Study of bound states in a thermal gas using the S-matrix formalism

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Motivation

- ⋆ Measurement of bound states, such as deuteron (d), helium-3 (^{3}He), tritium (^{3}H), helium-4 (^{4}He), hypertritium ($^3_\Lambda$H) and their antiparticles are going on in high energy collisions.
- ⋆ Binding energies of bound-states are typically much smaller than the temperature realized in such collisions. How they are produced in such a hot environment?
- ⋆ Hadrons in thermal model:

$$P_\pm = \pm T \int \frac{d^3k}{(2\pi)^3} \ln \left[1 \pm e^{-\beta\sqrt{k^2+m^2}} \right]$$

The $+(-)$ sign corresponds to fermions (bosons).
How to include BS in a thermal model?

$$P_B = -T \int \frac{d^3k}{(2\pi)^3} \ln \left[1 - e^{-\beta\sqrt{k^2+M_B^2}} \right] ?$$

- ⋆ Is the weight factor 1?
- ⋆ Is interaction important?

Scattering amplitudes

$$\mathcal{L} = \frac{1}{2}\left(\partial_\mu \varphi\right)^2 - \frac{1}{2} m^2 \varphi^2 - \frac{\lambda}{4!}\varphi^4,$$

$$\frac{d\sigma}{d\Omega} = \frac{|A(s,t,u)|^2}{64\pi^2 s},$$

- s, t and u are Mandelstam variables

$A(s,t,u)$ in terms of partial waves:

$$A(s,t,u) = A(s,\theta) = \sum_{l=0}^{\infty} (2l+1) A_l(s) P_l(\cos\theta) ,$$

- $P_l(\xi)$ with $\xi = \cos\theta$ are the Legendre polynomials:

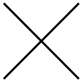
$$\int_{-1}^{+1} d\xi P_l(\xi) P_{l'}(\xi) = \frac{2}{2l+1}\delta_{ll'} .$$

l-th waves amplitude:

$$A_l(s) = \frac{1}{2}\int_{-1}^{+1} d\xi A(s,\theta) P_l(\xi)$$

Amplitudes and cross-section at the tree-level

Amplitude:



$$iA(s,t,u) = -i\lambda \quad \Rightarrow A(s,t,u) = A(s,\theta) = -\lambda$$

s-wave ($l=0$) amplitude:

$$A_0(s) = \frac{1}{2}\int_{-1}^{+1} d\xi A(s,\theta) = A(s,\theta) = -\lambda\,,$$

Amplitudes for $l > 0$: $\quad A_{l=1,2,..}(s) = 0$

Total cross-section:

$$\sigma(s) = \frac{1}{2} 2\pi \frac{1}{64\pi^2 s} \sum_{l=0}^{\infty} 2(2l+1)\left|A_l(s)\right|^2 = \frac{1}{32\pi s}\left|A_0(s)\right|^2 .$$

$$\sigma(s_{th} = 4m^2) = 8\pi \left|a_0^{\mathrm{SL}}\right|^2 , \quad a_0^{\mathrm{SL}} = \frac{1}{2}\frac{A_0(s=4m^2)}{8\pi\sqrt{4m^2}} = \frac{1}{2}\frac{-\lambda}{16\pi m} .$$

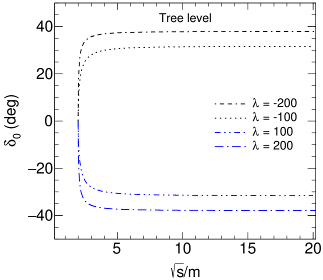
The factor 1/2 refers to identical particles.

Phase shifts

$$\frac{e^{2i\delta_l(s)}-1}{2i} = ka_l(s) = \frac{1}{2}\cdot\frac{k}{8\pi\sqrt{s}}A_l(s)\ ,$$

$k=\sqrt{\frac{s}{4}-m^2}$ is the modulus of the three-momentum of one of the ingoing (or outgoing) particles

$$\delta_0(s) = \frac{1}{2}\arg\left[1-\frac{1}{16\pi}\sqrt{\frac{4m^2}{s}-1}A_0(s)\right]$$



- ⋆ The asymptotic values $\delta_0(s\to\infty)$ do not tend to a multiple of $\pi \Rightarrow$ unitarization is required.
- ⋆ We can trust the results only when δ_0 is small.

Unitarization

We introduce the two-particle loop ($\Sigma(s)$) of the field φ: requirement (optical theorem): $\operatorname{Im}\Sigma(s)$ above threshold

$$I(s) = \operatorname{Im}\Sigma(s) = \frac{1}{2}\frac{\sqrt{\frac{s}{4} - m^2}}{8\pi\sqrt{s}} \text{for } \sqrt{s} > 2m.$$

$\Sigma(s)$ for complex values of the variable s reads

$$\Sigma(s) = \frac{1}{\pi}\int_{4m^2}^{\infty} ds' \frac{I(s')}{s' - s - i\epsilon} - C \ ,$$

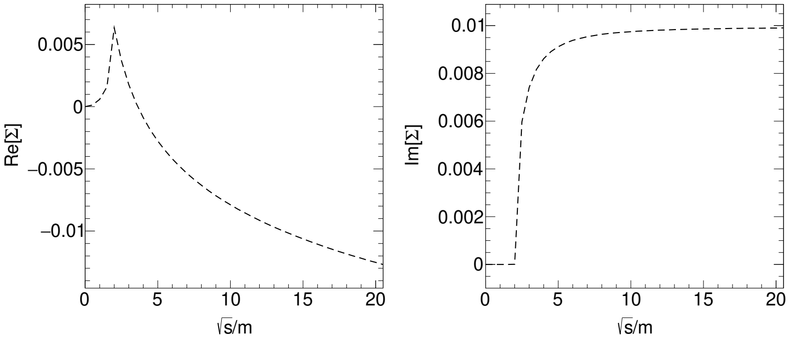
where the subtraction C guarantees convergence. Here, we make the choice $\Sigma(s \to 0) = 0$, hence

$$C = \frac{1}{\pi}\int_{4m^2}^{\infty} ds' \frac{I(s')}{s'}.$$

$$\boxed{\Sigma(s) = \frac{1}{2}\frac{1}{16\pi}\left(-\frac{1}{\pi}\sqrt{1 - \frac{4m^2}{s + i\epsilon}}\ln\frac{\sqrt{1 - \frac{4m^2}{s+i\epsilon}} + 1}{\sqrt{1 - \frac{4m^2}{s+i\epsilon}} - 1}\right) + \frac{1}{16\pi^2}}$$

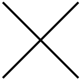
$$
\operatorname{Im}\Sigma(s) = \begin{cases} \frac{1}{2}\frac{\sqrt{\frac{s}{4}-m^2}}{8\pi\sqrt{s}} & \text{for } s > (2m)^2 \\ \varepsilon & \text{for } s < (2m)^2\,, \end{cases}
$$

$$
\operatorname{Re}\Sigma(s) = \frac{s}{\pi} P \int_{s_{th}}^{\infty} ds' \frac{I(s')}{(s'-s)s'}
$$



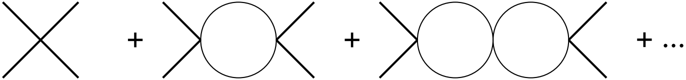
- $\star$ $\Sigma(s \to 0) = 0$
- $\star$ $\Sigma(s = 4m^2) = \frac{1}{16\pi^2}$

Tree level amplitude:



$$iA(s,t,u) = -i\lambda \quad \Rightarrow A(s,t,u) = A(s,\theta) = -\lambda$$

Unitarization (one-loop resumed approach):



$$A_k^U(s) = A_k(s) + A_k^2(s)\Sigma(s) + A_k^3(s)\Sigma^2(s) + A_k^4(s)\Sigma^3(s) + \ldots$$
$$= \frac{A_k(s)}{1 - A_k(s)\Sigma(s)}$$

$$\boxed{A_k^U(s) = [A_k^{-1}(s) - \Sigma(s)]^{-1}}$$

s-wave amplitude at tree-level

$A_0(s) = \frac{1}{2}\int_{-1}^{+1} d\xi A(s,\theta) = A(s,\theta) = -\lambda$

s-wave unitarized amplitude: $A_0^U(s) = \dfrac{-\lambda}{1+\lambda\Sigma}$

$$A_{l=1,2,..}^U(s) = 0$$

Scattering length

$$A_0^U(s) = \left[A_0^{-1}(s) - \Sigma(s)\right]^{-1} = \frac{-\lambda}{1+\lambda\Sigma(s)} \ ,$$

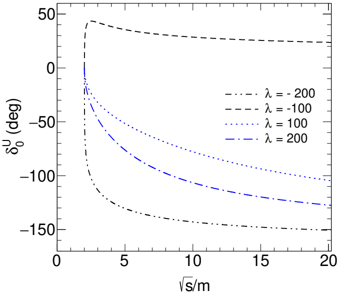
$$\begin{aligned} a_0^{U,\mathrm{SL}} &= \frac{1}{2}\frac{A_0^U(s=4m^2)}{8\pi\sqrt{4m^2}} = \frac{1}{2}\frac{1}{16\pi m}\frac{-\lambda}{1+\lambda\Sigma(4m^2)} \\ &= \frac{1}{2}\frac{1}{16\pi m}\frac{-\lambda}{1+\frac{\lambda}{16\pi^2}}, \quad \because \Sigma(s=4m^2) = \frac{1}{16\pi^2} \end{aligned}$$

- ⋆ $a_0^{U,\mathrm{SL}} < 0$ for $\lambda > 0$ (repulsion)
- ⋆ $a_0^{U,\mathrm{SL}} > 0$ for $\lambda \in (\lambda_c = -16\pi^2, 0)$ (attraction)
- ⋆ $a_0^{U,\mathrm{SL}} < 0$ for $\lambda < \lambda_c$, (repulsion sets in again) $\rightarrow$ bound-state below threshold emerges

Unitarized s-wave phase shift

$$\frac{e^{2i\delta_0^U(s)}-1}{2i}=\frac{1}{2}\cdot\frac{k}{8\pi\sqrt{s}}A_0^U(s)\ .$$

$$\boxed{\delta_0^U(s)=\frac{1}{2}\arg\left[1-\frac{1}{8\pi}\sqrt{\frac{m^2}{s}-\frac{1}{4}}A_0^U(s)\right]}$$



- ⋆ δ_0^U is negative when $\lambda > 0$ (blue lines)
- ⋆ δ_0^U is positive when $\lambda_c < \lambda < 0$ (see $\lambda = -100$)
- ⋆ δ_0^U becomes negative again when $\lambda < \lambda_c$
- ⋆ $\delta_0(s \to \infty)$ goes to a multiple of π

Bound state formation

$$A_0^U(s) = \frac{-\lambda}{1+\lambda\Sigma(s)} = \frac{1}{-\lambda^{-1}-\Sigma(s)}$$

for bound state: $A_0^U(s)^{-1} = \left[-\lambda^{-1} - \Sigma(s)\right] = 0$

$$\Rightarrow \quad \lambda = \frac{-1}{\Sigma(s)}$$

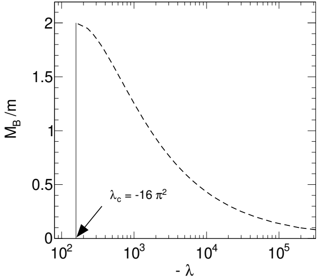
- ⋆ Σ is real for $s < 4m^2$

  At threshold: $\Sigma(s = 4m^2) = \Sigma_{max} = \dfrac{1}{16\pi^2}$

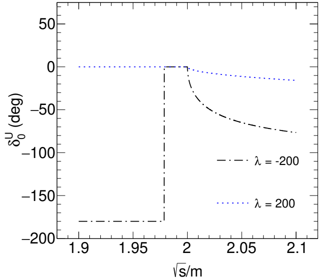
$$\Rightarrow \boxed{\lambda_c = -16\pi^2}$$

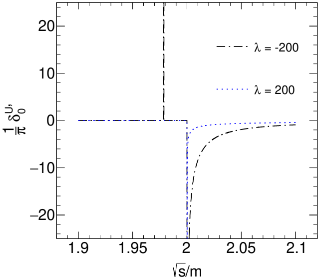
Bound state is present if $\lambda \leq \lambda_c = -16\pi^2$.

$$M_B(\lambda = \lambda_c) = 2m \ ,$$
$$M_B(\lambda \to -\infty) = 0 \ .$$



⋆ $M_B(\lambda = \lambda_c) = 2m$ and decreases with decreases of λ





⋆ Derivative of the phase shift shows a delta function near $\sqrt{s} = M_B$ for $\lambda = -200$ $\Rightarrow$ bound state

Thermodynamic properties

$$P_{\varphi,\text{free}} = -T \int_k \ln\left[1 - e^{-\beta\sqrt{k^2+m^2}}\right] \ , \quad \int_k \equiv \int \frac{d^3k}{(2\pi)^3}$$

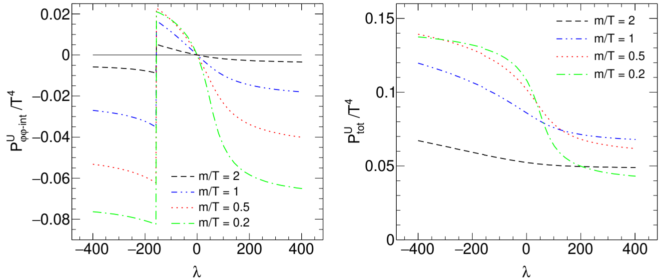
$$P_{\varphi\varphi\text{-int}} = -T \int_{2m}^{\infty} dx \frac{2l+1}{\pi} \sum_{l=0}^{\infty} \frac{d\delta_l(s=x^2)}{dx} \int_k \ln\left[1 - e^{-\beta\sqrt{k^2+x^2}}\right] \ ,$$

where $x = \sqrt{s}$.
In our specific case, only the s-wave contribution is nonzero:

$$P_{\varphi\varphi\text{-int}} = -T \int_{2m}^{\infty} dx \frac{1}{\pi} \frac{d\delta_0(s=x^2)}{dx} \int_k \ln\left[1 - e^{-\beta\sqrt{k^2+x^2}}\right] \ .$$

- ⋆ $P_{\varphi\varphi\text{-int}}$ depends on $d\delta_0/dx$
- ⋆ $P_{\varphi\varphi\text{-int}} > 0$ when derivative of the phase shift is positive
  $\Rightarrow$ Attractive interaction

P *vs* λ



- ⋆ For $\lambda > 0$, $P^U_{\phi\phi-int} < 0$ $\Rightarrow$ Repulsive interaction
- ⋆ $\lambda_c < \lambda < 0$ $P^U_{\phi\phi-int} > 0$ $\Rightarrow$ Attractive interaction
- ⋆ When $\lambda < \lambda_c$, $P^U_{\phi\phi-int}$ becomes negative. However, bound state is formed ($P_B > 0$).

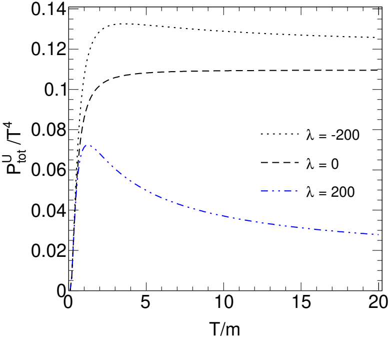
$$P_B = -\theta(\lambda_c - \lambda) T \int_k \ln\left[1 - e^{-\beta\sqrt{k^2+M_B^2}}\right]$$

The total pressure (in the presence of a bound state):

$$P^U_{tot} = P_B + P_{\varphi,\text{free}} + P^U_{\varphi\varphi\text{-int}} \text{ (unitarized, for any } \lambda).$$

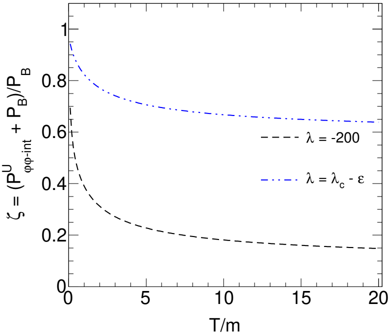
- ⋆ Total pressure (P^U_{tot}) is continuous

Temperature dependence of pressure



- ⋆ At $\lambda = 0$, $P^U_{tot} = P_{\varphi,free}$. At high T it goes to the mass less limit: $P_{\varphi,free}/T^4_{m=0} = \pi^2/90$
- ⋆ For $\lambda = -200$ ($< \lambda_c$), the bound state is present and, P^U_{tot} is larger than that of non-interacting particle.
- ⋆ For $\lambda = 200$, P^U_{tot} is smaller than that of non-interacting particle because of repulsion.
- ⋆ P^U_{tot}/T^4 saturates at high T in all the cases.

$$\zeta(T,\lambda) = \frac{P^U_{\varphi\varphi\text{-int}} + P_B}{P_B}$$



- ⋆ At small temperature $\zeta \approxeq 1$ for $\lambda \approxeq \lambda_c$. $P^U_{\varphi\varphi\text{-int}} \approxeq 0$
- ⋆ At high T, $P^U_{\varphi\varphi\text{-int}} < 0$ and $\zeta < 1$

Summary

- ⋆ Discussed how to include bound states in a thermal gas in the context of QFT.
- ⋆ A scalar QFT with φ^4 interacting is considered.
- ⋆ Analysis is based on a unitarized one-loop resumed approach.
- ⋆ A bound state of the φ-φ type is formed when the λ is negative and its modulus is larger than a certain critical value.
- ⋆ Contribution of this bound state to the pressure of the thermal gas is calculated using the S-matrix formalism.
- ⋆ Total pressure varies continuously with λ.
- ⋆ $\zeta \approx 1$ at low T but less than 1 at high T.

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Thank you

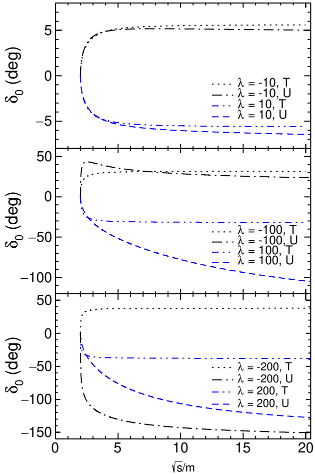
backup

Let us check the imaginary part of $\Sigma(s)$

$$\begin{aligned}
\operatorname{Im}\Sigma(s) &= \operatorname{Im}\left[\frac{1}{\pi}\int_{4m^2}^{\infty} ds' \frac{I(s')(s'-s+i\epsilon)}{(s'-s-i\epsilon)(s'-s+i\epsilon)} - C\right] \\
&= \frac{1}{\pi}\int_{4m^2}^{\infty} ds' \frac{I(s')\epsilon}{(s'-s)^2+\epsilon^2} \\
&= \int_{4m^2}^{\infty} ds'\, I(s')\, \delta(s'-s) \qquad \because\ \delta(s'-s) = \frac{1}{\pi}\frac{\epsilon}{(s'-s)^2+\epsilon^2} \\
&= I(s)
\end{aligned}$$

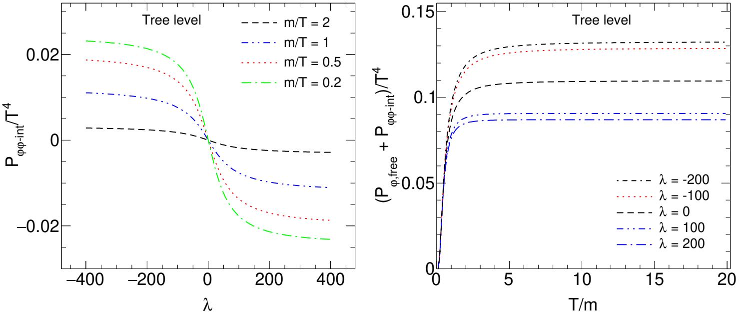
$$\boxed{\Sigma(s) = \frac{1}{2}\frac{1}{16\pi}\left(-\frac{1}{\pi}\sqrt{1-\frac{4m^2}{s+i\epsilon}}\ln\frac{\sqrt{1-\frac{4m^2}{s+i\epsilon}}+1}{\sqrt{1-\frac{4m^2}{s+i\epsilon}}-1}\right) + \frac{1}{16\pi^2}}$$

Phase shifts comparison



- ⋆ δ_0^T is always positive (negative) when $\lambda < 0$ ($\lambda > 0$)
- ⋆ δ_0^U is positive (negative) when $\lambda_c < \lambda < 0$ ($\lambda > 0$ or $\lambda \leq \lambda_c$)
- ⋆ $\delta_0(s \to \infty)$ goes to a multiple of π

Pressure at the tree-level



- ⋆ Interaction is repulsive when $\lambda > 0$. Attractive othersiwe.
- ⋆ At $\lambda = 0$, $P_{\varphi\varphi-int} = 0$. At high T total pressure goes to the mass less limit: $P/T^4_{m=0} = \pi^2/90$
- ⋆ No bound state formation.