Three-particle scattering from lattice QCD

Maxwell T. Hansen

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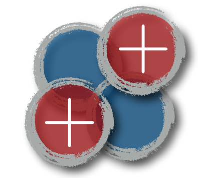
THE UNIVERSITY of EDINBURGH

Recipe for strong force predictions

1. Lagrangian defining QCD +
2. Formal / numerical machinery (lattice QCD) +
3. A few experimental inputs (e.g. M_π, M_K, M_Ω) =

$$\mathcal{L}_{\rm QCD} = \sum_f \overline{\Psi}_f (i\not{D} - m_f)\Psi_f - \frac{1}{4} G^a_{\mu\nu} G^{\mu\nu}_a$$

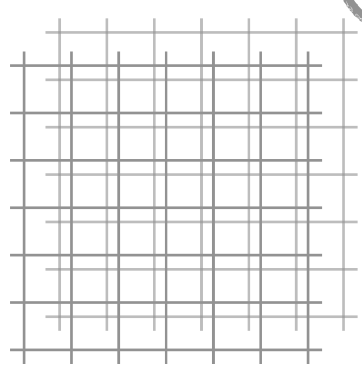


Wide range of precision pre-/post-dictions

$m_q^{\overline{\rm MS}}(\mu)$ $\alpha_{\overline{\rm MS}}(\mu)$

m_p

m_n



f_π

g_A

f_K

$(g-2)_\mu^{\rm HVP}$ f_D

f_B

FLAG 2019 Flavour Lattice Averaging Group

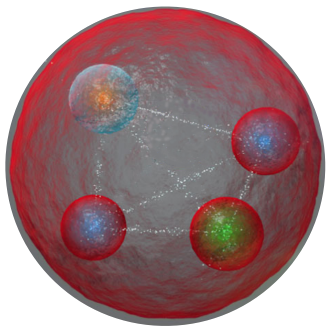
$\alpha^{(5)}_{\overline{\rm MS}}(M_Z) = 0.1182(8)$ lattice average

$\alpha^{(5)}_{\overline{\rm MS}}(M_Z) = 0.1174(16)$ PDG 18 (non-lattice)

Overwhelming evidence for QCD ✓ → *Tool for new physics searches*

Multi-hadron observables

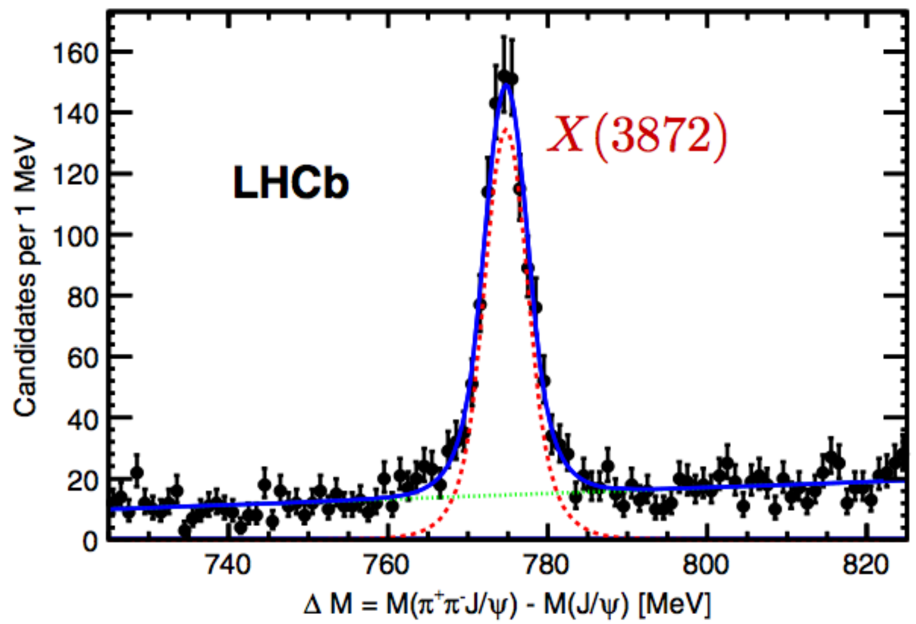
- Exotics, XYZs, tetra- and penta-quarks, *H* dibaryon



e.g. X(3872)

$$\sim |D^0\overline{D}^{*0} + \overline{D}^0 D^{*0}\rangle\,?$$

• LHCb (PRD92, 2015) •



- Eletroweak, CP violation, resonant enhancement

CP violation in charm $\quad D \to \pi\pi, K\overline{K}$

$f_0(1710)$ could enhance ΔA_{CP}
• Soni (2017) •

Resonant B decays $\quad B \to K^*\,\ell\ell \to K\pi\,\ell\ell$

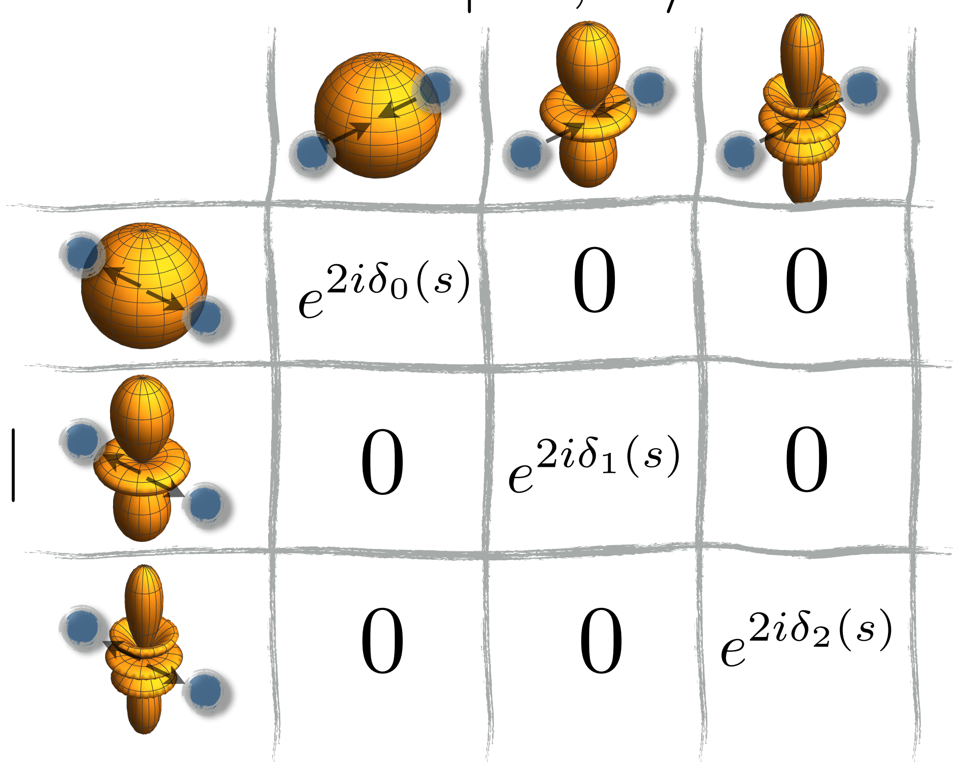
$|X\rangle, |\rho\rangle\,, |K^*\rangle\,, |f_0\rangle \quad \notin$ **QCD Fock space**

QCD Fock space

- At low-energies QCD = hadronic degrees of freedom $\pi \sim \bar{u}d,\ K \sim \bar{s}u,\ p \sim uud$
- Overlaps of multi-hadron ***asymptotic states*** → S matrix

$|\pi\pi, \text{in}\rangle$

$S(s) \equiv \langle \pi\pi, \text{out}|$



depends on $s = E_{\text{cm}}^2$
and angular variables

diagonal in angular momentum

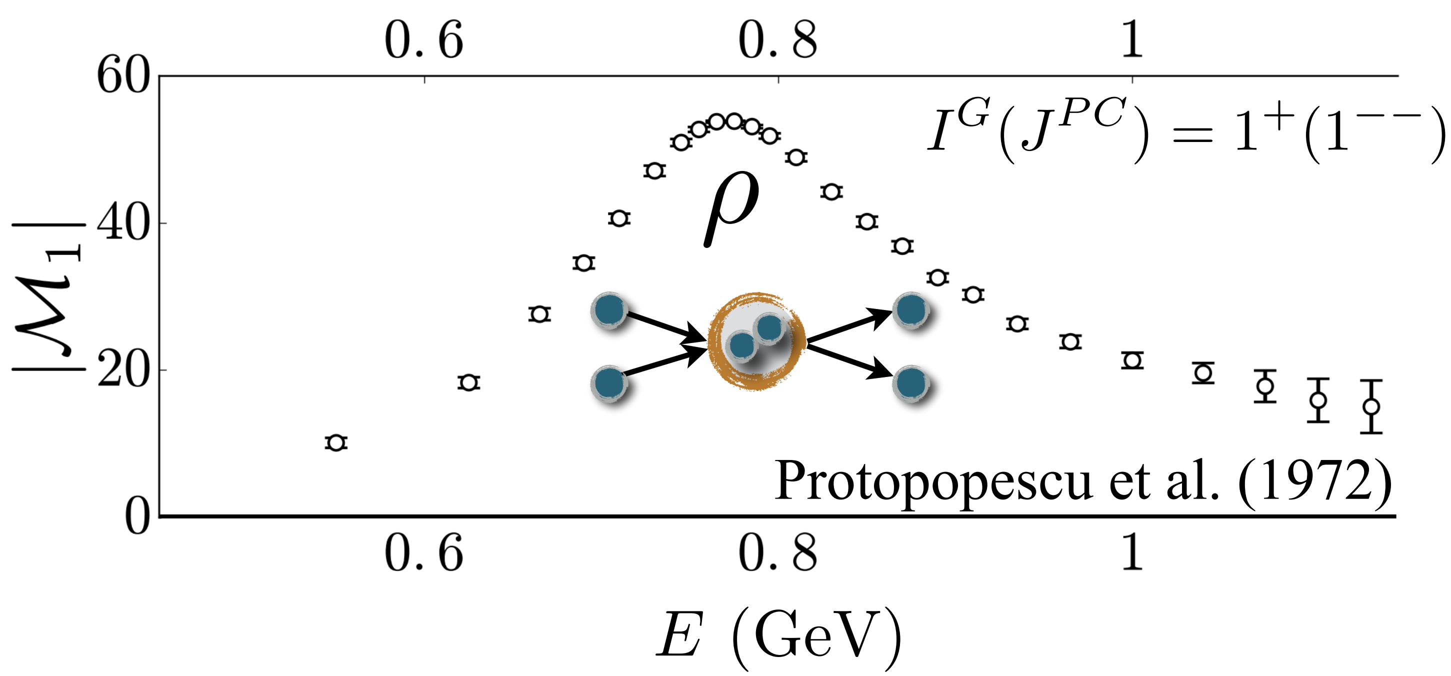
$$\mathcal{M}_\ell(s) \propto e^{2i\delta_\ell(s)} - 1$$

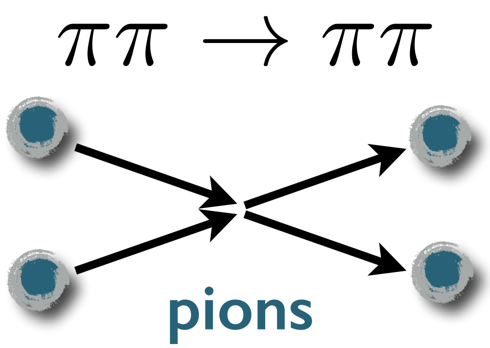
- ***An enormous space of information*** $|\pi\pi\pi\pi, \text{in}\rangle \quad |K\overline{K}, \text{in}\rangle \quad \cdots$

QCD resonances

- Roughly speaking, a bump in: $|\mathcal{M}_\ell(s)|^2 \propto |e^{2i\delta_\ell(s)} - 1|^2 \propto \sin^2 \delta_\ell(s)$

scattering rate





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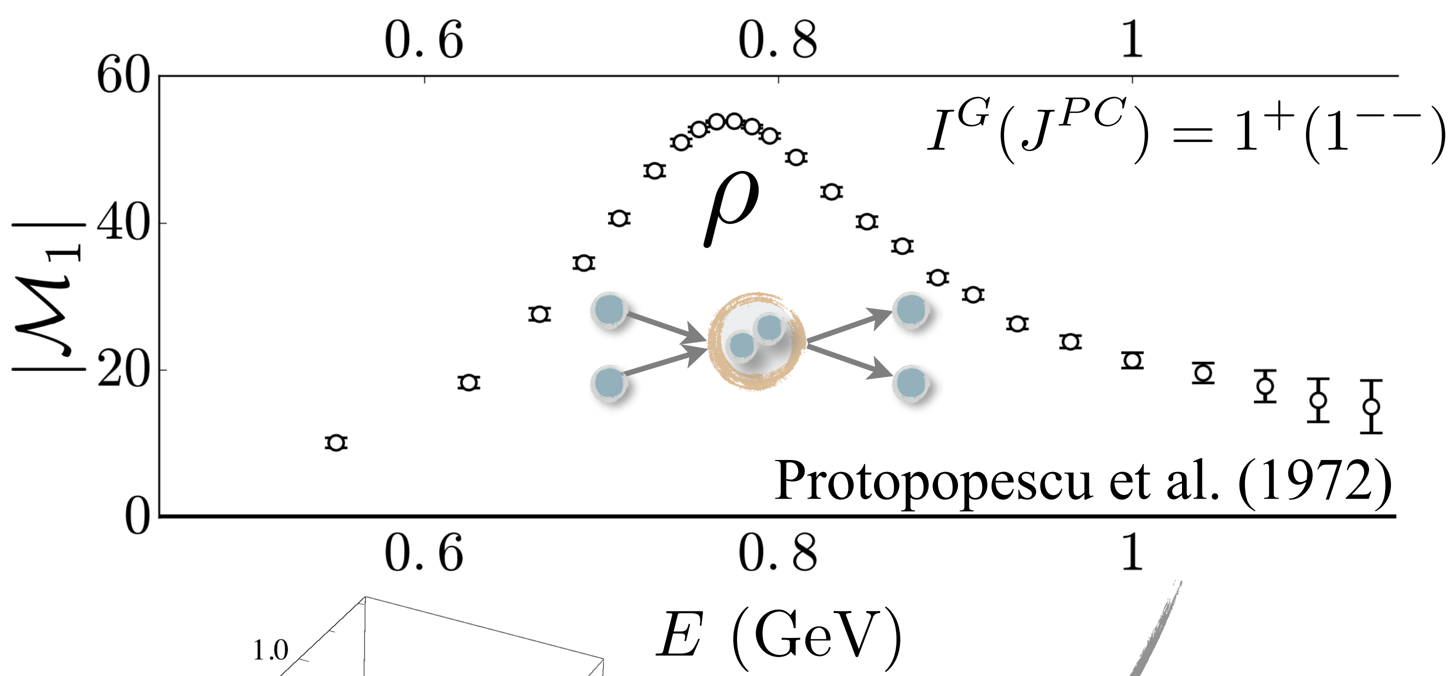
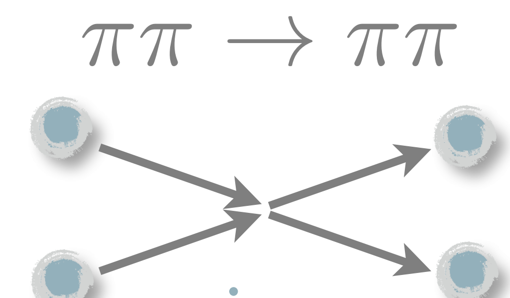
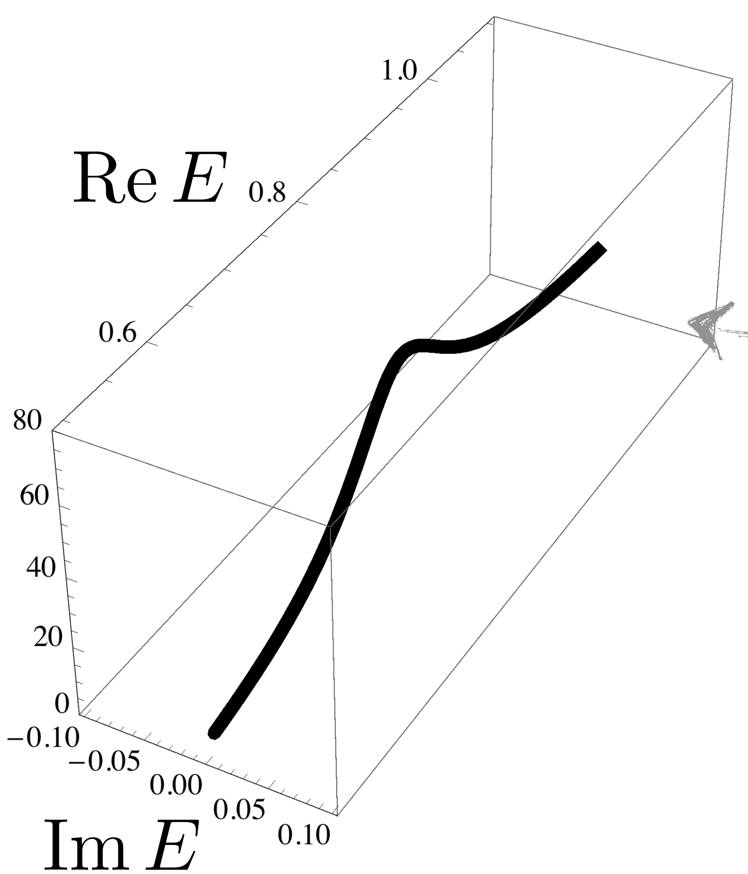




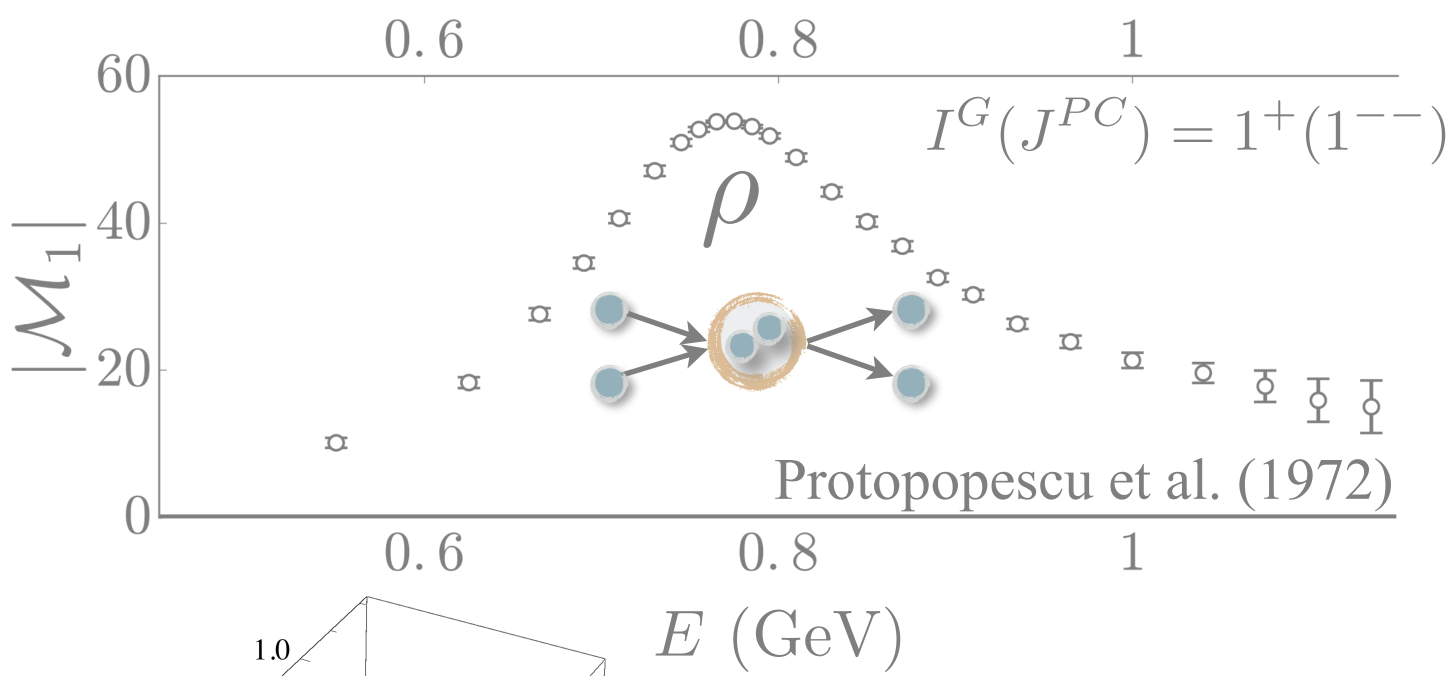


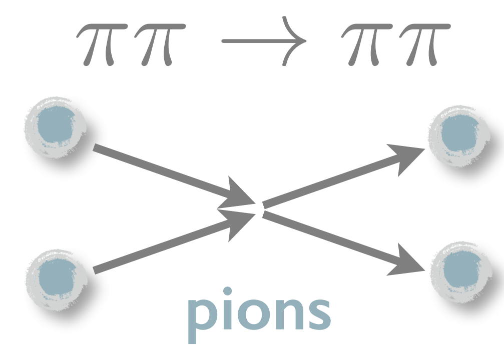
Extend to the
complex plane

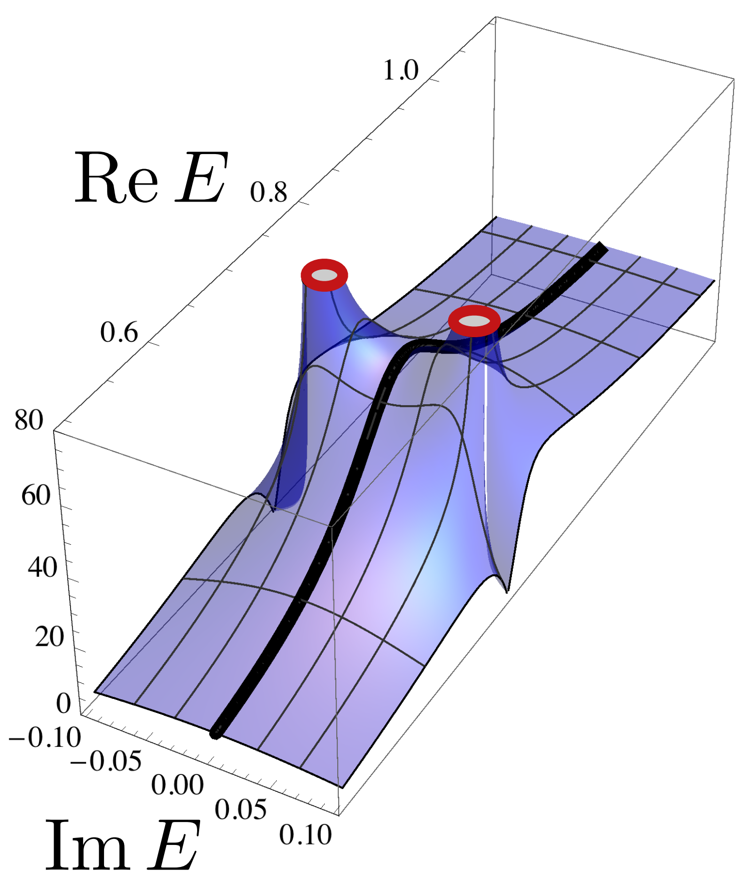
QCD resonances

- Roughly speaking, a bump in: $|\mathcal{M}_\ell(s)|^2 \propto |e^{2i\delta_\ell(s)} - 1|^2 \propto \sin^2\delta_\ell(s)$

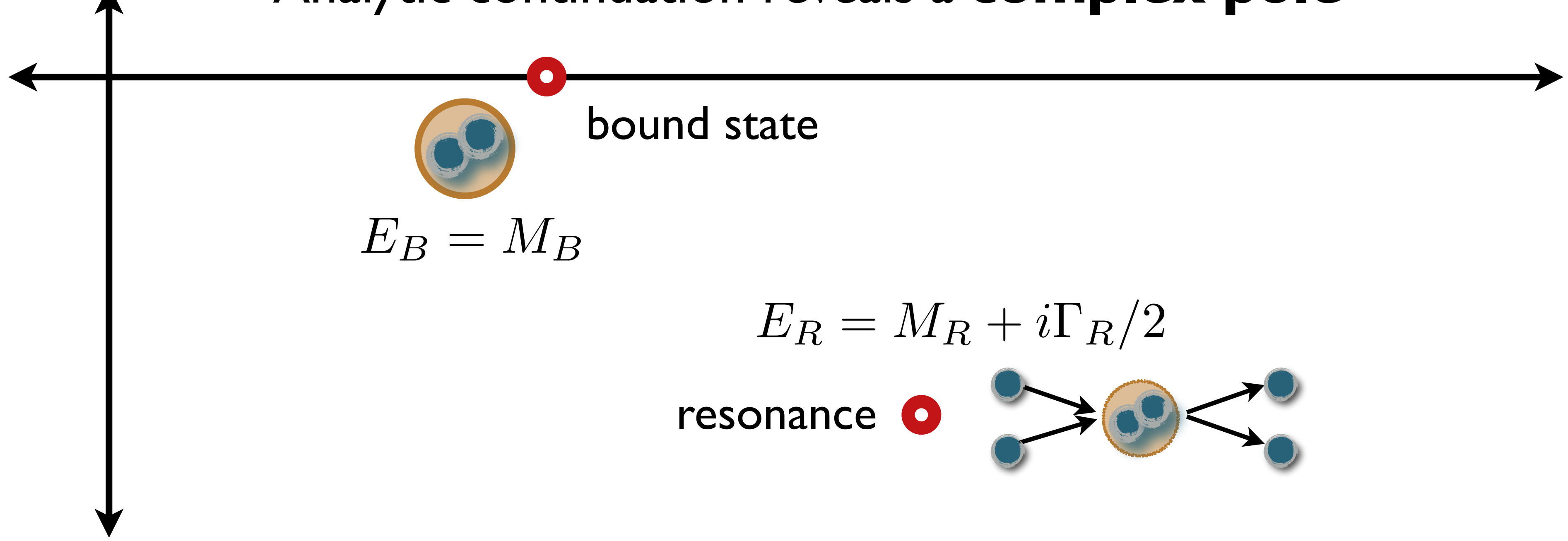
  scattering rate







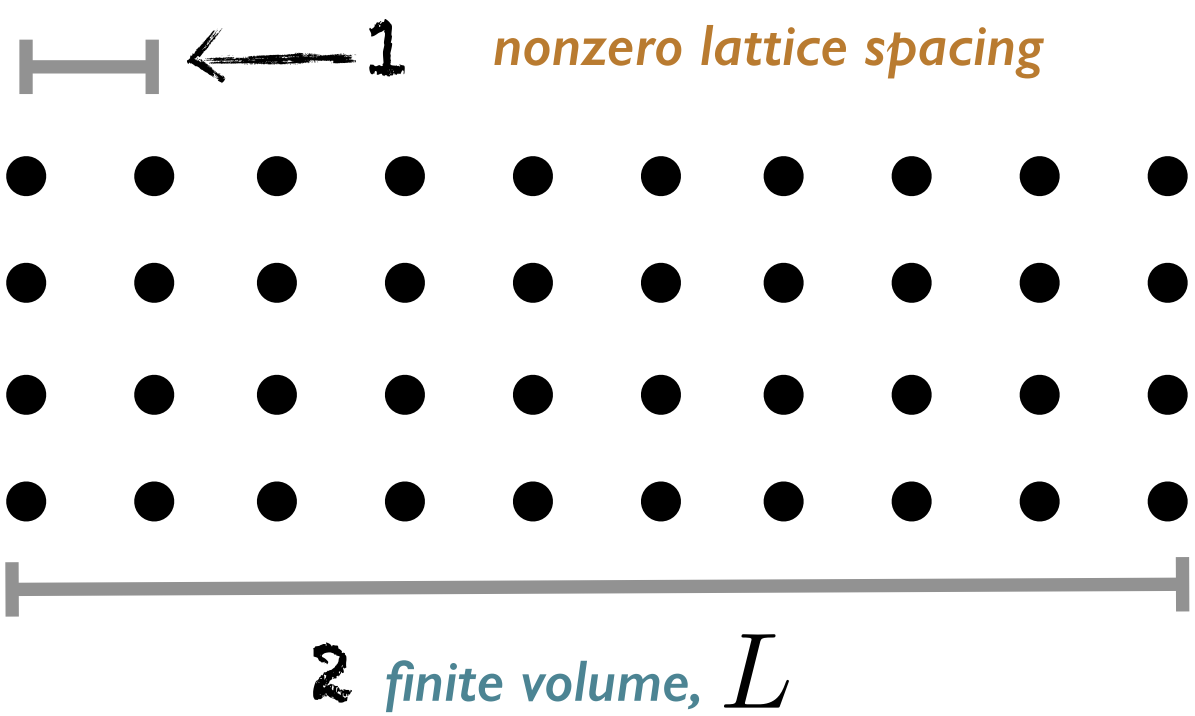
Analytic continuation reveals a **complex pole**

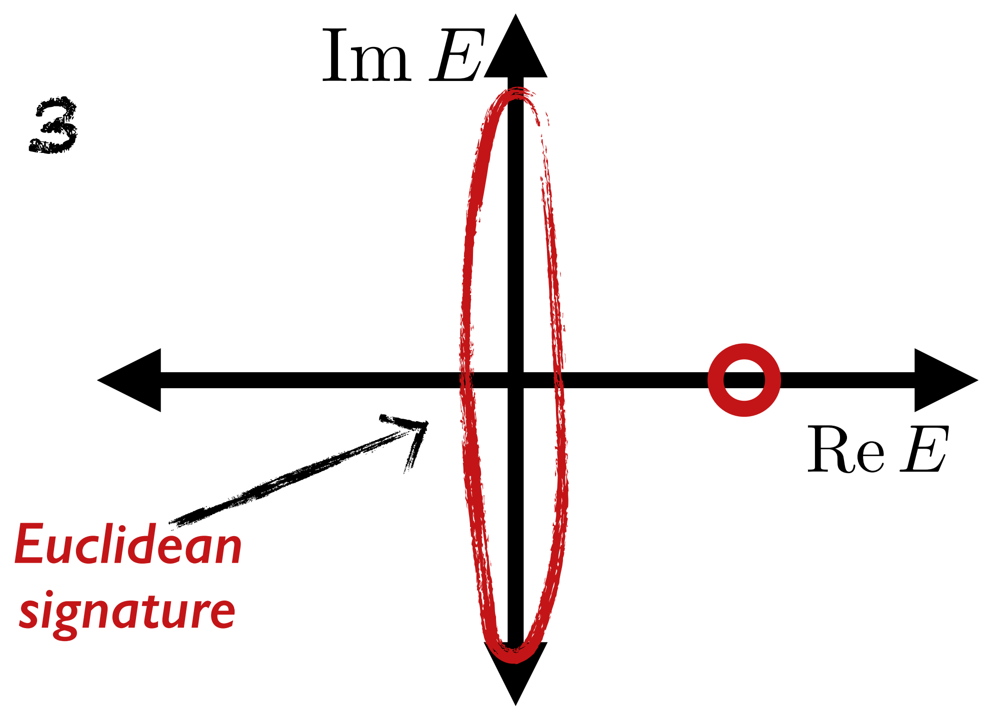


Lattice QCD

$$\text{observable?} = \int d^N\phi \, e^{-S} \left[\begin{array}{c} \textbf{interpolator} \\ \textbf{for observable} \end{array} \right]$$

To proceed we have to make *three modifications*

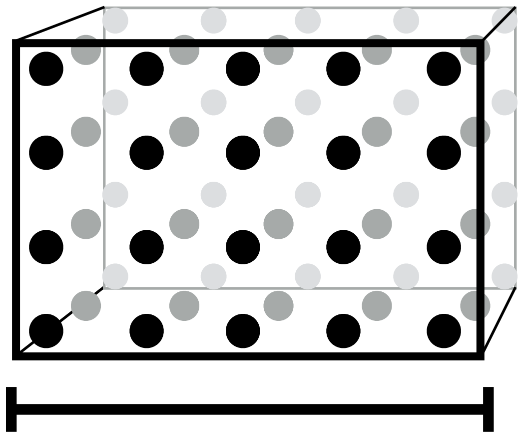




Also… $M_{\pi,\text{lattice}} > M_{\pi,\text{our universe}}$

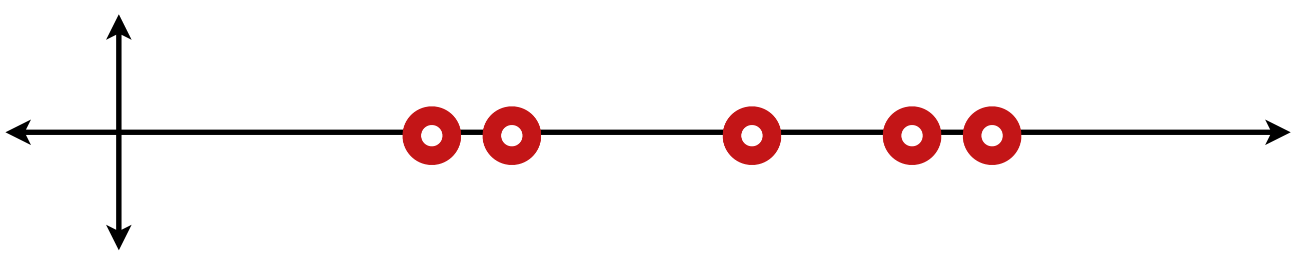
(but physical masses → increasingly common)

Difficulties for multi-hadron observables

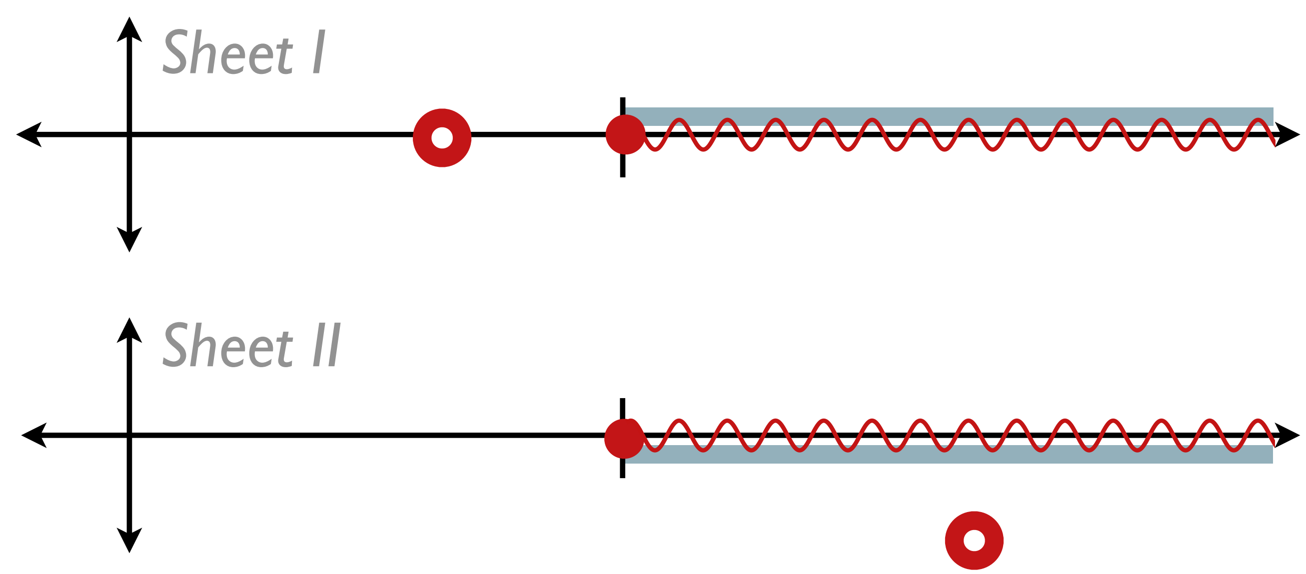


- The ***finite volume***…
  - ***Discretizes*** the spectrum
  - ***Eliminates*** the branch cuts and extra sheets
  - ***Hides*** the resonance poles

Finite-volume analytic structure



Infinite-volume analytic structure



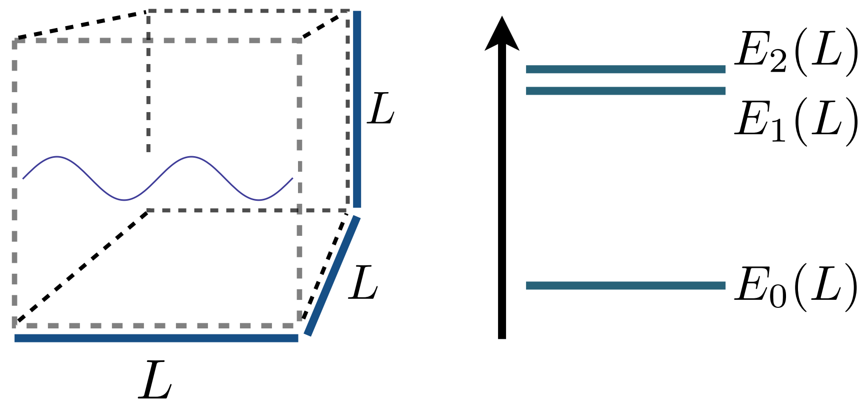
- LQCD → Energies and matrix elements

$$\langle \mathcal{O}_j(\tau)\mathcal{O}_i^\dagger(0)\rangle = \sum_n \langle 0|\mathcal{O}_j(\tau)|E_n\rangle\langle E_n|\mathcal{O}_i^\dagger(0)|0\rangle = \sum_n e^{-E_n(L)\tau} Z_{n,j} Z_{n,i}^*$$

- Our task is relate $E_n(L)$ and $\langle E_{m'}|\mathcal{J}(0)|E_m\rangle$ to ***experimental observables***

The finite-volume as a tool

- Finite-volume set-up

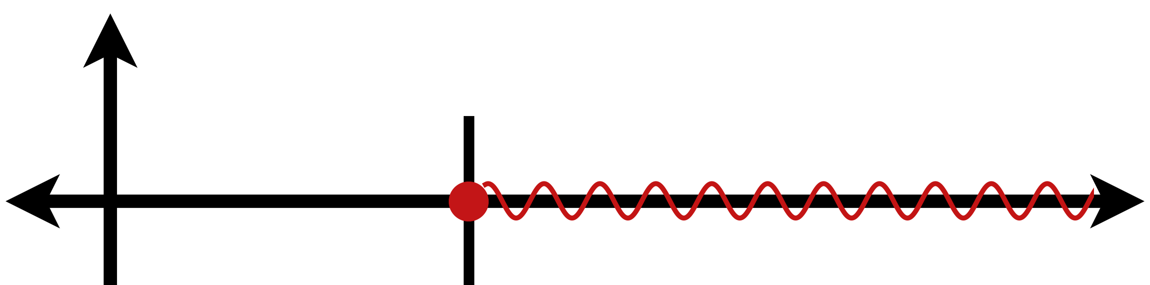


- **cubic**, spatial volume (extent *L*)
- **periodic**

$$\vec{p} = \frac{2\pi}{L}\vec{n}\,, \qquad \vec{n} \in \mathbb{Z}^3$$

- *L* is large enough to neglect $e^{-M_\pi L}$
- *T* and lattice also negligible

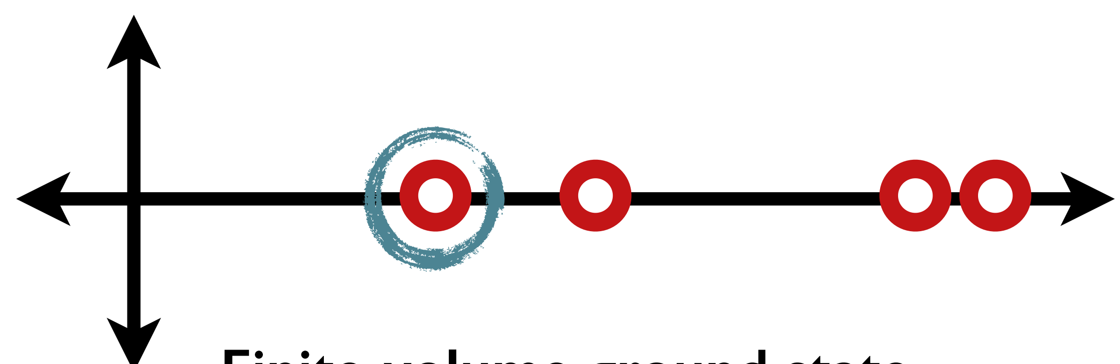
- Scattering leaves an ***imprint*** on finite-volume quantities



$$E_0 = 2M_\pi$$

Infinite-volume threshold

$$\mathcal{M}_{\ell=0}(2M_\pi) = -32\pi M_\pi a$$



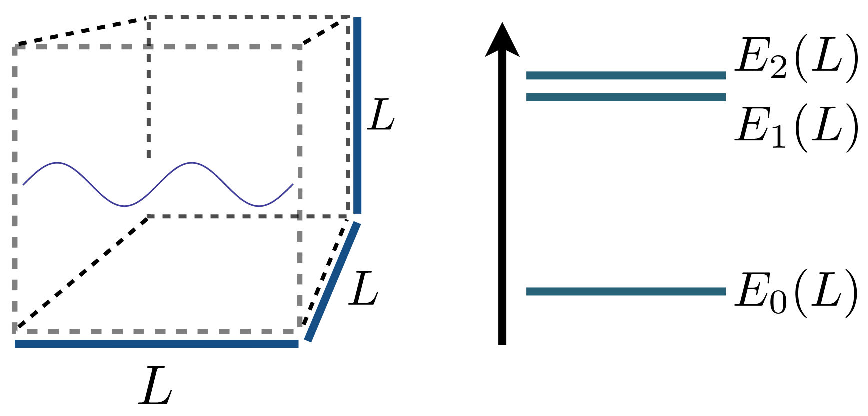
Finite-volume ground state

$$E_0(L) = 2M_\pi + \frac{4\pi a}{M_\pi L^3} + \mathcal{O}(1/L^4)$$

• Huang, Yang (1958) •

The finite-volume as a tool

- Finite-volume set-up



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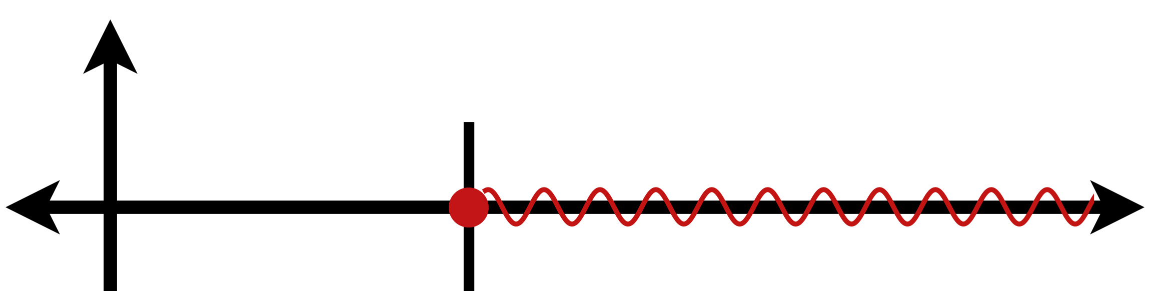

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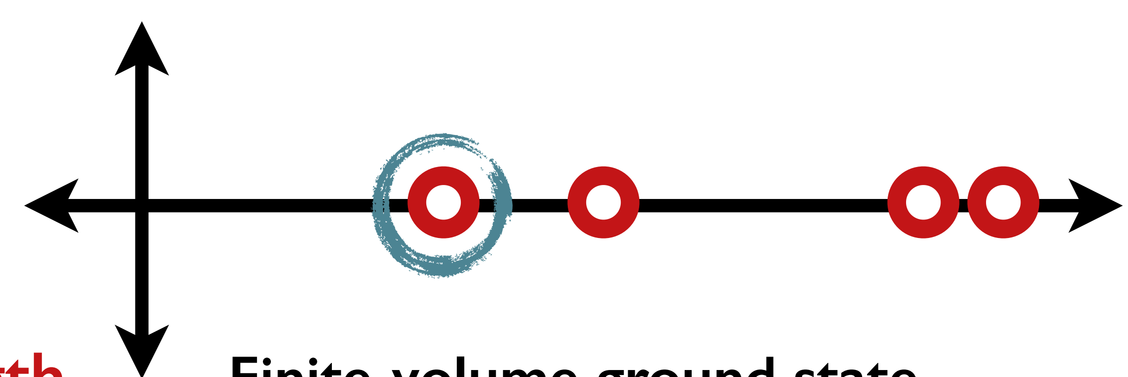

Finite-volume ground state

scattering length

$$\mathcal{M}_{\ell=0}(2M_\pi) = -32\pi M_\pi a$$

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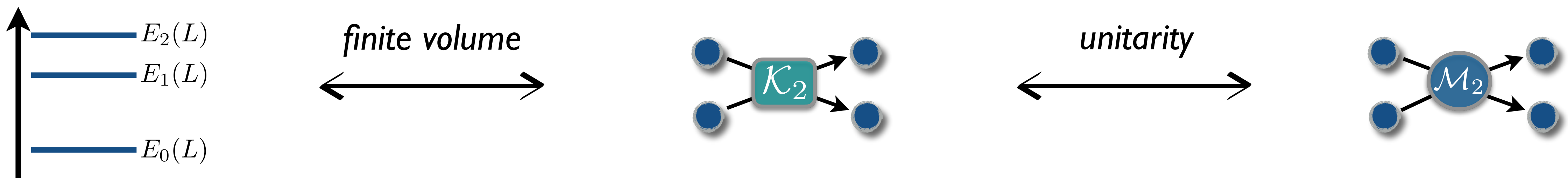
General method

$$\det[\mathcal{K}^{-1}(s) + F(P,L)] = 0$$

$F(P,L) \equiv$ Matrix of known geometric functions



Holds only for two-particle energies $s < (4m)^2$

Neglects e^{-mL}

Generalized to *non-degenerate masses, multiple channels, spinning particles*

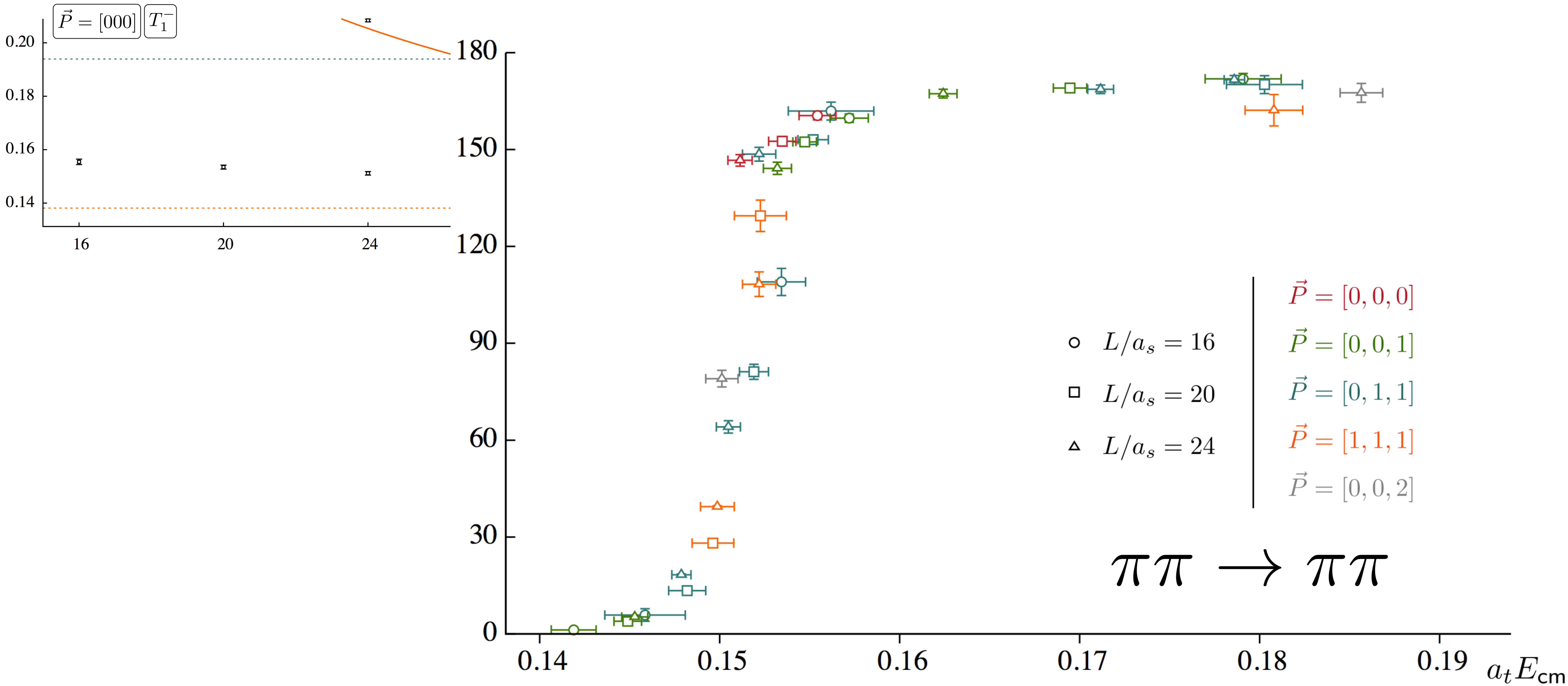
Encodes angular momentum mixing

Huang, Yang (1958) • Lüscher (1986, 1991) • Rummukainen, Gottlieb (1995)
Kim, Sachrajda, Sharpe (2005) • Christ, Kim, Yamazaki (2005) • He, Feng, Liu (2005)
Leskovec, Prelovsek (2012) • Bernard *et. al.* (2012) • MTH, Sharpe (2012) • Briceño, Davoudi (2012)
Li, Liu (2013) • Briceño (2014)

Using the result

- Single-channel case *(pions in a p-wave)*

$$\mathcal{K}(s_n)^{-1} = \rho \cot \delta(s_n) = -F(E_n, \vec{P}, L)$$



• Dudek, Edwards, Thomas in *Phys.Rev.* D87 (2013) 034505 •

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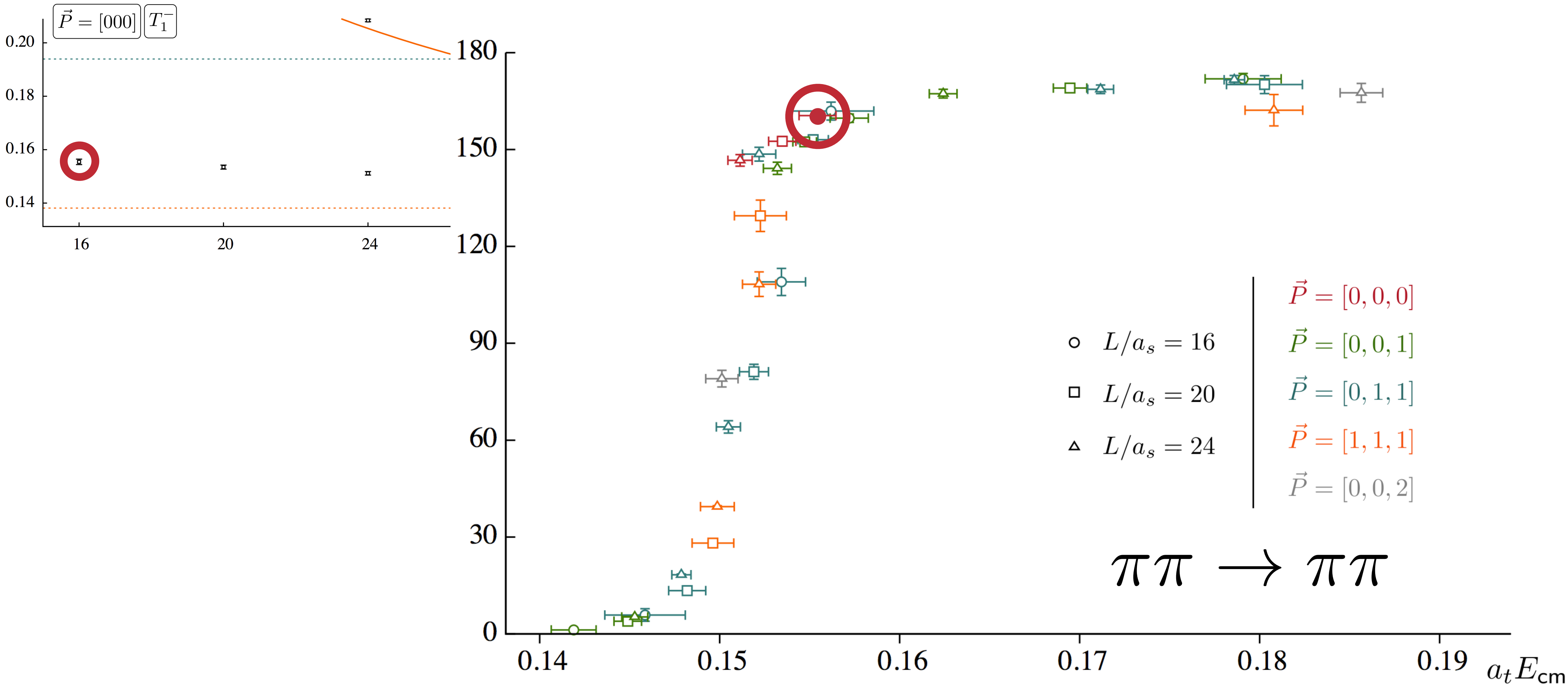

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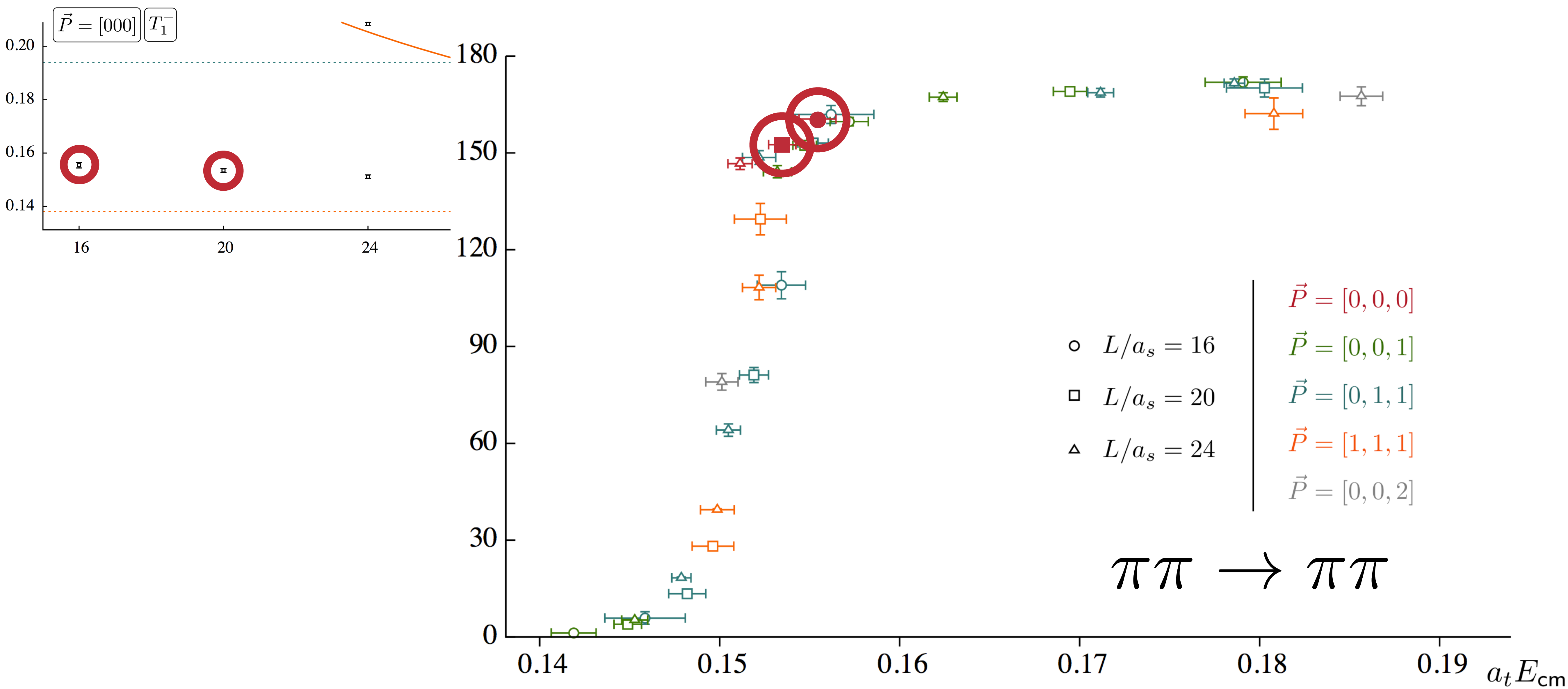

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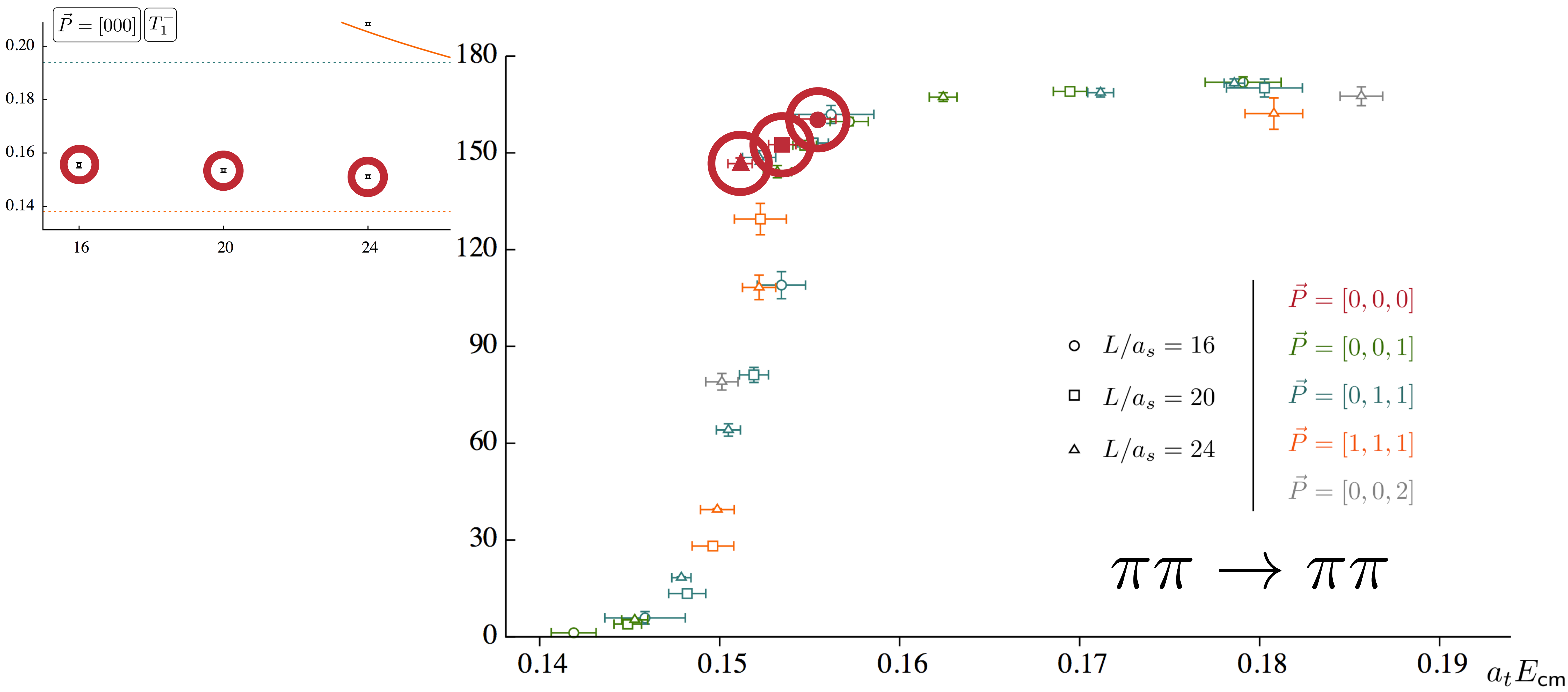

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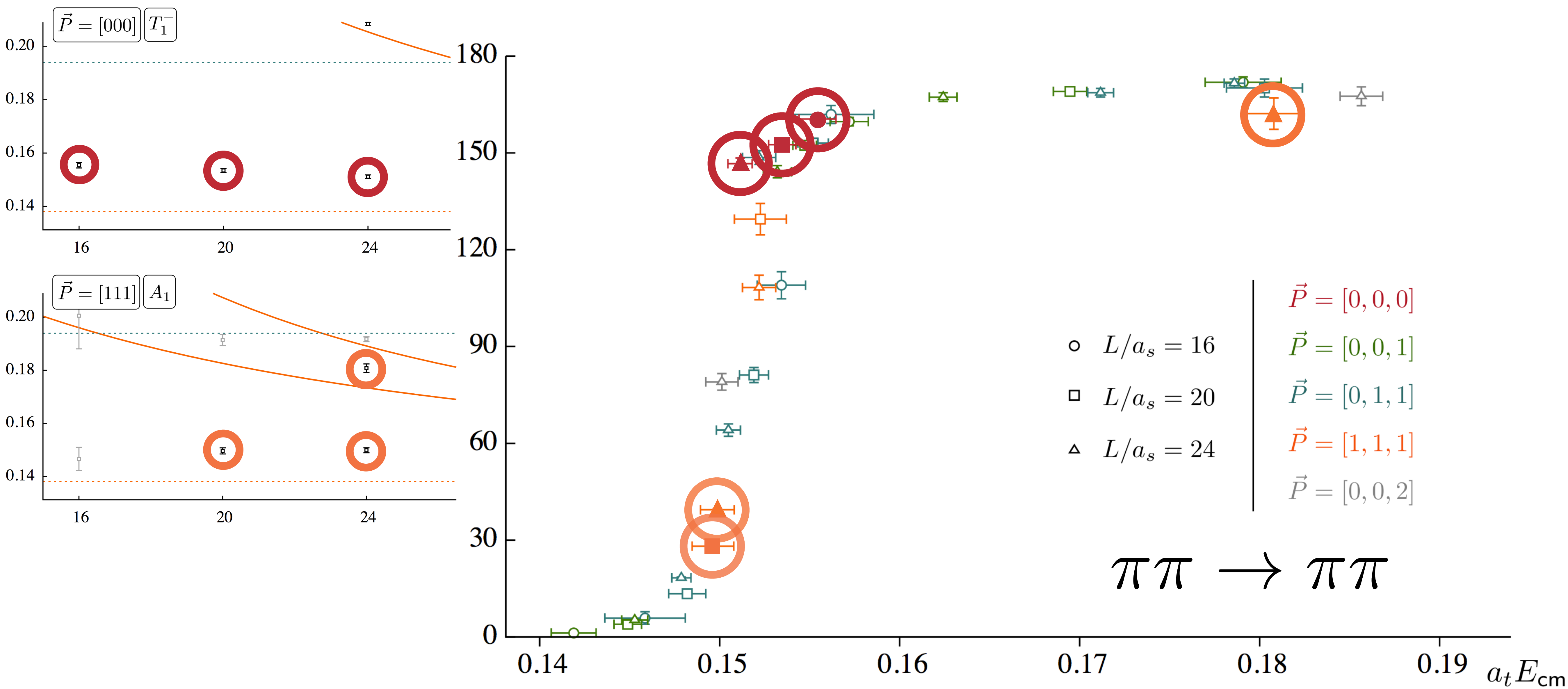
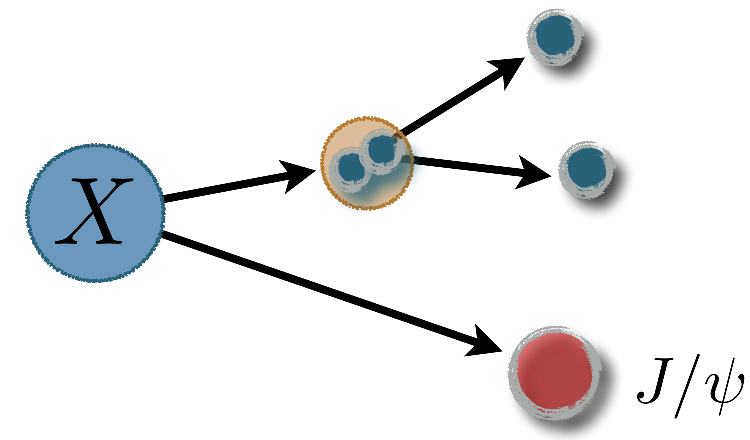


- Dudek, Edwards, Thomas in *Phys.Rev.* D87 (2013) 034505 -

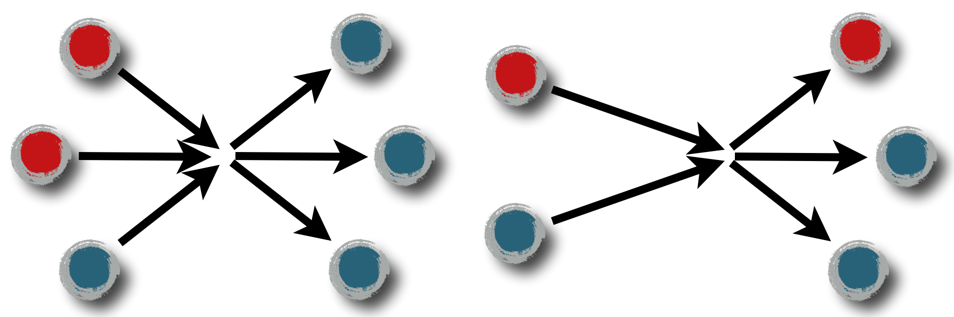
3-particle amplitudes

2-to-2 only samples J^P 0^+ 1^- $2^+ \cdots$

many interesting resonances have significant 3-body decays



Goal: ***finite-volume + unitarity formalism*** for generic two- and three-particle systems



Applications…

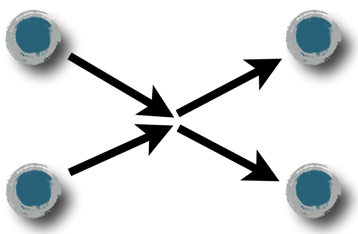
exotic resonance pole positions, couplings, quantum numbers

$\omega(782)$, $a_1(1420) \to \pi\pi\pi$ $X(3872) \to J/\psi\pi\pi$ $X(3915)[Y(3940)] \to J/\psi\pi\pi$

form factors and transitions

and much more!… *(3-body forces, weak transitions, gluons content)*

Complication: degrees of freedom

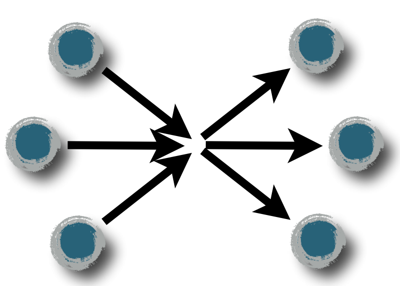


12 momentum components

-10 Poincaré generators

2 degrees of freedom

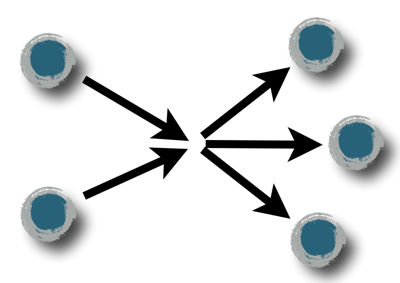
$\vec{p}_1 + \vec{p}_2 \ \rightarrow \ \vec{p}_3 + \vec{p}_4 \longrightarrow$ Mandelstam s, t



18 momentum components

-10 Poincaré generators

8 degrees of freedom



15 momentum components

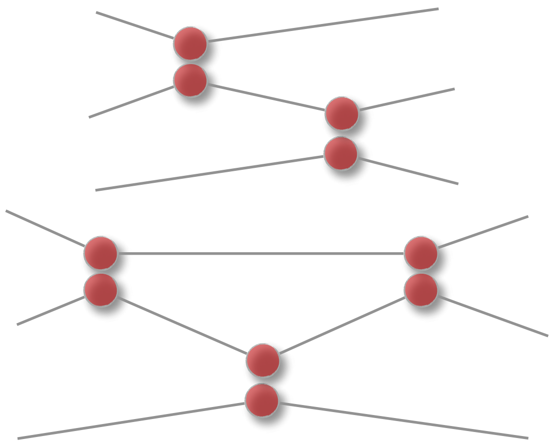
-10 Poincaré generators

5 degrees of freedom

Complication: on-shell states

- Classical pairwise scattering

for $m_1 = m_2 = m_3$ up to ***3***
binary collisions are possible



Dispersion Relations for Three-Particle Scattering Amplitudes. I*

MORTON RUBIN
Physics Department, University of Wisconsin, Madison, Wisconsin
AND
ROBERT SUGAR
Physics Department, Columbia University, New York, New York
AND
GEORGE TIKTOPOULOS
Palmer Physical Laboratory, Princeton University, Princeton, New Jersey
(Received 31 January 1966)

$$b=\frac{(m_1+m_3)(m_2+m_3)}{m_1m_2}$$

It follows that if

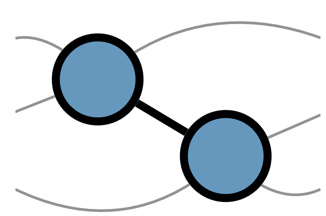
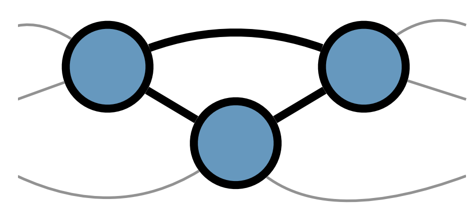
$$b^{n-2}(b-1)>1\,, \qquad \text{(IV.18)}$$

then $2n+1$ successive binary collisions are kinematically impossible.

$m_1 = m_2 = m_3 - \varepsilon$:
4 collisions possible
$\pi\pi K$

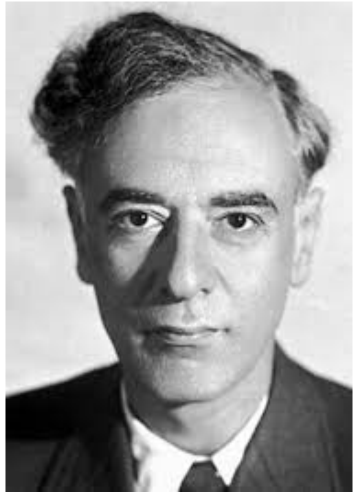
$b < 2$
5 collisions possible
$\pi K K$

- Correspond to Landau singularities

$i\mathcal{M}_{3\to3} \equiv$ fully connected correlator $=$  $+$  $+\cdots$

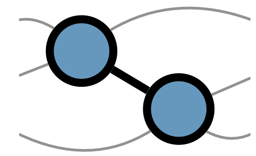
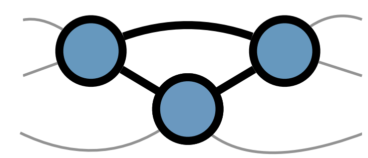
complicate analyticity & unitarity

difficult to disentangle kinematic
singularities from resonance poles



Two key observations

- Intermediate $K_{\mathrm{df},3}$ removes singularities

$\mathcal{K}_{\mathrm{df},3} \equiv$ fully connected diagrams w/ ***PV pole prescription*** $-$  $+$  $+\cdots$

same degrees of freedom as *M*₃ → same degrees of freedom as M_3

smooth real function

relation to M_3 = known

- $K_{\mathrm{df},3}$ has a systematic low-energy expansion

$$\mathcal{K}_{\mathrm{df},3}(p_3,p_2,p_1;k_3,k_2,k_1) = \mathcal{K}^{\mathrm{iso},0}_{\mathrm{df},3} + \mathcal{K}^{\mathrm{iso},1}_{\mathrm{df},3}\,\Delta + \cdots \qquad \Delta = \frac{s-(3m)^2}{(3m)^2}$$

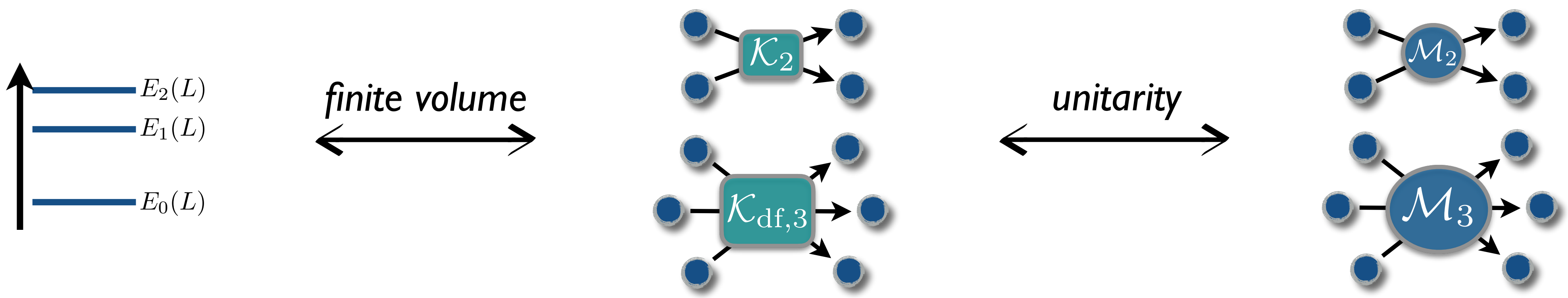
smooth real function

analogous to effective range expansion

$$p\cot\delta = -\frac{1}{a} + \frac{1}{2}\,r\,p^2 + \mathcal{O}(p^4)$$

gives handle on many *degrees of freedom*
(DOFs enter order by order)

Status…



- [x] Identical spin-zero, no 2-to-3, no K2 poles • MTH, Sharpe (2014, 2015) •
- [x] as above… but including 2-to-3 • Briceño, MTH, Sharpe (2017) •
- [x] as above… but including K2 poles • Briceño, MTH, Sharpe (2018) •
- [x] Non-identical, non-degenerate spin-zero $\pi\pi\pi \to \rho\pi \to \omega \to \rho\pi \to \pi\pi\pi$
  • MTH, Romero-López, Sharpe (2020) • Blanton, Sharpe (2020, 2021)
- [ ] Multiple three-particle channels… Spin!

Related work

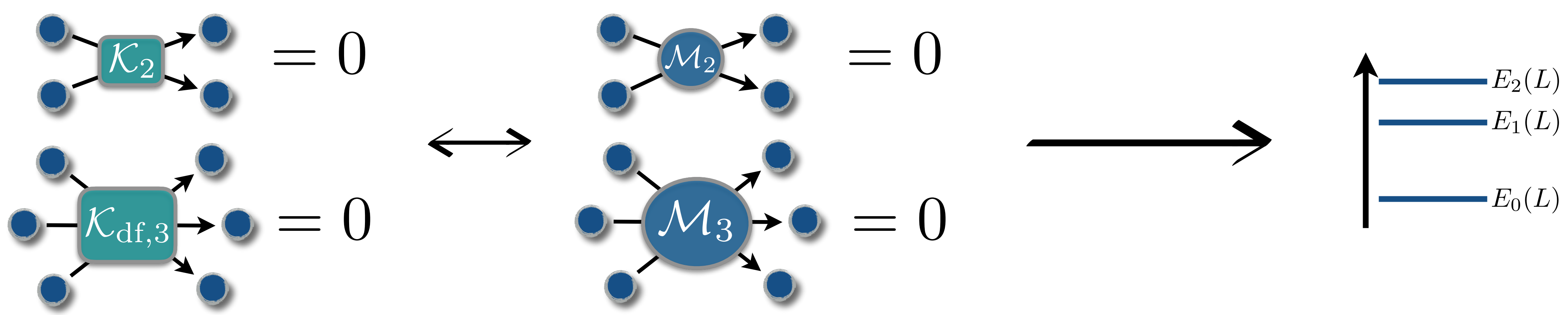
- Finite-volume unitarity method
  Döring, Mai (2016,2017)
  Gives connection to unitarity relations

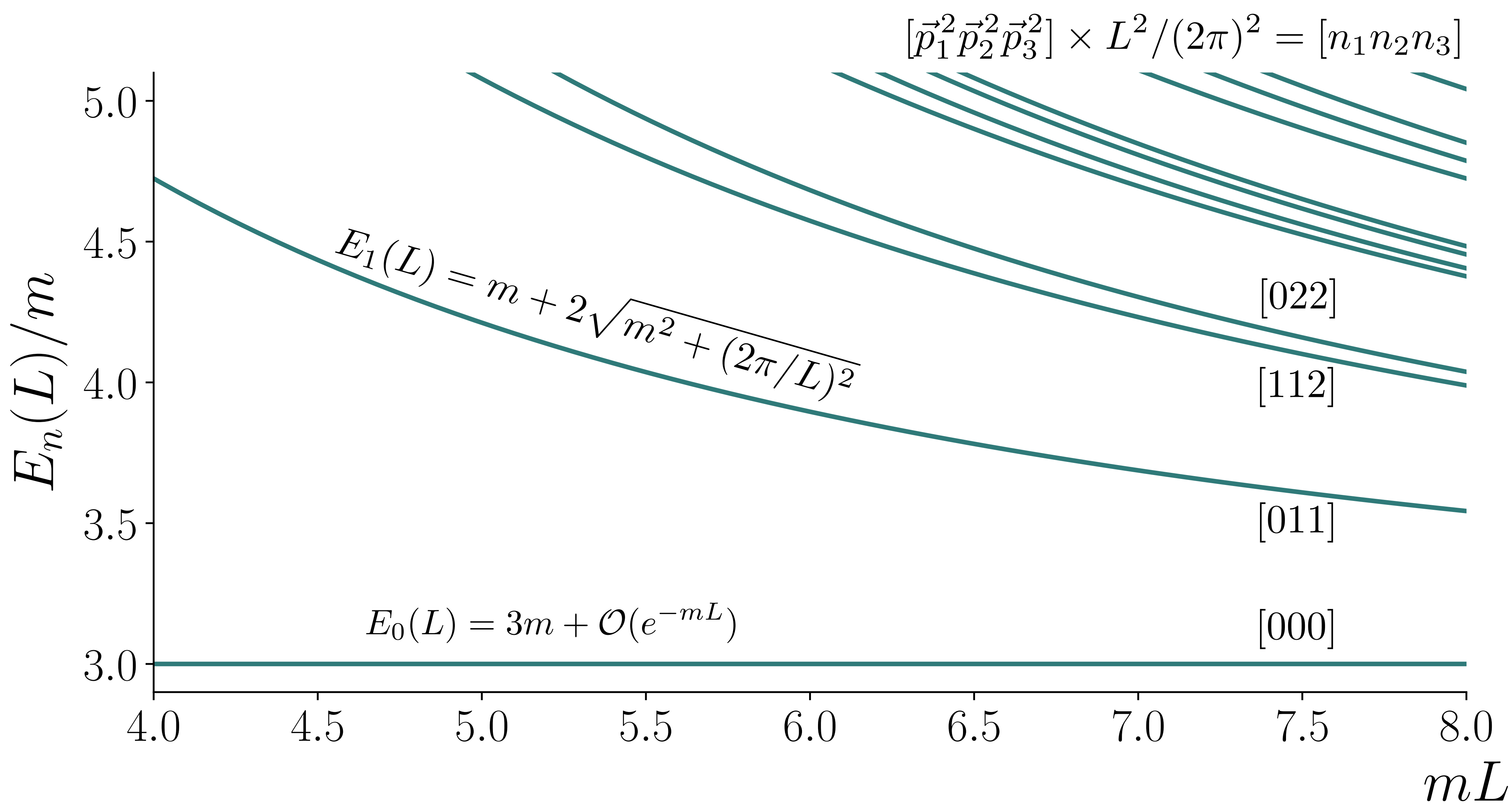
- Non-relativistic EFT method
  Hammer, Pang, Rusetsky (2017)
  Simplified derivation + integral equations
  Do not yet include non-degenerate, 2-to-3

- All methods
  Rely on intermediate, scheme-dependent quantity
  Hold up to e^{-mL} and for $E_3^\star < 5m_\pi$
  Equivalent where comparable

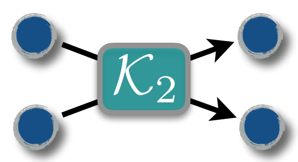
- Review articles
  MTH and Sharpe, 1901.00483 • Rusetsky, 1911.01253 • Mai, Döring, Rusetsky, 2103.00577

Non-interacting energies

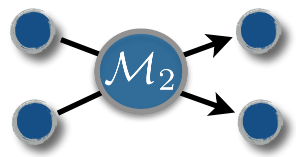




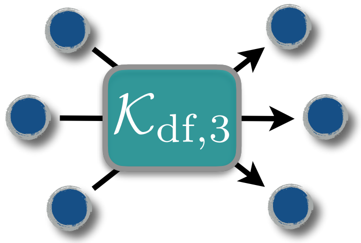
Two-particle interactions



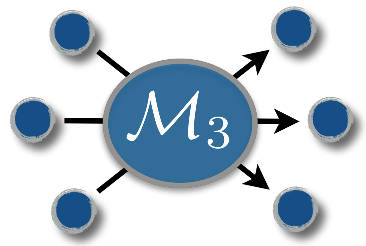
$= -16\pi\sqrt{s}\,a$



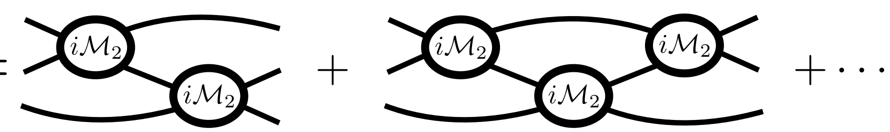
$= \dfrac{16\pi\sqrt{s}}{-1/a - ip}$

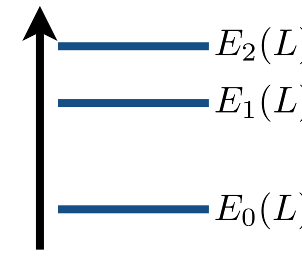


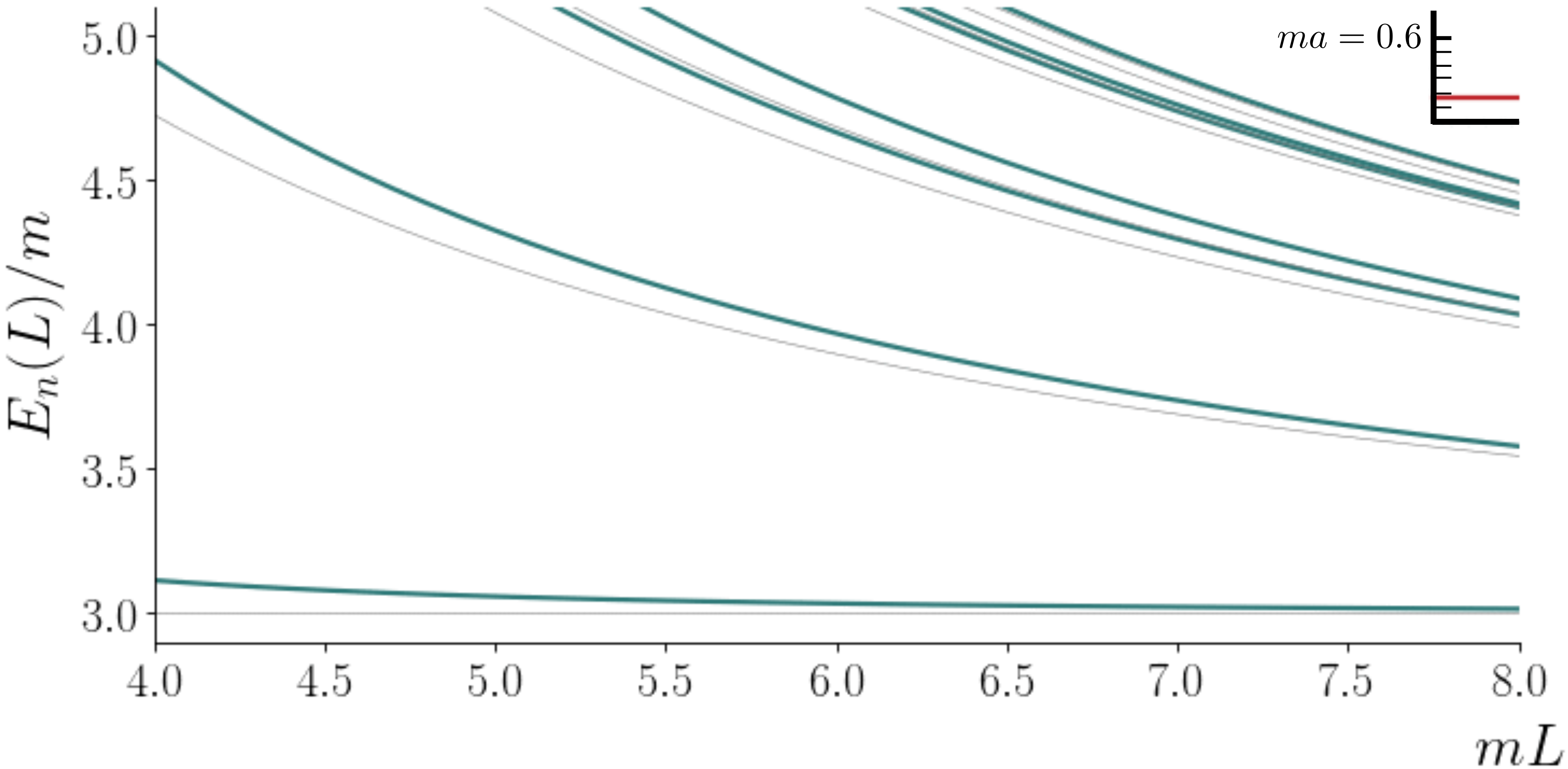
$= 0$



$=$







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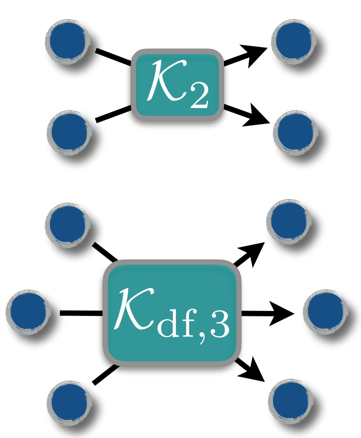
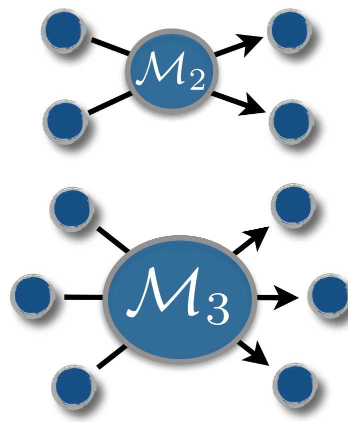


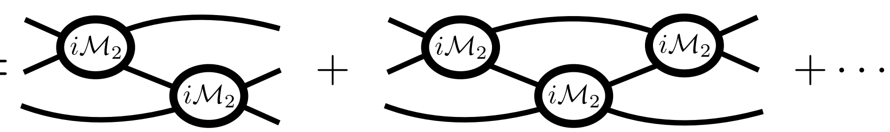
$$= -16\pi\sqrt{s}\, a$$

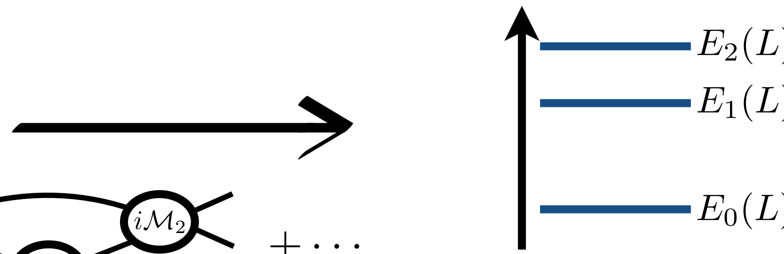
$$= 0$$

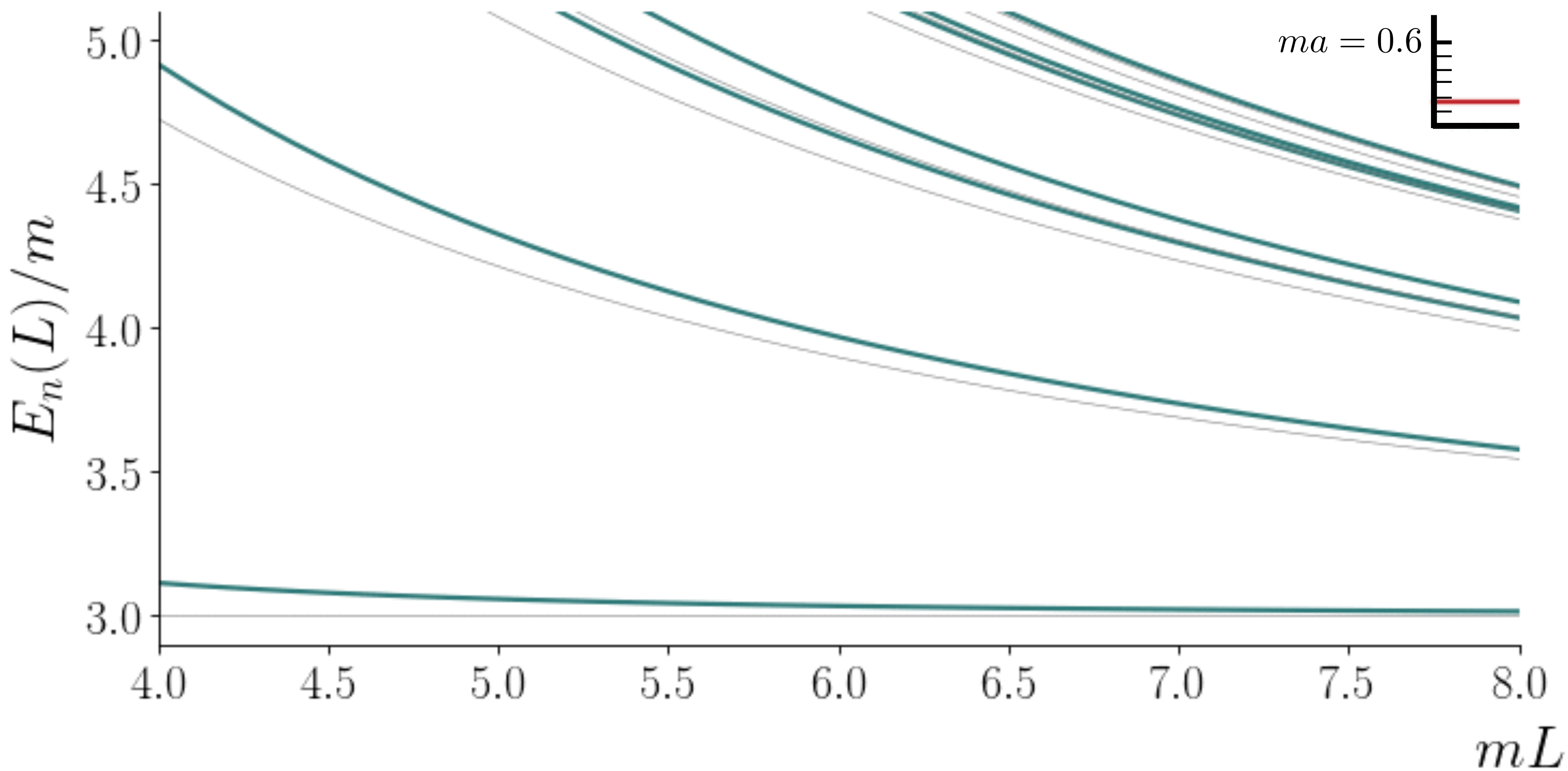


$$= \frac{16\pi\sqrt{s}}{-1/a - ip}$$

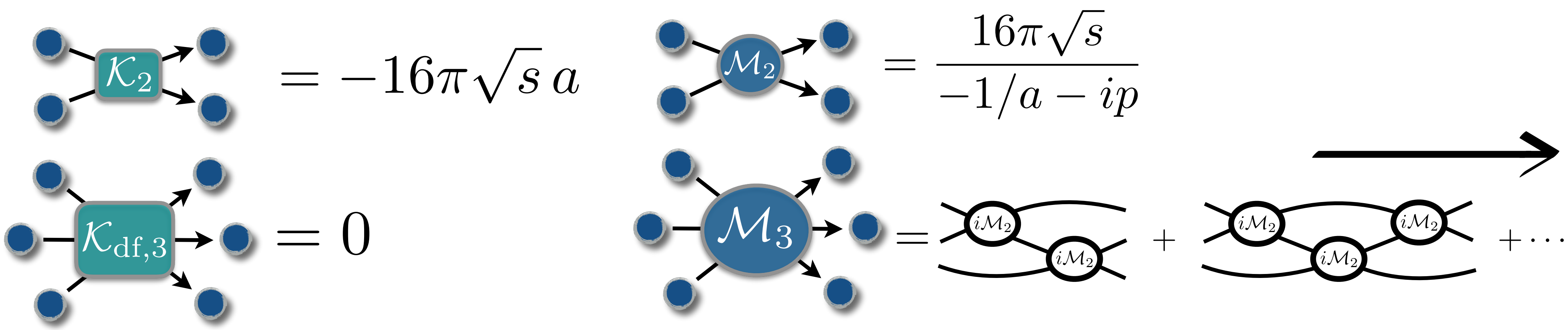
$$=$$

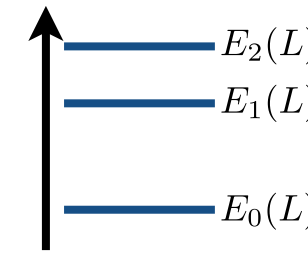


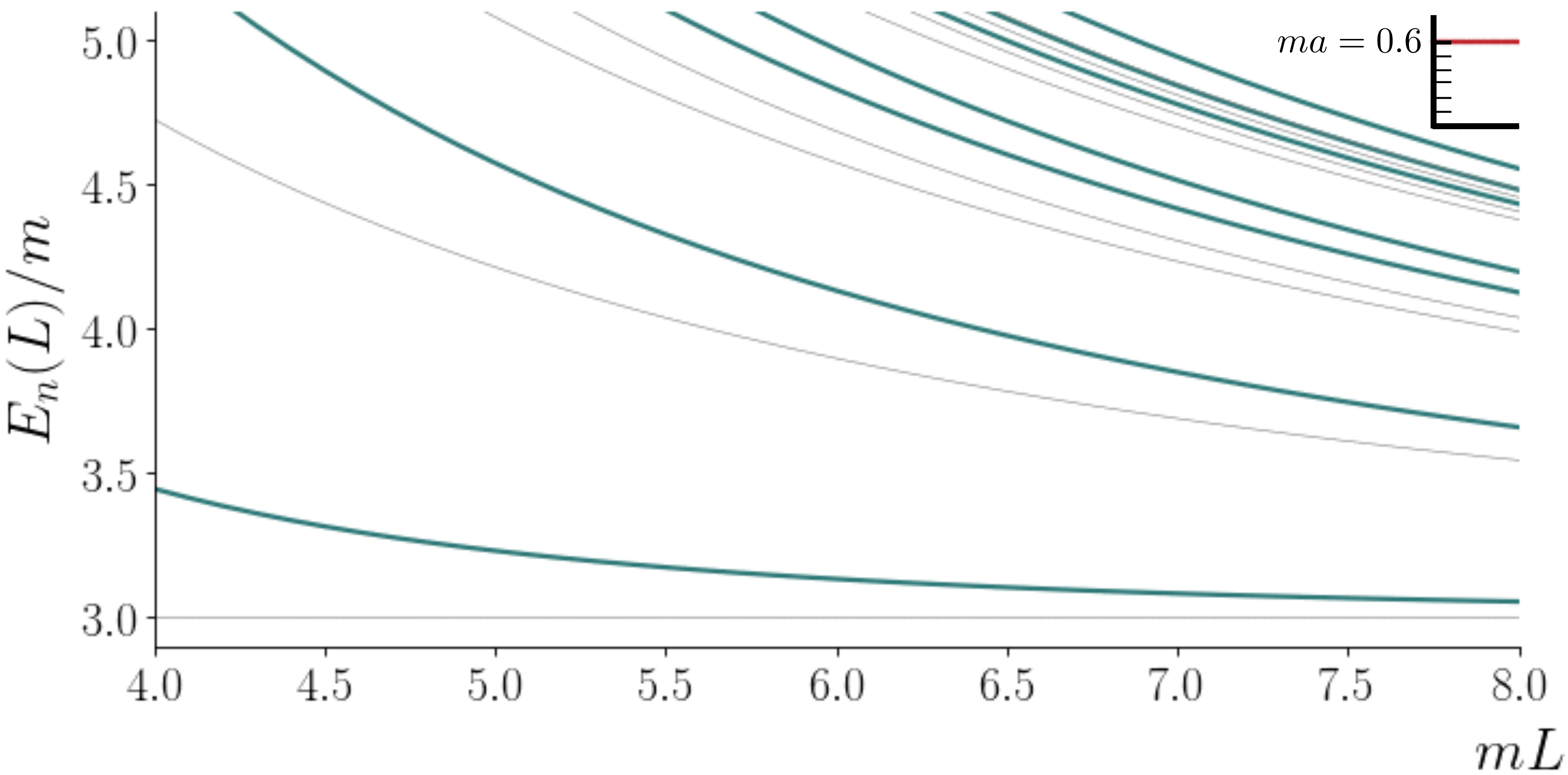




Two-particle interactions







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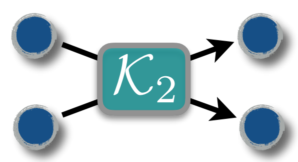

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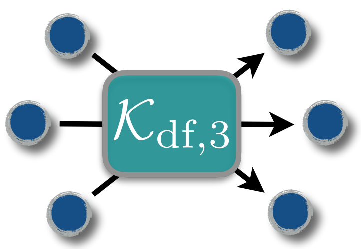

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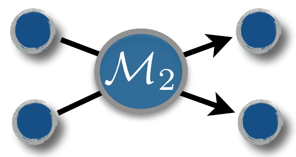

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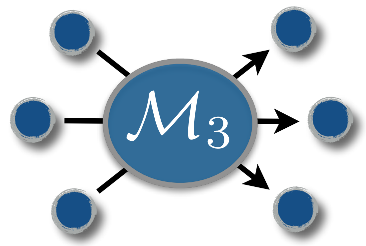

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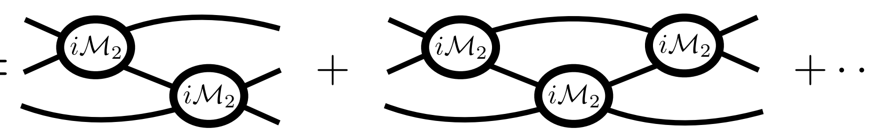
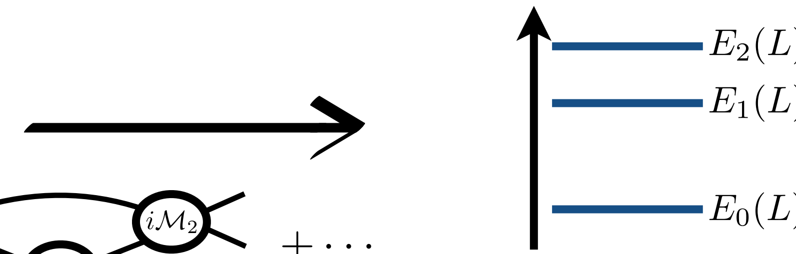
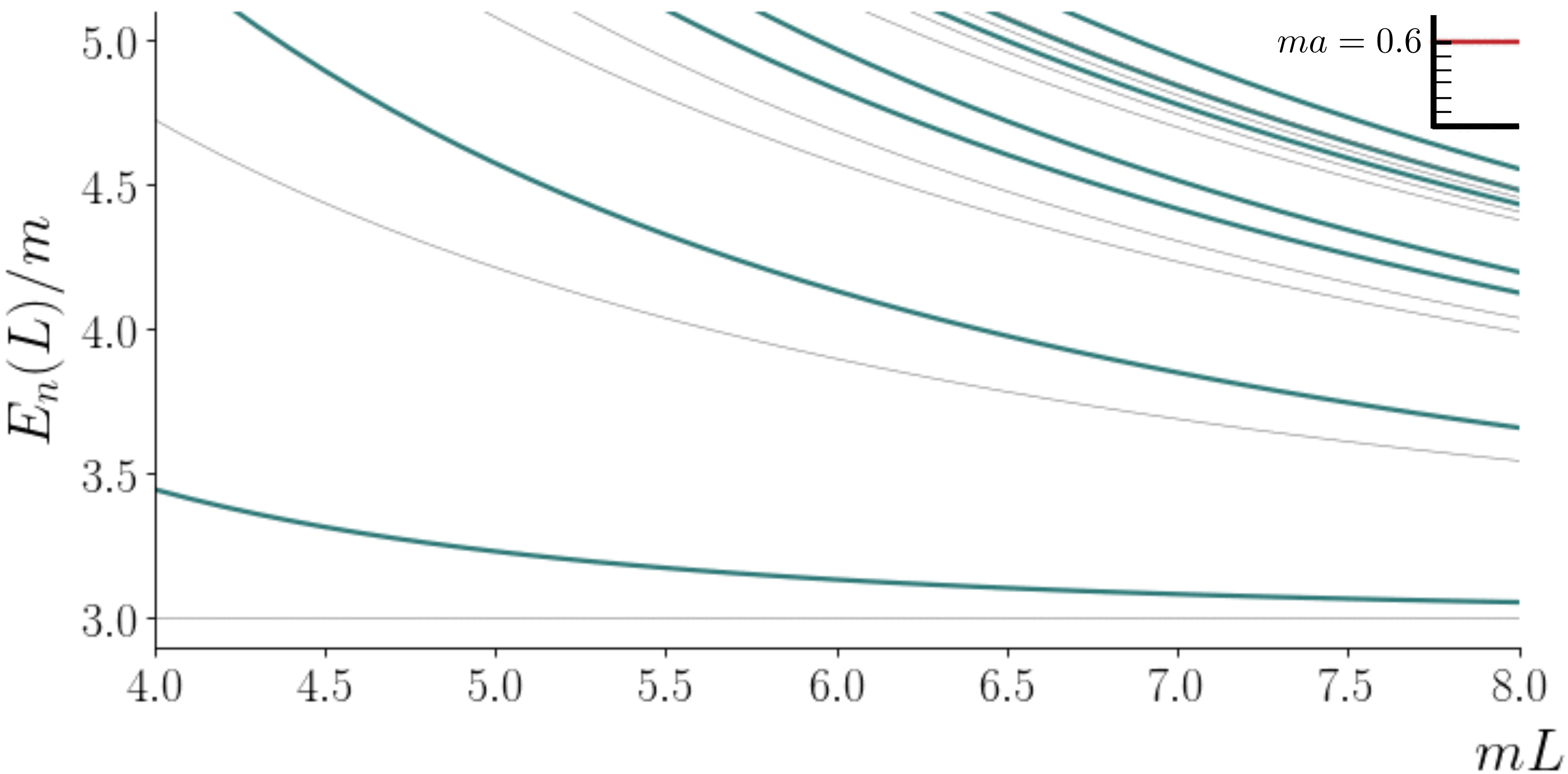
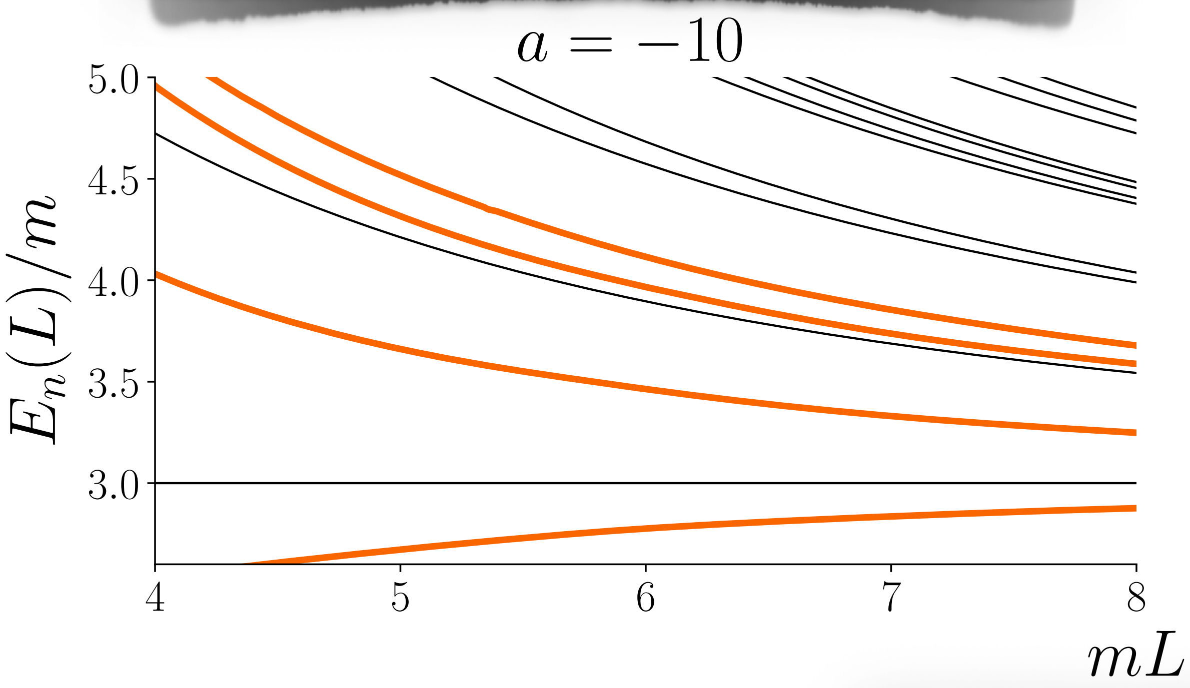




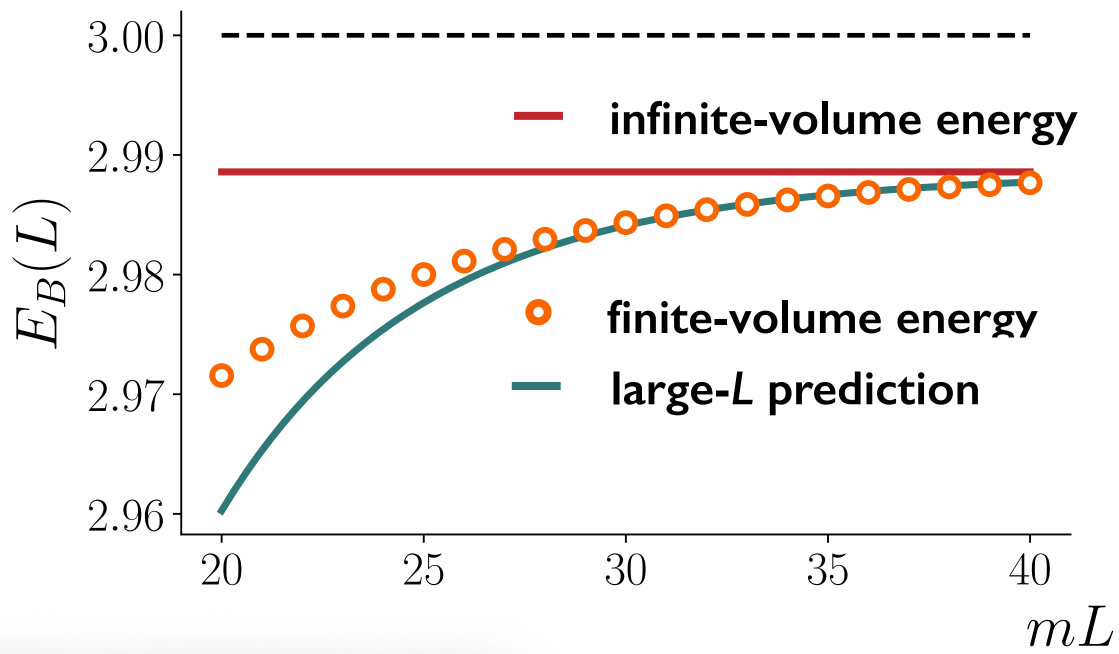


Many toy results

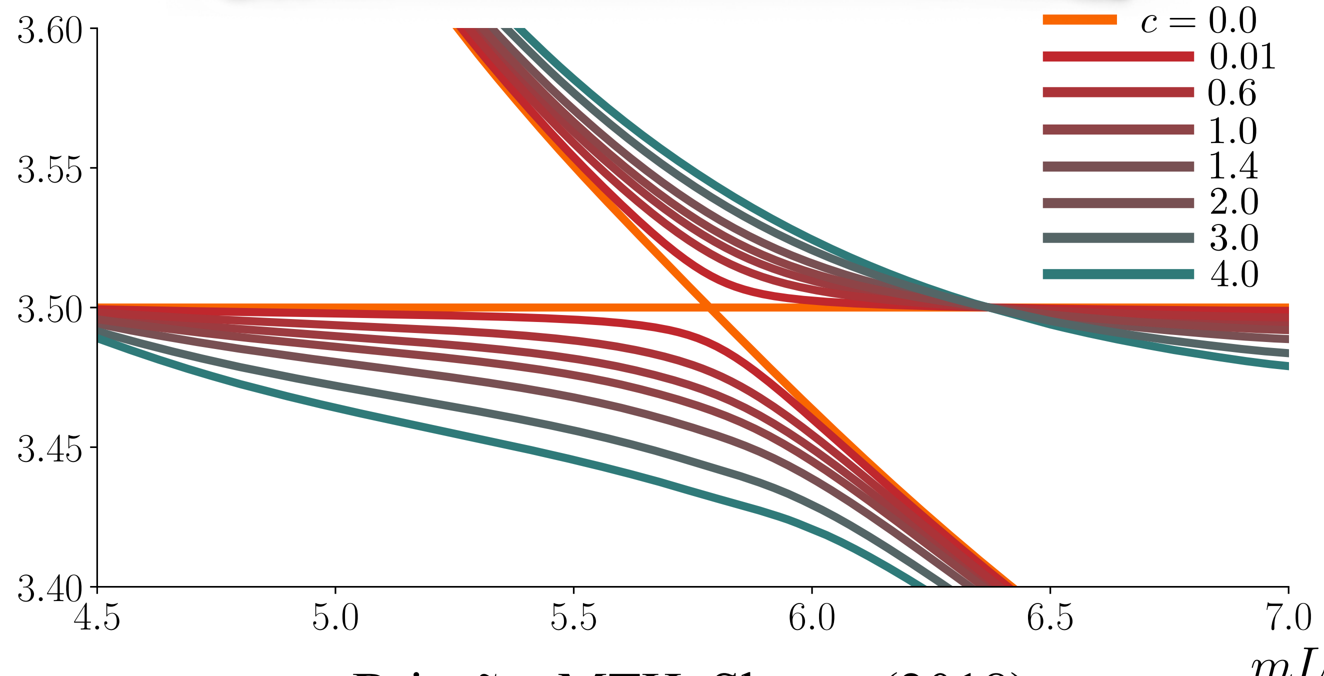
Spectrum with no 3-particle interaction



Finite-volume effects on a 3-particle bound state



Model of a 3-particle resonance

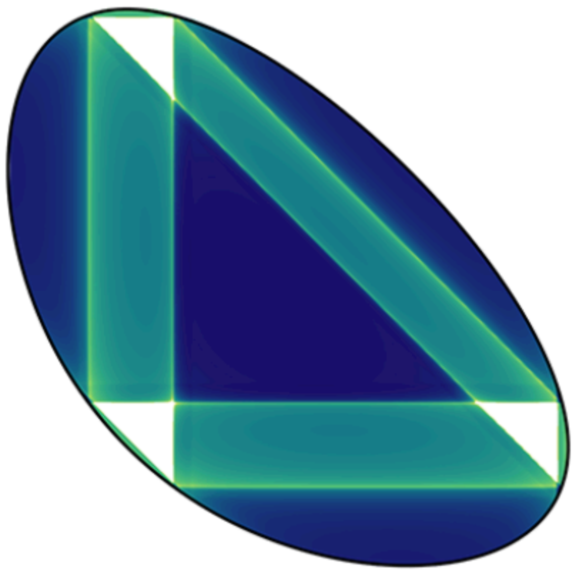


• Briceño, MTH, Sharpe (2018) •

Energy-Dependent $\pi^+\pi^+\pi^+$ Scattering Amplitude from QCD

Maxwell T. Hansen,[1,2,*] Raul A. Briceño,[3,4,†] Robert G. Edwards,[3,‡]
Christopher E. Thomas,[5,§] and David J. Wilson[5,∥]

(for the Hadron Spectrum Collaboration)



EDITORS' SUGGESTION

Energy-Dependent $\pi^+ \pi^+ \pi^+$ Scattering Amplitude from QCD

A three-hadron scattering amplitude is computed using lattice QCD for the first time.

Maxwell T. Hansen *et al.*

Phys. Rev. Lett. **126**, 012001 (2021)

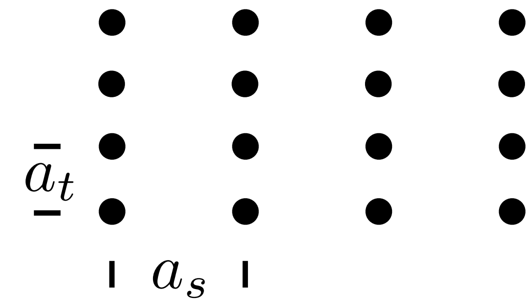
$\pi^+\pi^+\pi^+ \to \pi^+\pi^+\pi^+$

lattice details

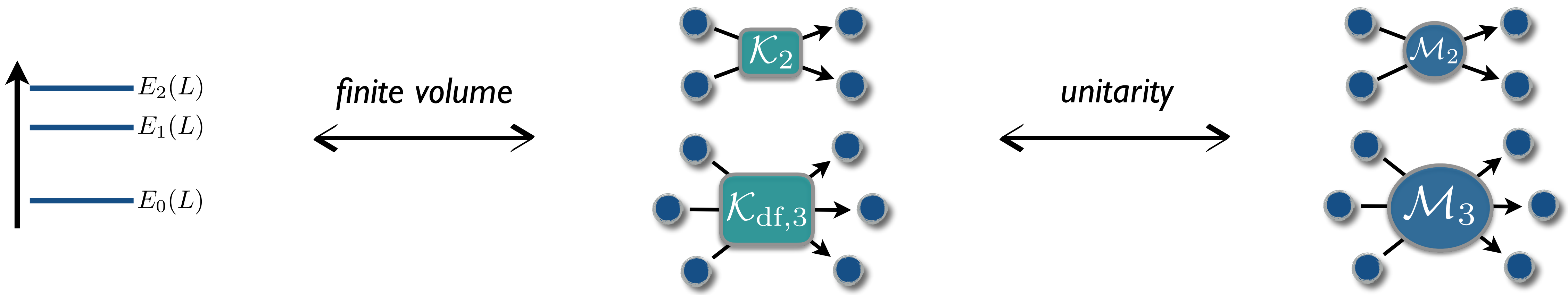
$N_f = 2 + 1 \qquad a_s/a_t = 3.444(6)$

$m_\pi \approx 400\text{MeV} \qquad a_s \approx 0.12\text{fm}$

$L_s/a_s = 20, 24$



- Workflow outline



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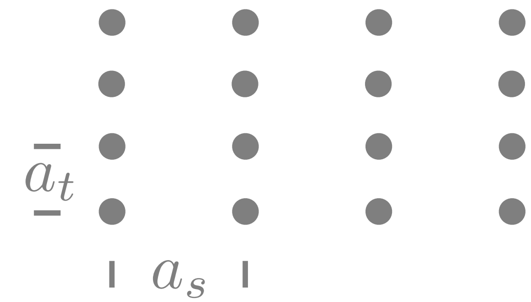

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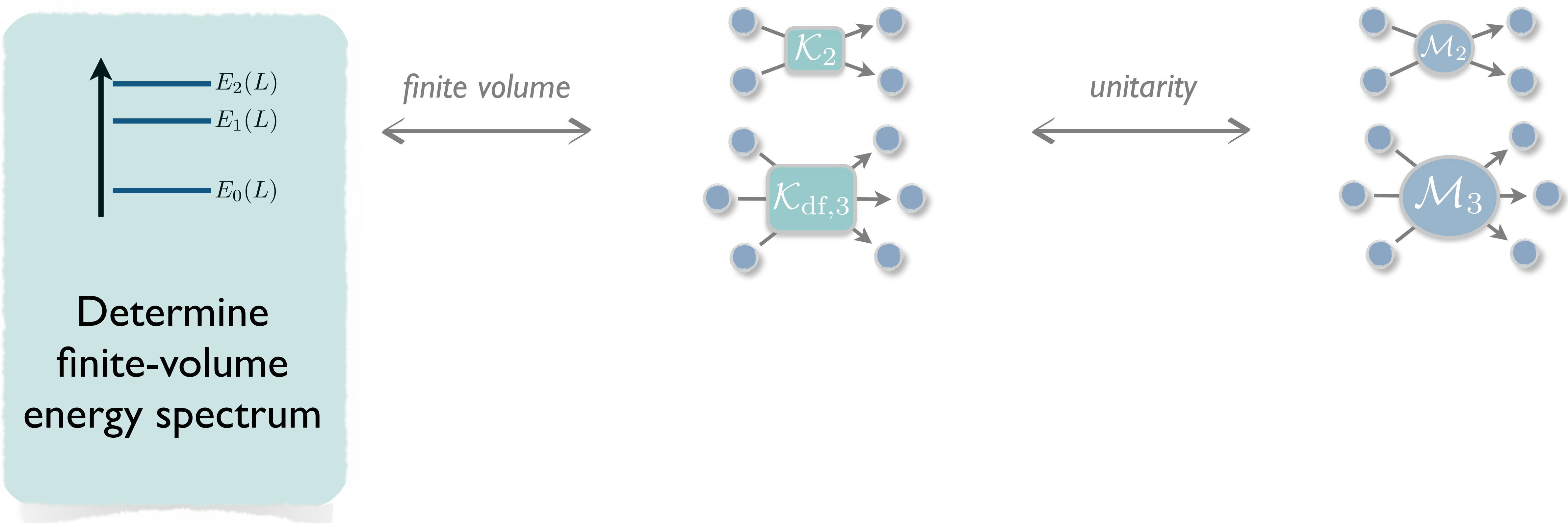


Operators

- Variational method → *optimized single hadron* $\pi = c_1\,\overline{q}\Gamma q + c_2\,\overline{q}\Gamma D q + \cdots$

- Build two-pion operators in the usual way…

$$\pi\pi(\boldsymbol{p}_{\pi\pi}, \Lambda_{\pi\pi}) = \sum \mathrm{CG}\,\pi(\boldsymbol{p}_1)\,\pi(\boldsymbol{p}_2)$$

… three-pion operators from there

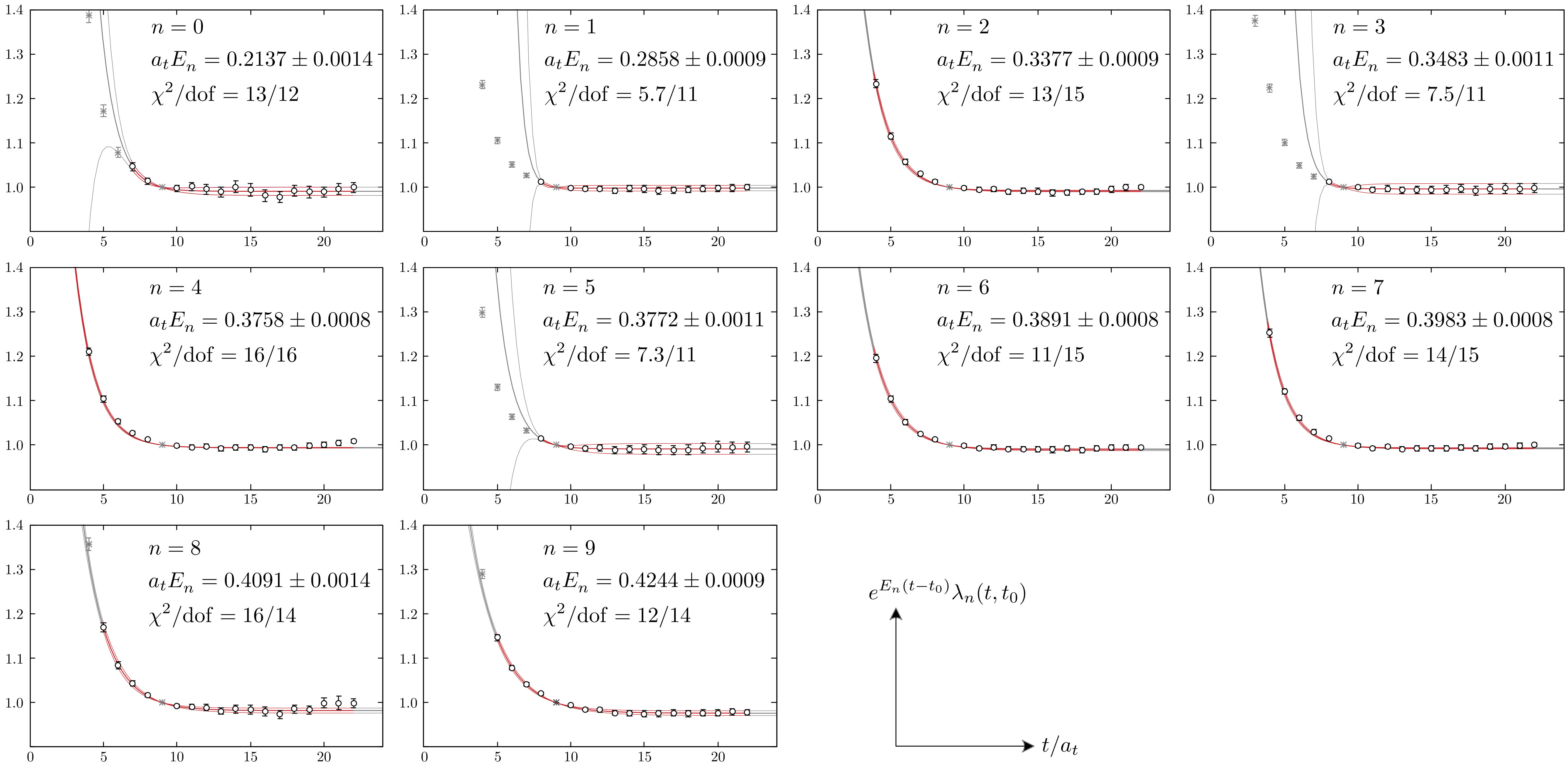
$$(\pi\pi\pi)(\boldsymbol{P}, \Lambda) = \sum \mathrm{CG}\,\pi(\boldsymbol{p}_3)\,\pi\pi(\boldsymbol{p}_{\pi\pi}, \Lambda_{\pi\pi})$$

- Note: No optimization of $\pi\pi(\boldsymbol{p}_{\pi\pi}, \Lambda)$ operators

- Analogous to…

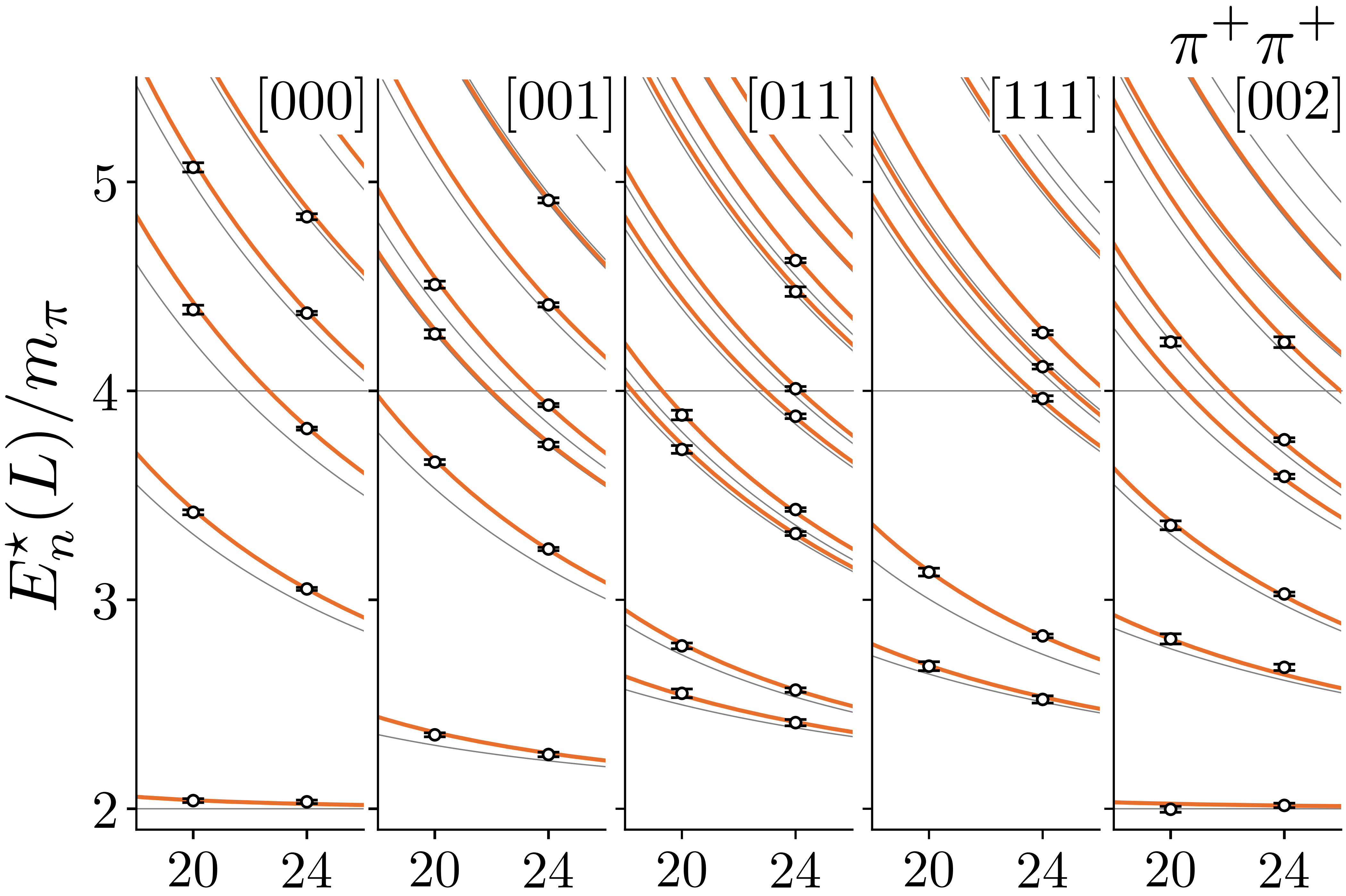
$$\left(\frac{1}{2} \otimes \frac{1}{2}\right) \otimes \frac{1}{2} = (0 \oplus 1) \otimes \frac{1}{2} = \frac{1}{2} \oplus \frac{1}{2} \oplus \frac{3}{2}$$

$I = 3\,(\pi^+\pi^+\pi^+),\ \boldsymbol{P} = [000],\ \Lambda = A_1^-,\ L/a_s = 24$



MTH, Briceño, Edwards, Thomas, Wilson, *Phys.Rev.Lett.* 126 (2021) 012001

$\pi^+\pi^+$ energies



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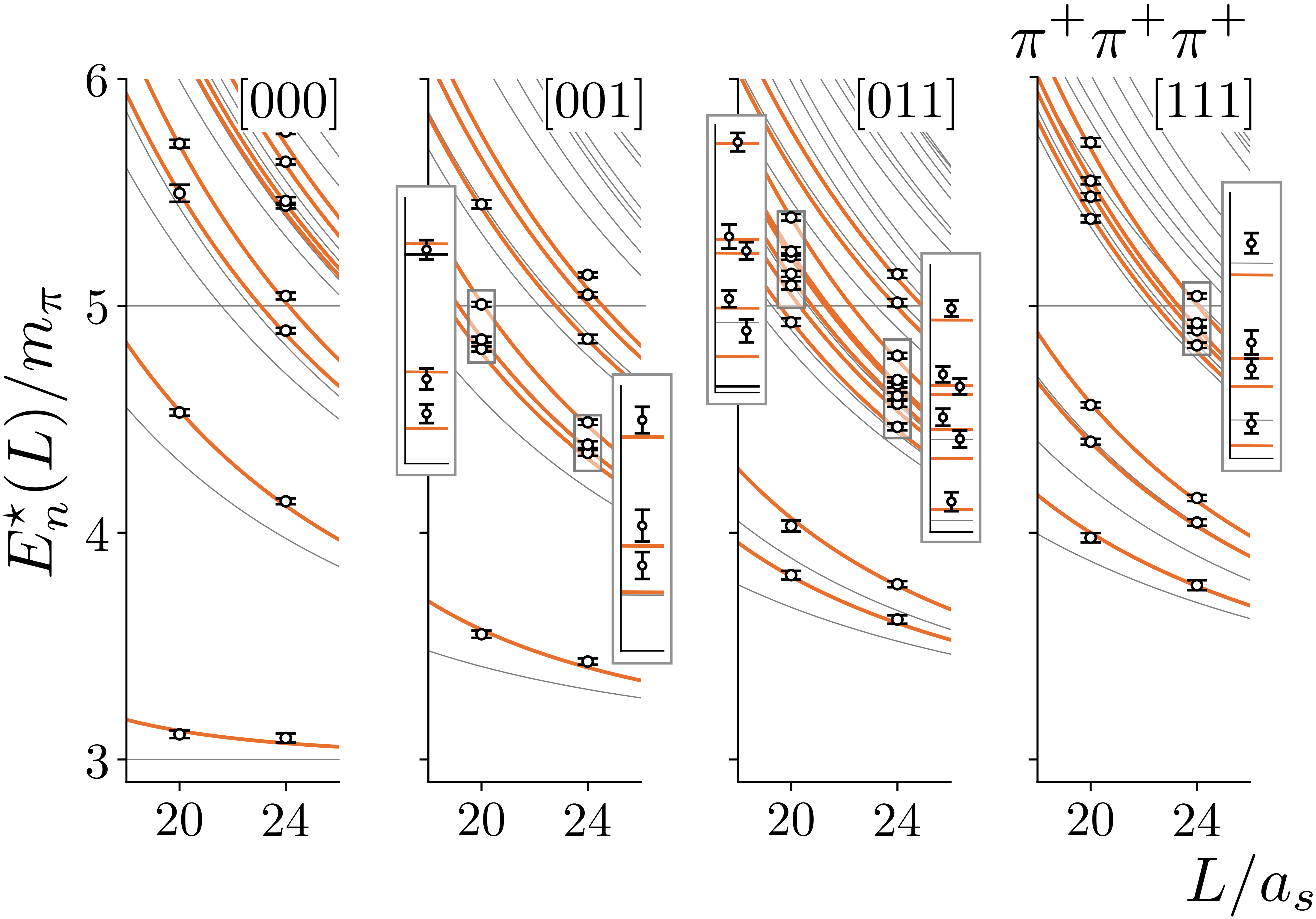

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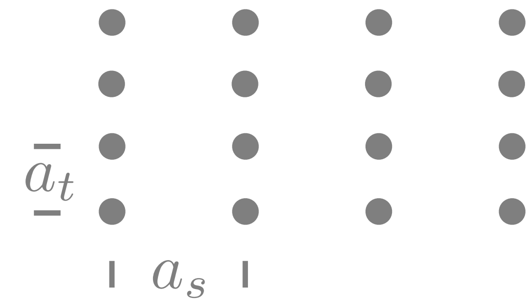

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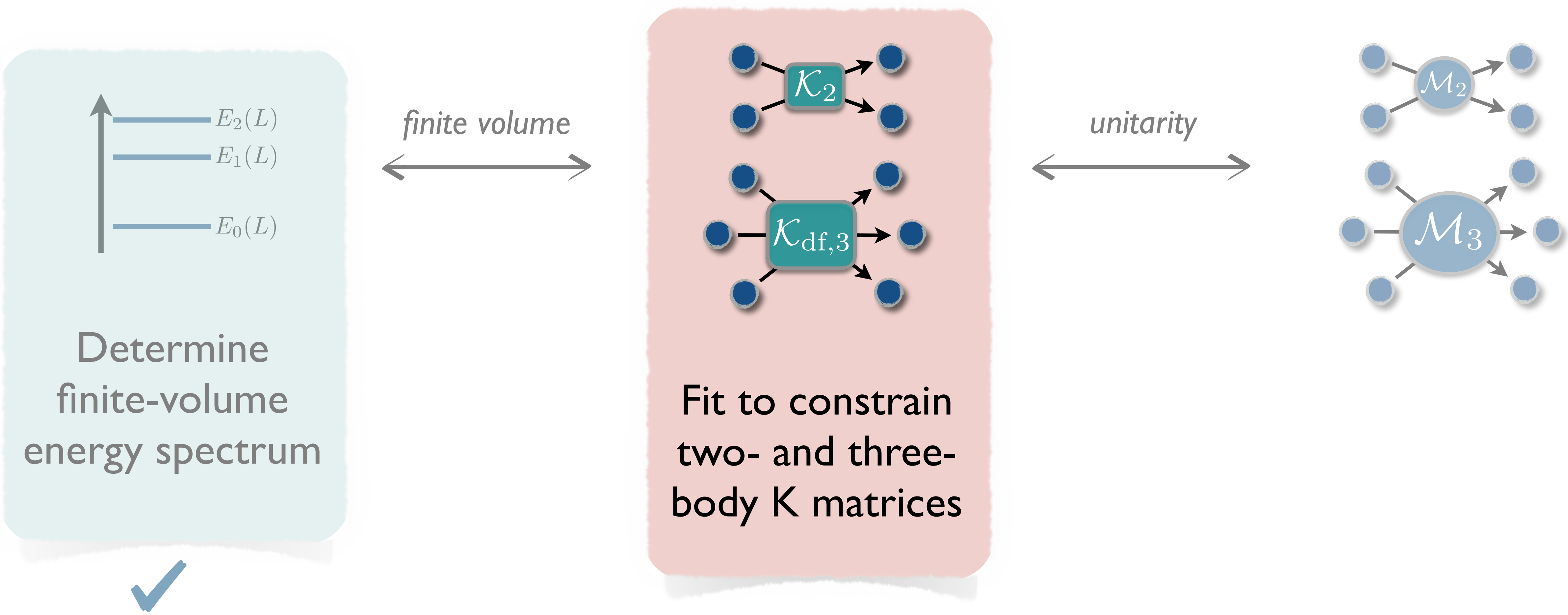
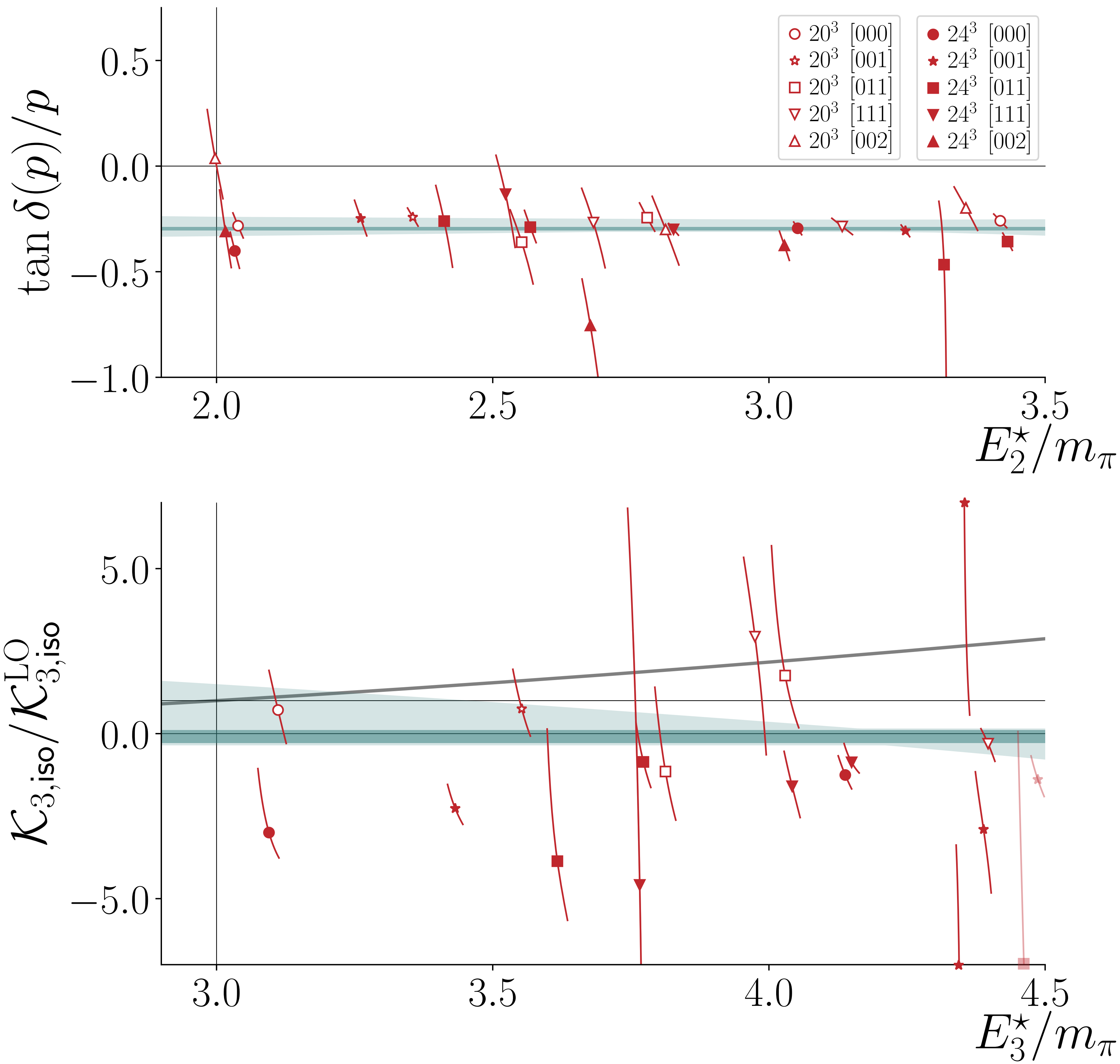


MTH, Briceño, Edwards, Thomas, Wilson, *Phys.Rev.Lett.* 126 (2021) 012001

K matrix fits



Finite-volume formalism relates energies to K matrices

One-to-one for $K_{\text{df},3}$ depending only on $E_{\text{cm}} = E^\star$

Fit both two and three-body K to various polynomials

Cut on the CM energy in the fits

$K_{\text{df},3}$ is scheme dependent (removed upon converting to $\mathcal{M}_3$)

$\pi^+\pi^+\pi^+ \to \pi^+\pi^+\pi^+$

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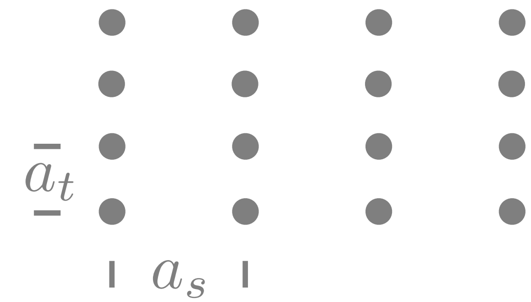

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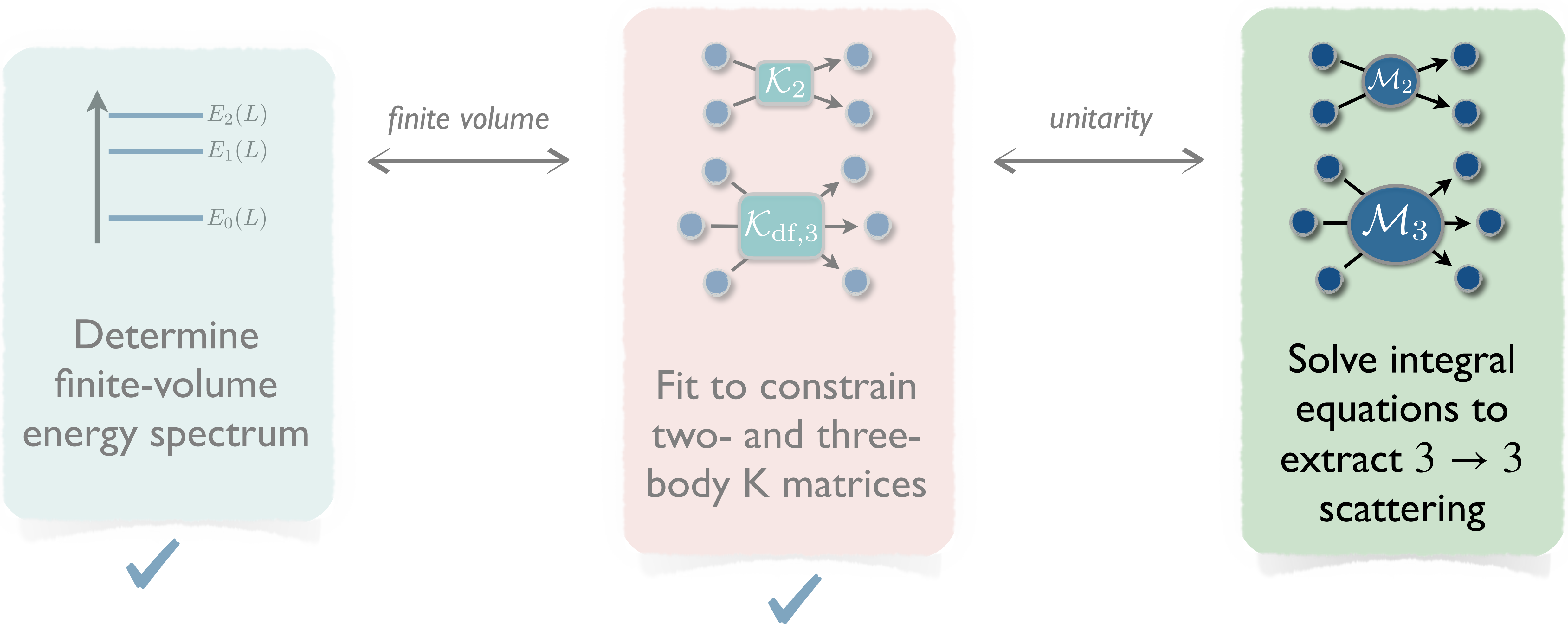

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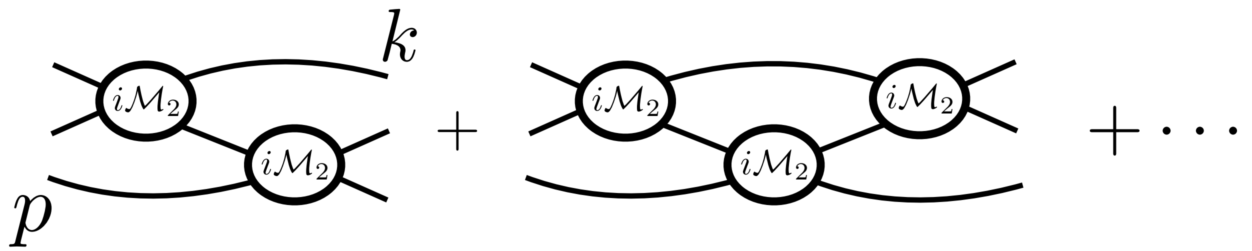
Integral equation

$$\mathcal{M}_3^{\mathsf{un}}(E_3^\star, \boldsymbol{p}, \boldsymbol{k}) = \mathcal{D}^{\mathsf{un}}(E_3^\star, \boldsymbol{p}, \boldsymbol{k}) + \mathcal{E}^{\mathsf{un}}(E_3^\star, \boldsymbol{p})\mathcal{T}(E_3^\star)\mathcal{E}^{\mathsf{un}}(E_3^\star, \boldsymbol{k})$$

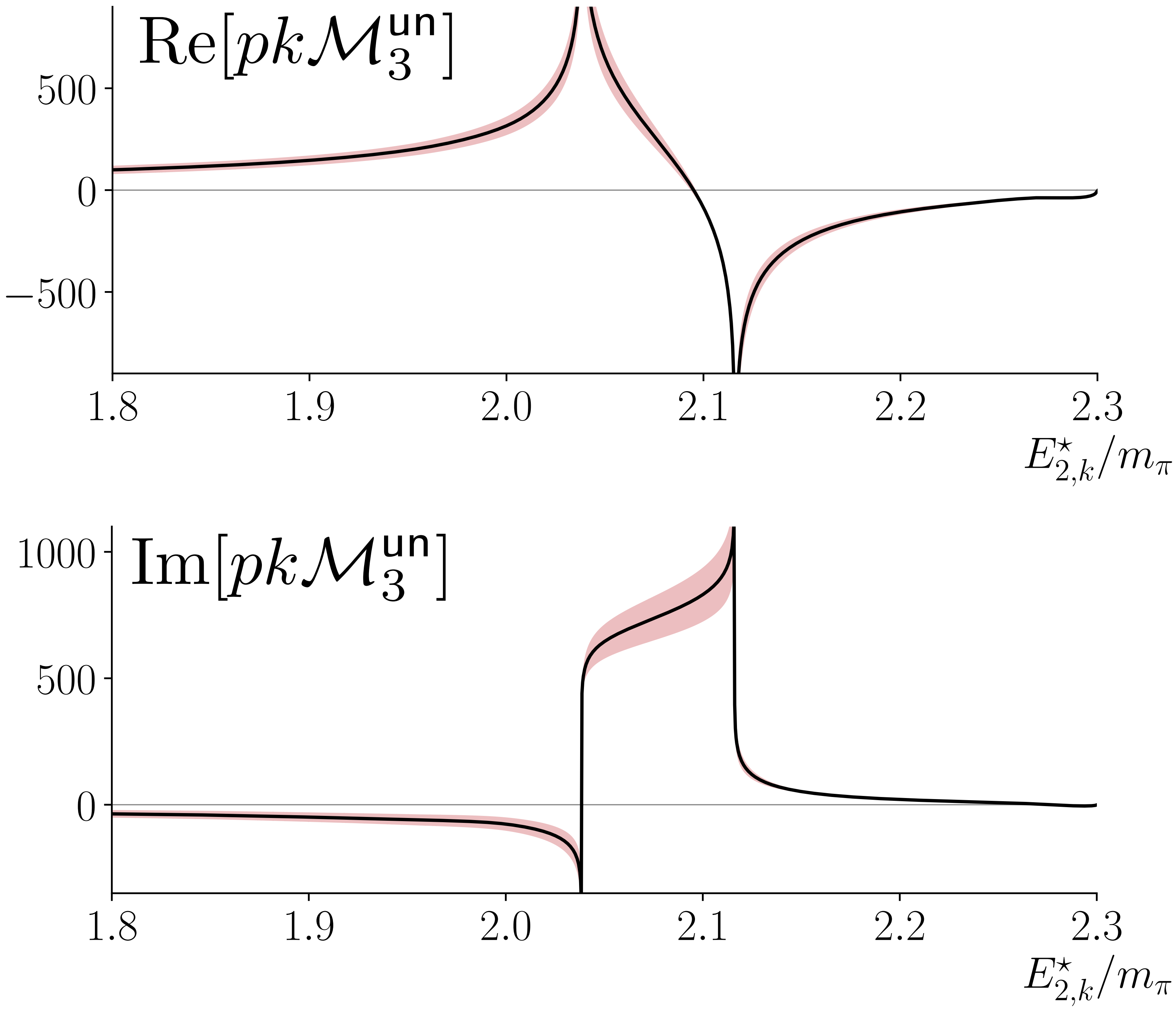


Vanishes for $K_{\mathrm{df},3} = 0$

$$\boldsymbol{D}(N, \epsilon) = -\boldsymbol{\mathcal{M}} \cdot \boldsymbol{G}(\epsilon) \cdot \boldsymbol{\mathcal{M}} - \boldsymbol{\mathcal{M}} \cdot \boldsymbol{G}(\epsilon) \cdot \boldsymbol{P} \cdot \boldsymbol{D}(N, \epsilon)$$

$$\mathcal{D}^{\mathsf{un}}(E_3^\star, \boldsymbol{p}, \boldsymbol{k}) = \lim_{\epsilon \to 0} \lim_{N \to \infty} \boldsymbol{D}_{pk}(N, \epsilon)$$

Integral equation



Total angular momentum = 0

Two-particle sub-system angular momentum = 0

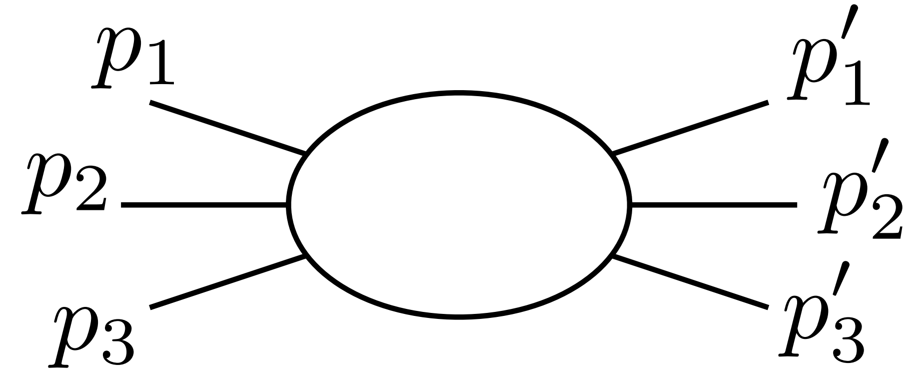
Plot at fixed $E_3^\star$ and p

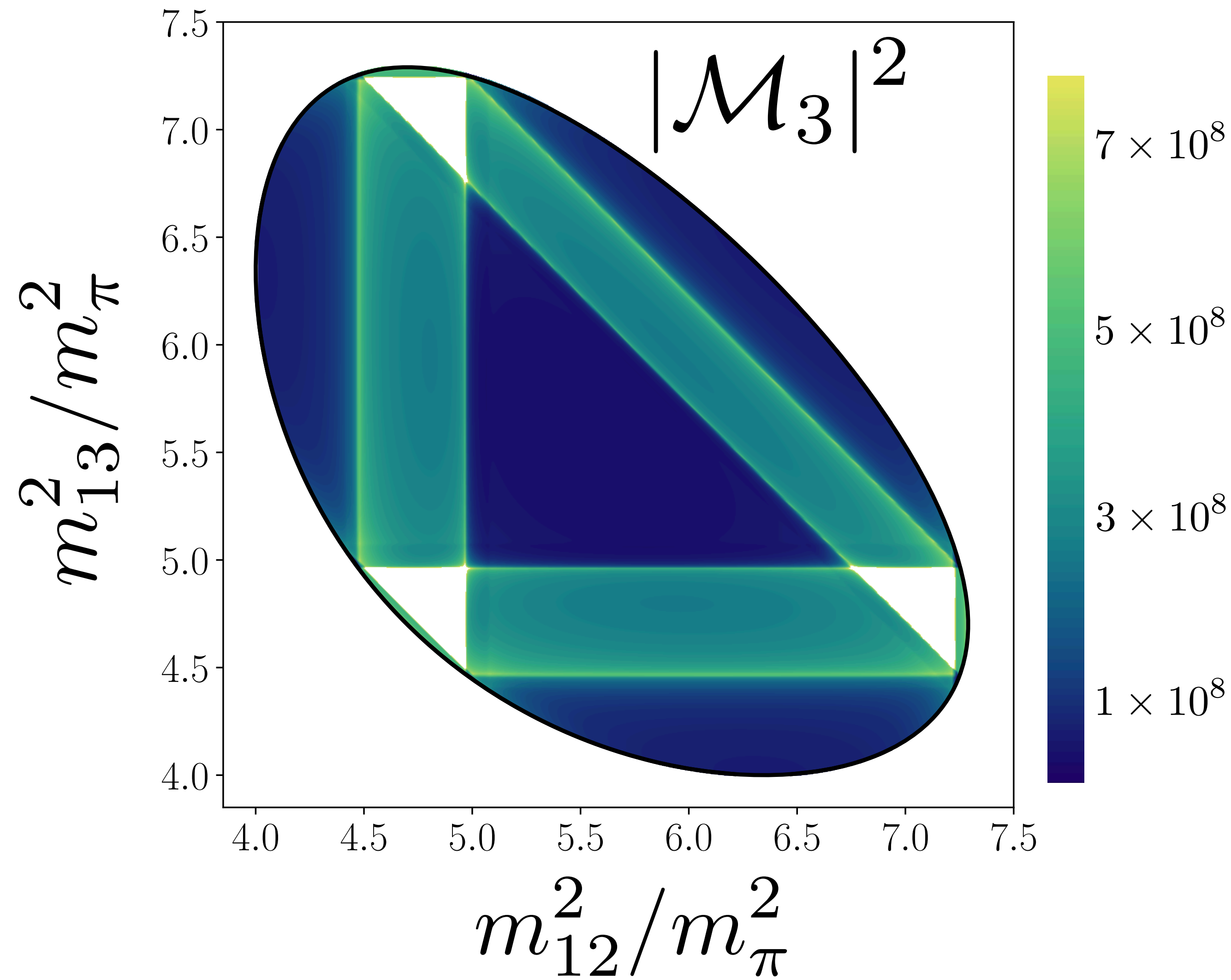
Both two- and three-body uncertainties estimated

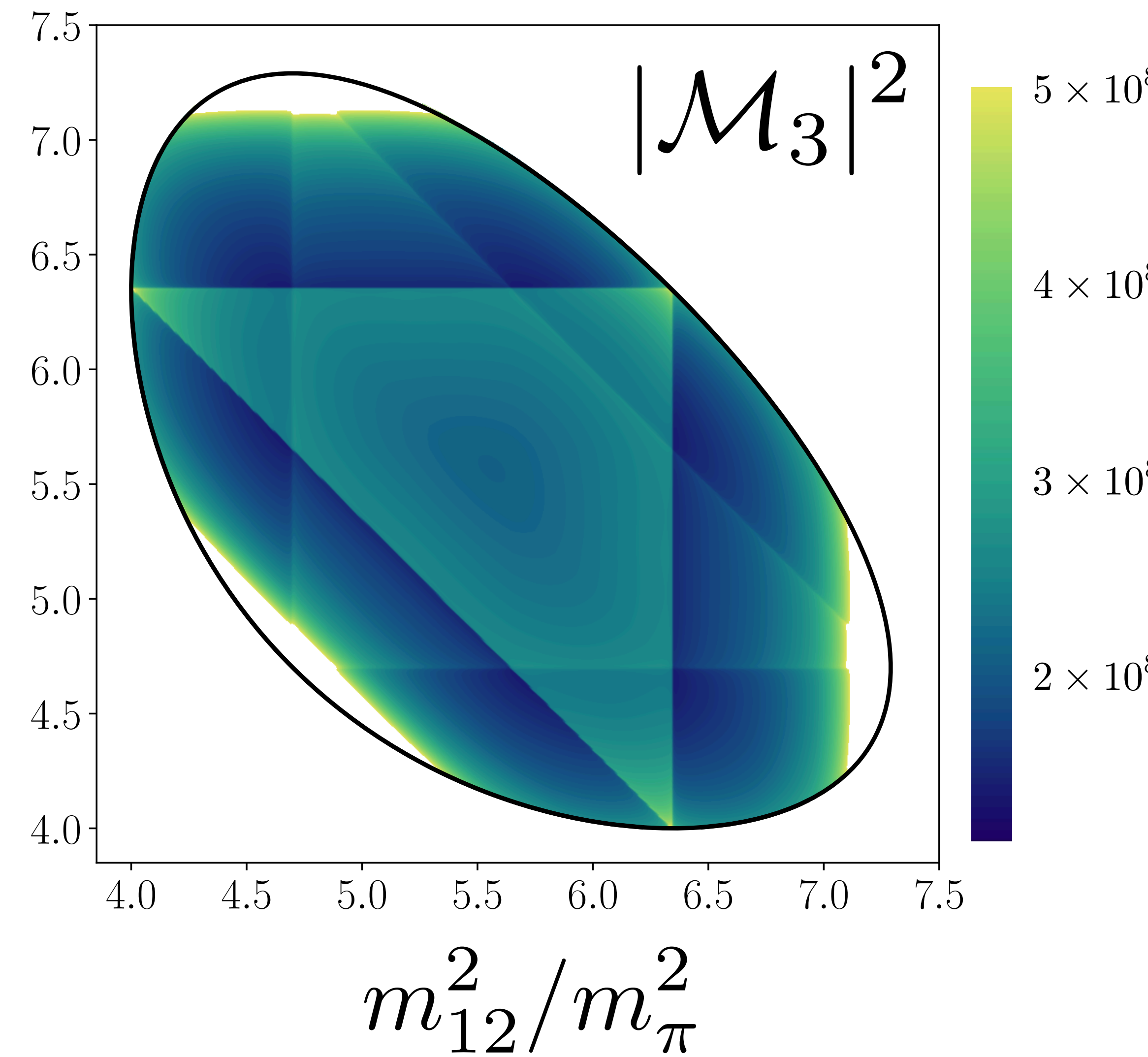
Still need to symmetrize

MTH, Briceño, Edwards, Thomas, Wilson, *Phys.Rev.Lett.* 126 (2021) 012001

$$\mathcal{M}_3 = \sum_{i,j\in\{1,2,3\}} \mathcal{M}_3^{\mathsf{un}}(p_i', p_j)$$







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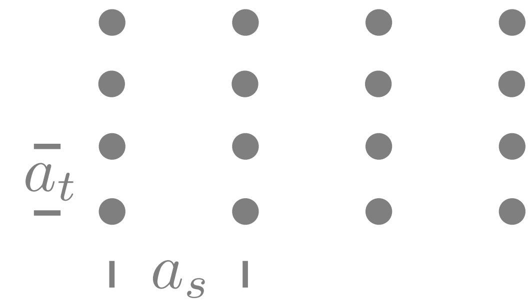

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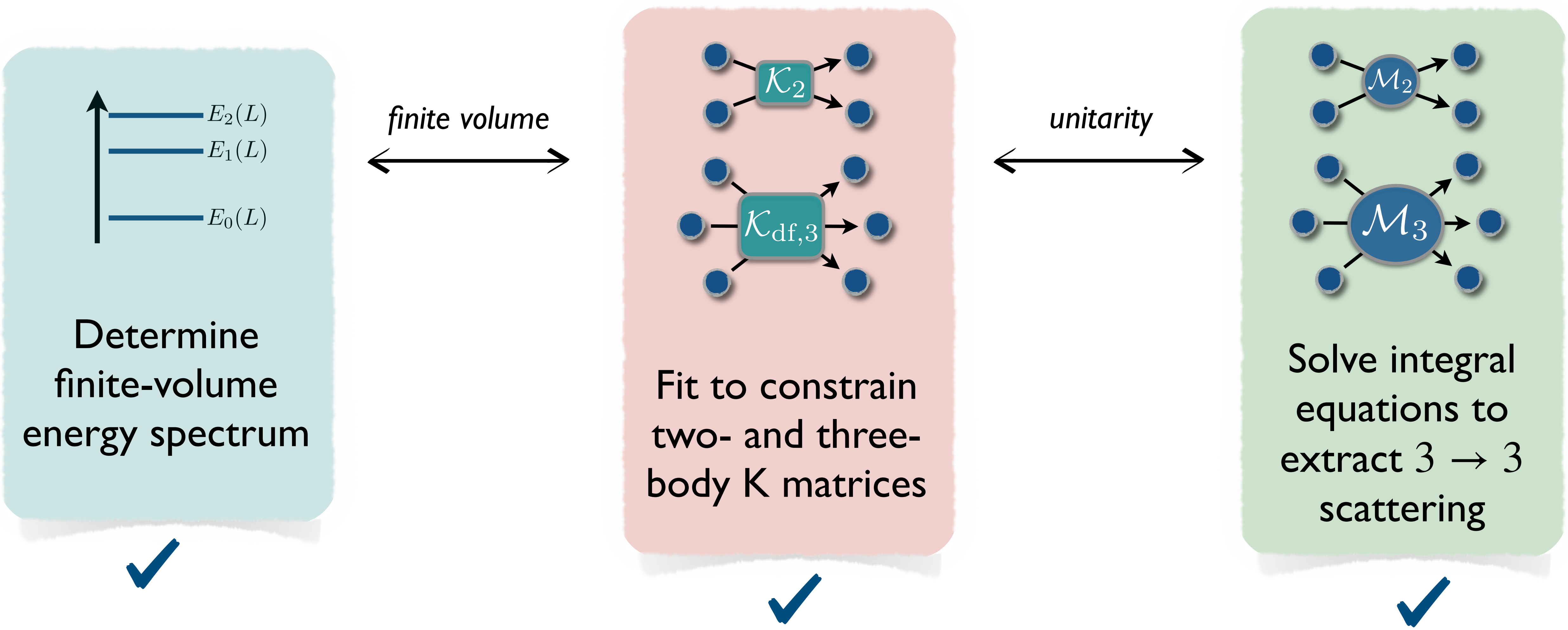

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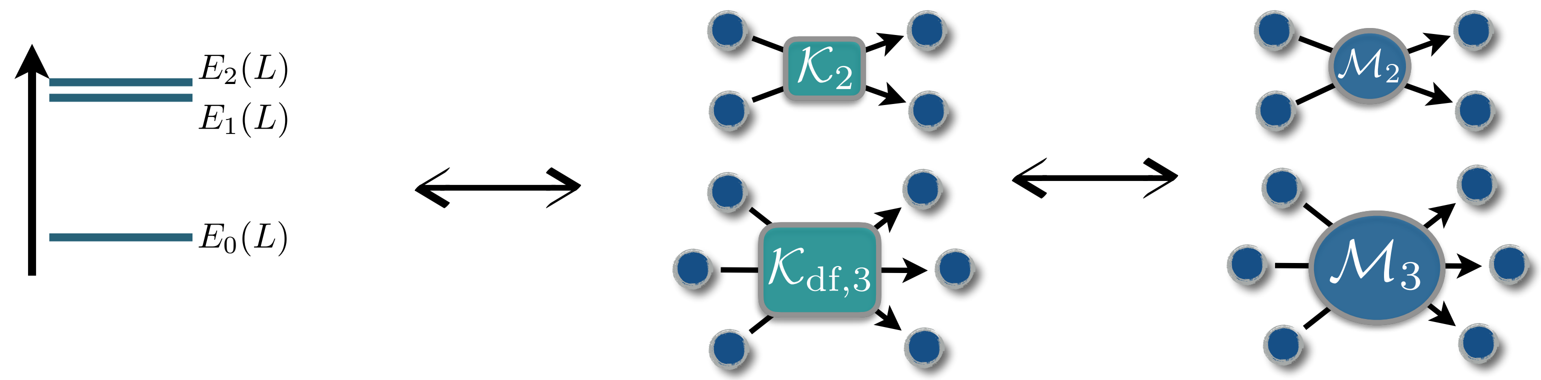
Conclusions

- The finite-volume = ***a useful tool***



- First lattice QCD applications in 2019… ***much more to come***

- Challenges and progress

  formal analysis was technical → *ground work is now set*

  3-body amplitude is highly singular → *intermediate K matrix overcomes this*

  many-body demands high precision excited states → *advanced algorithms make this possible*

- Many groups are now involved… this is great! *Thanks for listening!*