Generalizing the Lellouch-Luscher formula to three-particle decays

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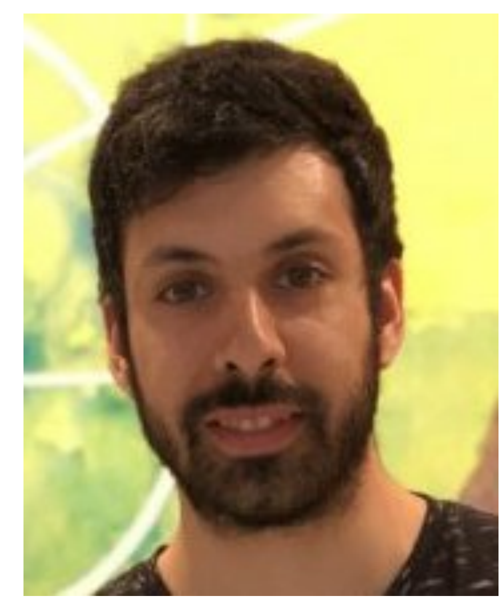
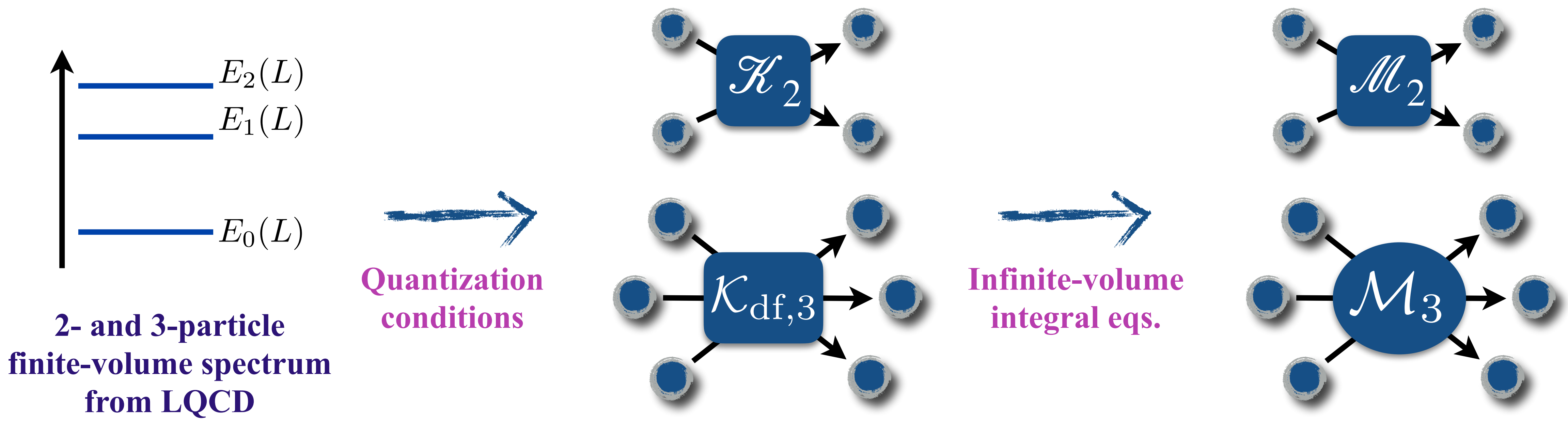
Based on work with Max Hansen and Fernando Romero-López:
[arXiv: 2101.10246] (JHEP)



Summary

- We derive a general formalism allowing the calculation of $1 \xrightarrow{\mathscr{H}_W} 3$ decay amplitudes using lattice QCD (LQCD)
  - Formalism for $1 \xrightarrow{\mathscr{H}_W} 2$ (e.g. $K \to \pi\pi$) is a standard LQCD tool [Lellouch & Lüscher, 2001]
  - Recently, considerable progress made in determining $3 \to 3$ amplitudes from the spectrum of 3 particle states obtained using LQCD [Refs in backup slides]
    - We use the generic effective field theory (RFT) approach [Hansen & SS, `14, `15; Hansen, Romero-López & SS, `00]
  - We extend the $3 \to 3$ formalism to $0 \xrightarrow{J} 3$ and $1 \xrightarrow{\mathscr{H}_W} 3$ processes involving 3 degenerate (but not necessarily identical) spinless particles in the final state
- Several phenomenologically relevant applications
  - $K \to 3\pi$: LQCD can now (in principle) determine CP-conserving and violating amplitudes
  - $\eta \to 3\pi$ (at first order in isospin breaking): alternative determination of $m_u - m_d$
  - $\gamma^* \to 3\pi$: component of hadronic contributions to muonic $g-2$

Schematic of $3 \to 3$ formalism

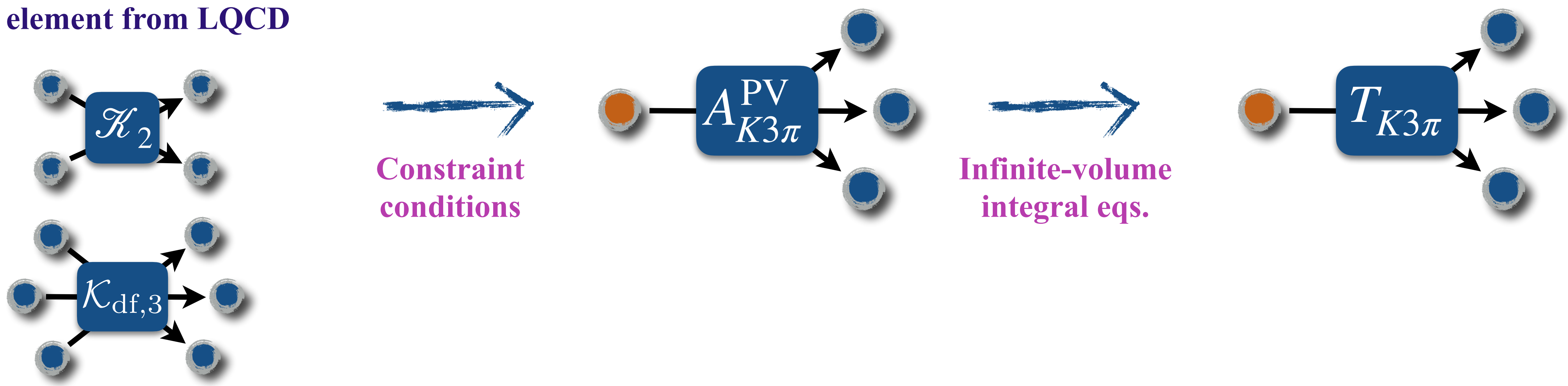


- $\mathcal{K}_{\mathrm{df},3}$ is an infinite-volume (but scheme dependent) $3 \to 3$ K matrix
- It is real, and smooth aside from possible 3-particle resonance poles
- LQCD applications require a parametrization of $\mathcal{K}_{\mathrm{df},3}$, e.g. a threshold expansion
- Integral equations ensure unitarity of $\mathcal{M}_3$, and incorporate initial- and final-state interactions

Schematic of $1 \xrightarrow{\mathscr{H}_W} 3$ formalism

$\langle 3\pi, L | \mathscr{H}_W | K, L \rangle$

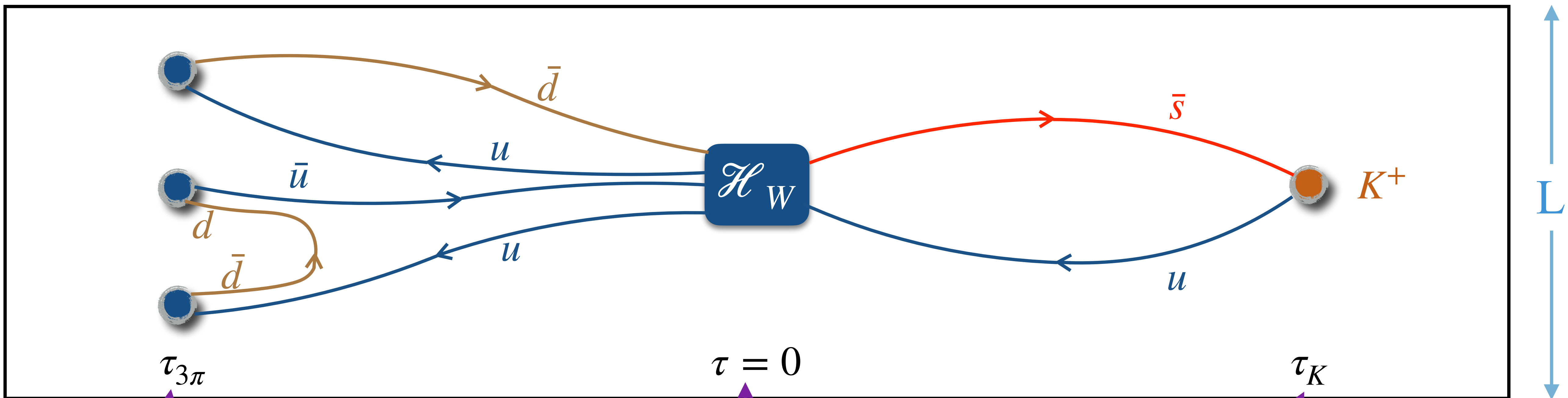
finite-volume matrix element from LQCD



- $A^{\mathrm{PV}}_{K3\pi}$ is a Lorentz-invariant infinite-volume (but scheme dependent) $1 \xrightarrow{\mathscr{H}_W} 3$ amplitude
- It is real (aside from phases in $\mathscr{H}_W$) and smooth
- LQCD applications require a parametrization of $A^{\mathrm{PV}}_{K3\pi}$
- Integral equations incorporate final-state interactions into $T_{K3\pi}$

Ingredients from LQCD

(I) Finite-volume matrix element: $\langle 3\pi, L | \mathscr{H}_W | K, L \rangle$



Use 3π operator determined from spectrum calculation to project onto desired finite-volume state with energy $E_n(\mathbf{P}, L)$

Add parts of $\mathscr{H}_W$ one at a time

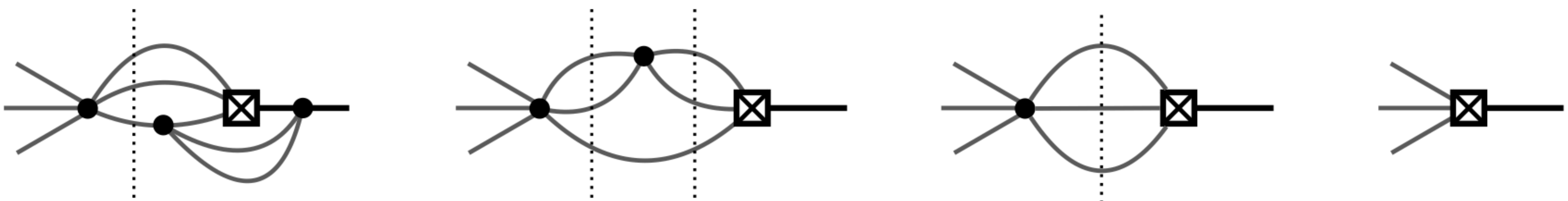
Send $\tau_K \ll 0$ to project onto kaon, with energy $E_K(\mathbf{P})$

Adjust L so that $E_n(\mathbf{P}, L) = E_K(\mathbf{P})$

(II) $\mathscr{K}_2$ and $\mathscr{K}_{\mathrm{df},3}$ from 2- and 3-particle spectrum

Method of derivation

- Determine all-orders expressions for $3 \to 3$ and $1 \xrightarrow{\mathscr{H}_W} 3$ finite-volume correlators in generic relativistic EFT in which d.o.f. are πs and Ks



- Dominant $1/L^n$ finite-volume dependence arises from 3π cuts
- Cuts connected by short-distance infinite-volume amplitudes in which the 3π poles regulated by PV prescription

$$C_{K3\pi,L}(P) = C_{K3\pi,\infty}(P) - A_{3\pi}^{\mathrm{PV}} \frac{1}{F_3^{-1} + \mathcal{K}_{\mathrm{df},3}} A_{K3\pi}^{\mathrm{PV}}$$

Can determine and cancel from $3 \to 3$ correlator

Matrix known from $3 \to 3$ analysis; has poles at finite-volume energies

This is what we want to extract

Results (step 1)

- Residue of finite-volume matrix determines vector v

$$\mathcal{R}_{\Lambda\mu}(E_n^\Lambda, \boldsymbol{P}, L) = \lim_{P_4 \to iE_n^\Lambda} -(E_n^\Lambda + iP_4)\, \mathbb{P}_{\Lambda\mu} \cdot \frac{1}{F_3^{-1} + \mathcal{K}_{\mathrm{df},3}} \cdot \mathbb{P}_{\Lambda\mu} \quad = v(E_n^\Lambda, \boldsymbol{P}, \Lambda\mu, L) v^\dagger(E_n^\Lambda, \boldsymbol{P}, \Lambda\mu, L)\,.$$

- Each finite-volume matrix element determines a projection of $A^{\mathrm{PV}}_{K3\pi}$

$$\sqrt{2E_K(\boldsymbol{P})} L^3 \langle E_n, \boldsymbol{P}, \Lambda\mu, L | \mathcal{H}_W(0) | K, \boldsymbol{P}, L\rangle = v^\dagger A^{\mathrm{PV}}_{K3\pi}\,.$$

- With enough matrix elements can determine parameters in $A^{\mathrm{PV}}_{K3\pi}$

$$A^{\mathrm{PV}}_{K3\pi} = A^{\mathsf{iso}} + A^{(2)} \sum_i \Delta_i^2 + A^{(3)} \sum_i \Delta_i^3 + A^{(4)} \sum_i \Delta_i^4 + \mathcal{O}(\Delta^5)\,.$$

For simplified case with no isospin; Constrained by Lorentz invariance, particle-interchange symmetry

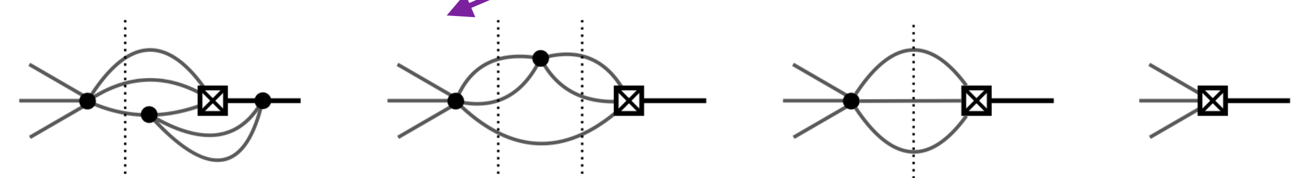
$$A^{\mathsf{iso}} = \sum_{n=0}^{\infty} \Delta^n A^{\mathsf{iso,n}}\,, \qquad \Delta = \frac{m_K^2 - 9m_\pi^2}{9m_\pi^2}\,. \qquad \Delta_i = \frac{s_i - 4m_\pi^2}{9m_\pi^2}\,, \qquad s_i = (p_j + p_k)^2 = (P - p_i)^2$$

Results (step 2)

Now amputated
(and asymmetric)

- Consider finite-volume decay matrix element in EFT

$$T^{(u)}_{K3\pi,L} = \mathcal{L}^{(u)}_L \frac{1}{1+\mathcal{K}_{\mathrm{df},3}F_3} A^{\mathrm{PV}}_{K3\pi} \;=$$



- Take appropriate $L \to \infty$ limit (with $i\epsilon$ prescription) and obtain decay amplitude

$$T^{(u)}_{K3\pi}(\boldsymbol{k})_{\ell m} = \lim_{\epsilon\to 0^+} \lim_{L\to\infty} T^{(u)}_{K3\pi,L}(\boldsymbol{k})_{\ell m}\Big|_{E\to E+i\epsilon}$$

$$\begin{aligned} T_{K3\pi}(\boldsymbol{k},\hat{\boldsymbol{a}}^*) &\equiv \mathcal{S}\left\{T_{K3\pi}(\boldsymbol{k})_{\ell m}\right\} , \\ &= T^{(u)}_{K3\pi}(\boldsymbol{k},\hat{\boldsymbol{a}}^*) + T^{(u)}_{K3\pi}(\boldsymbol{a},\hat{\boldsymbol{b}}^*) + T^{(u)}_{K3\pi}(\boldsymbol{b},\hat{\boldsymbol{k}}^*) , \end{aligned}$$

- Resulting integral equations depend on $\mathscr{K}_{\mathrm{df},3}$ and $\mathscr{K}_2$, and connect $A^{\mathrm{PV}}_{K3\pi}$ to $T_{K3\pi}$
- Similar to integral equations arising in $3 \to 3$ scattering, solutions to which have recently been obtained numerically [Hansen et al., 2009.04931; Jackura et al. 2010.09820]

Isotropic approximation

- $A_{K3\pi}^{\mathrm{PV}}$ and $\mathscr{K}_{\mathrm{df},3}$ are independent of momenta, and $\mathscr{K}_2$ is pure s-wave
  - Only a single finite-volume matrix element from LQCD is needed to determine $A_{K3\pi}^{\mathrm{PV}}$
  - Integral equations still needed, but simplify considerably
  - Can combine two steps & give single expression (ignoring isospin)

$$|T_{K3\pi}^{\mathsf{iso}}(E^*, m_{12}^2, m_{23}^2)|^2 = 2E_K(\boldsymbol{P})L^6 \Big| \langle E_n, \boldsymbol{P}, A_1, L| \mathcal{H}_W(0) |K, \boldsymbol{P}, L\rangle \Big|^2$$

$$\times \left| \mathcal{L}^{\mathsf{iso}}(E^*, m_{12}^2, m_{23}^2) \frac{1}{1 + \mathcal{K}_{\mathrm{df},3}^{\mathsf{iso}}(E^*) F_3^{\infty,\mathsf{iso}}(E^*)} \right|^2 \left(\frac{\partial F_3^{\mathsf{iso}}(E, \boldsymbol{P}, L)^{-1}}{\partial E} + \frac{\partial \mathcal{K}_{\mathrm{df},3}^{\mathsf{iso}}(E^*)}{\partial E} \right)$$

Analog of Lellouch-Lüscher factor

Obtain by solving single integral equation involving; Incorporates two-particle final-state interactions

Incorporates three-particle final-state interactions

Analogous to expression obtained using leading-order NREFT in [Müller & Rusetsky, 2012.13957]

Comparison with LL

- $1 \to 3$ result in isotropic approximation (and without isospin)

$$|T^{\mathsf{iso}}_{K3\pi}(E^*, m_{12}^2, m_{23}^2)|^2 = 2E_K(\boldsymbol{P})L^6 \Big| \langle E_n, \boldsymbol{P}, A_1, L | \mathcal{H}_W(0) | K, \boldsymbol{P}, L \rangle \Big|^2$$
$$\times \left| \mathcal{L}^{\mathsf{iso}}(E^*, m_{12}^2, m_{23}^2) \frac{1}{1 + \mathcal{K}^{\mathsf{iso}}_{\mathrm{df},3}(E^*) F_3^{\infty,\mathsf{iso}}(E^*)} \right|^2 \left(\frac{\partial F_3^{\mathsf{iso}}(E, \boldsymbol{P}, L)^{-1}}{\partial E} + \frac{\partial \mathcal{K}^{\mathsf{iso}}_{\mathrm{df},3}(E^*)}{\partial E} \right)$$

- $1 \to 2$ result in s-wave approximation (only case used in practice to date)

$$|T_{K2\pi}(E)|^2 = 2M_K L^6 | \langle E_n, A_1, L | \mathcal{H}_W(0) | K, L \rangle |^2$$
$$\times \left| \frac{1}{1 - i\mathcal{K}_2(E)\rho(E)} \right|^2 \left(\frac{\partial F(E, L)^{-1}}{\partial E} + \frac{\partial \mathcal{K}_2(E)}{\partial E} \right)$$

Alternative form of Lellouch-Lüscher result

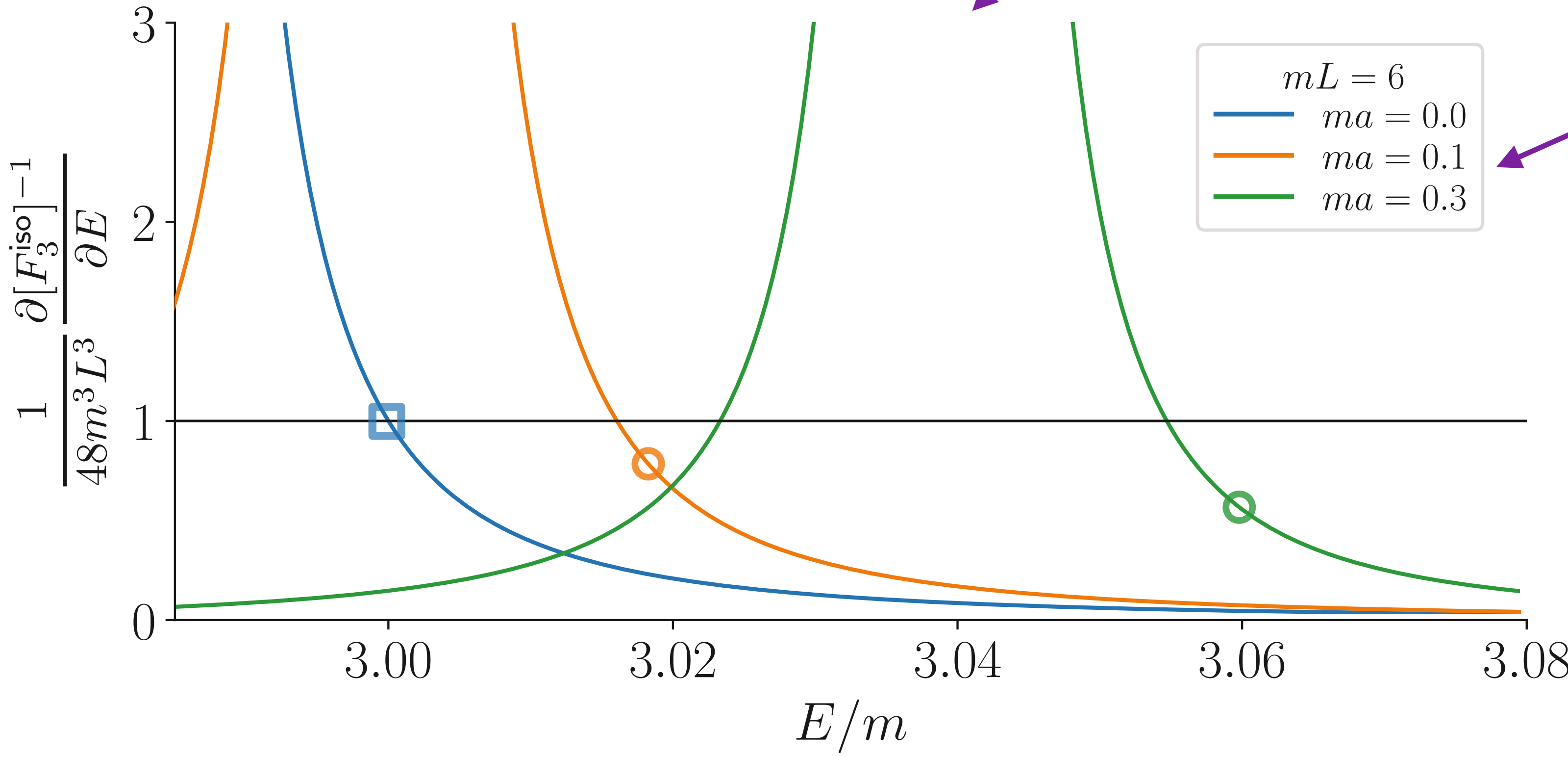
Includes Watson phase

Size of finite-volume corrections

- $1 \to 3$ result in isotropic approximation (and without isospin)

$$|T^{\text{iso}}_{K3\pi}(E^*, m_{12}^2, m_{23}^2)|^2 = 2E_K(\boldsymbol{P})L^6 \Big| \langle E_n, \boldsymbol{P}, A_1, L| \mathcal{H}_W(0) |K, \boldsymbol{P}, L\rangle \Big|^2$$

$$\times \left| \mathcal{L}^{\text{iso}}(E^*, m_{12}^2, m_{23}^2) \frac{1}{1+\mathcal{K}^{\text{iso}}_{\text{df},3}(E^*)F_3^{\infty,\text{iso}}(E^*)} \right|^2 \left(\frac{\partial F_3^{\text{iso}}(E,\boldsymbol{P},L)^{-1}}{\partial E} + \frac{\partial \mathcal{K}^{\text{iso}}_{\text{df},3}(E^*)}{\partial E} \right)$$



$\mathcal{K}_2$ given in terms of scattering length a

$\mathcal{K}^{\text{iso}}_{\text{df},3}$ assumed energy independent

Summary & Outlook

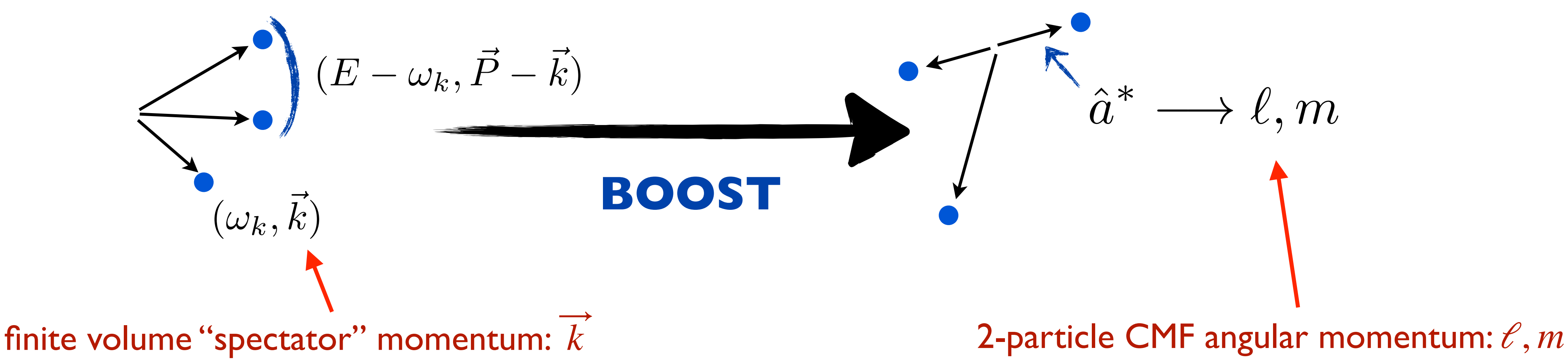
- We have derived a general, relativistically-invariant, formalism allowing the calculation of $1 \xrightarrow{\mathscr{H}_W} 3$ and $0 \xrightarrow{J} 3$ decay/transition amplitudes using LQCD
  - It piggybacks on the recent progress on $3 \to 3$ amplitudes
  - It holds for any such process involving three degenerate spinless particles in the final state
  - It requires two steps, the first accounting for finite-volume effects, and the second incorporating the effects of final-state interactions
- We hope that applications will be possible in the near future
  - Distillation and other algorithmic advances allow the calculation of the necessary quark contractions
  - $\gamma^* \to 3\pi$ is the simplest to study; isoscalar part of current couples to I=0 (ω) channel
  - $\eta \to 3\pi$ is next simplest, as it involves insertion of quark bilinear; couples to I=1
  - $K \to 3\pi$ is most challenging, with $\mathscr{H}_W$ a four-fermion operator, leading to more complicated contractions, and I=0, 1 and 2 final states

Thanks
Any questions?

Backup slides

Scope & Notation

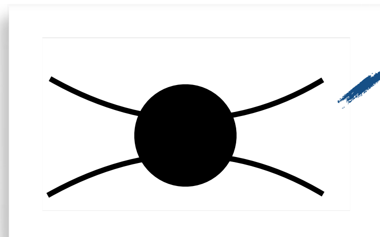
- Identical spinless particles of mass m (e.g. $3\pi^+$)
- Z_2 symmetry — no $2 \to 3$ transitions
- All quantities in QC3 are infinite-dimensional matrices with indices $\{\vec{k}, \ell, m\}$ describing 3 on-shell particles with total energy-momentum $(E, \vec{P})$

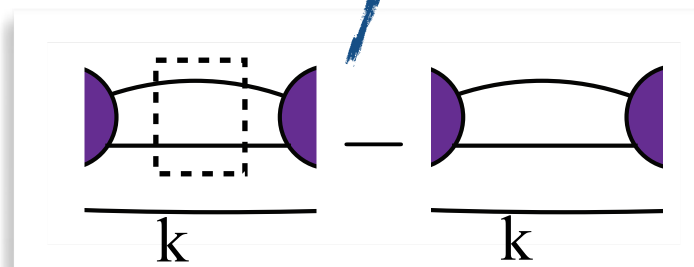


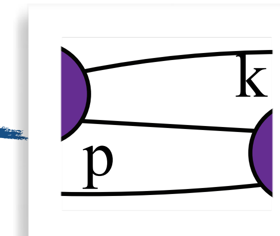
e.g. $\left[\mathscr{K}_{\text{df},3}^{(u,u)}\right]_{k\ell m;p\ell'm'}$

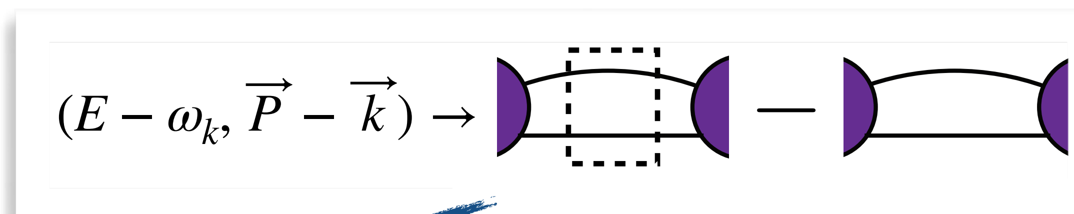
F_3 collects 2-particle interactions

$$F_3 = \left[\frac{\widetilde{F}}{3} - \widetilde{F} \frac{1}{(2\omega L^3 \mathscr{K}_2)^{-1} + \widetilde{F} + \widetilde{G}} \widetilde{F} \right]$$









- F & G are known geometrical functions, containing cutoff function H

$$\widetilde{F}_{p\ell'm';k\ell m} = \frac{1}{2\omega_k L^3} \delta_{pk} \; H(\vec{k}) \; F_{\mathrm{PV},\ell'm';\ell m}(E - \omega_k, \vec{P} - \vec{k}, L)$$

$$\widetilde{G}_{p\ell'm';k\ell m} = \frac{1}{2\omega_p L^3} \left(\frac{k^*}{q_p^*} \right)^{\ell'} \frac{4\pi Y_{\ell'm'}(\hat{k}^*) H(\vec{p}) H(\vec{k}) Y^*_{\ell m}(\hat{p}^*)}{(P-k-p)^2 - m^2} \left(\frac{p^*}{q_k^*} \right)^{\ell} \frac{1}{2\omega_k L^3}$$

RFT 3-particle papers

Max Hansen & SRS:

"Relativistic, model-independent, three-particle quantization condition,"

arXiv:1408.5933 (PRD) [HS14]

"Expressing the 3-particle finite-volume spectrum in terms of the 3-to-3 scattering amplitude,"

arXiv:1504.04028 (PRD) [HS15]

"Perturbative results for 2- & 3-particle threshold energies in finite volume,"

arXiv:1509.07929 (PRD) [HSPT15]

"Threshold expansion of the 3-particle quantization condition,"

arXiv:1602.00324 (PRD) [HSTH15]

"Applying the relativistic quantization condition to a 3-particle bound state in a periodic box,"

arXiv: 1609.04317 (PRD) [HSBS16]

"Lattice QCD and three-particle decays of Resonances,"

arXiv: 1901.00483 (Ann. Rev. Nucl. Part. Science) [HSREV19]

Raúl Briceño, Max Hansen & SRS:

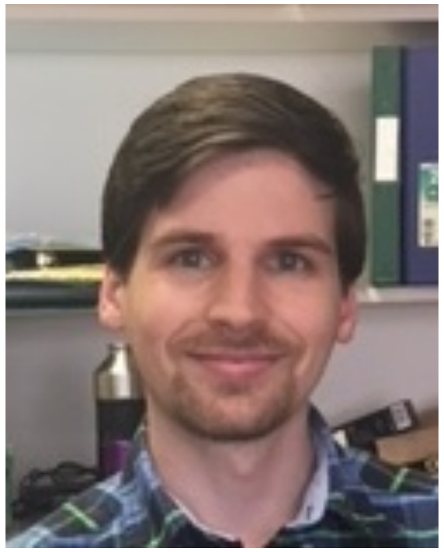
"Relating the finite-volume spectrum and the 2-and-3-particle S-matrix for relativistic systems of identical scalar particles," arXiv:1701.07465 (PRD) [BHS17]

"Numerical study of the relativistic three-body quantization condition in the isotropic approximation," arXiv:1803.04169 (PRD) [BHS18]

"Three-particle systems with resonant sub-processes in a finite volume," arXiv:1810.01429 (PRD 19) [BHS19]

SRS

"Testing the threshold expansion for three-particle energies at fourth order in φ^4 theory," arXiv:1707.04279 (PRD) [SPT17]



Tyler Blanton, Fernando Romero-López & SRS:

"Implementing the three-particle quantization condition including higher partial waves," arXiv:1901.07095 (JHEP) [BRS19]

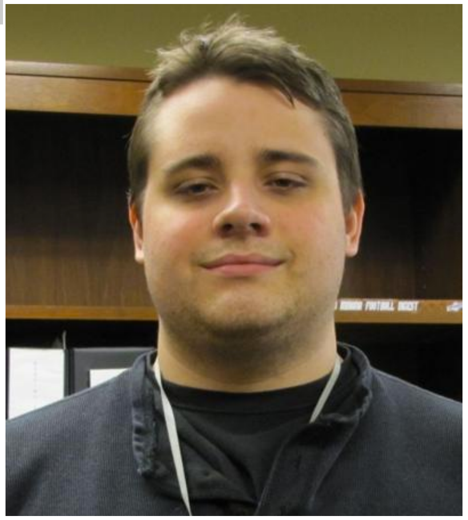
"I=3 three-pion scattering amplitude from lattice QCD," arXiv:1909.02973 (PRL) [BRS-PRL19]

Tyler Blanton, Raúl Briceño, Max Hansen, Fernando Romero-López, SRS:

"Numerical exploration of three relativistic particles in a finite volume including two-particle resonances and bound states", arXiv:1908.02411 (JHEP) [BBHRS19]

Raúl Briceño, Max Hansen, SRS & Adam Szczepaniak:

"Unitarity of the infinite-volume three-particle scattering amplitude arising from a finite-volume formalism," arXiv:1905.11188 (PRD)



Andrew Jackura, S. Dawid, C. Fernández-Ramírez, V. Mathieu, M. Mikhasenko, A. Pilloni, SRS & A. Szczepaniak:

"On the Equivalence of Three-Particle Scattering Formalisms," arXiv:1905.12007 (PRD)

Max Hansen, Fernando Romero-López, SRS:

"Generalizing the relativistic quantization condition to include all three-pion isospin channels", arXiv:2003.10974 (JHEP) [HRS20]

"Decay amplitudes to three particles from finite-volume matrix elements," arXiv: 2101.10246 (JHEP)

<u>Tyler Blanton & SRS:</u>

“Alternative derivation of the relativistic three-particle quantization condition,”

arXiv:2007.16188 (PRD) [BS20a]

“Equivalence of relativistic three-particle quantization conditions,”

arXiv:2007.16190 (PRD) [BS20b]

“Relativistic three-particle quantization condition for nondegenerate scalars,”

arXiv:2011.05520 (PRD)

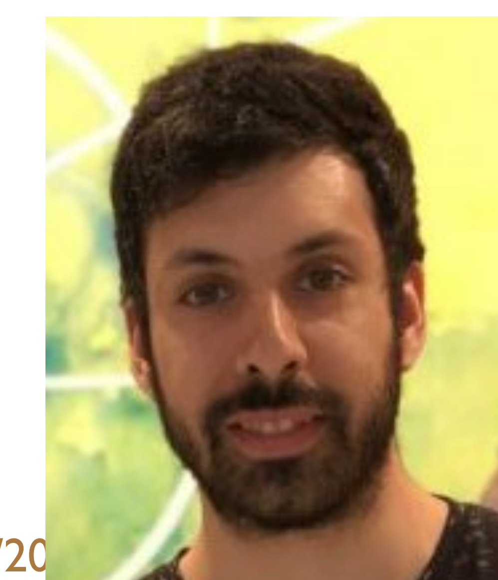
“Three-particle finite-volume formalism for $\pi^+\pi^+K^+$ and related systems,”

arXiv:2105.12904 (PRD under review)

<u>Tyler Blanton, Drew Hanlon, Ben Hörz, Colin Morningstar, Fernando Romero-López & SRS</u>

“$3\pi^+$ & $3K^+$ interactions beyond leading order from lattice QCD,”

arXiv:2106.05590 (JHEP under review)



Other work

★ **Implementing RFT integral equations**

- A. Jackura et al., 2010.09820 [Solving s-wave RFT integral equations in presence of bound states]
- M.T. Hansen et al. (HADSPEC), 2009.04931, PRL [Calculating $3\pi^+$ spectrum and using to determine three-particle scattering amplitude]

★ **Reviews**

- A. Rusetsky, 1911.01253 [LATTICE 2019 plenary]
- M. Mai, M. Döring and A. Rusetsky, 2103.00577 [Review of formalisms and chiral extrapolations]

★ **NREFT approach**

- H.-W. Hammer, J.-Y. Pang & A. Rusetsky, 1706.07700, JHEP & 1707.02176, JHEP [Formalism & examples]
- M. Döring et al., 1802.03362, PRD [Numerical implementation]
- J.-Y. Pang et al., 1902.01111, PRD [large volume expansion for excited levels]
- F. Müller, T. Yu & A. Rusetsky, 2011.14178, PRD [large volume expansion for I=1 three pion ground state]
- F. Romero-López, A. Rusetsky, N. Schlage & C. Urbach, 2010.11715, JHEP [generalized large-volume exps]
- F. Müller & A. Rusetsky, 2012.13957, JHEP [Three-particle analog of Lellouch-Lüscher formula]

Alternate 3-particle approaches

★ **Finite-volume unitarity (FVU) approach**

- M. Mai & M. Döring, 1709.08222, EPJA [formalism]
- M. Mai et al., 1706.06118, EPJA [unitary parametrization of M_3 involving R matrix; used in FVU approach]
- A. Jackura et al., 1809.10523, EPJC [further analysis of R matrix parametrization]
- M. Mai & M. Döring, 1807.04746, PRL [3 pion spectrum at finite-volume from FVU]
- M. Mai et al., 1909.05749,PRD [applying FVU approach to $3\pi^+$spectrum from Hanlon & Hörz]
- C. Culver et al., 1911.09047, PRD [calculating $3\pi^+$ spectrum and comparing with FVU predictions]
- A. Alexandru et al., 2009.12358, PRD [calculating $3K^-$ spectrum and comparing with FVU predictions]
- R. Brett et al., 2101.06144 [determining $3\pi^+$ interaction from LQCD spectrum]

★ **HALQCD approach**

- T. Doi et al. (HALQCD collab.), 1106.2276, Prog.Theor.Phys. [3 nucleon potentials in NR regime]