Determination of complete experiments using graphs

Talk based on work done in collaboration with:

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HISKP, University of Bonn

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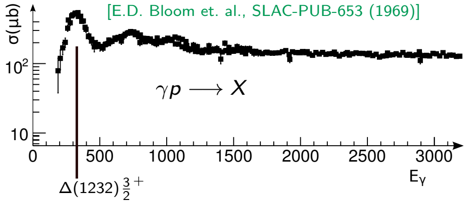
Introduction: why spin-amplitudes?

$*$) Photoproduction is a generic reaction used to study baryon resonances:



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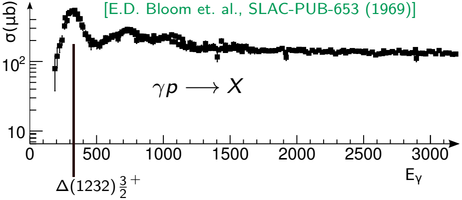

∗) Baryon resonances $\left(\Delta(1232)\frac{3}{2}^+, N(1440)\frac{1}{2}^+, \ldots\right)$ are Fermions

↪ Scatter particles with spin to excite systems with half-integer J

∗) 'T-matrix' $\mathcal{T}_{fi}$ parameterized by N spin-amplitudes $\{b_i, i = 1, \ldots, N\}$

∗) The usual reactions under study are:

- Pion-Nucleon (πN-) scattering: $\pi N \longrightarrow \pi N$ (2 spin-amplitudes)
- Pion photoproduction: $\gamma N \longrightarrow \pi N$ (4 spin-amplitudes)
- Pion electroproduction: $eN \longrightarrow e'\pi N$ (6 spin-amplitudes)
- 2-Pion photoproduction: $\gamma N \longrightarrow \pi\pi N$ (8 spin-amplitudes)
- ...

Algebraic starting point I

∗) Generic problem with N amplitudes $\{b_i, i = 1, \ldots, N\}$: the N^2 (polarization-) observables are bilinear hermitean forms (def. via orthogonal matrices $\tilde{\Gamma}^{\alpha}$):

$$\mathcal{O}^{\alpha} = \boldsymbol{c}^{\alpha} \sum_{i,j=1}^{N} b_i^{*} \tilde{\Gamma}^{\alpha}_{ij} b_j, \text{ for } \alpha = 1, \ldots, N^2.$$

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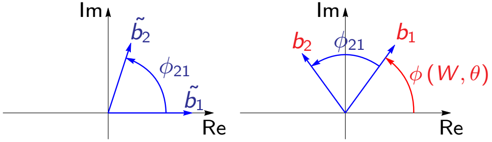
$\hookrightarrow$ Complete-experiment problem:

What are the minimal subsets of the observables $\mathcal{O}^{\alpha}$, which allow for the unique extraction of the amplitudes b_i up to one unknown overall phase $\phi(W, \theta)$?

- $*)$ Analysis operates on each bin in (W, θ) individually.
- $*)$ Consider idealized (academic) case without measurement uncertainty!



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$*)$ Expression $\mathcal{O}^{\alpha} = \boldsymbol{c}^{\alpha} \sum_{i,j=1}^{N} b_i^{*} \tilde{\Gamma}_{ij}^{\alpha} b_j$ can be 'inverted' (using the *completeness* of the $\tilde{\Gamma}$-matrices):

$$b_i^{*} b_j = \frac{1}{\tilde{N}} \sum_{\alpha=1}^{N^2} \left(\tilde{\Gamma}_{ij}^{\alpha} \right)^{*} \left(\frac{\mathcal{O}^{\alpha}}{\boldsymbol{c}^{\alpha}} \right) .$$

$\Rightarrow$ Determine the real- and imaginary parts of a 'minimal' set of $b_i^{*} b_j$
$\Rightarrow$ Obtain (quite large) over-complete set $\{\mathcal{O}^{\alpha}\}$ determined via the RHS

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$*$) Finding a generic solution for such problems, for arbitrary N, can be quite tough in the $\mathcal{O}^{\alpha}$-basis.
<u>However:</u> In the $b_i^{*} b_j$-basis, a general solution exists:

<u>Moravcsik's Theorem</u>!

Moravcsik's Theorem (modified form)

From [YW, P. Kroenert, F. Afzal, A. Thiel, Phys. Rev. C **102**, no.3, 034605 (2020)], based on [Moravcsik, J. Math. Phys. **26**, 211 (1985).]:

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'Geometrical (graphical) analog': Represent every amplitude $b_1, \ldots, b_N$ by a *point* and every product $b_j^* b_i$, or rel.-phase ϕ_{ij}, by a *line connecting points* 'i' and 'j'.

Furthermore: $\hookrightarrow$ Represent every $\mathrm{Re}\left[b_i^* b_j\right] \propto \cos \phi_{ij}$ by a *solid line*,

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  - all continuous ambiguities are resolved,
  - existence of *consistency relation* is ensured.
    $\hookrightarrow$ crucial for resolving discrete ambiguities

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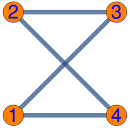

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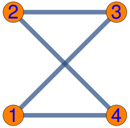

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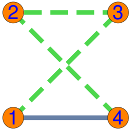
(ii) the graph has to have an *odd* number of dashed lines, as well as *any* number of solid lines:

- all discrete ambiguities are resolved.

E.g.:



is complete ✓

Example 1: pion photoproduction (formalism)

$*)$ Consider the reaction: $\vec{\gamma}\vec{N} \longrightarrow \pi\vec{N}$.

$\hookrightarrow$ Number of spin-amplitudes $N = \underbrace{2}_{\gamma} * \underbrace{2}_{N} * \underbrace{2}_{N} / \underbrace{2}_{\text{Parity}} = 4$.

E.g. CGLN amplitudes: $F_1(W,\theta),\ldots,F_4(W,\theta)$.

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$$b_1(W,\theta),\ b_2(W,\theta),\ b_3(W,\theta),\ b_4(W,\theta).$$

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$\hookrightarrow$ The $N^2 = 16$ polarization observables (or polarization asymmetries)

$$\mathcal{O} = \left[\left(\frac{d\sigma}{d\Omega}\right)^{(B_1, T_1, R_1)} - \left(\frac{d\sigma}{d\Omega}\right)^{(B_2, T_2, R_2)} \right],$$

take a particularly convenient form in the transversity basis.

Example 1: photoproduction (observables)

Observable	Bilinear form	Shape-class
$\sigma_0 = \frac{1}{2}\left(\lvert b_1\rvert^2 + \lvert b_2\rvert^2 + \lvert b_3\rvert^2 + \lvert b_4\rvert^2\right)$	$\frac{1}{2}\langle b\rvert \tilde{\Gamma}^1 \lvert b\rangle$	
$-\check{\Sigma} = \frac{1}{2}\left(\lvert b_1\rvert^2 + \lvert b_2\rvert^2 - \lvert b_3\rvert^2 - \lvert b_4\rvert^2\right)$	$\frac{1}{2}\langle b\rvert \tilde{\Gamma}^4 \lvert b\rangle$	$\mathcal{S} = \mathrm{D}$
$-\check{T} = \frac{1}{2}\left(-\lvert b_1\rvert^2 + \lvert b_2\rvert^2 + \lvert b_3\rvert^2 - \lvert b_4\rvert^2\right)$	$\frac{1}{2}\langle b\rvert \tilde{\Gamma}^{10} \lvert b\rangle$	
$\check{P} = \frac{1}{2}\left(-\lvert b_1\rvert^2 + \lvert b_2\rvert^2 - \lvert b_3\rvert^2 + \lvert b_4\rvert^2\right)$	$\frac{1}{2}\langle b\rvert \tilde{\Gamma}^{12} \lvert b\rangle$	
$\mathcal{O}^a_{1+} = \lvert b_1\rvert\lvert b_3\rvert \sin\phi_{13} + \lvert b_2\rvert\lvert b_4\rvert \sin\phi_{24} = \mathrm{Im}\left[b_3^* b_1 + b_4^* b_2\right] = -\check{G}$	$\frac{1}{2}\langle b\rvert \tilde{\Gamma}^3 \lvert b\rangle$	
$\mathcal{O}^a_{1-} = \lvert b_1\rvert\lvert b_3\rvert \sin\phi_{13} - \lvert b_2\rvert\lvert b_4\rvert \sin\phi_{24} = \mathrm{Im}\left[b_3^* b_1 - b_4^* b_2\right] = \check{F}$	$\frac{1}{2}\langle b\rvert \tilde{\Gamma}^{11} \lvert b\rangle$	$a = \mathcal{BT} = \mathrm{PR}$
$\mathcal{O}^a_{2+} = \lvert b_1\rvert\lvert b_3\rvert \cos\phi_{13} + \lvert b_2\rvert\lvert b_4\rvert \cos\phi_{24} = \mathrm{Re}\left[b_3^* b_1 + b_4^* b_2\right] = -\check{E}$	$\frac{1}{2}\langle b\rvert \tilde{\Gamma}^9 \lvert b\rangle$	
$\mathcal{O}^a_{2-} = \lvert b_1\rvert\lvert b_3\rvert \cos\phi_{13} - \lvert b_2\rvert\lvert b_4\rvert \cos\phi_{24} = \mathrm{Re}\left[b_3^* b_1 - b_4^* b_2\right] = \check{H}$	$\frac{1}{2}\langle b\rvert \tilde{\Gamma}^5 \lvert b\rangle$	
$\mathcal{O}^b_{1+} = \lvert b_1\rvert\lvert b_4\rvert \sin\phi_{14} + \lvert b_2\rvert\lvert b_3\rvert \sin\phi_{23} = \mathrm{Im}\left[b_4^* b_1 + b_3^* b_2\right] = \check{O}_{z'}$	$\frac{1}{2}\langle b\rvert \tilde{\Gamma}^7 \lvert b\rangle$	
$\mathcal{O}^b_{1-} = \lvert b_1\rvert\lvert b_4\rvert \sin\phi_{14} - \lvert b_2\rvert\lvert b_3\rvert \sin\phi_{23} = \mathrm{Im}\left[b_4^* b_1 - b_3^* b_2\right] = -\check{C}_{x'}$	$\frac{1}{2}\langle b\rvert \tilde{\Gamma}^{16} \lvert b\rangle$	$b = \mathcal{BR} = \mathrm{AD}$
$\mathcal{O}^b_{2+} = \lvert b_1\rvert\lvert b_4\rvert \cos\phi_{14} + \lvert b_2\rvert\lvert b_3\rvert \cos\phi_{23} = \mathrm{Re}\left[b_4^* b_1 + b_3^* b_2\right] = -\check{C}_{z'}$	$\frac{1}{2}\langle b\rvert \tilde{\Gamma}^2 \lvert b\rangle$	
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Example 1: photoproduction (further preliminaries)

$*$) Standard assumption: moduli are known from group $\mathcal{S}$ observables:

$$|b_1| = \frac{1}{2}\left(\sigma_0 - \check{\Sigma} + \check{T} - \check{P}\right), |b_2| = \frac{1}{2}\left(\sigma_0 - \check{\Sigma} - \check{T} + \check{P}\right),$$
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$\ast$) Define a basis of 'decoupled' observables $\tilde{\mathcal{O}}^n_{\nu\pm}$, which isolate the real- and imaginary parts of the bilinear products $b_j^* b_i$:

$$\tilde{\mathcal{O}}^n_{1\pm} := \frac{1}{2}\left(\mathcal{O}^n_{1+} \pm \mathcal{O}^n_{1-}\right), \; n = a, b, c,$$
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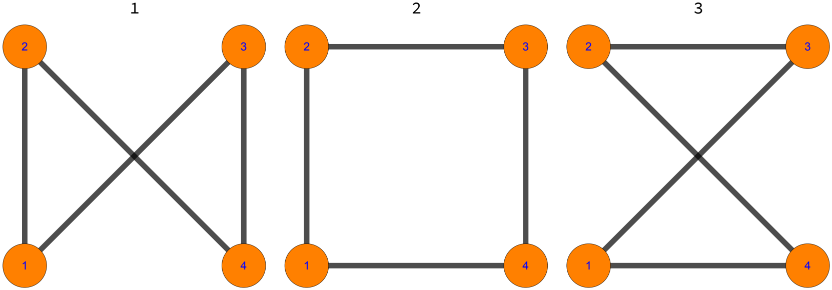
- Example:

$$\mathrm{Im}\left[b_4^\ast b_2\right] = |b_2|\,|b_4| \sin\phi_{24} = \tilde{\mathcal{O}}^a_{1-} = \tfrac{1}{2}\left(\mathcal{O}^a_{1+} - \mathcal{O}^a_{1-}\right) = \tfrac{1}{2}\left(-\check{G} - \check{F}\right).$$

Example 1: photoproduction (à la Moravcsik) I

$*)$ For $N = 4$ amplitudes, one gets $\frac{(N-1)!}{2} = \frac{3!}{2} = 3$ possible graph-topologies :



$\hookrightarrow$ *Each* of these topologies can be used as a *starting point* to derive complete sets of observables, by inserting *odd* numbers of dashed lines ...

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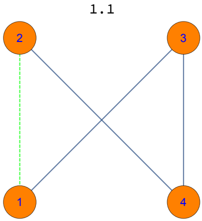

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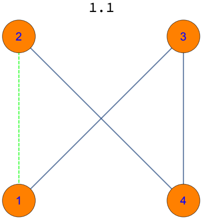

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↪ Map this result to observables (in $\tilde{\mathcal{O}}$- and $\mathcal{O}$-basis):

$$|b_1|\,|b_2|\sin\phi_{12} = \tilde{\mathcal{O}}^c_{1+} = (1/2)\left[\mathcal{O}^c_{1+} + \mathcal{O}^c_{1-}\right] = (1/2)\left[-\check{L}_{x'} - \check{T}_{z'}\right],$$

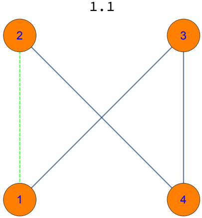
$$|b_2|\,|b_4|\cos\phi_{24} = \tilde{\mathcal{O}}^a_{2-} = (1/2)\left[\mathcal{O}^a_{2+} - \mathcal{O}^a_{2-}\right] = (1/2)\left[-\check{E} - \check{H}\right],$$

$$|b_3|\,|b_4|\cos\phi_{34} = \tilde{\mathcal{O}}^c_{2-} = (1/2)\left[\mathcal{O}^c_{2+} - \mathcal{O}^c_{2-}\right] = (1/2)\left[-\check{L}_{z'} - \check{T}_{x'}\right],$$

$$|b_1|\,|b_3|\cos\phi_{13} = \tilde{\mathcal{O}}^a_{2+} = (1/2)\left[\mathcal{O}^a_{2+} + \mathcal{O}^a_{2-}\right] = (1/2)\left[-\check{E} + \check{H}\right].$$

Example 1: photoproduction (à la Moravcsik) II

$*)$ Example (1.1) (fully complete):



$\longrightarrow \{\sin\phi_{12}, \cos\phi_{24}, \cos\phi_{34}, \cos\phi_{13}\}$

$\hookrightarrow$ Map this result to observables (in $\tilde{\mathcal{O}}$- and $\mathcal{O}$-basis):

$$|b_1|\,|b_2| \sin\phi_{12} = \tilde{\mathcal{O}}^{c}_{1+} = (1/2)\left[\mathcal{O}^{c}_{1+} + \mathcal{O}^{c}_{1-}\right] = (1/2)\left[-\check{L}_{x'} - \check{T}_{z'}\right],$$
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$\Rightarrow$ Extract the 'Moravcsik-complete' set (combined with $\{\sigma_0, \check{\Sigma}, \check{T}, \check{P}\}$):

$$\left\{\mathcal{O}^{a}_{2+}, \mathcal{O}^{a}_{2-}, \mathcal{O}^{c}_{1+}, \mathcal{O}^{c}_{1-}, \mathcal{O}^{c}_{2+}, \mathcal{O}^{c}_{2-}\right\} \equiv \left\{\check{E}, \check{H}, \check{L}_{x'}, \check{T}_{z'}, \check{L}_{z'}, \check{T}_{x'}\right\}.$$

Example 1: photoproduction (à la Moravcsik) III

$*)$ Similar procedure, applied to all the remaining relevant graphs, leads to 12 non-redundant 'Moravcsik-complete' sets for photoproduction (always in combination with $\left\{ \sigma_0, \check{\Sigma}, \check{T}, \check{P} \right\}$):

Set-Nr.	Observables			Set-Nr.	Observables		
1	$\mathcal{O}^a_{2\pm}$	$\mathcal{O}^c_{1\pm}$	$\mathcal{O}^c_{2\pm}$	7	$\mathcal{O}^b_{1\pm}$	$\mathcal{O}^c_{1\pm}$	$\mathcal{O}^c_{2\pm}$
2	$\mathcal{O}^a_{1\pm}$	$\mathcal{O}^a_{2\pm}$	$\mathcal{O}^c_{2\pm}$	8	$\mathcal{O}^b_{1\pm}$	$\mathcal{O}^b_{2\pm}$	$\mathcal{O}^c_{1\pm}$
3	$\mathcal{O}^a_{1\pm}$	$\mathcal{O}^c_{1\pm}$	$\mathcal{O}^c_{2\pm}$	9	$\mathcal{O}^a_{1\pm}$	$\mathcal{O}^a_{2\pm}$	$\mathcal{O}^b_{2\pm}$
4	$\mathcal{O}^a_{1\pm}$	$\mathcal{O}^a_{2\pm}$	$\mathcal{O}^c_{1\pm}$	10	$\mathcal{O}^a_{2\pm}$	$\mathcal{O}^b_{1\pm}$	$\mathcal{O}^b_{2\pm}$
5	$\mathcal{O}^b_{2\pm}$	$\mathcal{O}^c_{1\pm}$	$\mathcal{O}^c_{2\pm}$	11	$\mathcal{O}^a_{1\pm}$	$\mathcal{O}^a_{2\pm}$	$\mathcal{O}^b_{1\pm}$
6	$\mathcal{O}^b_{1\pm}$	$\mathcal{O}^b_{2\pm}$	$\mathcal{O}^c_{2\pm}$	12	$\mathcal{O}^a_{1\pm}$	$\mathcal{O}^b_{1\pm}$	$\mathcal{O}^b_{2\pm}$

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Observation: Moravcsik-complete sets contain 2 observables more than complete sets with an absolutely minimal amount of observables, i.e. with $2N = 8$ observables [Chiang & Tabakin (1997), Nakayama (2018)].

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$\leftrightarrow$ What happened?!

- Not fully clear yet. Possible method to reduce this mismatch $\rightarrow$ new graphs

Interlude: new 'directional' graphs

<u>Observation:</u> Moravcsik-complete sets tend to be slightly over-complete, i.e. to contain *more than* $2N$ observables, for problems with $N \geq 4$ amplitudes

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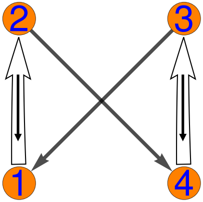
$*$) Example:



$\Leftrightarrow$ complete photoproduction-set ($2N = 8$ obs.'s in combination with 4 'diagonal' obs.'s $\left\{ \sigma_0, \check{\Sigma}, \check{T}, \check{P} \right\}$):
$\left\{ \mathcal{O}^a_{2+}, \mathcal{O}^a_{2-}, \mathcal{O}^c_{1+}, \mathcal{O}^c_{2-} \right\} = \left\{ \check{E}, \check{H}, \check{L}_{x'}, \check{T}_{x'} \right\}$.

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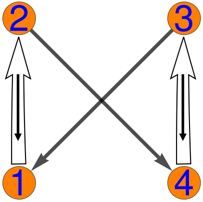

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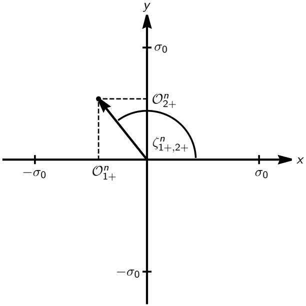
- Single-lined arrows: same as in Moravcsik's Theorem
- Double-lined arrows: 'crossed' selection $\mathcal{O}^c_{1\pm} \oplus \mathcal{O}^c_{2\pm}$
- 'Outer' direction $\Leftrightarrow$ 'directional convention' for consistency rel.: $\phi_{12} + \phi_{24} + \phi_{43} + \phi_{31} = 0$.
- Direction of 'inner' arrows: sign of 'ζ-angle' (cf. Figure on the right) in discrete-ambiguity formulas



$\hookrightarrow$ Confirm photoprod.; new sets for e^--production

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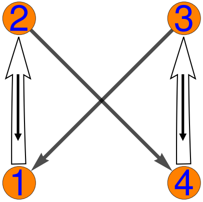

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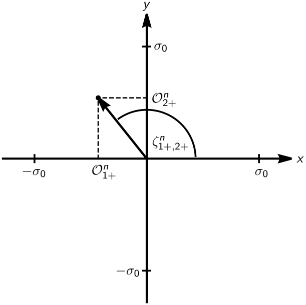

Now: e^--production with usual Moravcsik-Theorem ...

Example 2: electroproduction (formalism)

*) Reaction: $eN \longrightarrow e'\pi N \Rightarrow$ no. of amplitudes $= \underbrace{3}_{\gamma^*} * \underbrace{2}_{N} * \underbrace{2}_{N} / \underbrace{2}_{\text{Parity}} = 6$.

One has: 6 amplitudes $b_1, \ldots, b_6$ vs. 36 observables.

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One has: <u>6</u> amplitudes $b_1, \ldots, b_6$ vs. <u>36</u> observables.

Observable	Bilinear form	Shape-class
$R_T^{00} = \frac{1}{2} \left(\lvert b_1 \rvert^2 + \lvert b_2 \rvert^2 + \lvert b_3 \rvert^2 + \lvert b_4 \rvert^2 \right)$	$\frac{1}{2} \left< b \right\rvert \tilde{\Gamma}^1 \left\lvert b \right>$	
$- {}^c R_{TT}^{00} = \frac{1}{2} \left(\lvert b_1 \rvert^2 + \lvert b_2 \rvert^2 - \lvert b_3 \rvert^2 - \lvert b_4 \rvert^2 \right)$	$\frac{1}{2} \left< b \right\rvert \tilde{\Gamma}^4 \left\lvert b \right>$	D1
$-R_T^{0y} = \frac{1}{2} \left(- \lvert b_1 \rvert^2 + \lvert b_2 \rvert^2 + \lvert b_3 \rvert^2 - \lvert b_4 \rvert^2 \right)$	$\frac{1}{2} \left< b \right\rvert \tilde{\Gamma}^{10} \left\lvert b \right>$	
$-R_T^{y'0} = \frac{1}{2} \left(- \lvert b_1 \rvert^2 + \lvert b_2 \rvert^2 - \lvert b_3 \rvert^2 + \lvert b_4 \rvert^2 \right)$	$\frac{1}{2} \left< b \right\rvert \tilde{\Gamma}^{12} \left\lvert b \right>$	
$\mathcal{O}_{1+}^a = \lvert b_1 \rvert \lvert b_3 \rvert \sin \phi_{13} + \lvert b_2 \rvert \lvert b_4 \rvert \sin \phi_{24} = \mathrm{Im} \left[b_3^* b_1 + b_4^* b_2 \right] = - {}^s R_{TT}^{0z}$	$\frac{1}{2} \left< b \right\rvert \tilde{\Gamma}^3 \left\lvert b \right>$	
$\mathcal{O}_{1-}^a = \lvert b_1 \rvert \lvert b_3 \rvert \sin \phi_{13} - \lvert b_2 \rvert \lvert b_4 \rvert \sin \phi_{24} = \mathrm{Im} \left[b_3^* b_1 - b_4^* b_2 \right] = R_{TT'}^{0x}$	$\frac{1}{2} \left< b \right\rvert \tilde{\Gamma}^{11} \left\lvert b \right>$	$a = \mathrm{PR1}$
$\mathcal{O}_{2+}^a = \lvert b_1 \rvert \lvert b_3 \rvert \cos \phi_{13} + \lvert b_2 \rvert \lvert b_4 \rvert \cos \phi_{24} = \mathrm{Re} \left[b_3^* b_1 + b_4^* b_2 \right] = R_{TT'}^{0z}$	$\frac{1}{2} \left< b \right\rvert \tilde{\Gamma}^9 \left\lvert b \right>$	
$\mathcal{O}_{2-}^a = \lvert b_1 \rvert \lvert b_3 \rvert \cos \phi_{13} - \lvert b_2 \rvert \lvert b_4 \rvert \cos \phi_{24} = \mathrm{Re} \left[b_3^* b_1 - b_4^* b_2 \right] = {}^s R_{TT}^{0x}$	$\frac{1}{2} \left< b \right\rvert \tilde{\Gamma}^5 \left\lvert b \right>$	
$\mathcal{O}_{1+}^b = \lvert b_1 \rvert \lvert b_4 \rvert \sin \phi_{14} + \lvert b_2 \rvert \lvert b_3 \rvert \sin \phi_{23} = \mathrm{Im} \left[b_4^* b_1 + b_3^* b_2 \right] = - {}^s R_{TT}^{z'0}$	$\frac{1}{2} \left< b \right\rvert \tilde{\Gamma}^7 \left\lvert b \right>$	
$\mathcal{O}_{1-}^b = \lvert b_1 \rvert \lvert b_4 \rvert \sin \phi_{14} - \lvert b_2 \rvert \lvert b_3 \rvert \sin \phi_{23} = \mathrm{Im} \left[b_4^* b_1 - b_3^* b_2 \right] = -R_{TT'}^{x'0}$	$\frac{1}{2} \left< b \right\rvert \tilde{\Gamma}^{16} \left\lvert b \right>$	$b = \mathrm{AD1}$
$\mathcal{O}_{2+}^b = \lvert b_1 \rvert \lvert b_4 \rvert \cos \phi_{14} + \lvert b_2 \rvert \lvert b_3 \rvert \cos \phi_{23} = \mathrm{Re} \left[b_4^* b_1 + b_3^* b_2 \right] = R_{TT'}^{z'0}$	$\frac{1}{2} \left< b \right\rvert \tilde{\Gamma}^2 \left\lvert b \right>$	
$\mathcal{O}_{2-}^b = \lvert b_1 \rvert \lvert b_4 \rvert \cos \phi_{14} - \lvert b_2 \rvert \lvert b_3 \rvert \cos \phi_{23} = \mathrm{Re} \left[b_4^* b_1 - b_3^* b_2 \right] = - {}^s R_{TT}^{x'0}$	$\frac{1}{2} \left< b \right\rvert \tilde{\Gamma}^{14} \left\lvert b \right>$	

Example 2: electroproduction (formalism)

$\mathcal{O}^{c}_{1+} = \lvert b_1\rvert \lvert b_2\rvert \sin\phi_{12} + \lvert b_3\rvert \lvert b_4\rvert \sin\phi_{34} = \mathrm{Im}\left[b_2^* b_1 + b_4^* b_3\right] = -R_T^{x'z}$	$\frac{1}{2}\langle b\rvert \tilde{\Gamma}^{8}\lvert b\rangle$	
$\mathcal{O}^{c}_{1-} = \lvert b_1\rvert \lvert b_2\rvert \sin\phi_{12} - \lvert b_3\rvert \lvert b_4\rvert \sin\phi_{34} = \mathrm{Im}\left[b_2^* b_1 - b_4^* b_3\right] = R_T^{z'x}$	$\frac{1}{2}\langle b\rvert \tilde{\Gamma}^{13}\lvert b\rangle$	$c = \mathrm{PL1}$
$\mathcal{O}^{c}_{2+} = \lvert b_1\rvert \lvert b_2\rvert \cos\phi_{12} + \lvert b_3\rvert \lvert b_4\rvert \cos\phi_{34} = \mathrm{Re}\left[b_2^* b_1 + b_4^* b_3\right] = R_T^{z'z}$	$\frac{1}{2}\langle b\rvert \tilde{\Gamma}^{15}\lvert b\rangle$	
$\mathcal{O}^{c}_{2-} = \lvert b_1\rvert \lvert b_2\rvert \cos\phi_{12} - \lvert b_3\rvert \lvert b_4\rvert \cos\phi_{34} = \mathrm{Re}\left[b_2^* b_1 - b_4^* b_3\right] = R_T^{x'x}$	$\frac{1}{2}\langle b\rvert \tilde{\Gamma}^{6}\lvert b\rangle$	
$R_L^{00} = \lvert b_5\rvert^2 + \lvert b_6\rvert^2$	$\frac{1}{\sqrt{2}}\langle b\rvert \tilde{\Gamma}^{17}\lvert b\rangle$	$\mathrm{D2}$
$R_L^{0y} = \lvert b_5\rvert^2 - \lvert b_6\rvert^2$	$\frac{1}{\sqrt{2}}\langle b\rvert \tilde{\Gamma}^{18}\lvert b\rangle$	
$\mathcal{O}^{d}_{1} = 2\lvert b_5\rvert \lvert b_6\rvert \sin\phi_{56} = 2\mathrm{Im}\left[b_6^* b_5\right] = R_L^{z'x}$	$\frac{1}{\sqrt{2}}\langle b\rvert \tilde{\Gamma}^{20}\lvert b\rangle$	$d = \mathrm{AD2}$
$\mathcal{O}^{d}_{2} = 2\lvert b_5\rvert \lvert b_6\rvert \cos\phi_{56} = 2\mathrm{Re}\left[b_6^* b_5\right] = -R_L^{x'x}$	$\frac{1}{\sqrt{2}}\langle b\rvert \tilde{\Gamma}^{19}\lvert b\rangle$	
$\mathcal{O}^{e}_{1+} = \lvert b_3\rvert \lvert b_6\rvert \sin\phi_{36} + \lvert b_4\rvert \lvert b_5\rvert \sin\phi_{45} = \mathrm{Im}\left[b_6^* b_3 + b_5^* b_4\right] = -\,{}^{s}R_{LT'}^{00}$	$\frac{1}{2}\langle b\rvert \tilde{\Gamma}^{31}\lvert b\rangle$	
$\mathcal{O}^{e}_{1-} = \lvert b_3\rvert \lvert b_6\rvert \sin\phi_{36} - \lvert b_4\rvert \lvert b_5\rvert \sin\phi_{45} = \mathrm{Im}\left[b_6^* b_3 - b_5^* b_4\right] = {}^{s}R_{LT'}^{0y}$	$\frac{1}{2}\langle b\rvert \tilde{\Gamma}^{29}\lvert b\rangle$	$e = \mathrm{AD3}$
$\mathcal{O}^{e}_{2+} = \lvert b_3\rvert \lvert b_6\rvert \cos\phi_{36} + \lvert b_4\rvert \lvert b_5\rvert \cos\phi_{45} = \mathrm{Re}\left[b_6^* b_3 + b_5^* b_4\right] = {}^{c}R_{LT}^{00}$	$\frac{1}{2}\langle b\rvert \tilde{\Gamma}^{21}\lvert b\rangle$	
$\mathcal{O}^{e}_{2-} = \lvert b_3\rvert \lvert b_6\rvert \cos\phi_{36} - \lvert b_4\rvert \lvert b_5\rvert \cos\phi_{45} = \mathrm{Re}\left[b_6^* b_3 - b_5^* b_4\right] = -\,{}^{c}R_{LT}^{0y}$	$\frac{1}{2}\langle b\rvert \tilde{\Gamma}^{23}\lvert b\rangle$	
$\mathcal{O}^{f}_{1+} = \lvert b_1\rvert \lvert b_6\rvert \sin\phi_{16} + \lvert b_2\rvert \lvert b_5\rvert \sin\phi_{25} = \mathrm{Im}\left[b_6^* b_1 + b_5^* b_2\right] = -\,{}^{s}R_{LT}^{0z}$	$\frac{1}{2}\langle b\rvert \tilde{\Gamma}^{30}\lvert b\rangle$	
$\mathcal{O}^{f}_{1-} = \lvert b_1\rvert \lvert b_6\rvert \sin\phi_{16} - \lvert b_2\rvert \lvert b_5\rvert \sin\phi_{25} = \mathrm{Im}\left[b_6^* b_1 - b_5^* b_2\right] = {}^{c}R_{LT'}^{0x}$	$\frac{1}{2}\langle b\rvert \tilde{\Gamma}^{24}\lvert b\rangle$	$f = \mathrm{AD4}$
$\mathcal{O}^{f}_{2+} = \lvert b_1\rvert \lvert b_6\rvert \cos\phi_{16} + \lvert b_2\rvert \lvert b_5\rvert \cos\phi_{25} = \mathrm{Re}\left[b_6^* b_1 + b_5^* b_2\right] = {}^{c}R_{LT'}^{0z}$	$\frac{1}{2}\langle b\rvert \tilde{\Gamma}^{32}\lvert b\rangle$	
$\mathcal{O}^{f}_{2-} = \lvert b_1\rvert \lvert b_6\rvert \cos\phi_{16} - \lvert b_2\rvert \lvert b_5\rvert \cos\phi_{25} = \mathrm{Re}\left[b_6^* b_1 - b_5^* b_2\right] = {}^{s}R_{LT}^{0x}$	$\frac{1}{2}\langle b\rvert \tilde{\Gamma}^{22}\lvert b\rangle$	

Example 2: electroproduction (formalism)

$\mathcal{O}^{g}_{1+} = \lvert b_1 \rvert \lvert b_5 \rvert \sin\phi_{15} + \lvert b_2 \rvert \lvert b_6 \rvert \sin\phi_{26} = \mathrm{Im}\left[b_5^{*} b_1 + b_6^{*} b_2 \right] = -\,{}^{s}R^{z'0}_{LT}$	$\frac{1}{2} \langle b \rvert \tilde{\Gamma}^{33} \lvert b \rangle$	
$\mathcal{O}^{g}_{1-} = \lvert b_1 \rvert \lvert b_5 \rvert \sin\phi_{15} - \lvert b_2 \rvert \lvert b_6 \rvert \sin\phi_{26} = \mathrm{Im}\left[b_5^{*} b_1 - b_6^{*} b_2 \right] = -\,{}^{c}R^{x'0}_{LT'}$	$\frac{1}{2} \langle b \rvert \tilde{\Gamma}^{26} \lvert b \rangle$	$g = \mathrm{PR2}$
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Example 2: electroproduction (formalism)

$\mathcal{O}^{g}_{1+} = \lvert b_1 \rvert \lvert b_5 \rvert \sin\phi_{15} + \lvert b_2 \rvert \lvert b_6 \rvert \sin\phi_{26} = \mathrm{Im}\left[b_5^* b_1 + b_6^* b_2\right] = -\,{}^{s}R^{z'0}_{LT}$	$\frac{1}{2}\left\langle b \right\rvert \tilde{\Gamma}^{33} \left\lvert b \right\rangle$	
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$\hookrightarrow$ For $N = 6$: quite many observables/inteference terms. Possible algebraic dependencies among the observables are manifold and quite complicated!

$\Rightarrow$ It is quite tough to solve this problem 'by hand'.

However: Moravcsik's Theorem comes to the rescue, since its application is very systematic and quite easily automated!

Example 3: electroproduction (graph topologies)

∗) For $N = 6$, we have $(N-1)!/2 = 5!/2 = 60$ possible graph-topologies:

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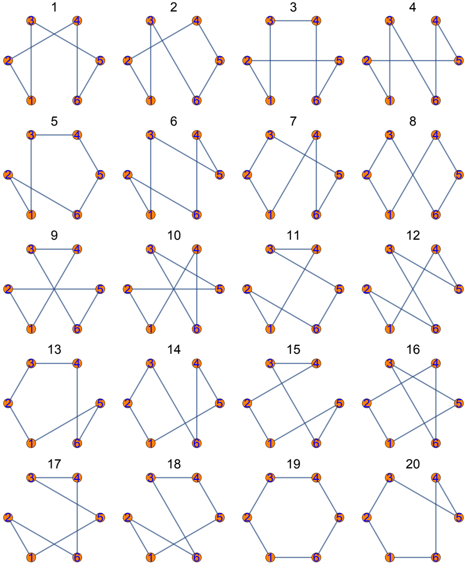
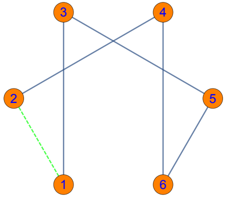


. . .

$*)$ Example (1.1) (fully complete):

1.1



$\longrightarrow \{\sin\phi_{12}, \cos\phi_{24}, \cos\phi_{46}, \cos\phi_{56}, \cos\phi_{35}, \cos\phi_{13}\}$

$\hookrightarrow$ In the same way as before, extract the 'Moravcsik-complete' set (combined with $\left\{R_T^{00},\ {}^cR_{TT}^{00}, R_T^{0y}, R_T^{y'0}, R_L^{00}, R_L^{0y}\right\}$):

$$\left\{\mathcal{O}_{2+}^a, \mathcal{O}_{2-}^a, \mathcal{O}_{1+}^c, \mathcal{O}_{1-}^c, \mathcal{O}_{2}^d, \mathcal{O}_{2+}^h, \mathcal{O}_{2-}^h\right\}$$
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Example 3: electroproduction (fully complete graphs)

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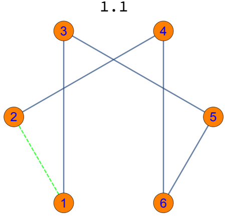

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*) In total, we obtain for the first time (!):

- 64 non-redundant Moravcsik-complete sets composed of 13 observables
  ↪ Only one observable more than the minimal number of $2N = 12$ observables!

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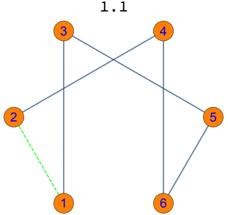

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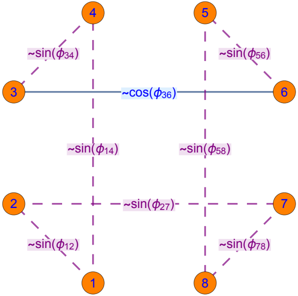
$\hookrightarrow$ What about problems with large numbers of amplitudes (i.e. $N > 6$)?

Cases with larger numbers of $N > 6$ amplitudes

Two-meson photoproduction

$\ast$) 8 amplitudes vs. 64 observables

$\ast$) Typical complete graph:



[P. Kroenert et al. (2021)]

$\ast$) [Phys. Rev. C **103**, 1, 014607 (2021)]

$\rightarrow$ see also next talk by P. Kroenert

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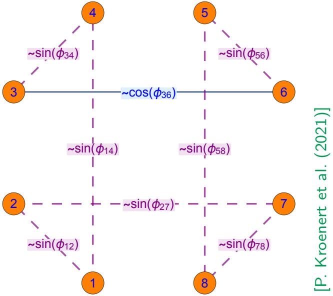

∗) [Phys. Rev. C **103**, 1, 014607 (2021)]
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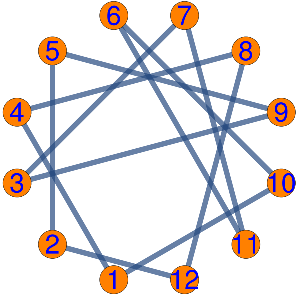
Vector-meson photoproduction

∗) 12 amp.'s vs. 244 observables

∗) Example for start-topology:



∗) No. of start-topologies:

$$\frac{(N-1)!}{2} = 19\,958\,400$$

⇒ Numerically very demanding problem!

. . .

Conclusion and Outlook

For a reaction involving particles with spin:
N (transversity) amplitudes b_i vs. N^2 pol.-observables $\check{\mathcal{O}}^{\alpha} \propto \langle b | \tilde{\Gamma}^{\alpha} | b \rangle$.

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*) (Re-) derived a modified version of Moravcsik's Theorem

$\hookrightarrow$ Useful solution-tool for *any* number of amplitudes N

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$\hookrightarrow$ Moravcsik-complete sets with 10 obs.'s vs. minimal sets with $2N = 8$ obs.'s

*) Shown the application to the (tougher!) problem of electroproduction

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*) Further possible directions of research:

- Two-meson photoproduction [P. Kroenert et al. (2021)] → next talk!
- First ever treatment of vector-meson photoproduction ($N = 12$ amplitudes: tough!!)
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  ⇒ new 'directional' graphs [YW, [arXiv:2106.00486 [nucl-th]] (2021)]

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We have:

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Thank You for your attention!

References

[Moravcsik (1985)]: M. J. Moravcsik, J. Math. Phys. **26**, 211 (1985).

[W. K. A. T. (2020)]: YW, P. Kroenert, F. Afzal and A. Thiel, Phys. Rev. C **102**, no.3, 034605 (2020) [arXiv:2004.14483 [nucl-th]].

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[YW (2021)]: YW, [arXiv:2106.00486 [nucl-th]] (2021).

Additional Slides

Discrete ambiguities

'Cosine-type' ambiguities:

The real part

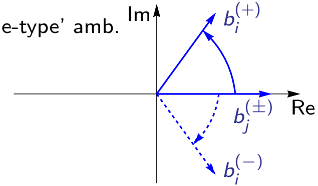
$$\begin{aligned} \mathrm{Re}\left[b_j^* b_i\right] &= |b_i|\,|b_j|\,\mathrm{Re}\left[e^{i\phi_{ij}}\right] \\ &= |b_i|\,|b_j| \cos\phi_{ij}, \end{aligned}$$

fixes the relative phase ϕ_{ij} up to the discrete ambiguity:

$$\phi_{ij} \longrightarrow \phi_{ij}^{\pm} = \begin{cases} +\alpha_{ij}, \\ -\alpha_{ij}, \end{cases}$$

with a unique $\alpha_{ij} \in [0, \pi]$.

'Cosine-type' amb.



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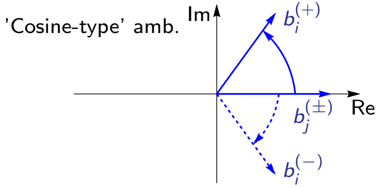

'Sine-type' ambiguities:

The imaginary part

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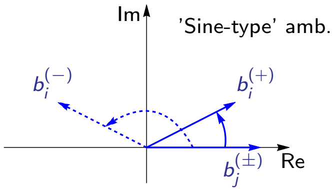

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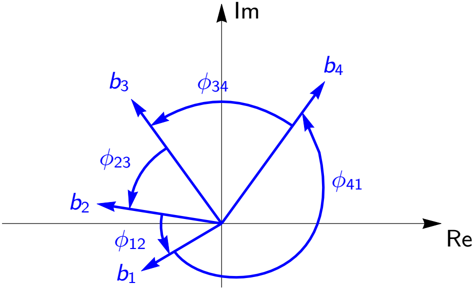
$\hookrightarrow$ Discrete ambiguities for a subset of real-and imaginary parts of bilinear products $b_j^* b_i$, defined by N amplitudes $\{b_i, i = 1, \ldots, N\}$, are 'direct (or Kronecker-) products' of these fundamental discrete ambiguities.

$\Rightarrow$ Such ambiguities turn up time and again in the discussion of complete experiments! Is there help? Yes! $\rightarrow$ Consistency Relations

Consistency relations

$*$) Consider amplitude-arrangement in the complex plane (e.g.: $N = 4$):

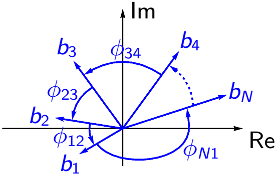


$*$) Natural constraint satisfied by this constellation: <u>consistency relation</u>

$$\phi_{12} + \phi_{23} + \phi_{34} + \phi_{41} = 0 \text{ (up to add. of multiples of } 2\pi\text{).}$$

Consistency relations

∗) Consider amplitude-arrangement in the complex plane (arbitrary N):



∗) Fundamental consistency relation for a problem with N amplitudes:

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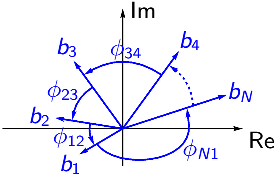

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∗) Consistency relations may look trivial, but they are very important for the resolution of discrete ambiguities: in case all the possible cases

$$\phi_{12}^{\pm} + \phi_{23}^{\pm} + \ldots + \phi_{N-1,N}^{\pm} + \phi_{N1}^{\pm} = 0,$$

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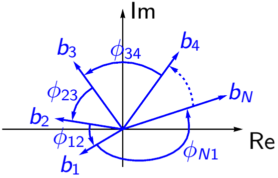

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↪ Moravcsik's Theorem is a systematic study of all cases where such non-degeneracies are obtained, in the $b_j^* b_i$-basis.