Sudakov resummation from BFKL[1]

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Outline

1. TMD PDFs, Sudakov double-logarithms and rapidity divergences
2. High-Energy factorization
3. "Mathematical" derivation of Sudakov formfactor from HEF
4. More intuitive derivation

Evolution of TMD PDFs

TMD factorization is applicable for q_T-dependent processes with $q_T \ll Q$, up to terms $O(q_T/Q)$.

Evolution of TMD PDFs $f_{q/g}(x, \mathbf{x}_T, \mu^2, \mu_Y^2)$ ($\mathbf{x}_T$ is Fourier-conjugate to $\mathbf{q}_T$) is governed by *Collins-Soper-Sterman equations:*

$$\frac{df_{q/g}(\mu,\mu_Y)}{d\ln\mu^2} = \frac{\gamma_{q/g}(\mu,\mu_Y)}{2} f_{q/g}(\mu,\mu_Y),$$

$$\frac{df_{q/g}(\mu,\mu_Y)}{d\ln\mu_Y^2} = -\frac{\mathcal{D}_{q/g}(\mu,\mathbf{x}_T)}{2} f_{q/g}(\mu,\mu_Y),$$

$$\frac{d\mathcal{D}_{q/g}(\mu,\mathbf{x}_T)}{d\ln\mu^2} = \frac{\Gamma^{\rm cusp}_{q/g}}{2} = -\frac{d\gamma_{q/g}(\mu,\mu_Y)}{d\ln\mu_Y^2},$$

where μ_Y – **rapidity scale**, and

$$\boxed{\Gamma^{\rm cusp}_{q/g} = \frac{\alpha_s C_{F/A}}{\pi} + \ldots}$$

Solution in LLA w.r.t. $\ln(\mu^2_{(Y)}\mathbf{x}_T^2)$:

$$\mathcal{D}_{q/g}(\mu,\mathbf{x}_T) = \frac{\alpha_s C_{F/A}}{2\pi}\ln(\mu^2\mathbf{x}_T^2) \Rightarrow$$

$$f_{q/g}(x,\mathbf{x}_T,\mu=\mu_Y,\mu_Y) = \exp\left[-\frac{\alpha_s C_{F/A}}{4\pi}\ln^2(\mu_Y^2\mathbf{x}_T^2)\right] \times f_{q/g}(x,\mathbf{x}_T,\mu=|\mathbf{x}_T|^{-1},\mu_Y=|\mathbf{x}_T|^{-1})$$

In $\mathbf{q}_T$-space:

$$\frac{\alpha_s C_{F/A}}{2\pi}\frac{1}{\mathbf{q}_T^2}\exp\left[-\frac{\alpha_s C_{F/A}}{4\pi}\ln^2\left(\frac{\mu_Y^2}{\mathbf{q}_T^2}\right)\right]$$

Rapidity divergences and rapidity scale

Evolution with μ_Y:

$$\frac{df_{q/g}(x, \mathbf{x}_T, \mu, \mu_Y)}{d\ln\mu_Y^2} = -\frac{\mathcal{D}_{q/g}(\mu, \mathbf{x}_T)}{2} f_{q/g}(x, \mathbf{x}_T, \mu, \mu_Y),$$

comes from the **rapidity divergences** in TMD-operators containing **light-like Wilson lines**.

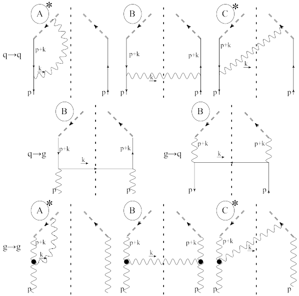
If rapidity of gluons is cut-off as $y > Y_\mu$ then:

$$\mu_Y = xP_+ e^{-Y_\mu},$$

where $xP_+ = (p+k)_+$ in the diagrams →

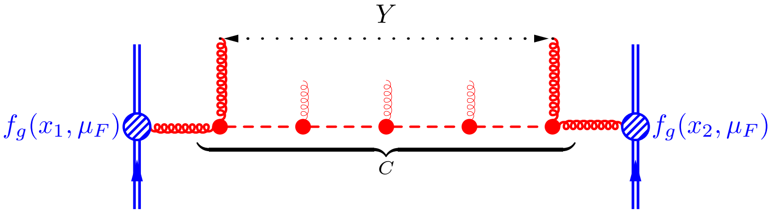
Typically $\mu_Y \sim M_T = \sqrt{M^2 + p_T^2} \sim M$ if $p_T \ll M$.

Diagrams for quark/gluon TMD PDF at NLO (fig. from [1604.07869]):



High-Energy factorization in a nutshell

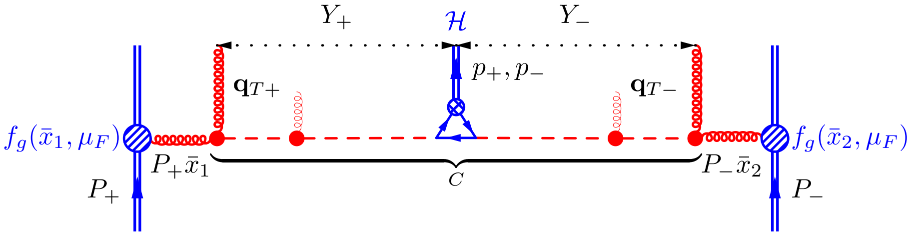
Reminder: Müller-Navelet dijet production (p_T of both jets is fixed):



Hard-scattering coefficient C contains higher-order corrections $\propto (\alpha_s Y)^n$ (LLA) or $\alpha_s(\alpha_s Y)^n$ (NLLA), which can be resummed at leading power w.r.t. e^{-Y} using Balitsky-Fadin-Kuraev-Lipatov (BFKL)-formalism.

High-Energy factorization in a nutshell

High-Energy Factorization [Collins, Ellis, 91'; Catani, Ciafaloni, Hautmann, 91',94']:



Using the same formalism one can resum corrections to C enhanced by

$$Y_\pm = \ln\left(\frac{\mu_Y}{|\mathbf{q}_{T\pm}|}\frac{1-z_\pm}{z_\pm}\right) \simeq \ln\frac{\mu_Y}{|\mathbf{q}_{T\pm}|} + \ln\frac{1}{z_\pm}, \text{ in LP w.r.t. } \frac{|\mathbf{q}_{T\pm}|}{\mu_Y}\frac{z_\pm}{1-z_\pm}$$

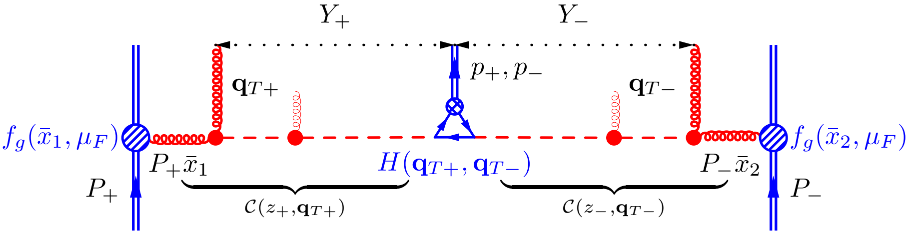
in inclusive observables (e.g. inclusive quarkonium production). Here

$$z_+ = \frac{p_+}{P_+\bar{x}_1},\ z_- = \frac{p_-}{P_-\bar{x}_2} \text{ and } \mu_Y = p_+ e^{-y_\mathcal{H}} = p_- e^{y_\mathcal{H}},$$

e.g. $\mu_Y^2 = m_\mathcal{H}^2 + \mathbf{p}_T^2$.

High-Energy factorization in a nutshell

High-Energy Factorization [Collins, Ellis, 91'; Catani, Ciafaloni, Hautmann, 91',94']:



Hard-scattering coefficient is re-factorized, *unintegrated PDF* is introduced:

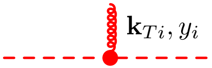
$$\Phi_g(x, \mathbf{q}_T, \mu_Y) = f_g\left(\frac{x}{z}, \mu_F\right) \otimes \mathcal{C}(z, \mathbf{q}_T, \mu_F, \mu_Y).$$

- *Collinear divergences* from additional emissions are subtracted inside UPDF.
- New coefficient function H depends on $x_{1,2}$ as well as $\mathbf{q}_{T\pm}$ (k_T*-factorization*).
- Factorization with single type of factors $\mathcal{C}$ and H is proven at LL and NLL approximation [Fadin *et.al.*, early 2000s], and known to be violated at N^2LL. Factorization with several types of $\mathcal{C}$ and H should be introduced then.

Building blocks of BFKL Green's function

For the squared amplitude:

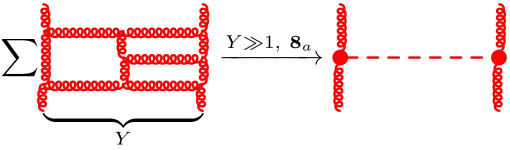
- Real emisssion – squared *Lipatov's vertex:*



$$= \hat{\alpha}_s \frac{(2\pi)^{2\epsilon}}{\pi \mathbf{k}_{Ti}^2} d^2\mathbf{k}_{Ti} dy_i,$$

where $\hat{\alpha}_s = \frac{\alpha_s C_A}{\pi}$.

- Virtual corrections – *Regge factors*:



$$\propto \exp\left[2\omega_g(\mathbf{p}_T^2)Y\right],$$

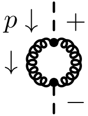
where $\omega_g(\mathbf{p}_T^2)$ – one-loop gluon Regge trajectory:

$$\omega_g(\mathbf{p}_T^2) = -\frac{\hat{\alpha}_s}{4} \int \frac{d^{2-2\epsilon}\mathbf{k}_T}{\pi(2\pi)^{-2\epsilon}} \frac{\mathbf{p}_T^2}{\mathbf{k}_T^2(\mathbf{p}_T - \mathbf{k}_T)^2} = \frac{\hat{\alpha}_s}{2\epsilon} (\mathbf{p}_T^2)^{-\epsilon} \frac{(4\pi)^\epsilon \Gamma(1+\epsilon)\Gamma^2(1-\epsilon)}{\Gamma(1-2\epsilon)}.$$

Digression: Regge trajectory and RDs

Due to the presence of the $1/q^{\pm}$-factors in the induced vertices, loop integrals in Lipatov's High-energy EFT contain the light-cone (Rapidity) divergences:

$$\Pi^{(1)}_{ab} = q \downarrow$$



$$= g_s^2 C_A \delta_{ab} \int \frac{d^d q}{(2\pi)^D} \frac{\left(\mathbf{p}_T^2 (n_+ n_-)\right)^2}{q^2 (p-q)^2 q^+ q^-}$$

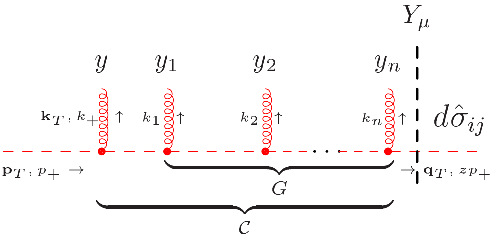
The regularization by explicit cutoff in rapidity was proposed by Lipatov [Lipatov, 1995] ($q^{\pm} = \sqrt{q^2 + \mathbf{q}_T^2} e^{\pm y}$, $p^+ = p^- = 0$):

$$\int \frac{dq^+ dq^-}{q^+ q^-} = \int\limits_{y_1}^{y_2} dy \int \frac{dq^2}{q^2 + \mathbf{q}_T^2},$$

then

$$\Pi^{(1)}_{ab} \sim \delta_{ab} \mathbf{p}_T^2 \times \boxed{\underbrace{\frac{C_A g_s^2}{2(2\pi)^3} \int \frac{\mathbf{p}_T^2 d^{D-2}\mathbf{q}_T}{\mathbf{q}_T^2 (\mathbf{p}_T - \mathbf{q}_T)^2}}_{\omega_g(\mathbf{p}_T^2)}} \times (y_2 - y_1) + \text{finite terms}$$

Resummation factor



$$y > y_1 > y_2 > \ldots > y_n > Y_\mu$$

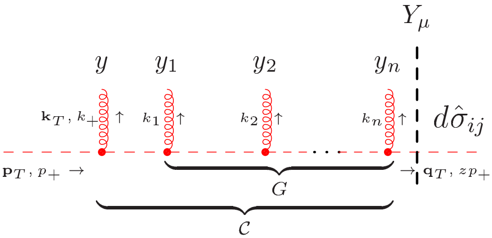
Collinearly un-subtracted resummation factor:

$$\tilde{\mathcal{C}}(z, \mathbf{q}_T^2, \mu_Y | \mathbf{p}_T^2) = \delta(z-1)\delta(\mathbf{q}_T^2 - \mathbf{p}_T^2)$$

$$+\hat{\alpha}_s \int\limits_{Y_\mu}^{+\infty} dy \int \frac{d^2\mathbf{k}_T}{\pi \mathbf{k}_T^2} \, G\Big(\mathbf{q}_T^2, zp_+ \Big| \, y - Y_\mu, (\mathbf{p}_T - \mathbf{k}_T)^2, p_+ - k_+\Big),$$

where $\boxed{\hat{\alpha}_s = \alpha_s C_A/\pi}$, y and $\mathbf{k}_T$ – rapidity and transverse momentum of the *rebounded gluon* (or *ricochet gluon* ☺), G – (modified) BFKL Green's function with longitudinal-momentum dependence, and $\mathbf{p}_T^2 \neq 0$ regularises collinear divergences.

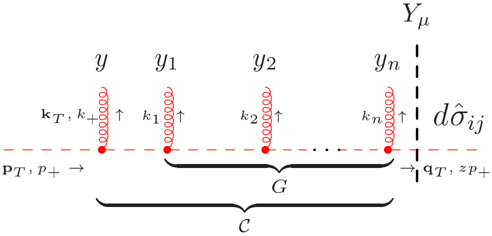
Longitudinal-momentum conservation



$$\delta\left(p_+ - k_+ - k_1^+ - \ldots - k_n^+ - zp_+\right) = \int_{-\infty}^{+\infty} \frac{dx_-}{2\pi} e^{ix_-(p_+(1-z)-k_+)} \prod_{i=1}^{n} e^{-ix_-k_i^+},$$

where $k_i^+ = |\mathbf{k}_{Ti}|e^{y_i}$. So we introduce: $G\left(\mathbf{q}_T^2 \middle| Y, \mathbf{p}_T^2, x_-\right)$

Evolution for Green's function



The LL(Y) BFKL evolution with x_--dependence in integral form reads:

$$G\Big(\mathbf{q}_T^2\Big|\; Y, \mathbf{p}_T^2, x_-\Big) = G_0\Big(\mathbf{q}_T^2\Big|\; \mathbf{p}_T^2, x_-\Big) + \int\limits_0^Y dy\; \Big\{2\omega_g(\mathbf{p}_T^2) G\Big(\mathbf{q}_T^2\Big|\; y, \mathbf{p}_T^2, x_-\Big)$$

$$+\hat{\alpha}_s \int \frac{d^{2-2\epsilon}\mathbf{k}_T}{\pi(2\pi)^{-2\epsilon}\mathbf{k}_T^2} \exp[-ix_-|\mathbf{k}_T|e^y] G\Big(\mathbf{q}_T^2\Big|\; y, (\mathbf{p}_T - \mathbf{k}_T)^2, x_-\Big)\Big\},$$

Differential form of evolution

$$\frac{\partial G\left(\mathbf{q}_T^2\middle|\, Y, \mathbf{p}_T^2, x_-\right)}{\partial Y} = \hat{\alpha}_s \int d^{2-2\epsilon}\mathbf{k}_T\; K(\mathbf{k}_T^2, \mathbf{p}_T^2, x_-, Y) G\left(\mathbf{q}_T^2\middle|\, Y, (\mathbf{p}_T - \mathbf{k}_T)^2, x_-\right),$$

with

$$K(\mathbf{k}_T^2, \mathbf{p}_T^2, x_-, Y) = \delta^{(2-2\epsilon)}(\mathbf{k}_T)\, \frac{(\mathbf{p}_T^2)^{-\epsilon}}{\epsilon} \frac{(4\pi)^\epsilon \Gamma(1+\epsilon)\Gamma^2(1-\epsilon)}{\Gamma(1-2\epsilon)} + \frac{\exp[-ix_-|\mathbf{k}_T|e^Y]}{\pi(2\pi)^{-2\epsilon}\mathbf{k}_T^2}.$$

Mellin transform:

$$G\left(\mathbf{q}_T^2\middle|\, Y, \mathbf{p}_T^2, x_-\right) = \int \frac{d\gamma}{2\pi i} \frac{1}{\mathbf{p}_T^2} \left(\frac{\mathbf{p}_T^2}{\mu_Y^2}\right)^\gamma G\left(\mathbf{q}_T^2\middle|\, Y, \gamma, x_-\right),$$

does not diagonalize $\mathbf{p}_T^2$-dependence, because:

$$\lim_{\varepsilon\to 0} \int d^{2-2\epsilon}\mathbf{k}_T\; K(\mathbf{k}_T^2, \mathbf{p}_T^2, x_-, Y)((\mathbf{p}_T - \mathbf{k}_T)^2)^{-1+\gamma} = (\mathbf{p}_T^2)^{-1+\gamma} \widetilde{\chi}(\gamma, x_-|\mathbf{p}_T|e^Y),$$

Compare with standard BFKL case:

$$\int d^2\mathbf{k}_T\; K(\mathbf{k}_T^2, \mathbf{p}_T^2)((\mathbf{p}_T - \mathbf{k}_T)^2)^{-1+\gamma} = (\mathbf{p}_T^2)^{-1+\gamma}\chi(\gamma).$$

where $\chi(\gamma) = 2\psi(1) - \psi(\gamma) - \psi(1-\gamma)$ – *Lipatov's characteristic function.*

Modified characteristic function

To factorize-out power-like dependence on $\mathbf{p}_T^2$, we introduce another Mellin-transform w.r.t. $\tilde{x}_- = x_- |\mathbf{p}_T| e^Y$:

$$\widetilde{\chi}(\gamma, \tilde{x}_-) = \chi(\gamma) + \int \frac{d\lambda}{2\pi i}\ \tilde{x}_-^{\lambda} \Delta\chi(\gamma, \lambda),$$

where $\chi(\gamma) = 2\psi(1) - \psi(\gamma) - \psi(1-\gamma)$ – Lipatov's characteristic function and

$$\Delta\chi(\gamma, \lambda) = e^{\frac{i\pi}{2}\lambda} \frac{\Gamma(-\lambda)\Gamma\left(\frac{\lambda}{2}\right)\Gamma(\gamma)\Gamma\left(1 - \frac{\lambda}{2} - \gamma\right)}{\Gamma\left(1-\frac{\lambda}{2}\right)\Gamma(1-\gamma)\Gamma\left(\frac{\lambda}{2}+\gamma\right)}.$$

The BFKL equation now reads:

$$\begin{aligned} \frac{\partial G\left(\mathbf{q}_T^2 \middle| Y, \gamma, x_-\right)}{\partial Y} &= \hat{\alpha}_s \chi(\gamma) G\left(\mathbf{q}_T^2 \middle| Y, \gamma, x_-\right) \\ &+ 2\hat{\alpha}_s \int \frac{d\lambda}{2\pi i}\ (\mu_Y x_- e^Y)^{2(\gamma-\lambda)} \Delta\chi(\lambda, 2(\gamma-\lambda)) G\left(\mathbf{q}_T^2 \middle| Y, \lambda, x_-\right). \end{aligned}$$

We should do inverse Fourier transform in the end, so both $x_- \ll \mu_Y^{-1}$ (**BFKL-regime**) and $x_- \gg \mu_Y^{-1}$ (**Sudakov regime**) will contribute. The latter one is controled by singularities to the right of the λ-contour.

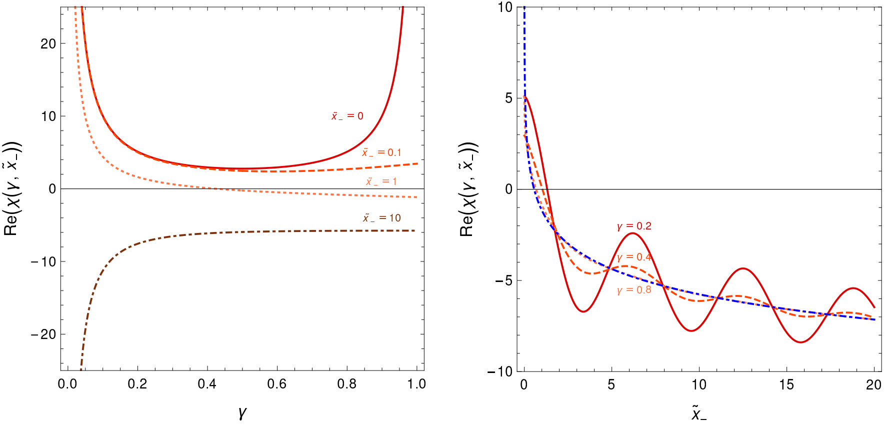
“Sudakov pole” of the modified characteristic function

$$\Delta\chi(\gamma,\lambda) = -\frac{2}{\lambda^2} - \frac{1}{\lambda}\left(2\gamma_E + i\pi + \chi(\gamma)\right) + O(1).$$

Is in one-to-one corresponcdence with logarithmic term in large-$\tilde{x}_-$ asymptotics for $\widetilde{\chi}(\gamma,\tilde{x}_-)$:

$$\begin{aligned}\widetilde{\chi}(\gamma,\tilde{x}_-) &= -2\gamma_E - i\pi - 2\ln\tilde{x}_- \\ &+ \tilde{x}_-^{-2\gamma} e^{-i\tilde{x}_-}\frac{2\cos(\pi\gamma)\Gamma(2\gamma)\Gamma\left(\frac{1}{2}-\gamma\right)}{\sqrt{\pi}\Gamma(1-\gamma)} + O(\tilde{x}_-^{-1}).\end{aligned}$$

Plots of $\mathrm{Re}\left[\widetilde{\chi}(\gamma, \tilde{x}_-)\right]$



Green's function for $x_- \gg \mu_Y^{-1}$

Using the singular part of $\widetilde{\chi}$ and boundary condition at $Y = 0$:

$$G_0\left(\mathbf{q}_T^2 \middle| \, \mathbf{p}_T^2, x_-\right) = \int_0^{+\infty} dq_+ e^{-ix_- q_+} \, \delta\left(\frac{zp_+}{q_+} - 1\right) \delta(\mathbf{q}_T^2 - \mathbf{p}_T^2)$$

$$= zp_+ e^{-izx_- p_+} \delta(\mathbf{q}_T^2 - \mathbf{p}_T^2) \to G_0\left(\mathbf{q}_T^2 \middle| \, \gamma, x_-\right) = zp_+ e^{-izx_- p_+} \left(\frac{\mathbf{q}_T^2}{\mu_Y^2}\right)^{-\gamma}$$

one obtains

$$G\left(\mathbf{q}_T^2 \middle| \, Y, \gamma, x_-\right) = zp_+ e^{-izx_- p_+} \left(\frac{\mathbf{q}_T^2}{\mu_Y^2}\right)^{-\gamma} \times \exp\left[-\hat{\alpha}_s Y \left(2\gamma_E + i\pi + \ln(\mathbf{q}_T^2 x_-^2)\right) - \hat{\alpha}_s Y^2\right].$$

To get the LLA Sudakov FF, it is enough to take the LL part:

$$G\left(\mathbf{q}_T^2 \middle| \, Y, \gamma, x_-\right) = zp_+ e^{-izx_- p_+} \left(\frac{\mathbf{q}_T^2}{\mu_Y^2}\right)^{-\gamma} \exp\left[-\hat{\alpha}_s Y^2\right]$$

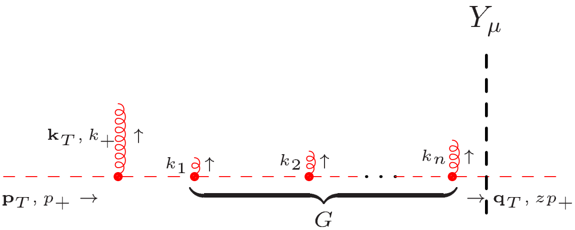
LLA Sudakov formfactor

Substituting the Green's function back to $\tilde{\mathcal{C}}$ one obtains:

$$\tilde{\mathcal{C}}(z,\mathbf{q}_T^2,\mu_Y|\mathbf{p}_T^2)=\delta(z-1)\delta(\mathbf{q}_T^2-\mathbf{p}_T^2)$$
$$+\hat{\alpha}_s\int\frac{d\gamma}{2\pi i}\int_0^\infty dY\int\frac{d\mathbf{k}_T^2}{\mathbf{k}_T^2}\left[\int_{-\infty}^{+\infty}\frac{dx_-}{2\pi}(zp_+)\exp\left(ix_-(p_+(1-z)-|\mathbf{k}_T|e^{Y_\mu+Y})\right)\right]$$
$$\times\frac{1}{(\mathbf{p}_T-\mathbf{k}_T)^2}\left(\frac{(\mathbf{p}_T-\mathbf{k}_T)^2}{\mathbf{q}_T^2}\right)^\gamma e^{-\hat{\alpha}_s Y^2},$$

Note, that limit $\mathbf{p}_T\to 0$ exists.

Sudakov cascade:



LLA Sudakov formfactor

Taking integrals over x_-, $\mathbf{k}_T^2$ and Y one obtains:

$$\tilde{\mathcal{C}}(z, \mathbf{q}_T^2, \mu_Y | \mathbf{p}_T^2 = 0) = \delta(z-1)\delta(\mathbf{q}_T^2) + \hat{\alpha}_s \frac{2z^3}{\mu_Y^2 (1-z)^3} \int \frac{d\gamma}{2\pi i} \left(\frac{\mu_Y^2}{\mathbf{q}_T^2} \frac{(1-z)^2}{z^2} \right)^{\gamma} J(\gamma, \hat{\alpha}_s),$$

where

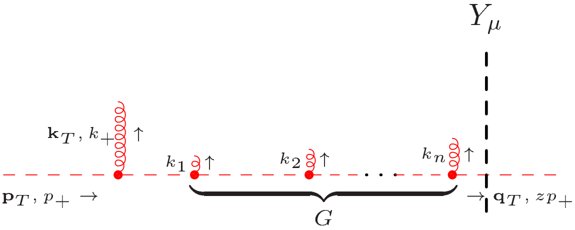
$$J(\gamma, \alpha) = \int_0^\infty dY \exp\left[-\alpha Y^2 + 2Y(1-\gamma)\right] = \frac{\sqrt{\pi}}{2\sqrt{\alpha}} e^{\frac{(1-\gamma)^2}{\alpha}} \left[1 + \mathrm{Erf}\left(\frac{1-\gamma}{\sqrt{\alpha}} \right) \right]$$

$$= \sqrt{\frac{\pi}{\alpha}} e^{\frac{(1-\gamma)^2}{\alpha}} + \frac{1}{2(\gamma-1)} + \sum_{n=1}^{\infty} \frac{(2n-1)!}{2^{2n}(n-1)!} \frac{(-1)^n \alpha^n}{(\gamma-1)^{2n+1}}.$$

taking residues at $\gamma = 1$ one gets:

$$\tilde{\mathcal{C}}(z, \mathbf{q}_T^2, \mu_Y | \mathbf{p}_T^2 = 0) = \delta(z-1)\delta(\mathbf{q}_T^2) + \frac{z}{1-z} \frac{\hat{\alpha}_s}{\mathbf{q}_T^2} \exp\left[-\frac{\hat{\alpha}_s}{4} \ln^2 \left(\frac{\mu_Y^2}{\mathbf{q}_T^2} \frac{(1-z)^2}{z^2} \right) \right].$$

Derivation from "Sudakov cascade" picture



In this region, real emissions inside Green's function are so soft, that they change **neither** transverse, **nor** longitudinal momentum. This is achieved via the cut:

$$\boxed{k_i^+ = |\mathbf{k}_{Ti}| e^{y_i} \ll |\mathbf{q}_T| = |\mathbf{k}_T|,}$$

since $|\mathbf{q}_T| \ll \mu_Y \sim p_+$. With logarithmic accuracy we replace $\ll \rightarrow <$

Derivation from "Sudakov cascade" picture

In the "Sudakov cascade" region, the BFKL Green's function is:

$$G = \sum_{n=0}^{\infty} \hat{\alpha}_s^n e^{2\omega_g(\mathbf{q}_T^2)Y} \int_0^Y dy_1 \int_{y_1}^Y dy_2 \ldots \int_{y_{n-1}}^Y dy_n$$

$$\times \prod_{i=1}^{n} \underbrace{\int \frac{d^{2-2\epsilon}\mathbf{k}_{Ti}}{\pi(2\pi)^{-2\epsilon}\mathbf{k}_{Ti}^2} \theta(|\mathbf{q}_T| e^{-y_i} - |\mathbf{k}_{Ti}|)}_{L_\epsilon - 2y_i}$$

$$= e^{-\hat{\alpha}_s L_\epsilon Y} \sum_{n=0}^{\infty} \frac{\hat{\alpha}_s^n}{n!} \left[Y(L_\epsilon - Y)\right]^n$$

$$\boxed{= \exp\left[-\hat{\alpha}_s Y^2\right],}$$

where $L_\epsilon = -\frac{1}{\epsilon} + \ln \mathbf{q}_T^2 + \gamma_E - \ln 4\pi$.

Conclusions

- BFKL evolution contains Sudakov effects if longitudinal-momentum conservation is included
- Effects beyond LLA ($\sim \hat{\alpha}_s Y^2$) are present. The exact solution for G provides some all-order constraints to rapidity anomalous dimension $\mathcal{D}_g$ and collinear matching functions of TMD factorization
- If one comes-up with appropriate generalization of the Regge trajectory term, the evolution equation with $\mathbf{k}_T$-dependent P_{gg}-splitting [Hentschinski, Kusina, Kutak, Serino, 2018], unifying DGLAP and BFKL can be obtained, and it will include TMD/Sudakov effects automatically
- Extension to quark case is also possible through *Reggeized quark* formalism [Fadin, Sherman, 1976; Lipatov, Vyazovsky, 2001]

Thank you for your attention!