Predicting the production of loosely bound di-baryons in bottomonium decays

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(Based on a paper in preparation)

Introduction

Introduction

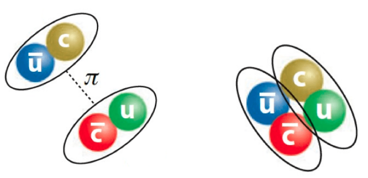
I will present new results from a **phenomenological study** on loosely-bound molecules formation in Υ decays and e^+e^- collisions at $\sqrt{s} = 10.58$ GeV, focusing on:

- **deuteron** (pn bound state)
- **H-dibaryon** ($\Lambda\Lambda$ bound state)

Description of (anti)deuteron production in HEP processes → **coalescence model**.

Understanding deuteron production has several fundamental consequences both in particle and astroparticle physics. Recently, **more refined models** have been suggested.

Deuteron as a benchmark for hadronic molecules



Hadronic molecules are composite systems made of hadrons bound together via strong interaction → fundamental to understand strong interaction beyond the quark model.

- Hadronic molecules share **universal properties** [1]
- Deuteron is the best known hadronic molecule → **ideal benchmark** for characterizing the main features of loosely-bound molecules in the simplest framework

Goal

Understanding deuteron formation gives important information about other molecules less known → **make predictions about H-dibaryon production rate**.

[1] Feng-Kun Guo et al., "Hadronic molecules", Reviews of Modern Physics 90 (2018).

Phenomenological models regarding $\bar{d}$ production

Simple coalescence model [2][3]

$$P(pn \to dX \mid k) = \Theta(k_{cut} - k) = \begin{cases} 1 & k < k_{cut} \\ 0 & k > k_{cut} \end{cases}$$

Cross section based model [4]

$$P(pn \to dX \mid k) = \frac{\sigma_{pn \to dX}(k)}{\sigma_0}$$

Advanced coalescence model [5]

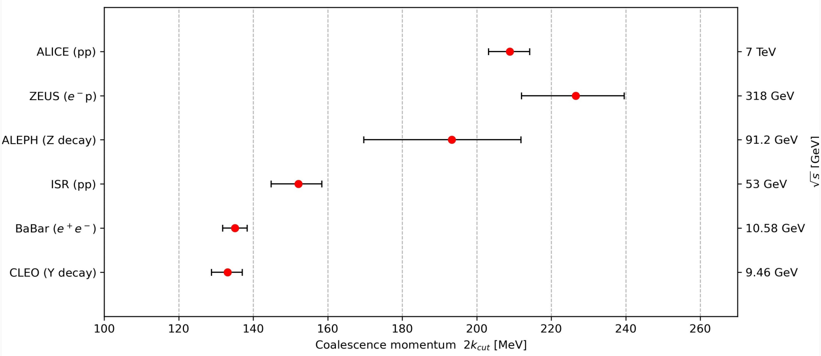
$$P(pn \to dX \mid k) = 3\left(\zeta_1(\sigma)\Delta e^{-k^2 d_1^2} + \zeta_2(\sigma)(1-\Delta)e^{-k^2 d_2^2}\right)$$

Common features:

- **single free parameter** (k_{cut}, σ_0, σ)
- $\bar{d}$ production depends on a **formation probability** function of $k = \frac{1}{2}(p_p - p_n)|_{CoM}$
- they must be fitted on experimental data to be predictive

[2] G. Gustafson and J. Häkkinen, Z. Phys. C Particles and Fields 61, (1994),
[3] P. Artoisenet and E. Braaten, Physical Review D 83 (2011),
[4] L. A. Dal and A. R. Raklev, Phys. Rev. D 91, 123536 (2015),
[5] M. Kachelriess et al., J. Eur. Phys. J. A 56, 4 (2020).

Phenomenological models regarding $\bar{d}$ production



- The simple coalescence model reveals a strong **process dependence**
- The more refined models aim at solving this problem. Do they succeed?

The open problem

Available experimental measurements in the bottomonium region:

Experiment	$\mathcal{B}(\Upsilon(1S) \to \bar{d}X)$	$\mathcal{B}(\Upsilon(2S) \to \bar{d}X)$	$\mathcal{B}(\Upsilon(3S) \to \bar{d}X)$	$\frac{\sigma(e^+e^- \to \bar{d}X)}{\sigma(e^+e^- \to hadrons)}$
BaBar [6]	$(2.81 \pm 0.49^{+0.20}_{-0.24}) \times 10^{-5}$	$(2.64 \pm 0.11^{+0.26}_{-0.21}) \times 10^{-5}$	$(2.33 \pm 0.15^{+0.31}_{-0.28}) \times 10^{-5}$	$(3.01 \pm 0.13^{+0.37}_{-0.31}) \times 10^{-6}$
CLEO [7]	$(2.86 \pm 0.19 \pm 0.21) \times 10^{-5}$			$< 10^{-5}$

Our knowledge of $\bar{d}$ production in the bottomonium region is still **incomplete**:

	$\mathcal{B}[\Upsilon(1S) \to \bar{d}X]$	$\mathcal{B}[\Upsilon(2S) \to \bar{d}X]$	$\mathcal{B}[\Upsilon(3S) \to \bar{d}X]$	$\frac{\sigma(e^+e^- \to \bar{d}X)}{\sigma(e^+e^- \to hadrons)}$
Simple coalescence model	✓	✗	✗	✓
Cross section based model	✓	✗	✗	✓
Advanced coalescence model	✗	✗	✗	✗

Goal

The **first comprehensive study** of $\bar{d}$ production in the bottomonium region.

[6] J. P. Lees et al. (BaBar Collab.), Phys. Rev. D 89, 111102 (2014), [7] D. M. Asner et al. (CLEO Collab.), Phys. Rev. D 75, 012009 (2007).

Monte Carlo generators

Monte Carlo generators

To get the two-nucleon spectra **we have to rely on MC simulations**.

Both Υ decays and $e^+e^- \to q\bar{q}$ events are generated using `Pythia8`.

`Pythia8`:

- simulates the **fragmentation** $\to$ explained and parametrized by the "Lund string model"
- comes with a **default version** $\to$ parameters optimized to simulate the majority of processes

We know that:

- Several papers suggest to tune the MC specifically toward $\bar{d}$ production
- `Pythia8` default version cannot well describe the $\Upsilon \to ggg$ decays

Tuning `Pythia8` toward $\bar{d}$ production and Υ decays $\to$ **Grid tuning.**

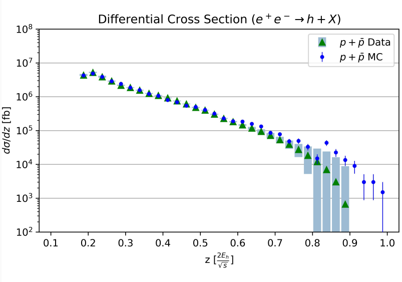
Tuning of Pythia8

We exploited all the relevant experimental data to perform the grid tuning:

- the $ggg/q\bar{q}$ **enhancement** of $\bar{p}$, p, Λ, ϕ as a function of the scaled momentum (p/E_{beam})
  - *R. A. Briere et al. (CLEO Collab.), Phys. Rev. D 76, 012005 (2007)*
- the **single hadron differential cross sections**
  - *R. Seidl et al. (Belle Collab.), Phys. Rev. D 92, 092007 (2015)*
  - *R. Seidl et al. (Belle Collab.), Phys. Rev. D 101, 092004 (2020)*
- the **total cross sections** of hyperons and charmed baryons
  - *M. Niiyama et al. (Belle Collab.), Phys. Rev. D 97, 072005 (2018)*

`Pythia8` **parameters optimized** in the grid tuning:

```
StringZ:aLund=0.22
StringZ:bLund=1.35
StringZ:aExtraDiquark=1.05
StringPT:sigma=0.238
StringFlav:probQQtoQ=0.091
StringFlav:probStoUD=0.32
StringFlav:probSQtoQQ=1.0
```



Study on antideuteron production

The antideuteron inclusive production

Given the programs and the tuned MC $\rightarrow$ **the models are fitted on available data.**

Experiment	$\mathcal{B}(\Upsilon(1S) \rightarrow \bar{d}X)$	$\mathcal{B}(\Upsilon(2S) \rightarrow \bar{d}X)$	$\mathcal{B}(\Upsilon(3S) \rightarrow \bar{d}X)$	$\frac{\sigma(e^+e^- \rightarrow \bar{d}X)}{\sigma(e^+e^- \rightarrow hadrons)}$
BaBar [6]	$(2.81 \pm 0.49^{+0.20}_{-0.24}) \times 10^{-5}$	$(2.64 \pm 0.11^{+0.26}_{-0.21}) \times 10^{-5}$	$(2.33 \pm 0.15^{+0.31}_{-0.28}) \times 10^{-5}$	$(3.01 \pm 0.13^{+0.37}_{-0.31}) \times 10^{-6}$
CLEO [7]	$(2.86 \pm 0.19 \pm 0.21) \times 10^{-5}$			$< 10^{-5}$

We have generated a **MC data sample** made by:

- 5×10^7 events of $\Upsilon(1S)$, $\Upsilon(2S)$, $\Upsilon(3S)$ decays
- 10^8 events of $e^+e^- \rightarrow q\bar{q}$ at $\sqrt{s} = 10.58$ GeV

We required that $\bar{p}$ and $\bar{n}$ must be close "enough" in coordinate space,

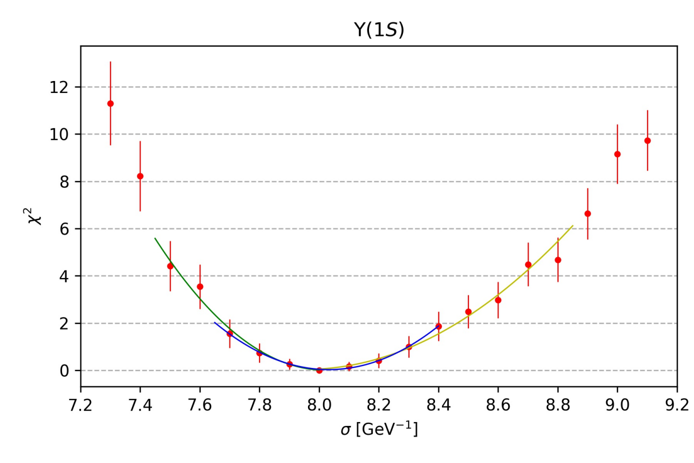
- **including** those directly produced or produced in strong decay, *e.g.* from Δ decay
- **removing** those produced in weak decays, *e.g.* from Λ, $\Sigma^{\pm}$, Λ_c^+, Ξ_c^+, Ξ_c^0 decays

[6] J. P. Lees et al. (BaBar Collab.), Phys. Rev. D 89, 111102 (2014).

[7] D. M. Asner et al. (CLEO Collab.), Phys. Rev. D 75, 012009 (2007).

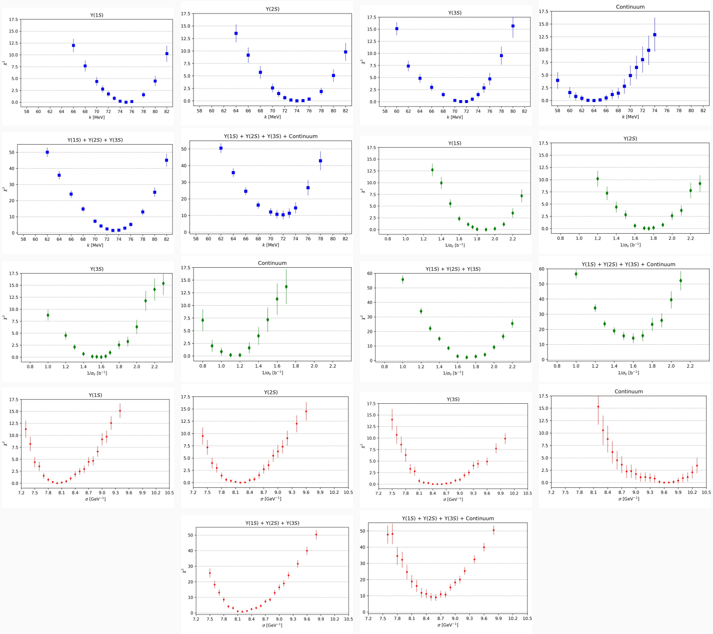
The scan procedure

We performed a **scan** of the free parameter followed by a χ^2 minimization.



Best-fit value and associated error of the free parameter $\rightarrow$ fits and interpolations.

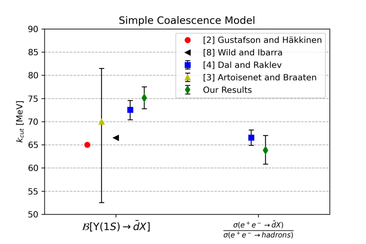
The scan procedure

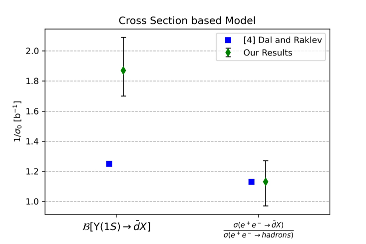


Best-fit free parameters

First extraction

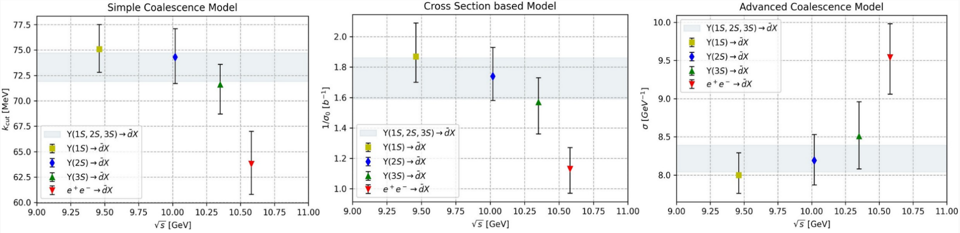
Model	$\Upsilon(1S) \to \bar{d}X$	$\Upsilon(2S) \to \bar{d}X$	$\Upsilon(3S) \to \bar{d}X$	$e^+e^- \to \bar{d}X$
Simple coalescence (k_{cut})	$75.1^{+2.4}_{-2.3}$ MeV	$74.3^{+2.8}_{-2.6}$ MeV	$71.6^{+2.0}_{-2.9}$ MeV	$63.8^{+3.2}_{-3.0}$ MeV
	$73.4^{+1.3}_{-1.5}$ MeV			
Cross section based ($\frac{1}{\sigma_0}$)	$1.87^{+0.22}_{-0.17}$ b^{-1}	$1.74^{+0.19}_{-0.16}$ b^{-1}	$1.57^{+0.16}_{-0.21}$ b^{-1}	$1.13^{+0.14}_{-0.16}$ b^{-1}
	$1.72^{+0.14}_{-0.13}$ b^{-1}			
Advanced coalescence (σ)	$8.00^{+0.29}_{-0.24}$ GeV^{-1}	$8.19^{+0.34}_{-0.32}$ GeV^{-1}	$8.51^{+0.45}_{-0.43}$ GeV^{-1}	$9.54^{+0.48}_{-0.44}$ GeV^{-1}
	$8.18^{+0.21}_{-0.14}$ GeV^{-1}			





[2] G. Gustafson and J. Häkkinen, Z. Phys. C Particles and Fields 61, (1994),
[4] L. A. Dal and A. R. Raklev, Phys. Rev. D 91, 123536 (2015),
[3] P. Artoisenet and E. Braaten, Physical Review D 83 (2011),
[8] A. Ibarra and S. Wild, J. Cosmol. Astropart. Phys. 21 (2013).

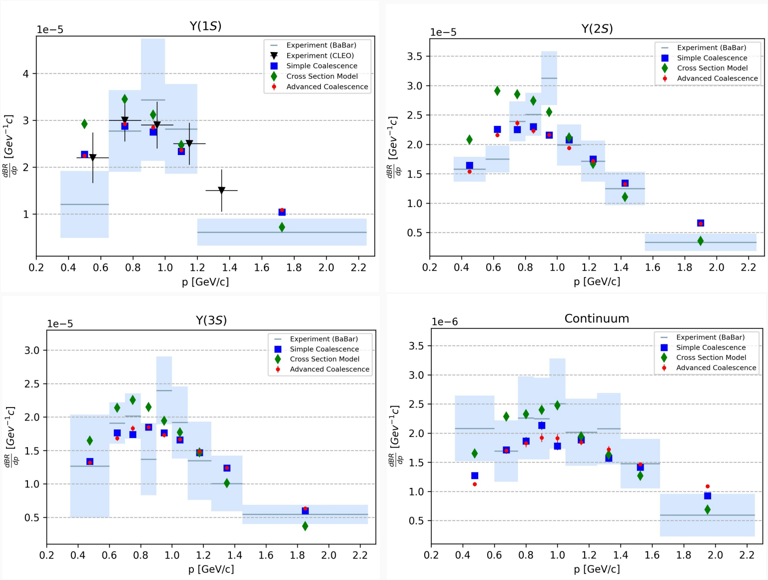
Best-fit free parameters



The three models show a similar behaviour → no clear winner,

- they can consistently describe Υ decays
- they cannot include the e^+e^- process in a unified treatment
- the "factorization hypothesis" do not hold → the initial parton state affects $\bar{d}$ formation

Antideuteron momentum spectrum



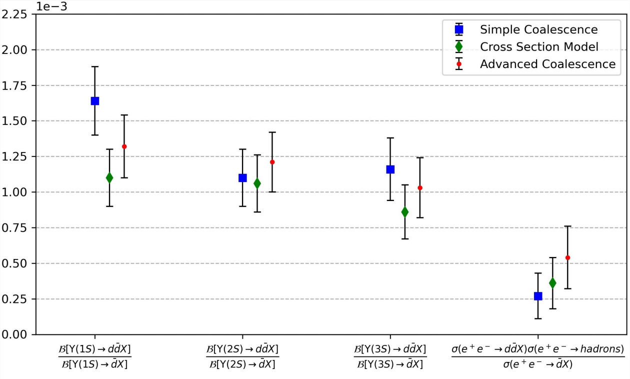
→ **No reasons** to consider models more refined than the simple coalescence one.

The deuteron-antideuteron associated production

Models with predictive power $\rightarrow$ **first prediction of $d\bar{d}$ associated production rate.**

- $d\bar{d}$ production $\rightarrow$ **sensitive to effects beyond the coalescence process**
- the CLEO collab. made a first rough estimation [3] $\rightarrow$ $\frac{\mathcal{B}(\Upsilon(1S)\rightarrow d\bar{d}X)}{\mathcal{B}(\Upsilon(1S)\rightarrow \bar{d}X)} \approx (9 \pm 6) \times 10^{-3}$

Predictions:



[3] P. Artoisenet and E. Braaten, Physical Review D 83 (2011).

Predictions on H-dibaryon

The H-dibaryon

- H-dibaryon was proposed in 1977 as a $\Lambda\Lambda$ long lived deeply-bound state with $B_H \approx 80$ MeV
- "Nagara event" [9] $\rightarrow$ excluded a deeply-bound H-dibaryon: $B_H < 7.66$ MeV
- **H-dibaryon remains a puzzle**: several LQCD predictions but no experimental signals observed

The Belle collab. has recently reported an analysis based on 10^8 $\Upsilon(1S, 2S)$ decays [10] and **they set a 90% C.L. upper limit on H-dibaryon production rate**:

$2m_\Lambda - m_H$ [MeV]	2	6	10	14	18	22	26	30
$\mathcal{B}(\Upsilon(1S, 2S) \rightarrow HX)\mathcal{B}(H \rightarrow \Lambda p \pi^-)(\times 10^{-7})$	15.0	9.7	7.1	6.3	1.5	5.2	1.7	4.6

How strong are these limits? **No predictions are available...**

(Note: see B. Scavino's talk at HADRON2021 on Belle II perspectives on deeply-bound dibaryons)

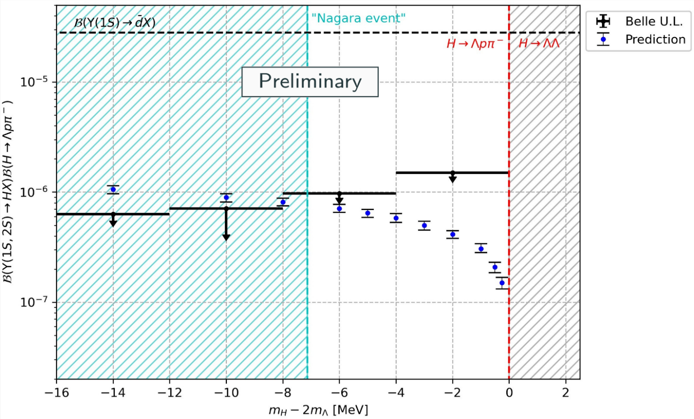
[9] H. Takahashi et al., Phys. Rev. Lett. 87 (2001), [10] B. H. Kim et al. (Belle collab.), Phys. Rev. Lett. 110 (2013).

Our predictions vs Belle upper limits

We extended the simple coalescence model to H-dibaryon to make **novel predictions**.

- for deuteron: $(2k_{cut})^3 \approx \frac{36}{\sqrt{\pi}}\sigma^{-2}\sqrt{m_p B_d} \quad \rightarrow \quad \sigma = 3.36^{+0.09}_{-0.10}$ fm
- for H-dibaryon: $(2k_{cut})^3 \approx \frac{45}{\sqrt{\pi}}\sigma^{-2}\sqrt{m_\Lambda B_H}, \quad$ if $\quad B_H \lesssim 1/(m_\Lambda \sigma^2) = 3.2$ MeV

Fixing $\sigma = 3.36$ fm:



Systematic errors

Systematic error on MC generator:

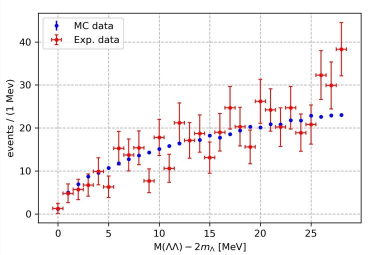
- our grid tuning includes data about Λ enhancement
- additional information about $\Lambda\Lambda$ spectrum from Belle collab. [10]

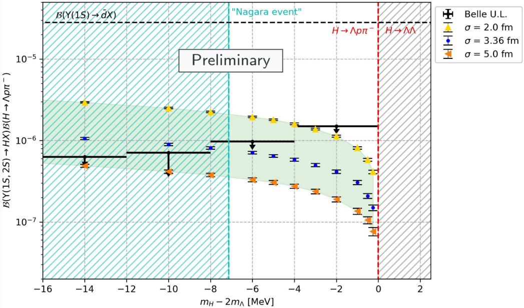
$\rightarrow$ expected to be **small**.

Systematic error on σ:

- the H-dibaryon production rate depends strongly on σ value
- we fixed $\sigma = 3.36$ fm, but it is constrained to [2, 5] fm

$\rightarrow$ expected to be **dominant**.





Belle U.L. is not strong enough $\rightarrow$ a higher statistical analysis at Belle II is suggested.

Conclusions

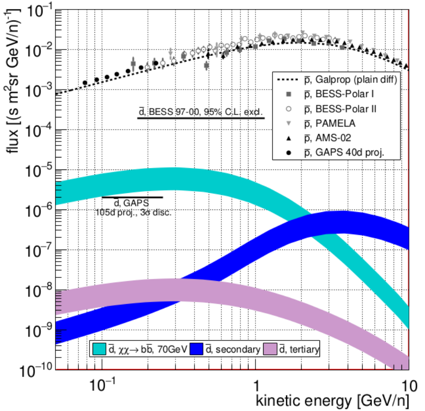
Conclusions

Results:

- First comprehensive study of $\bar{d}$ production in the bottomonium region
  - a unified treatment in the bottomonium region is not possible
  - all the models show a dependence from the initial partonic state
  - the simple coalescence model is still the best one
- We report the first prediction of $d\bar{d}$ production rate
- We report the first prediction of H-dibaryon production rate
- The current upper limits are not strong enough $\rightarrow$ future analysis at Belle II

Thank you

Antideuteron in Astrophysics

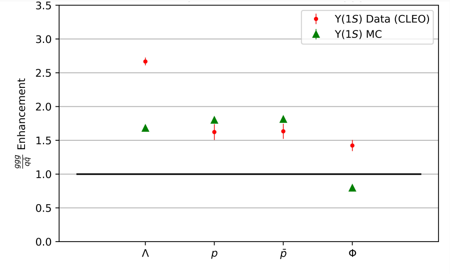


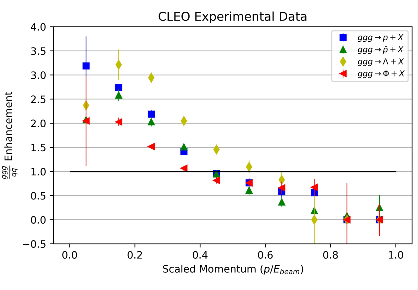
$\bar{d}$ → ultra-low background detection channel for **dark matter indirect searches** [1].

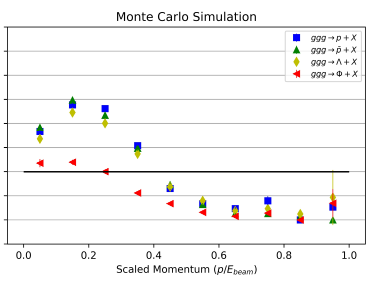
- Uncertainties in the $\bar{d}$ flux bounds from DM models → the antideuteron Formation Model
- Understanding $\bar{d}$ production at colliders → **reducing the uncertainty on flux predictions**

[1] F. Donato, N. Fornengo, P. Salati, Phys. Rev. D62, 043003 (2000).

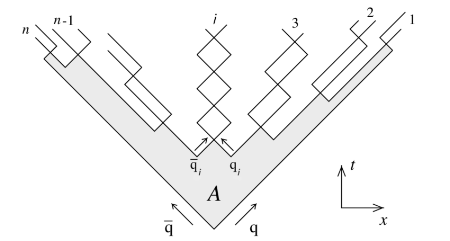
The Tuning Strategy







Tuning of Pythia8



Tunneling probability

The probability to generate the $q_i\bar{q}_i$ pairs that lead to string break-ups is given by:

$$\text{Prob.} \propto \exp\Big(-\frac{\pi m^2}{k}\Big) + \exp\Big(-\frac{\pi p_\perp^2}{k}\Big),$$

where k is the string tension, while $p_\perp$ and m are assumed the same for the $q_i\bar{q}_i$ pair.

$\mathrm{SU(3)_F}$ symmetry is unbroken $\rightarrow u\bar{u} : d\bar{d} : s\bar{s} = 1 : 1 : \gamma_s$, with $\gamma_s = 0.3$ by default.

Tuning of Pythia8

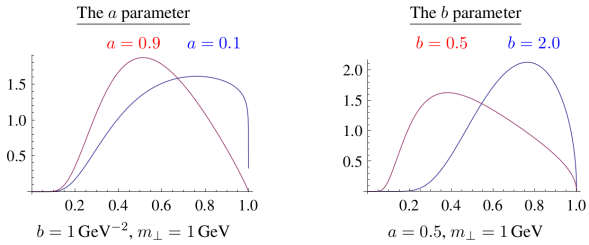
Lund symmetric fragmentation function

The longitudinal component of the energy carried by a hadrons formed in the string-breaking process **string** $\rightarrow$ **hadron + string'** is:

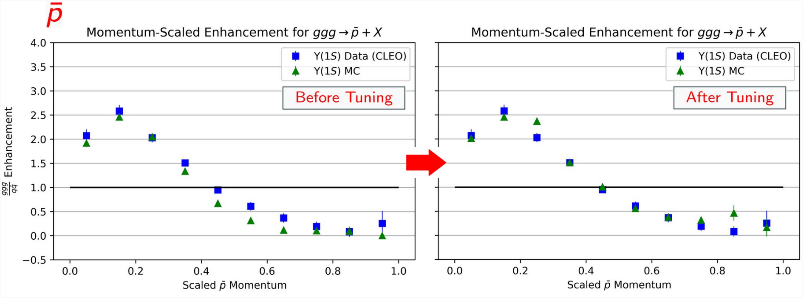
$$f(z) \propto \frac{(1-z)^a}{z} \exp\left(\frac{-bm_\perp^2}{z}\right),$$

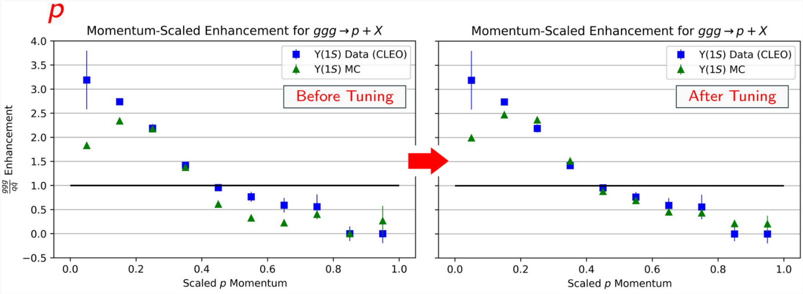
where z is the energy carried by the (ij) hadron, $m_\perp^2 = m_{had}^2 + p_{\perp,had}^2$ is the transverse mass of the produced (ij) hadron and a and b are free parameters.

- large $a \rightarrow$ suppressed hard region, $z \rightarrow 1$ and $f(z) \approx (1/z)\exp(-bm_\perp^2/z)$
- large $b \rightarrow$ suppressed soft region, $z \rightarrow 0$ and $f(z) \approx (1-z)^a$



Default vs Tuning settings





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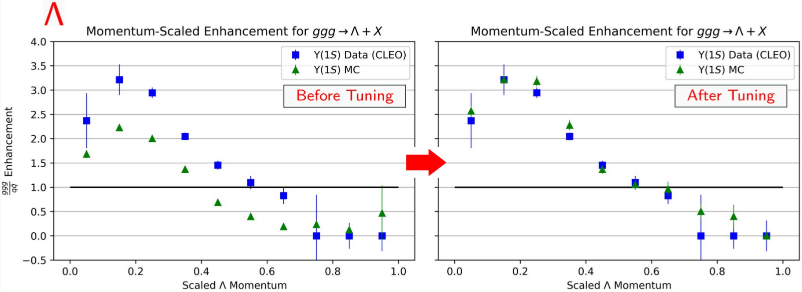
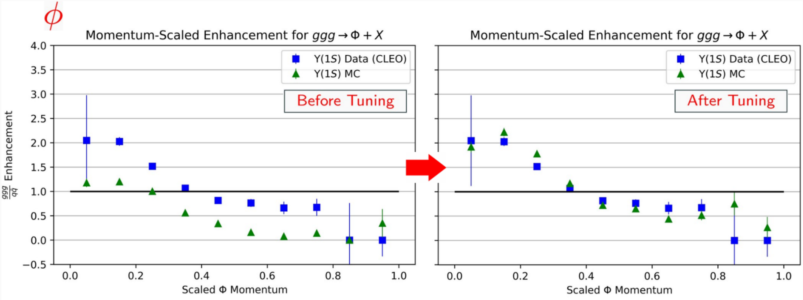




The Simple Coalescence Model

It is a **phenomenological model** which harks back to the 1960s [3][4]. While slightly modified over the years, it has been tested for decades and it is still state-of-the-art.

The invariant production rate

In coalescence processes, light nuclei formation is generally described by

$$E_A \frac{d^3 N_A}{dp_A^3} = B_A \left(E_p \frac{d^3 N_p}{dp_p^3} \right)^Z \left(E_n \frac{d^3 N_n}{dp_n^3} \right)^N . \tag{1}$$

In e^+e^- collisions, one usually imposes the **coalescence condition** in momentum space, requiring that the relative momentum of nucleons in their CoM system is smaller than some critical value p_0, which is the free parameter of the model.

In the limit of isotropic and equal proton and neutron yields, the so-called **coalescence momentum** p_0 is related to B_A via

$$B_A = A \left(\frac{4\pi}{3} \frac{p_0^3}{m_N} \right)^{A-1} , \tag{2}$$

where m_N denotes the nucleon mass.

[3] S. T. Butler and C. A. Pearson, Phys. Rev. 129, 836 (1963)
[4] A. Schwarzschild and C. Zupancic, Phys. Rev. 129, 854 (1963).

The Simple Coalescence Model

Focusing on deuteron formation [5],

$$E_d \frac{d^3 N_d}{dp_d^3} = B_d E_p \frac{d^3 N_p}{dp_p^3} E_n \frac{d^3 N_n}{dp_n^3} = B_d E \frac{m_p}{8} \frac{d^6 N_d}{d^3 p d^3 k}\bigg|_{\bar{k}=\bar{0}}, \tag{3}$$

where

$$\bar{k} = \frac{\bar{p}_p - \bar{p}_n}{2}\bigg|_{pn-CoM} \tag{4}$$

A deuteron is formed if the nucleons of the *pn*-pair have $|\bar{k}| < k_{cut}$, and

- $B_d \frac{m_p}{8} = \frac{4\pi}{3} k_{cut}^3$ and $\bar{p}_0 = 2\bar{k}_{cut}$,
- The deuteron spectrum scales roughly as k_{cut}^3,
- Factorised approximation $\rightarrow$ the momentum correlations between nucleons are not included.

The formation probability

Given a proton and a neutron, their probability to form a deuteron is equal to

$$P(pn \rightarrow d\gamma|k) = \Theta(k_{cut} - k) = \begin{cases} 1 & k < k_{cut}, \\ 0 & k > k_{cut}. \end{cases} \tag{5}$$

Where k_{cut} (or $p_0 = 2k_{cut}$) is the free parameter of the model.

[5] G. Gustafson, and J. Häkkinen, Z. Phys. C Particles and Fields 61, 683–687 (1994).

The Cross Section based Model

The antideuteron formation is described as a **probabilistic process** [6].

We expect this probability to be proportional to the **cross section** for $\bar{p}\bar{n} \to \bar{d}X$.

In this model, antideuterons can be produced at values of k well into the GeV range.

- For low values of k, the relevant process is still the $\bar{p}\bar{n} \to \bar{d}\gamma$ process,
- For CoM energies above the pion threshold, $\bar{p}\bar{n} \to \bar{d}(N\pi)^0$ processes dominate,
- At these energies, deuteron are actually more efficiently produced from $\bar{p}\bar{p}$ and $\bar{n}\bar{n}$ pairs.

The formation probability

Given a $\bar{p}\bar{n}$, $\bar{p}\bar{p}$ or $\bar{n}\bar{n}$ pair, the probability that it will form an antideuteron through a process i between those shown below:

$$P(\bar{N}_1\bar{N}_2 \to \bar{d}X_i|k) = \frac{\sigma_{\bar{N}_1\bar{N}_2 \to \bar{d}X_i}(k)}{\sigma_0}, \tag{6}$$

where $\bar{N}_1$ and $\bar{N}_2$ are the species of the two antinucleons, X_i represents the other final state particles in the given process and σ_0 is the free parameter of the model.

1) $\bar{p}\bar{n} \to \bar{d}\gamma$	3) $\bar{p}\bar{n} \to \bar{d}\pi^+\pi^-$	5) $\bar{p}\bar{p} \to \bar{d}\pi^-$	7) $\bar{p}\bar{p} \to \bar{d}\pi^-\pi^0$
2) $\bar{p}\bar{n} \to \bar{d}\pi^0$	4) $\bar{p}\bar{n} \to \bar{d}\pi^0\pi^0$	6) $\bar{n}\bar{n} \to \bar{d}\pi^+$	8) $\bar{n}\bar{n} \to \bar{d}\pi^+\pi^-$

[6] L. A. Dal, A. R. Raklev, Phys. Rev. D 91, 123536 (2015).

The Cross Section based Model

This model implies to know the cross sections as a function of k for **all** these processes.

To gather as much experimental data as possible to perform a good fit:

- The charge conjugate processes: $\sigma_{\bar{N_1}\bar{N_2}\to\bar{d}X} = \sigma_{N_1 N_2\to dX}$,
- The isospin invariance,
- The principle of detailed balance: $\sigma(Aa \to Bb) = \frac{g_B g_b}{g_A g_a}\frac{p_b^2}{p_a^2}\,\sigma(Bb \to Aa), \quad g_i = (2s_i + 1)$.

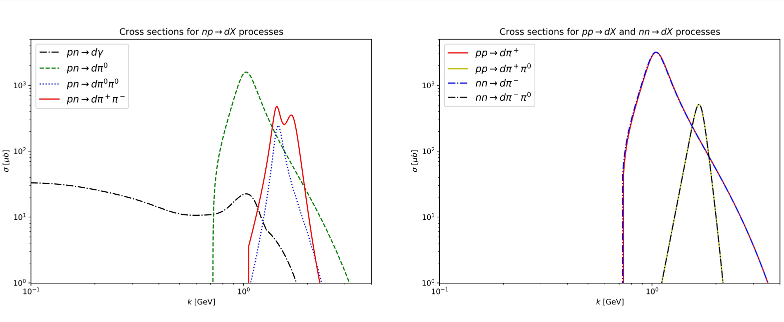


Figure 1: Fits to cross sections: each line is the result a least squares fit to experimental data using a properly defined function. For each process I have calculated the threshold in k, setting the cross sections to zero below it.

[6] L. A. Dal, A. R. Raklev, Phys. Rev. D 91, 123536 (2015).

The Advanced Coalescence Model

The advanced coalescence model combines the event-by-event MC simulations with a microscopic picture based on the **Wigner function** of produced states [7].

The density matrix formalism

In QM the statistical properties of a system can be described by its **density matrix** ρ. The number of observed deuterons d with a given momentum $\bar{p}_d$ is than given by:

$$\frac{d^3 N_d}{dp_d^3} = Tr\left(\rho_d \rho_{pn}\right), \tag{7}$$

where $\rho_d = |\phi_d\rangle\langle\phi_d|$ and $\rho_{pn} = |\psi_p \psi_n\rangle\langle\psi_n \psi_p|$, normalised as $\langle\psi_n \psi_p|\psi_p \psi_n\rangle = N_p N_n$.

Evaluating eq. 7 in coordination representation $|\bar{x}_1 \bar{x}_2\rangle$,

$$\frac{d^3 N_d}{dp_d^3} = g_d N_d \int d^3x_1 d^3x_2 d^3x_1' d^3x_2' \; \phi_d^*(\bar{x}_1, \bar{x}_2)\phi_d(\bar{x}_1', \bar{x}_2') \; \langle \psi_n^\dagger(\bar{x}_2')\psi_p^\dagger(\bar{x}_1')\psi_p(\bar{x}_1)\psi_n(\bar{x}_2)\rangle, \tag{8}$$

where $\phi_d(\bar{x}_1, \bar{x}_2)$ and $\psi_i(\bar{x})$ are the **wave functions** of the deuteron and the nucleon i.

[7] M. Kachelriess, S. Ostapchenko, J. Tjemslan, J. Eur. Phys. J. A 56, 4 (2020).

The Advanced Coalescence Model

I factorise $\phi_d(\bar{x}_1, \bar{x}_2)$ into a plane wave and an internal wave function φ_d,

$$\phi_d(\bar{x}_1, \bar{x}_2) = \frac{1}{(2\pi)^{3/2}} \, e^{\frac{i}{2}\bar{p}_d \cdot (\bar{x}_1 + \bar{x}_2)} \, \varphi_d(\bar{x}_1 - \bar{x}_2), \tag{9}$$

and I replace the two-nucleon density matrix by its **two-body Wigner function** W_{np},

$$\langle \psi_n^\dagger(\bar{x}_2') \psi_p^\dagger(\bar{x}_1') \psi_p(\bar{x}_1) \psi_n(\bar{x}_2) \rangle = \int \frac{d^3p_n}{(2\pi)^3} \frac{d^3p_p}{(2\pi)^3} \, W_{np}\left(\bar{p}_n, \bar{p}_p, \frac{\bar{x}_2 + \bar{x}_2'}{2}, \frac{\bar{x}_1 + \bar{x}_1'}{2}\right) \cdot \cdot \, e^{i\bar{p}_n \cdot (\bar{x}_2 - \bar{x}_2')} \, e^{i\bar{p}_p \cdot (\bar{x}_1 - \bar{x}_1')}. \tag{10}$$

Introducing the new coordinates $\bar{r}_p = \frac{(\bar{x}_1 + \bar{x}_1')}{2}$ and $\bar{r}_n = \frac{(\bar{x}_2 + \bar{x}_2')}{2}$, the separations $\bar{r} = \bar{r}_n - \bar{r}_p$, $\bar{\xi} = \bar{x}_1 - \bar{x}_1' - \bar{x}_2$ and $\bar{\rho} = \frac{(\bar{x}_1 - \bar{x}_1' + \bar{x}_2 - \bar{x}_2')}{2}$ and the new momentum integration variables $\bar{p} = \bar{p}_n + \bar{p}_p$ and $\bar{k} = \frac{(\bar{p}_n - \bar{p}_p)}{2}$, the eq. 8 becomes:

The production yield in terms of Wigner functions

$$\frac{d^3N_d}{dp_d^3} = g_d N_d \int d^3k \int d^3r_p d^3r_n \, \mathcal{D}(\bar{r}, \bar{k}) \, W_{np}\left(\frac{\bar{p}_d}{2} + \bar{k}, \frac{\bar{p}_d}{2} - \bar{k}, \bar{r}_n, \bar{r}_p\right), \tag{11}$$

where

$$\mathcal{D}(\bar{r}, \bar{k}) = \int d^3\xi \, e^{-i\bar{k}\cdot\bar{\xi}} \, \varphi_d\left(\frac{\bar{r} + \bar{\xi}}{2}\right) \varphi_d^*\left(\frac{\bar{r} - \bar{\xi}}{2}\right) \tag{12}$$

is the **deuteron Wigner function**.

The two-body Wigner Function

Starting from a complete QM treatment, I now simplify to a **semi-classical problem** assuming a factorization of the momentum and coordinate dependences:

$$W_{np}\left(\frac{\bar{p}_d}{2} + \bar{k}, \frac{\bar{p}_d}{2} - \bar{k}, \bar{r}_n, \bar{r}_p\right) = H_{np}(\bar{r}_n, \bar{r}_p)\ G_{np}\left(\frac{\bar{p}_d}{2} + \bar{k}, \frac{\bar{p}_d}{2} - \bar{k}\right). \tag{13}$$

I neglect spatial correlations between p and n, $H_{np}(\bar{r}_n, \bar{r}_p) = h(\bar{r}_n)h(\bar{r}_p)$, choosing a **Gaussian ansatz** for $h(\bar{r})$,

$$h(\bar{r}) = \frac{1}{(2\pi\sigma^2)^{3/2}}\ e^{-\frac{r^2}{2\sigma^2}}. \tag{14}$$

With these assumption,

$$W_{np}\left(\frac{\bar{p}_d}{2} + \bar{k}, \frac{\bar{p}_d}{2} - \bar{k}, \bar{r}_n, \bar{r}_p\right) = \frac{1}{(2\pi\sigma^2)^3}\ e^{-\frac{r_n^2}{2\sigma^2}}\, e^{-\frac{r_p^2}{2\sigma^2}}\ G_{np}\left(\frac{\bar{p}_d}{2} + \bar{k}, \frac{\bar{p}_d}{2} - \bar{k}\right), \tag{15}$$

where σ **is the same for** p **and** n **and represents the size of the formation region**.

The Deuteron Wigner Function

I chose the sum of **two Gaussians ansatz** as deuteron wave function,

$$\varphi_d(\bar{r}) = \pi^{-\frac{3}{4}} \left[\frac{\sqrt{\Delta}}{d_1^{3/2}}\, e^{-\frac{r^2}{2d_1^2}} + e^{i\alpha}\, \frac{\sqrt{1-\Delta}}{d_2^{3/2}}\, e^{-\frac{r^2}{2d_2^2}} \right], \tag{16}$$

and fitting the probability distribution $|\varphi_d(\bar{r})|^2$ to the Hulthen function I fix the physical parameters Δ, d_1, d_2. The **deuteron Wigner function** follows then as

$$\mathcal{D}(\bar{r}, \bar{k}) = 8\Big[\Delta\, e^{-\frac{r^2}{d_1^2}}\, e^{-k^2 d_1^2} + (1-\Delta)\, e^{-\frac{r^2}{d_2^2}}\, e^{-k^2 d_2^2}\Big] + A(\bar{r}\cdot\bar{k}). \tag{17}$$

Then, from

$$\frac{d^3 N_d}{dp_d^3} = g_d N_d \int d^3k \int d^3r_p d^3r_n\ \mathcal{D}(\bar{r}, \bar{k})\ W_{np}\Big(\frac{\bar{p}_d}{2} + \bar{k}, \frac{\bar{p}_d}{2} - \bar{k}, \bar{r}_n, \bar{r}_p\Big), \tag{18}$$

The formation probability

A given *pn*-pair with momentum difference $2k$ in its CoM has probability

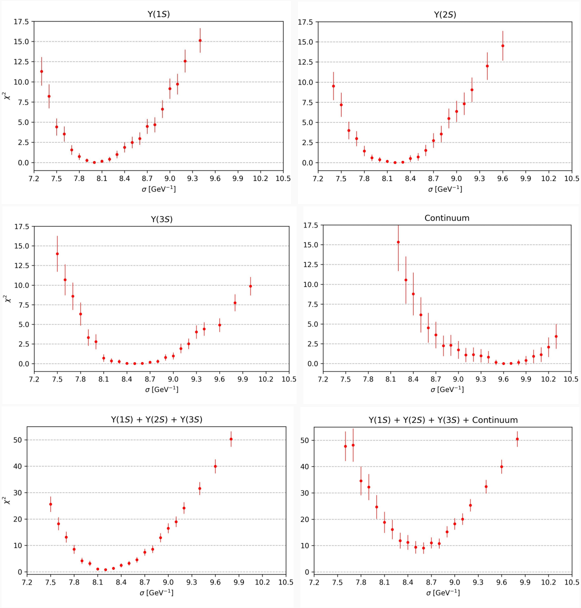
$$P(pn \to dX \mid k) = 3\,\Big(\zeta_1\, \Delta\, e^{-k^2 d_1^2} + \zeta_2\, (1-\Delta)\, e^{-k^2 d_2^2}\Big), \tag{19}$$

to form a deuteron, where the ζ_i are given by

$$\zeta_i = \left(\frac{d_i^2}{d_i^2 + 4\sigma^2}\right)^{3/2} \leq 1, \tag{20}$$

and σ represents the only free parameter of the model.

χ^2 minimization - Advanced Coalescence Model



H-dibaryon production rate