x 10^{-16} m

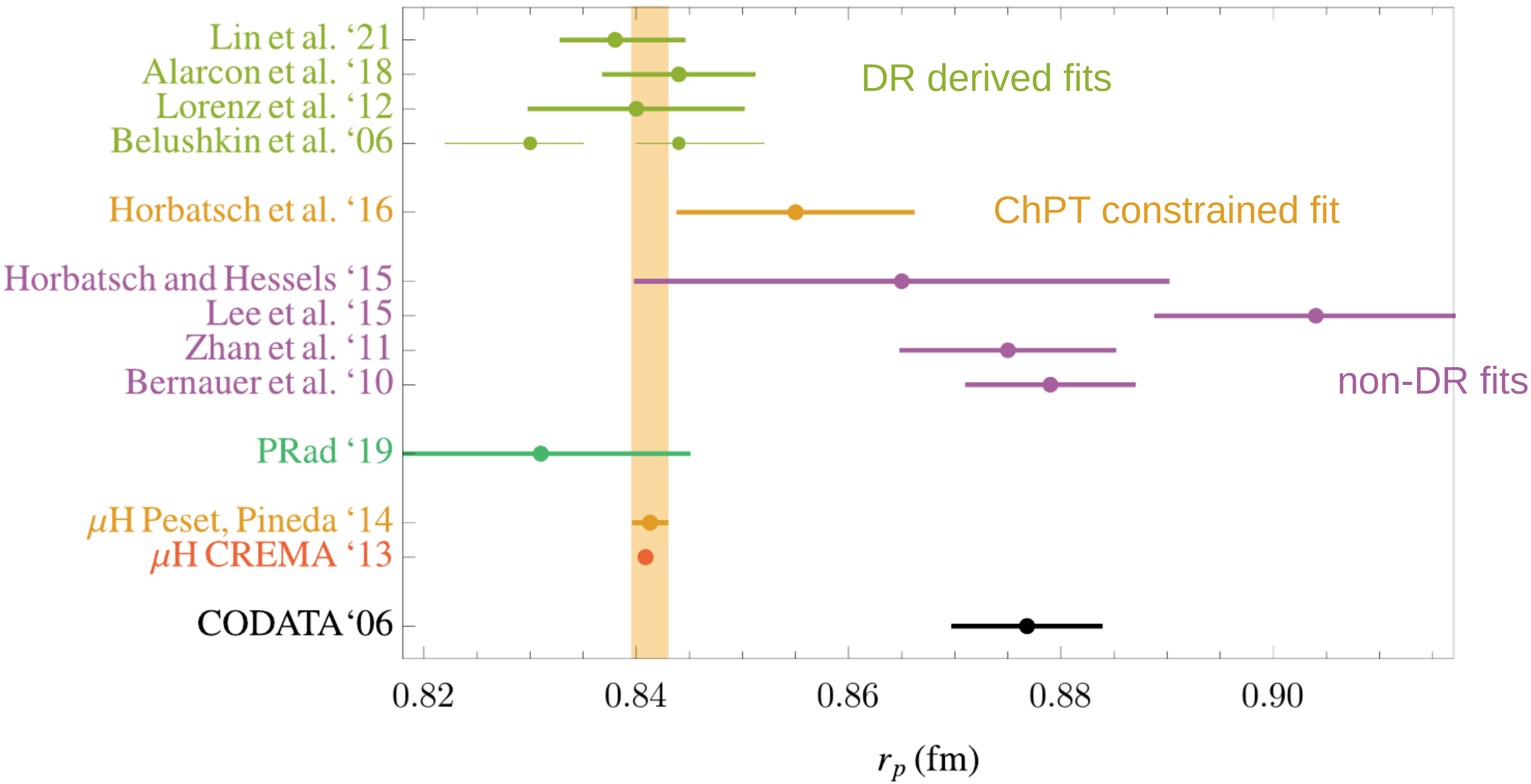
borrowed from Christian Dreisbach (TUM)



Status of the proton radius "puzzle"

- Determination from ep scattering

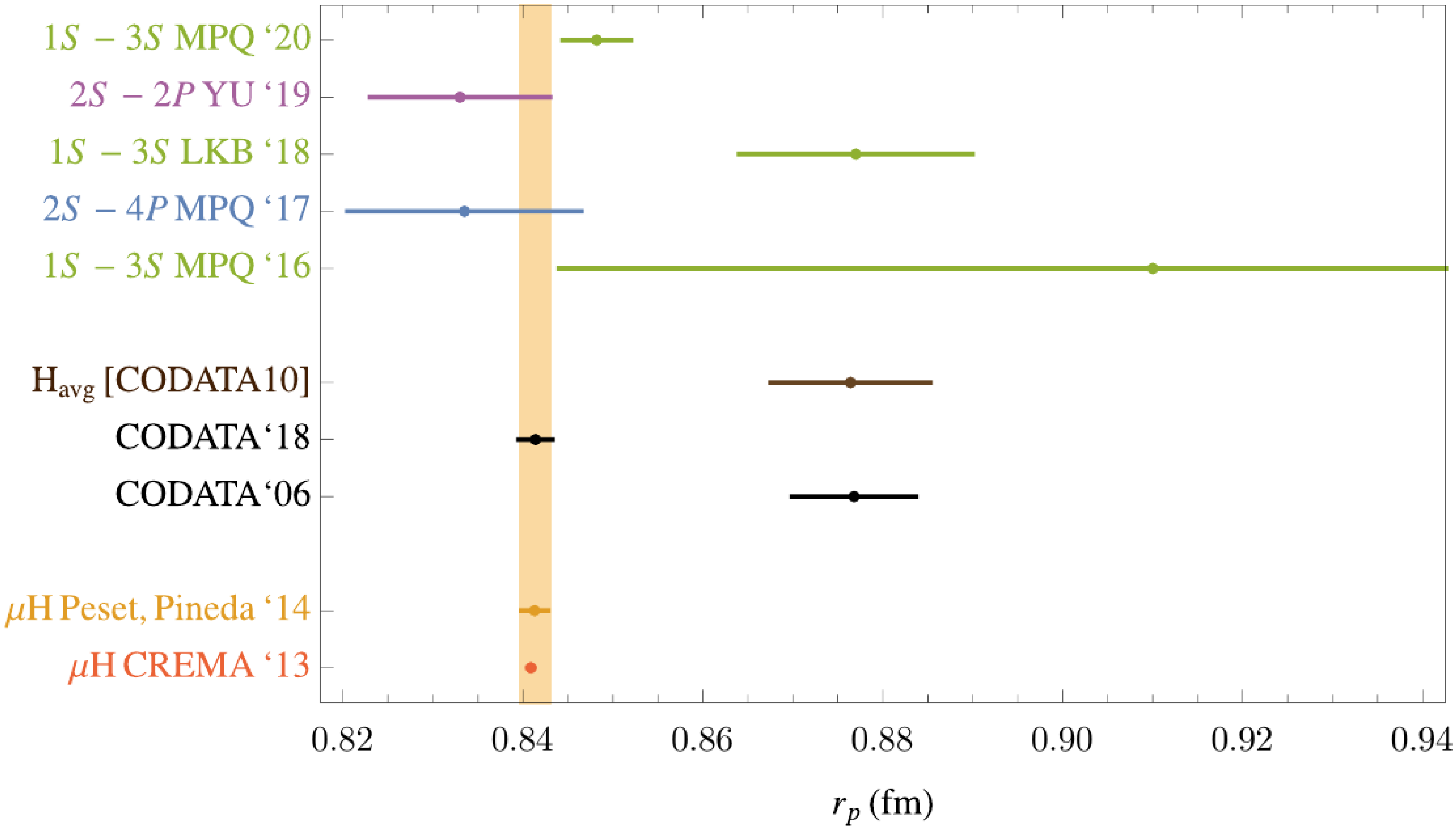
borrowed from C. Peset, *et al.* 2106.00695



Status of the proton radius "puzzle"

- Determination from the hydrogen energy shift

borrowed from C. Peset, *et al.* 2106.00695



From "puzzle" to precision

- Some discussions

H.-W. Hammer, *et al.* SciBull65, 257(2020)
C. Peset, *et al.* 2106.00695

$$(\Delta E_L)_{\text{measured}} = E_1 + E_2 C(r_p^2) + \mathcal{O}(m_r\alpha^6), \quad C(r_p^2) = c_1 + c_2 r_p^2 + \mathcal{O}(\alpha^2)$$

δ_{TPE} encoded in coefficients E_1, E_2, and $C \sim \mathcal{O}(m_l)$

$$\left(\frac{d\sigma}{d\Omega}\right)_{\text{measured}} = \frac{d\sigma_{\text{Mott}}}{d\Omega}\frac{1}{1+\tau_p}\left(G_E^2 + \frac{\tau_p}{\varepsilon}G_M^2\right)(1+\delta_{\text{TPE}}) + \mathcal{O}(\alpha^2)$$

$\langle r_p^2\rangle$, δ_{TPE}, higher moments($\langle r_p^n\rangle$) and polarizabilities

interwinded together when goes to the higher order corrections.

Only when the theory and experiment are at the same order of accuracy

can the same $\langle r_p^2\rangle$ be obtained.

Precision is the issue that really matters in the proton charge radius problem.

Summary

- We updated previous DR analysis on ep scattering data in the following aspects,
  - Included the unprecedented low-momentum transfer data by PRad
  - Included high precise $\pi\pi$ continuum $\mathrm{M.\,Hoferichter,}\ et\ al.\ \mathrm{EPJA52,331(2016)}$
  - Improved uncertainty estimation
- DR analysis on NFFs data provide robust and consistent r_p^E over decades.
- Newest electronic hydrogen measurements and DR determination definitely agree with ‘small’ μH-CREMA proton radius. Should not call it a puzzle anymore.

Thank you very much for your attention!

Proton charge radius

- In Breit frame (non-relativistic limit), $e^- + p \to e^- + p$

$$q = (0, \vec{q}),\ \ Q^2 = -q^2 = \vec{q}^{\,2} \geq 0$$

$$G_{\mathrm{E}}(q^2) = \int d^3\vec{r}\, \rho_{\mathrm{E}}(\vec{r})\, e^{-i\vec{q}\cdot\vec{r}}$$

Considering a spherical density,

$$G_{\mathrm{E}}(q^2) = 2\pi \int_0^\infty r^2 dr\, \rho_{\mathrm{E}}(r) \int_{-1}^{1} d\cos(\theta)\, e^{-i|\vec{q}|\, r\ \cos(\theta)}$$

Re-define the momentum dependence, $|\vec{q}| = Q$

$$G_{\mathrm{E}}(Q^2) = \frac{4\pi}{Q} \int_0^\infty r\rho_C(r)\sin(Qr)\, dr$$

Proton charge radius

Expanding at $Q^2 = 0$

$$G_{\mathrm{E}}(Q^2) = 4\pi \sum_{j=0}^{\infty} (-1)^n \frac{\left(Q^2\right)^n}{(2n+1)!} \int_0^{\infty} r^{2n+2} \rho_{\mathrm{E}}(r)\, dr$$

$$= G_{\mathrm{E}}(0) \sum_{n=0}^{\infty} (-1)^n \frac{\left\langle r^{2n} \right\rangle_{\mathrm{E}}}{(2n+1)!} \left(Q^2\right)^n$$

$$\left\langle r^{2n} \right\rangle_{\mathrm{E}} \equiv \frac{4\pi \int_0^{\infty} r^{2n+2} \rho_{\mathrm{E}}(r)\, dr}{4\pi \int_0^{\infty} r^2 \rho_{\mathrm{E}}(r)\, dr} = \frac{1}{G_{\mathrm{E}}(0)} 4\pi \int_0^{\infty} r^{2n+2} \rho_{\mathrm{E}}(r)\, dr\,, n \in \mathbb{N}$$

Comparing with the Taylor series of $G_{\mathrm{E}}(Q^2)$, around $Q^2 = 0$

$$G_{\mathrm{E}}(Q^2) = \sum_{n=0}^{\infty} \frac{1}{n!} \left. \frac{d^n G_{\mathrm{E}}}{d\left(Q^2\right)^n} \right|_{Q^2=0} \left(Q^2\right)^n$$

$$\left\langle r^{2n} \right\rangle_{\mathrm{E}} = (-1)^n \frac{(2n+1)!}{n!} \frac{1}{G_{\mathrm{E}}(0)} \left. \frac{d^n G_{\mathrm{E}}}{d(Q^2)^n} \right|_{Q^2=0}, n \in \mathbb{N}$$

Parametrization of NFFs

- Our spectral functions of NFFs read

$$\mathrm{Im}\, F_i^s(t) = \mathrm{Im}\, F_i^{(s,K\bar{K})}(t) + \mathrm{Im}\, F_i^{(s,\rho\pi)}(t) + \sum_{V=\omega,\phi,s_1,\ldots} \pi a_i^V \delta(M_V^2 - t)\,,$$

$$\mathrm{Im}\, F_i^v(t) = \mathrm{Im}\, F_i^{(v,2\pi)}(t) + \sum_{V=v_1,\ldots} \pi a_i^V \delta(M_V^2 - t)\,, \quad i = 1, 2\,.$$

- $\mathrm{Im}\, F_i^{(s,K\bar{K})}(t)$, $\mathrm{Im}\, F_i^{(s,\rho\pi)}(t)$, $\pi a_i^V \delta(M_V^2 - t)$

  can be converted into (after DR integrals)

$$F_i(t) = \frac{a_i^V}{M_V^2 - t}$$

Two-pion continuum

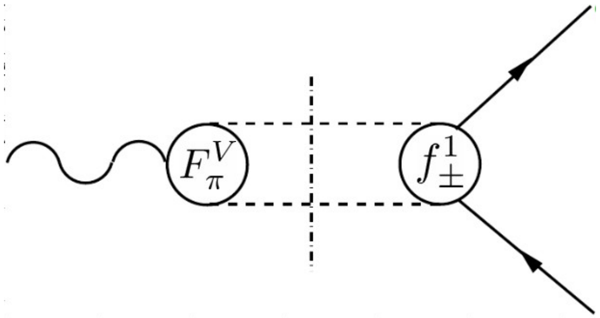
- Two-pion contribution to isovector spectral functions

M. Hoferichter, *et al.* EPJA52, 331(2016)

$$\text{Im}\ G_E^v(t) = \frac{q_t^3}{m\sqrt{t}}\, F_\pi^V(t)^*\, f_+^1(t)\, \theta\big(t - 4M_\pi^2\big)\ ,$$

$$\text{Im}\ G_M^v(t) = \frac{q_t^3}{\sqrt{2t}}\, F_\pi^V(t)^*\, f_-^1(t)\, \theta\big(t - 4M_\pi^2\big)\ ,$$

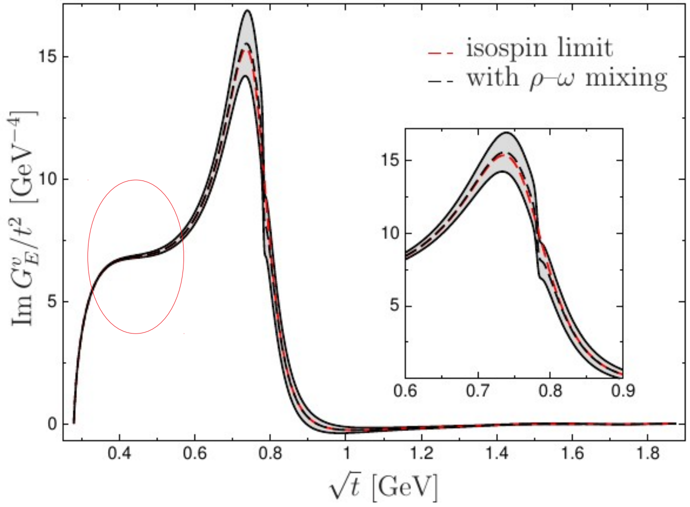
$$q_t = \sqrt{t/4 - M_\pi^2}$$



  - Pion FFs F_π : from $\pi\pi$ scattering phase shift
  - P-wave $\pi\pi \to N\bar{N}$ $f_\pm^1$: from analytic continuation of πN data

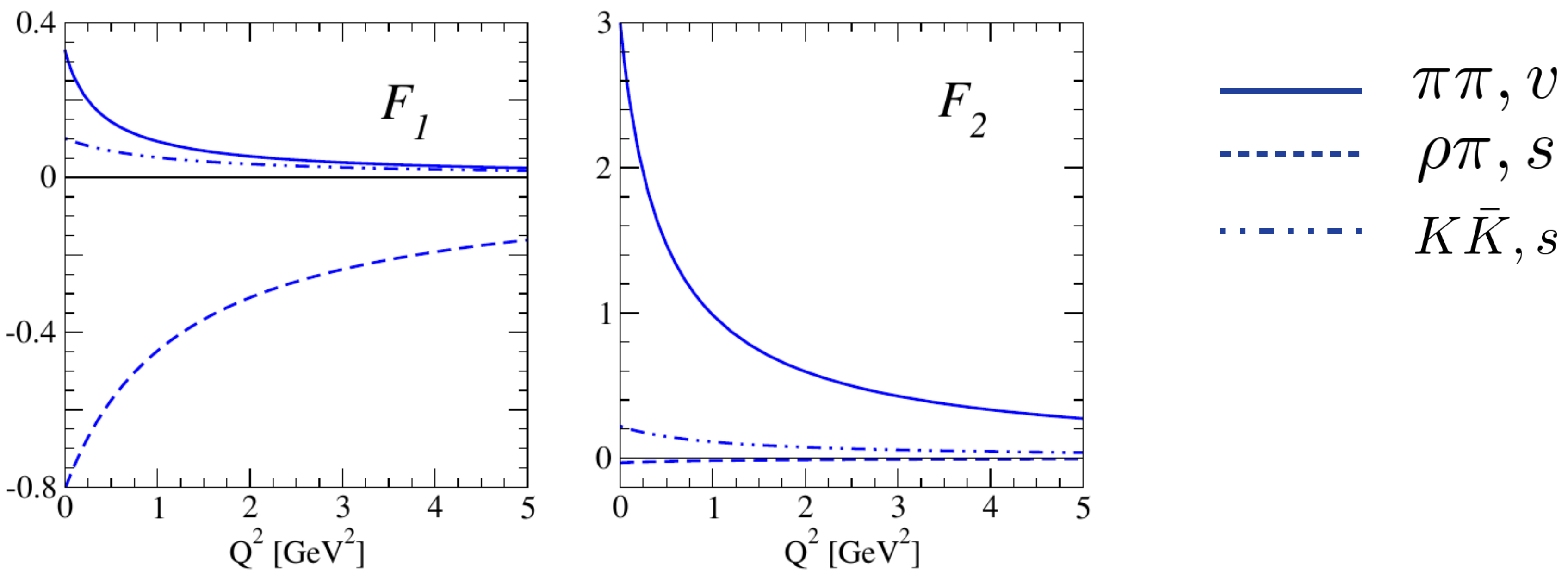
- Substantially different with the single ρ meson approximation

A visible enhancement on the left shoulder of ρ is found. $t_c = 3.98 m_\pi^2$ very close to threshold



Parametrization of NFFs

- Continuum contributions (not fitted to data)
  - $\pi\pi$ based on precise analysis of pion-nucleon scattering
    M. Hoferichter, *et al.* EPJA52, 331(2016)
  - $K\bar{K}$ from an analytic continuation of kaon-nucleon scattering data
    H.-W. Hammer, *et al.* PRC60, 045205(1999)
  - $\rho\pi$ from investigation of the Bonn-Jülich N-N interaction model
    U.-G.Meißner, *et al.* PLB633, 507(2006)



Two-photon-exchange correction

- Soft-photon approximation

$$\frac{d\sigma_{\mathrm{corr}}}{d\Omega} = \frac{d\sigma_{1\gamma}}{d\Omega}(1+\delta_{2\gamma}+\ldots)\ ,\ \delta_{2\gamma} \underbrace{\approx}_{\mathcal{O}(\alpha)} \frac{2\mathrm{Re}(\mathcal{M}_{1\gamma}^{\dagger}\mathcal{M}_{2\gamma})}{|\mathcal{M}_{1\gamma}|^2}$$

- One-gamma amplitude

$$\mathcal{M}_{1\gamma} = -\frac{e^2}{q^2}\bar{u}_e(p_3)\gamma_\mu u_e(p_1)\bar{u}_N(p_4)\Gamma^\nu u_N(p_2)$$

- Two-gamma amplitude

$$\mathcal{M}_{2\gamma}^{\mathrm{box}} = -ie^4\int\frac{d^4k}{(2\pi)^4}L_{\mu\nu}^{\mathrm{box}}(H_N^{\mu\nu}+H_\Delta^{\mu\nu})D(k)D(q-k)$$

$$L_{\mu\nu}^{\mathrm{box}} = \bar{u}_e(p_3)\gamma_\mu S_F(p_1-k,m_e)\gamma_\nu u_e(p_1)$$

$$\begin{aligned} H_N^{\mu\nu} &= \bar{u}_N(p_4)\Gamma^\mu(q-k)S_F(p_2+k,m_N)\Gamma^\nu(k)u_N(p_2) \\ H_\Delta^{\mu\nu} &= \bar{u}_N(p_4)(p_4)\Gamma^{\mu\alpha}_{\gamma\Delta\to N}(p_2+k,q-k)S_{\alpha\beta} \\ &\quad\times(p_2+k)\Gamma^{\beta\nu}_{\gamma N\to\Delta}(p_2+k,k)u_N(p_2), \end{aligned}$$

Values of the proton charge radius

- Historical DR determination

Ref.	r_E^p [fm]	r_M^p [fm]
Hohler:1976ax	0.836 ± 0.025	0.843 ± 0.025
Mergell:1995bf	0.847 ± 0.008	0.836 ± 0.008
Hammer:2003ai	0.848*	0.857*
Belushkin:2006qa	$0.844^{+0.008}_{-0.004}$	0.854 ± 0.005
Lorenz:2012tm	0.84 ± 0.01	$0.86^{+0.02}_{-0.03}$
Lorenz:2014yda	$0.840^{+0.015}_{-0.012}$	$0.848^{+0.06}_{-0.05}$
Lin:2021umk	$0.838^{+0.005+0.004}_{-0.004-0.003}$	$0.847 \pm 0.004 \pm 0.004$

Here * means that no error analysis has been performed.

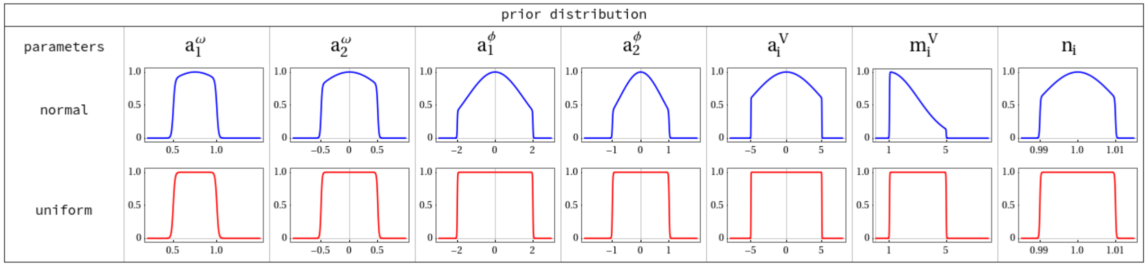
Bootstrap vs Bayesian in PRad fits

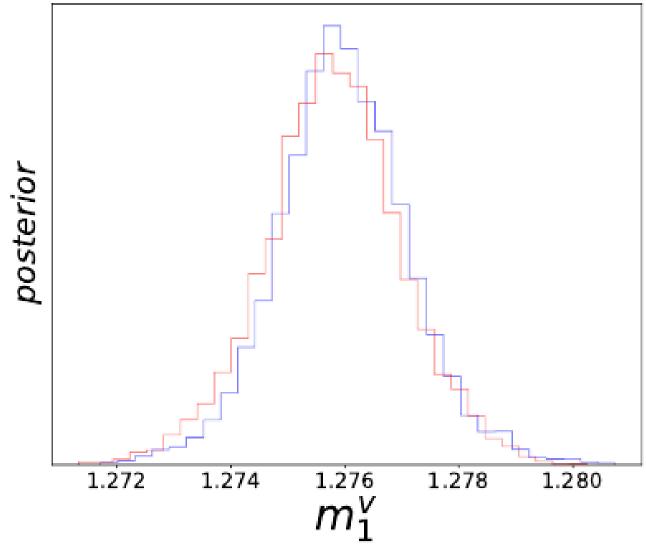
- Bayesian theorem

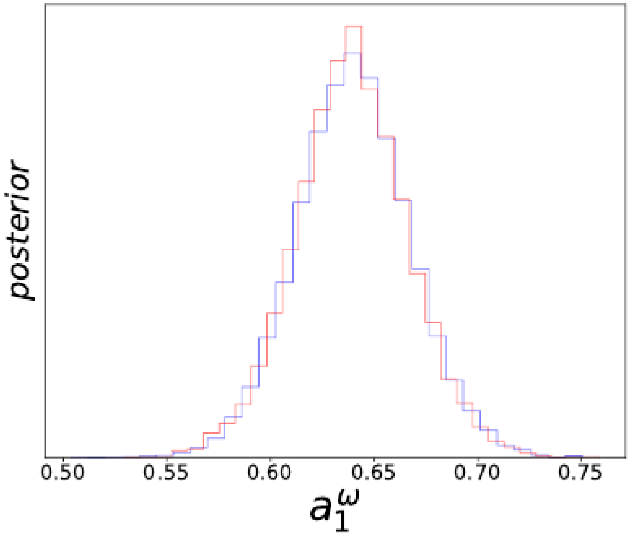
$$P(\text{parameters}|\text{data}) = \frac{P(\text{parameters})P(\text{data}|\text{parameters})}{P(\text{data})}$$

$$\text{posterior} \sim \text{prior} \times \text{likelihood}$$

- Bayesian analysis



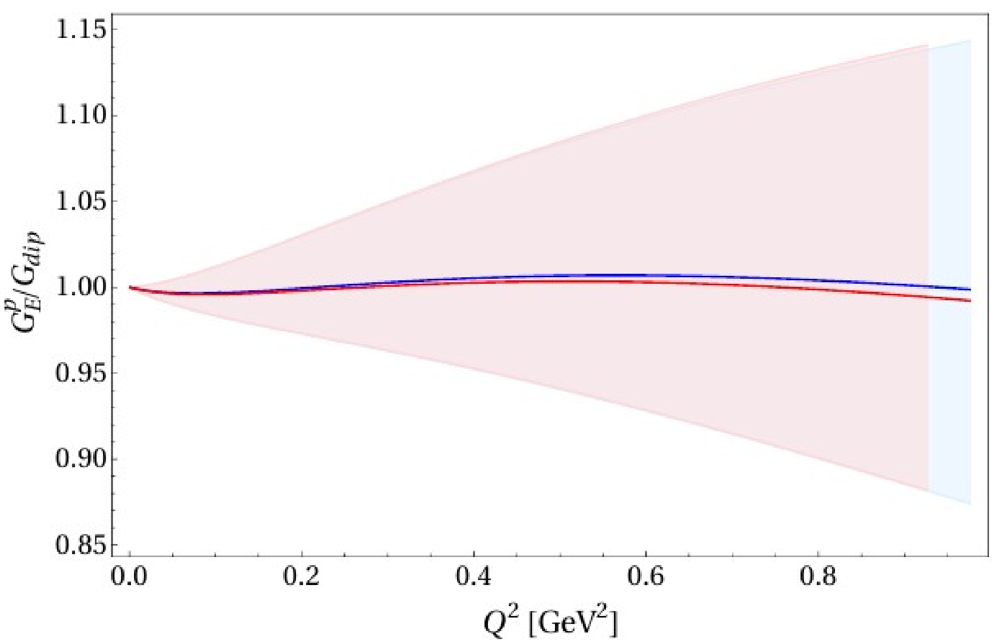


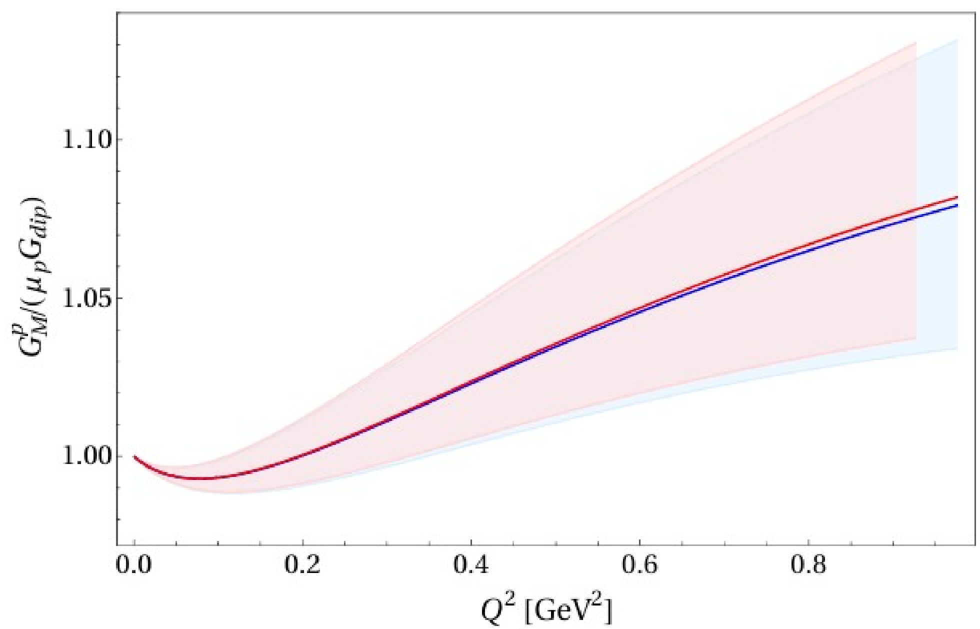


Bootstrap vs Bayesian in PRad fits

- Comparing with bootstrap sampling

Method	r_E^p [fm]	r_M^p [fm]
Bayesian normal	0.828 ± 0.011	0.843 ± 0.004
Bayesian uniform	0.828 ± 0.011	0.843 ± 0.004
Bootstrap	0.828 ± 0.012	0.843 ± 0.005





Parameters of best fit

- Best fit “6s+4v”

V_s	m_V	a_1^V	a_2^V	V_v	m_V	a_1^V	a_2^V
ω	0.7830	0.6893	0.0431	v_1	1.1222	1.0414	−0.6239
ϕ	1.0190	−0.0281	−0.4705	v_2	1.5147	−4.0062	−3.0365
s_1	1.8267	0.3768	0.5590	v_3	1.8062	4.8533	2.1897
s_2	4.0020	−1.2786	−4.882	v_4	2.2543	−2.0208	−0.0438
s_3	4.0713	1.8028	4.0681				
s_4	4.3075	−0.6576	0.4944				

n1	0.9965	n2	1.0061	n3	1.0028	n4	1.0010	n5	1.0035	n6	0.9914
n7	0.9982	n8	0.9929	n9	1.0076	n10	1.0000	n11	1.0000	n12	1.0037
n13	1.0030	n14	1.0044	n15	1.0055	n16	1.0027	n17	1.0048	n18	1.0013
n19	0.9995	n20	1.0029	n21	0.9977	n22	0.9905	n23	0.9985	n24	1.0100
n25	1.0080	n26	1.0069	n27	0.9999	n28	1.0100	n29	1.0066	n30	0.9999
n31	1.0100	ñ1	0.9989	ñ2	1.0059						