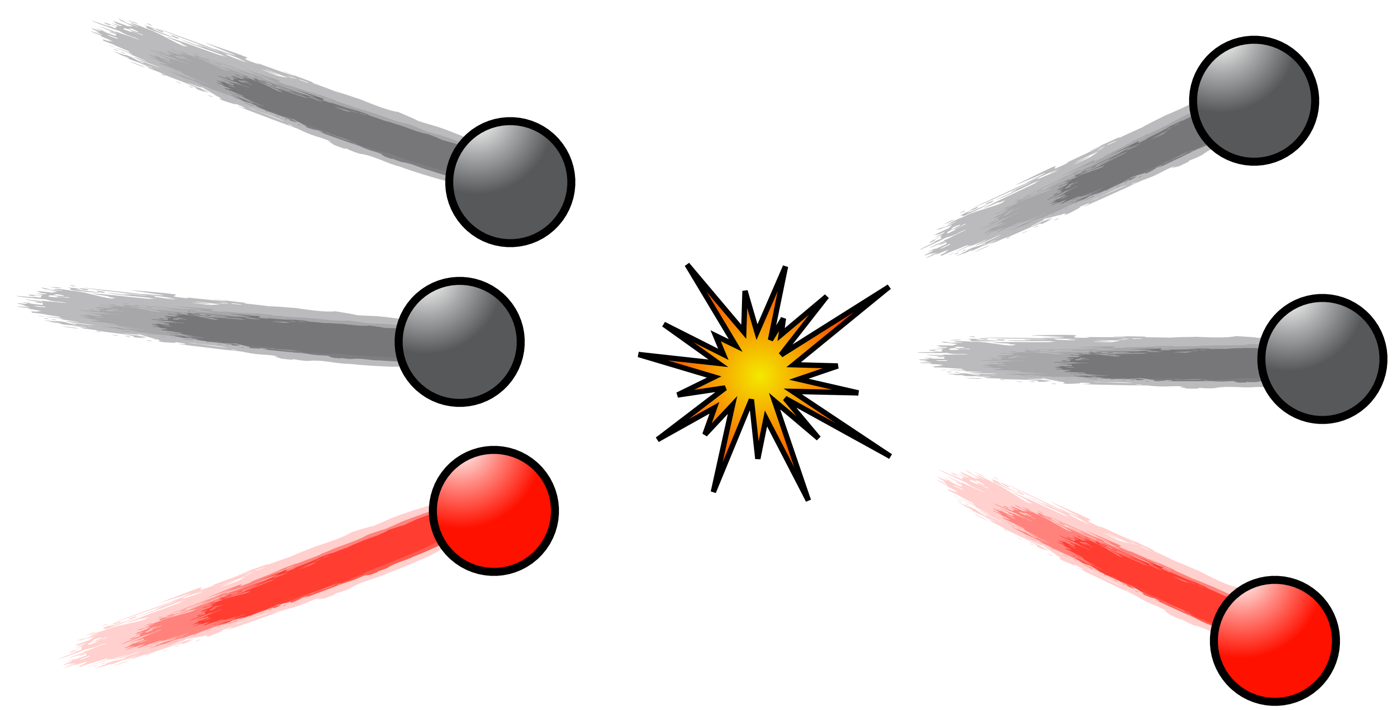
Progress in relativistic three-hadron scattering from lattice QCD

Andrew W. Jackura

19th International Conference on Hadron Spectroscopy and Structure
26th-31st, July 2021



Old Dominion University
Jefferson Lab
Thomas Jefferson National Accelerator Facility

HADRON 2021
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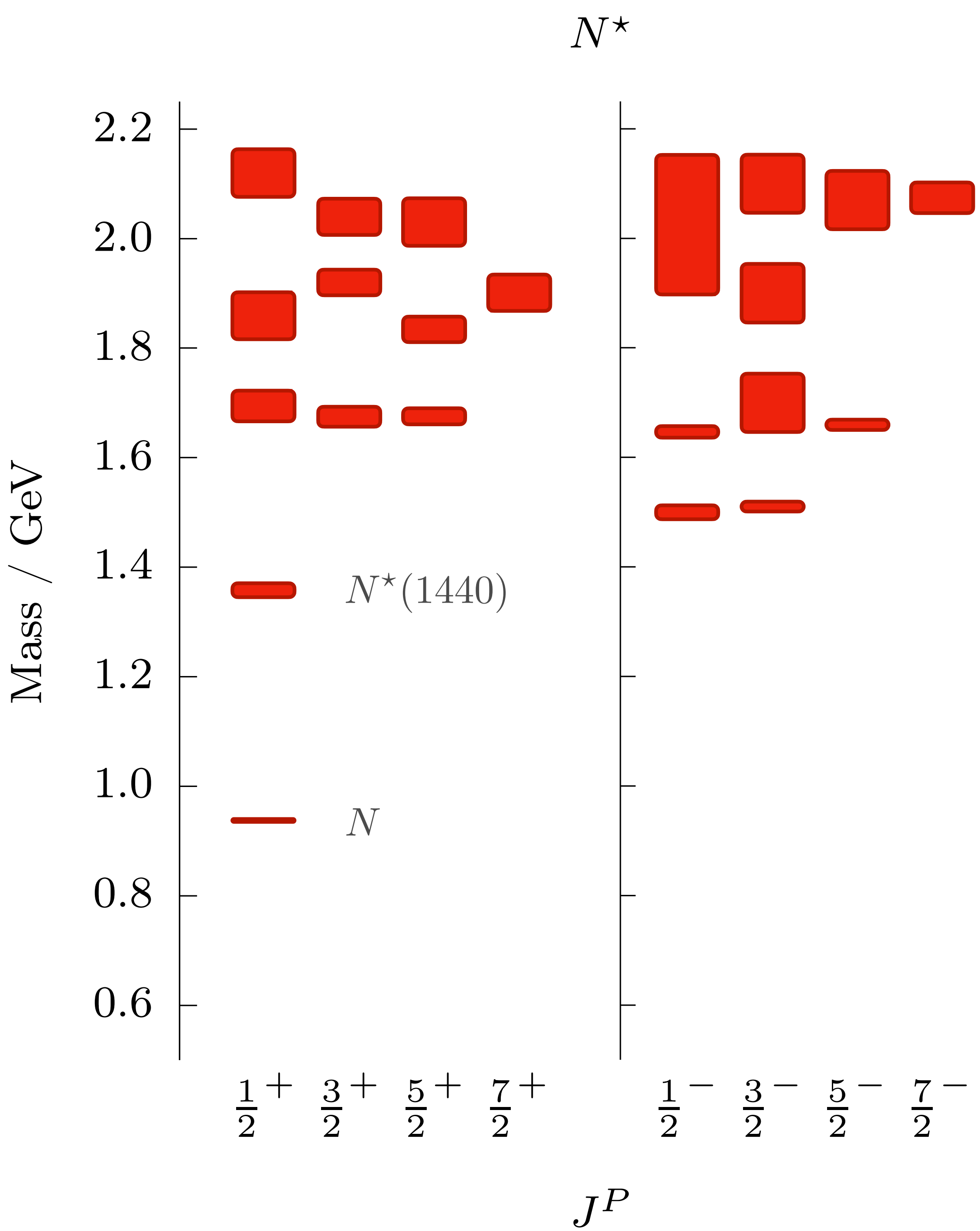
Hadronic Physics, Lattice QCD, and Three-body Interactions

Goal: Compute hadronic/nuclear properties from first principles QCD

- *e.g.* ***Excited nucleon spectrum***



PDG listings

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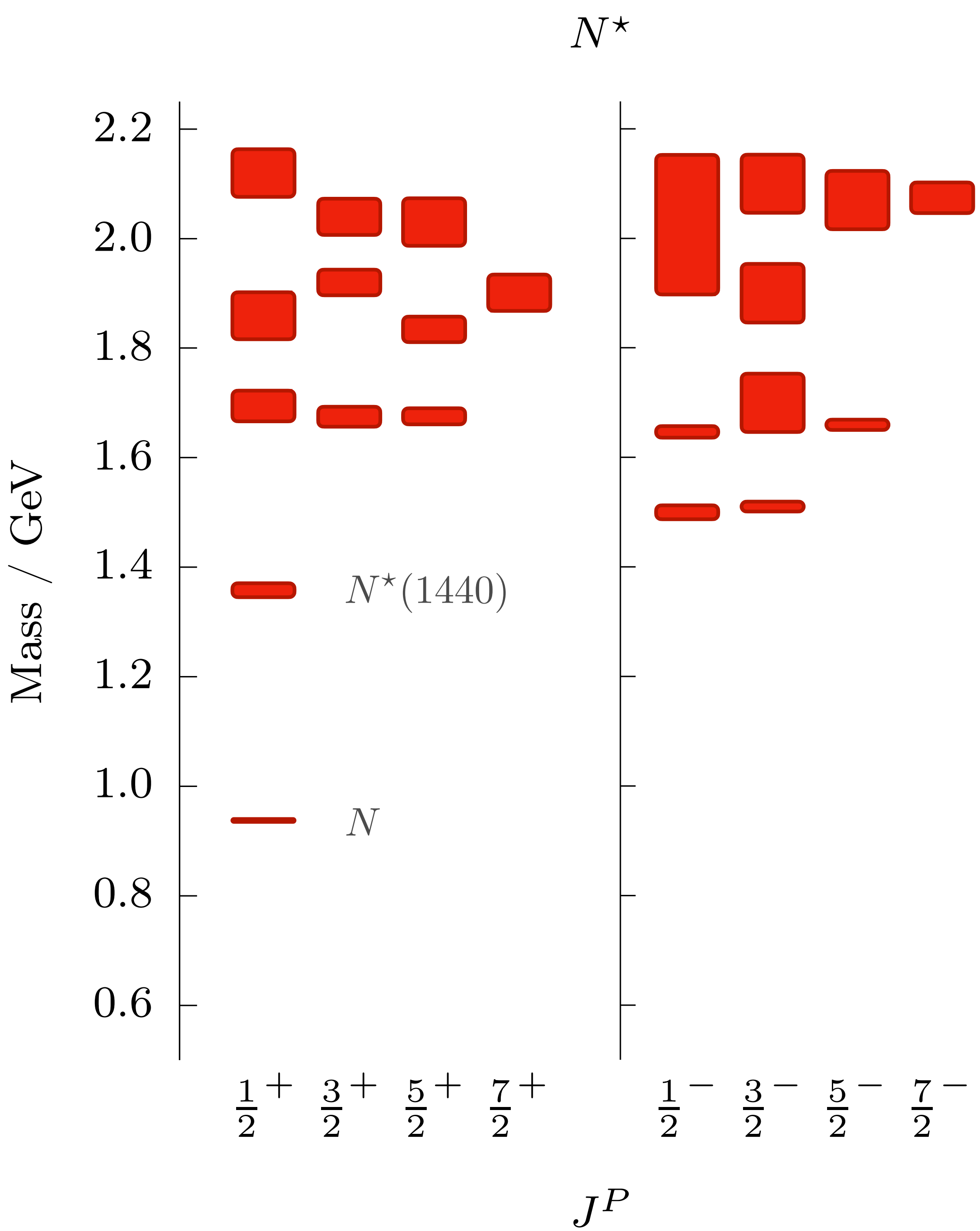
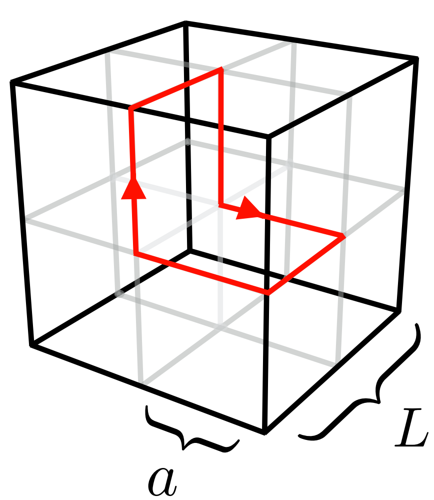


Lattice QCD offers a systematic approach to compute hadrons from QCD

- Numerically evaluate QCD path integral
  - *Euclidean spacetime,* $t \to -i\tau$
  - *Finite volume,* L
  - *Discrete spacetime,* a
  - *Heavier than physical quark mass,* $m > m_{\text{phys.}}$



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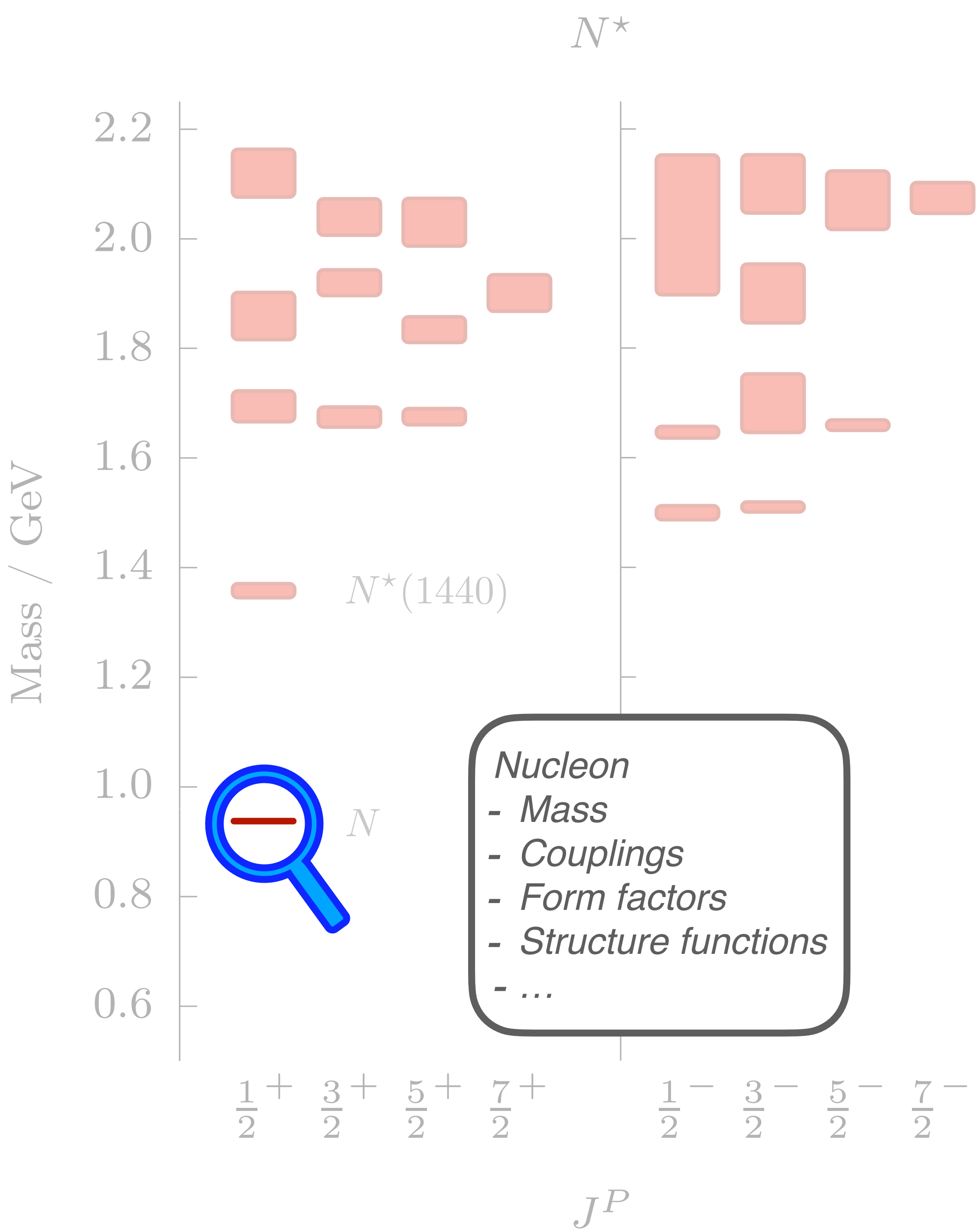

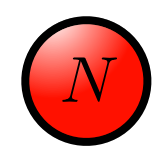

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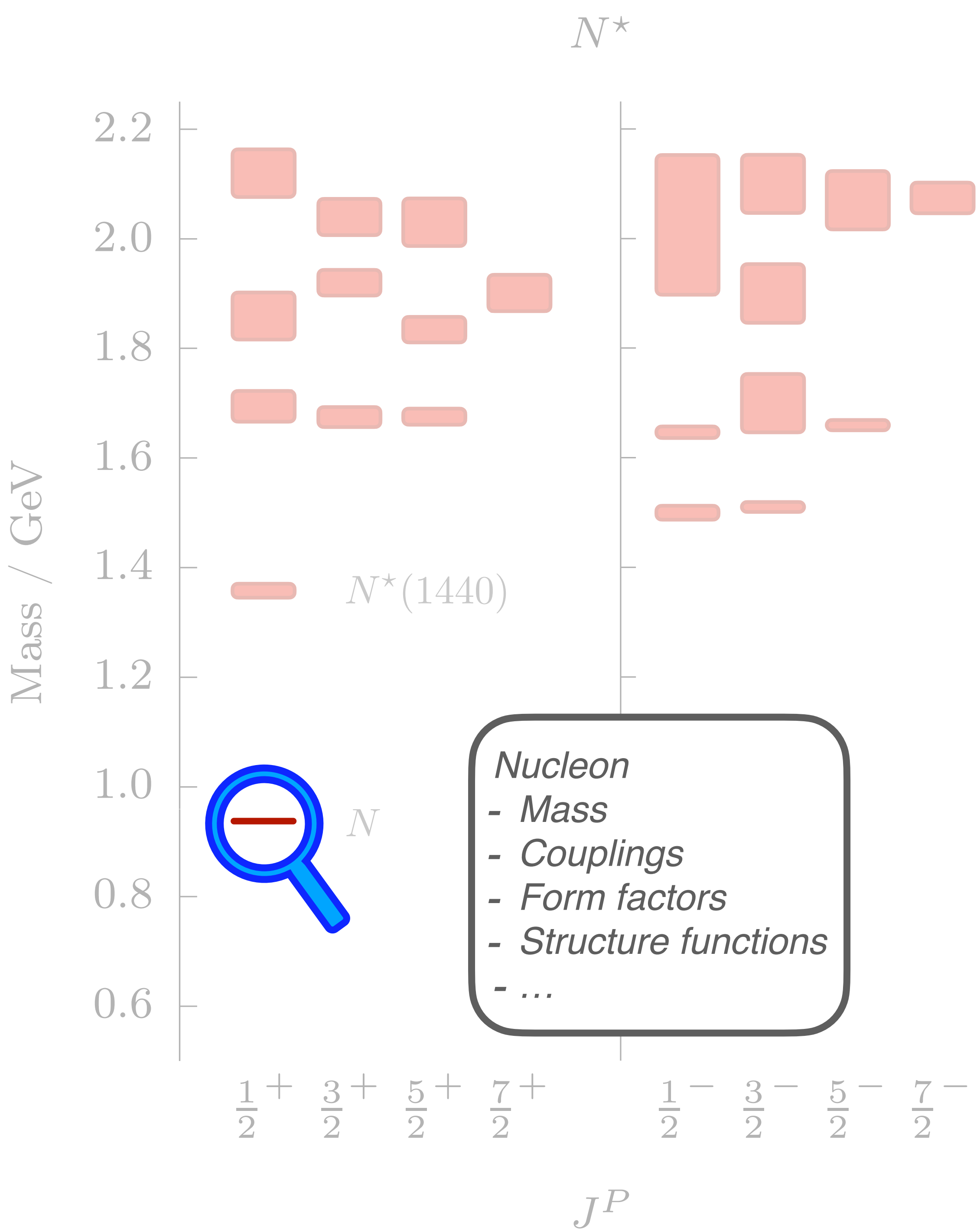

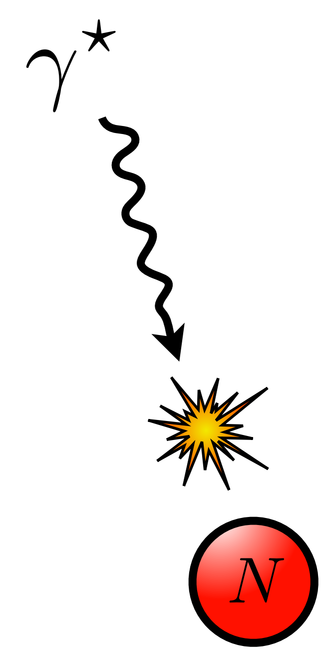

@JLab
- *Clas12*
- *GlueX*

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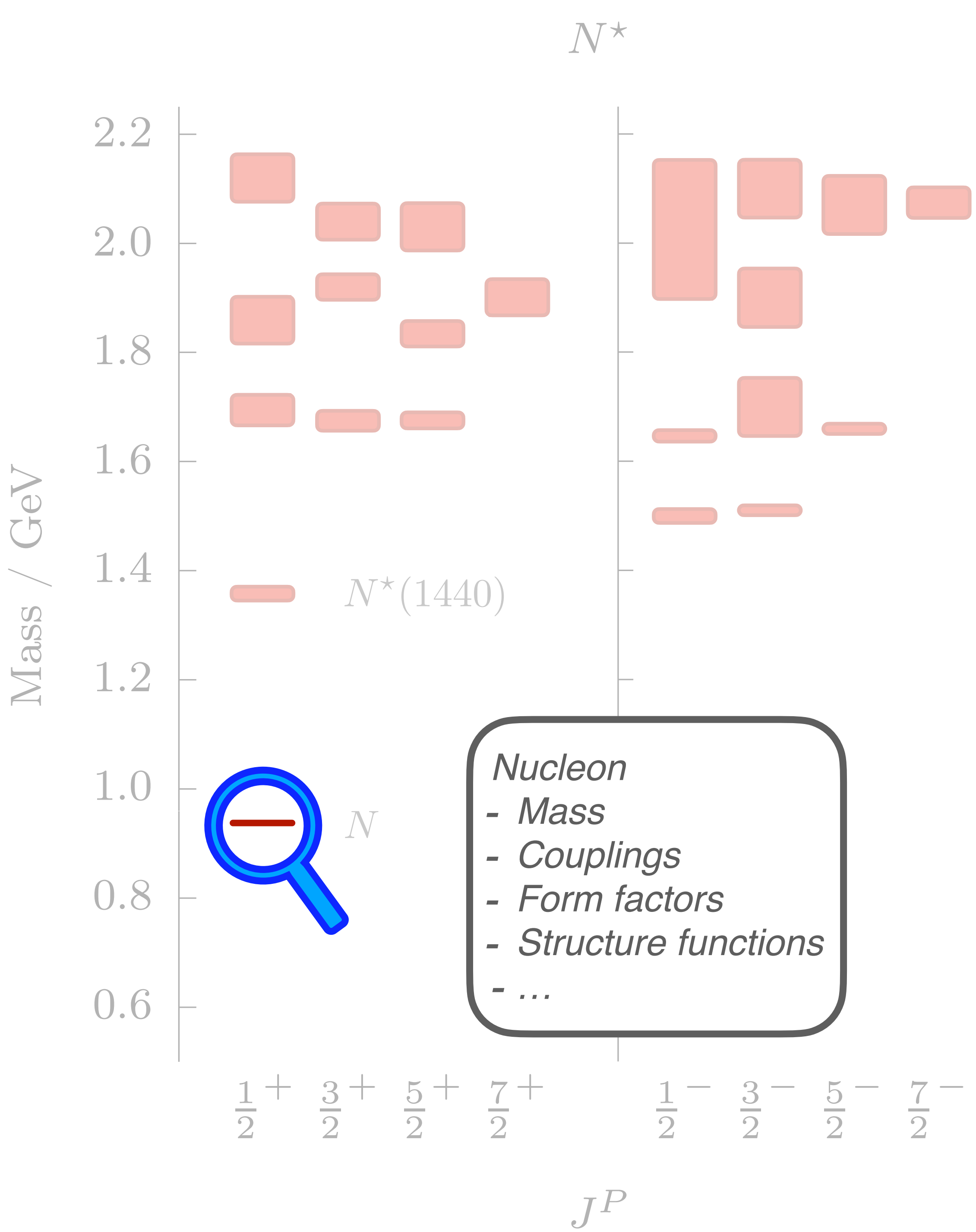

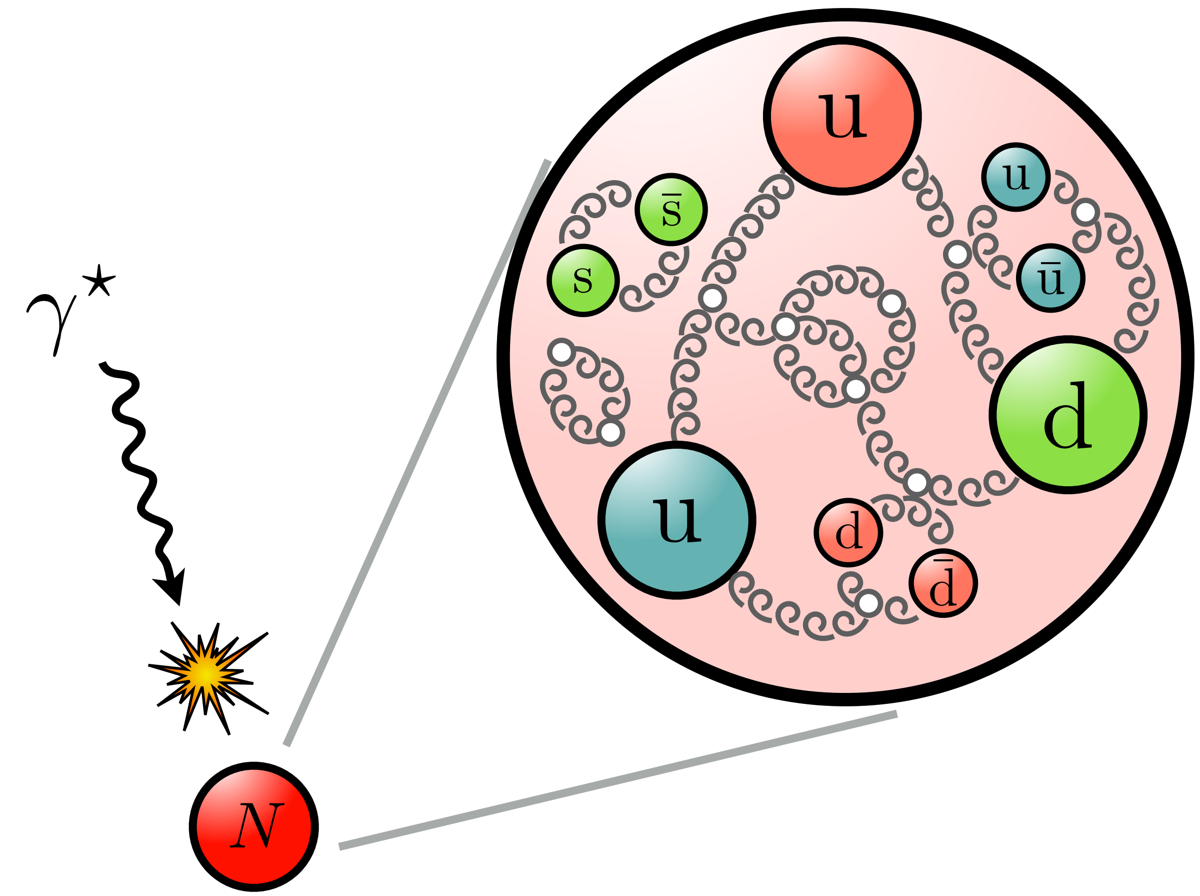

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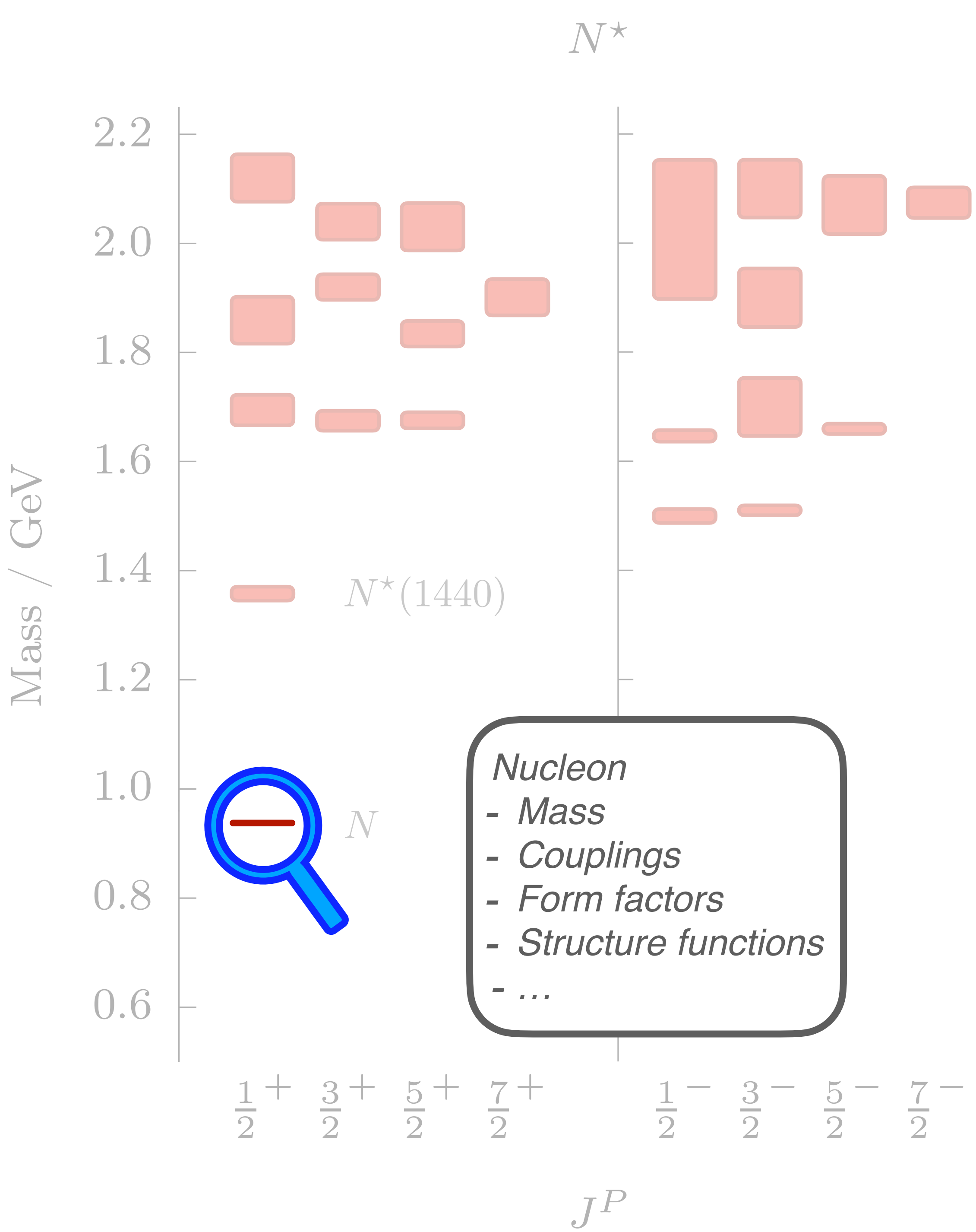
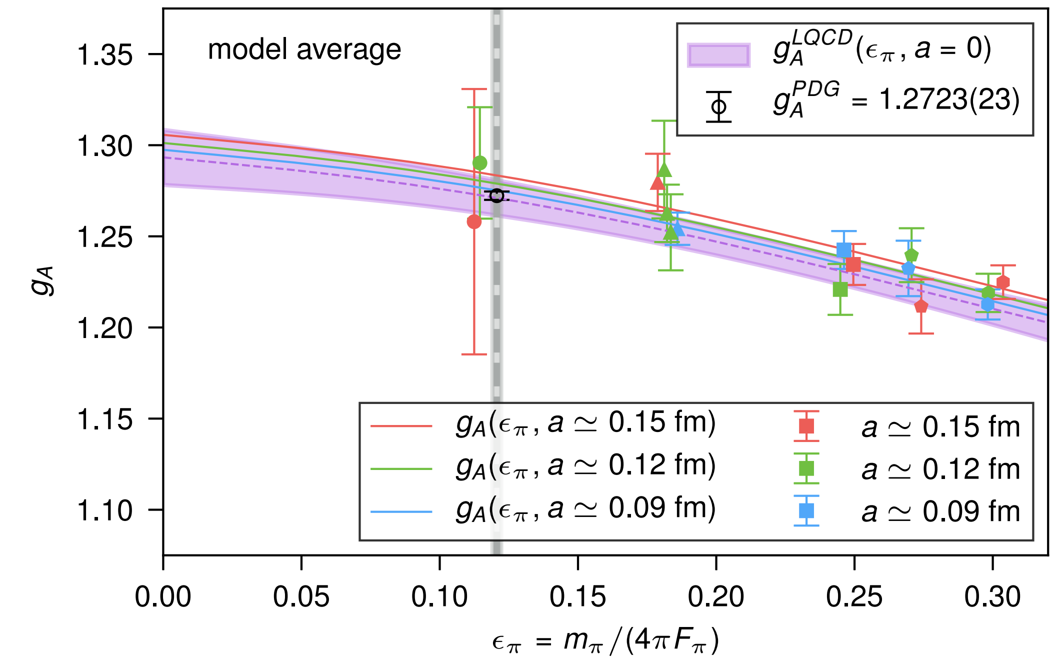
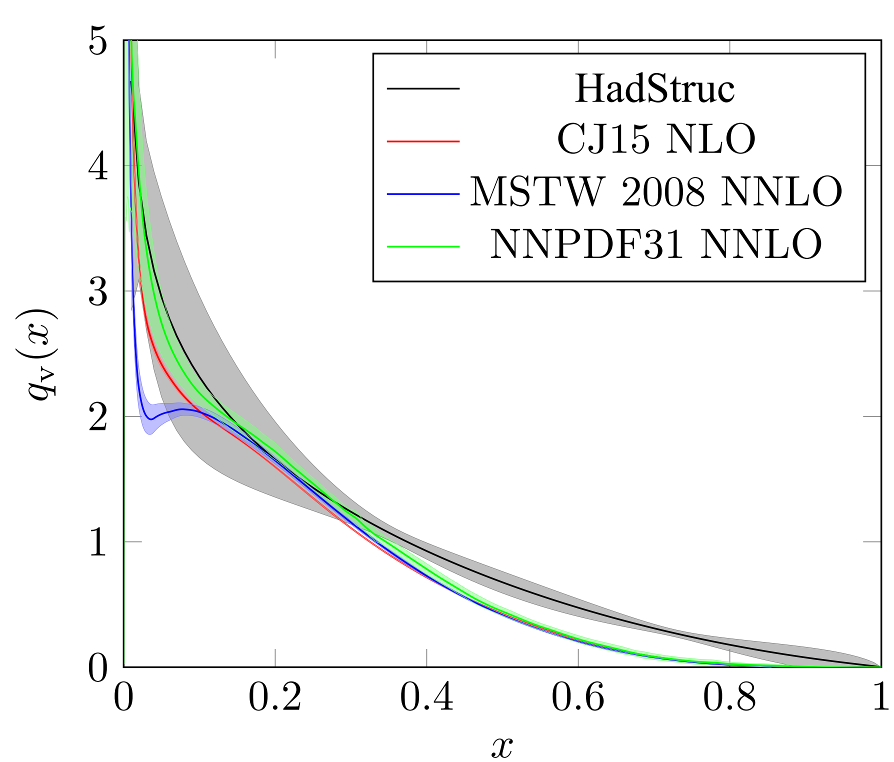


PDG listings



C. Chang et al.
Nature **558**, (2018) 7708



B. Joó et al. [HadStruc]
Phys. Rev. Lett. **125**, (2020) 232003

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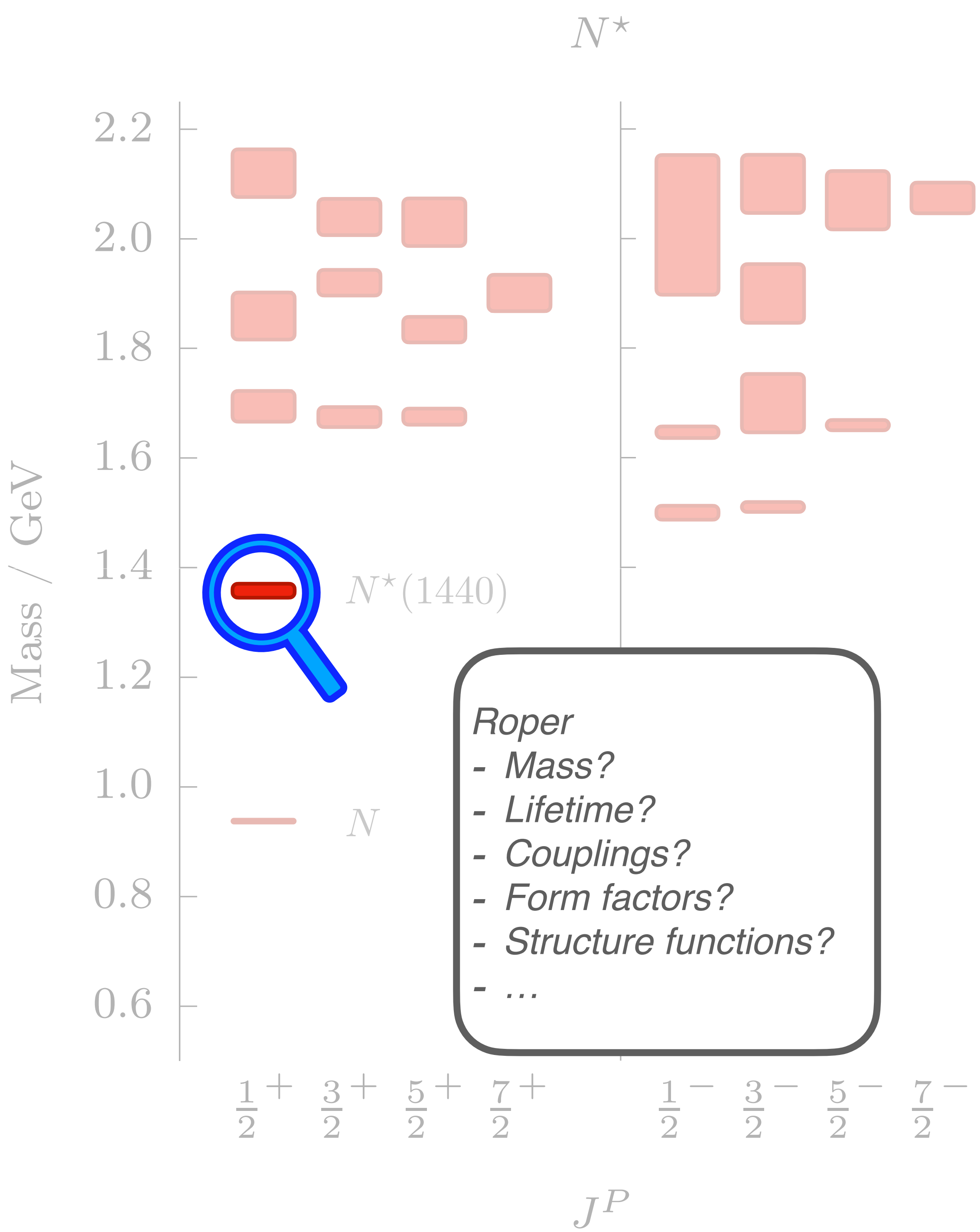

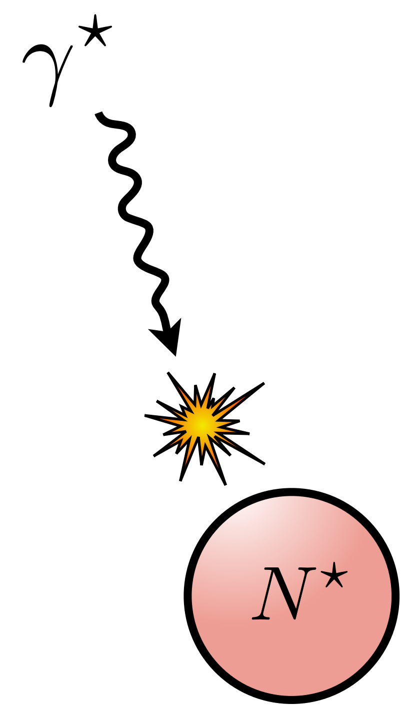

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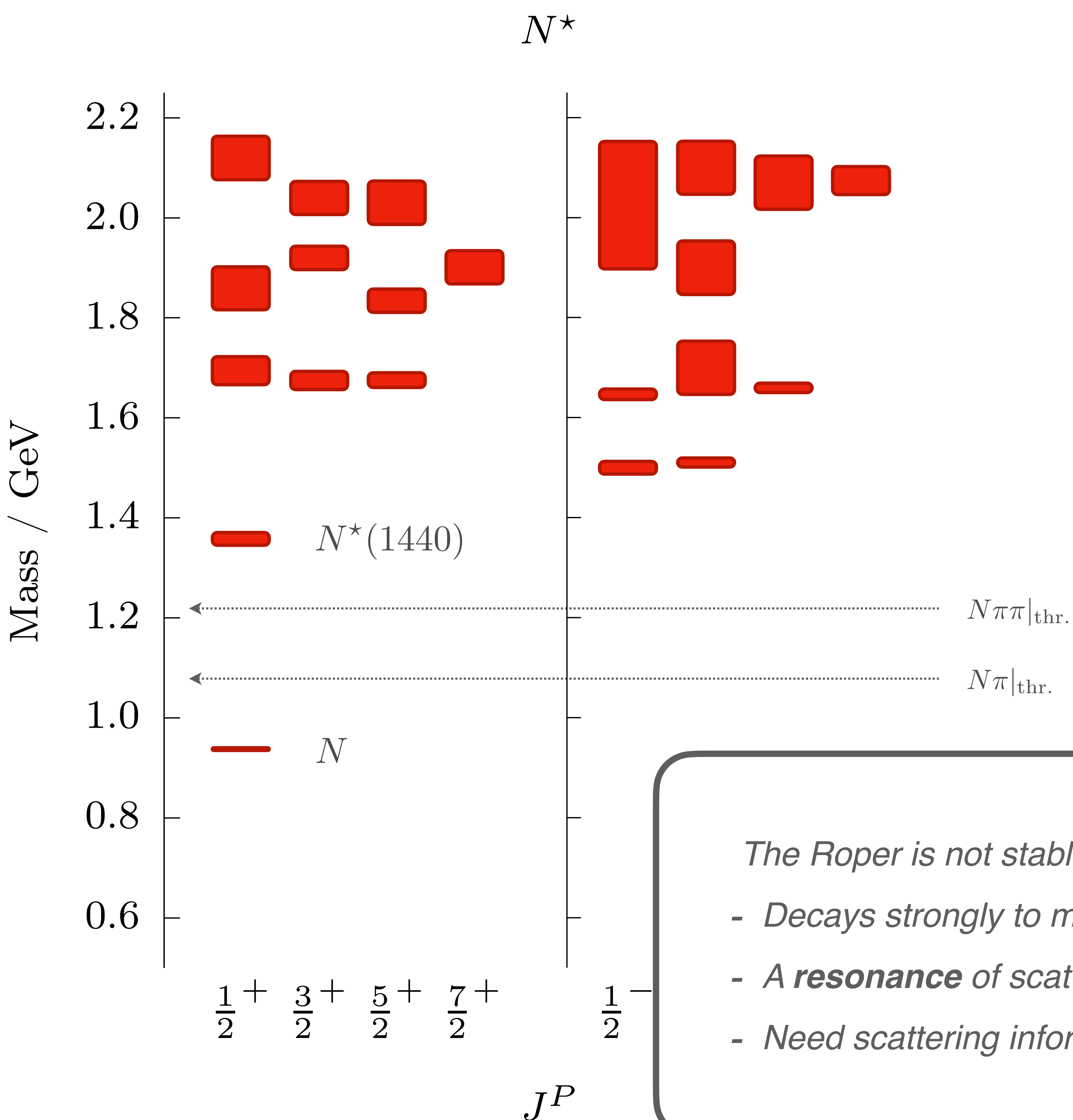
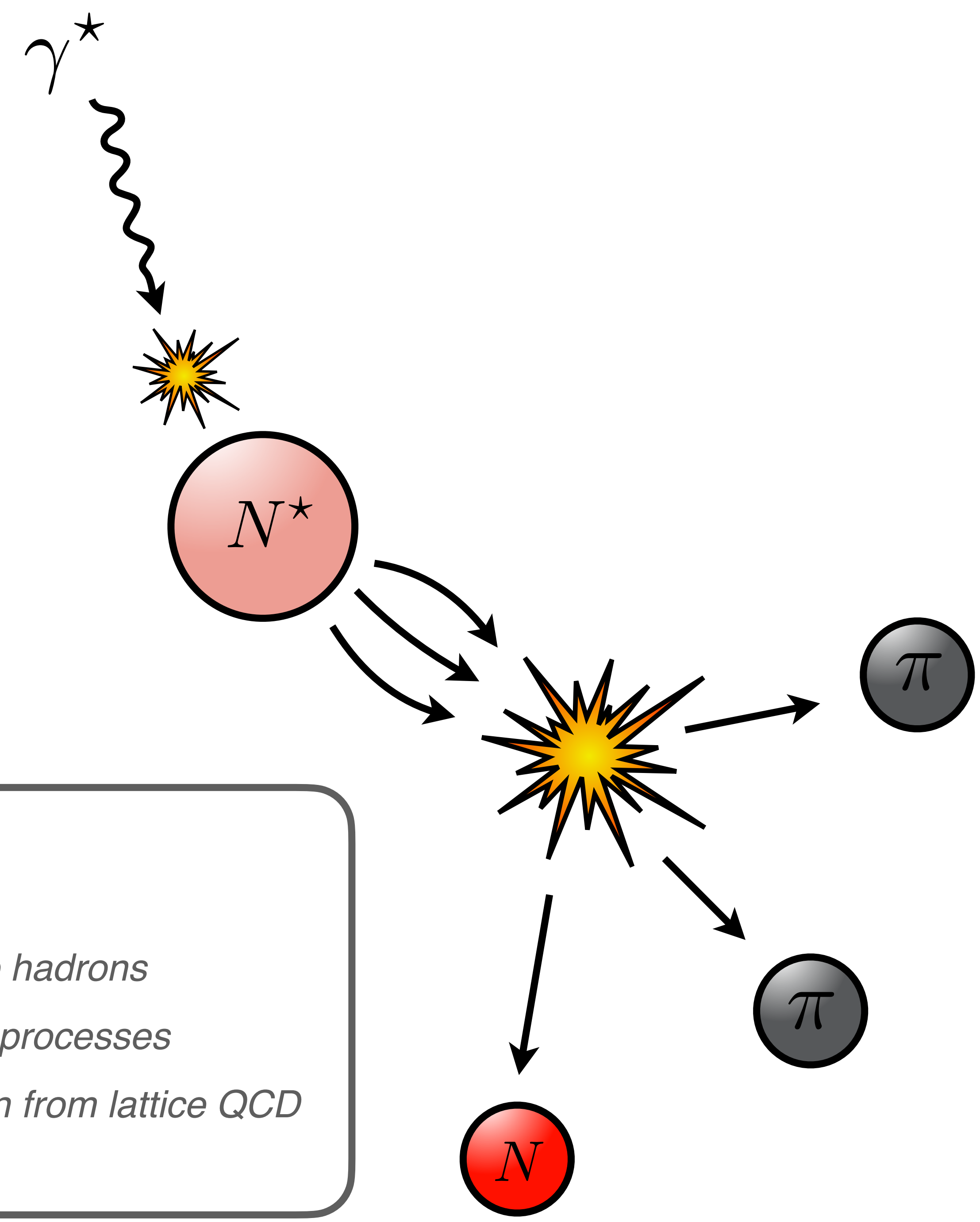


The Roper is not stable!

- *Decays strongly to multiple hadrons*
- *A **resonance** of scattering processes*
- *Need scattering information from lattice QCD*

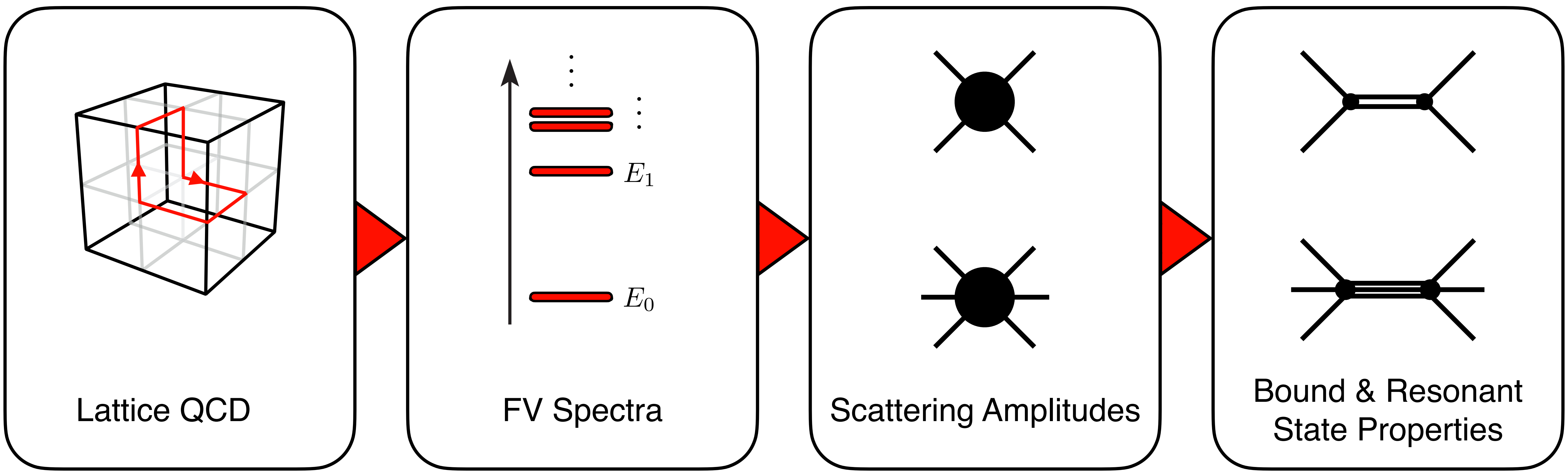


PDG listings

Path to scattering physics from QCD

Use Lüscher methodology to connect lattice QCD observables to scattering amplitudes

- Non-perturbative, model-independent, systematically improvable

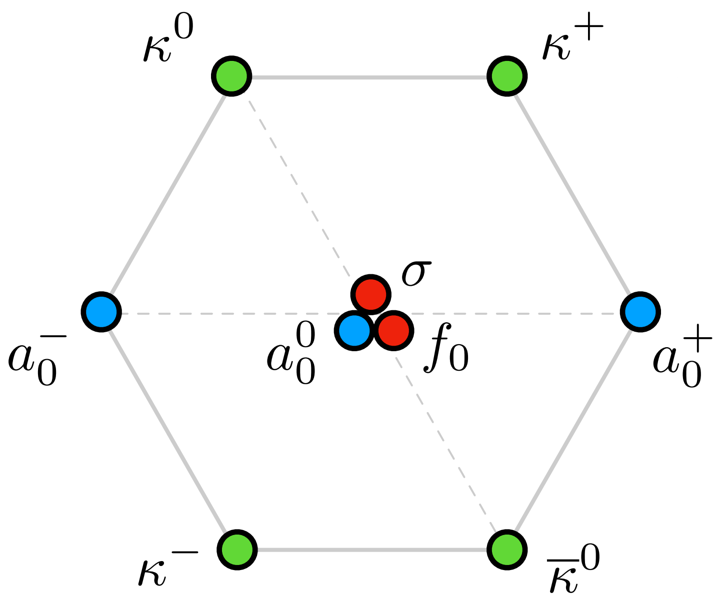


R. Briceño, J. Dudek, and R. Young
Rev.Mod.Phys. **90**, (2018) 025001

Path to scattering physics from QCD

Much success in two-body sector

- e.g., ***the Scalar nonet***



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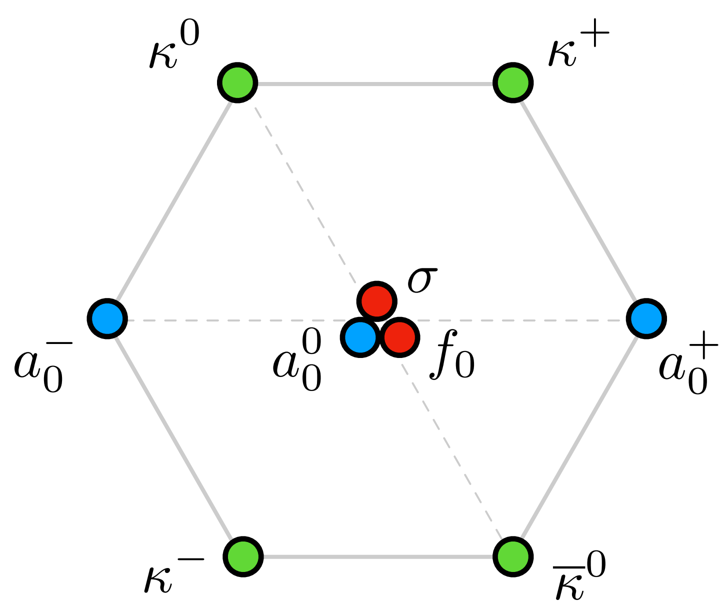
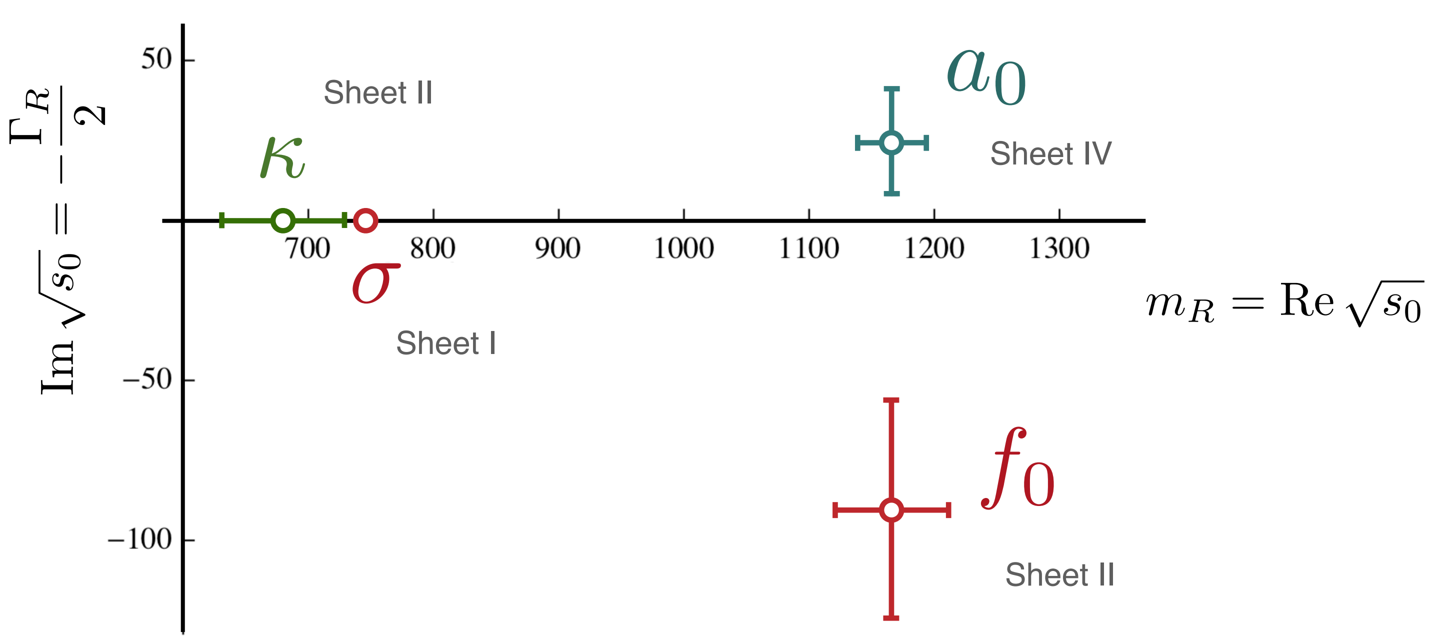
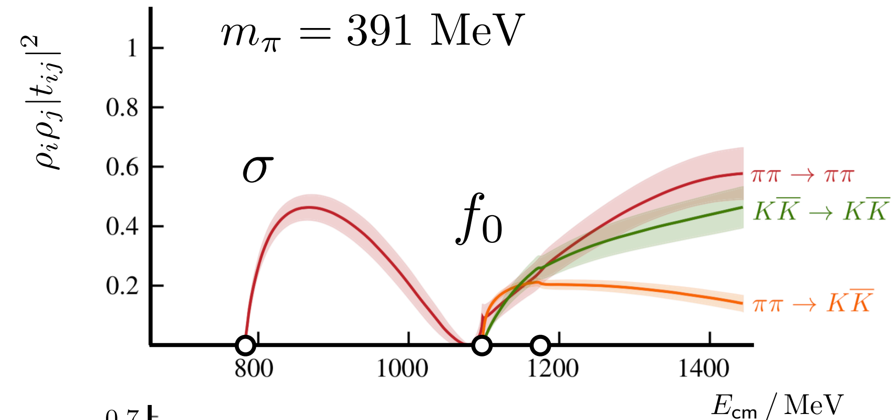
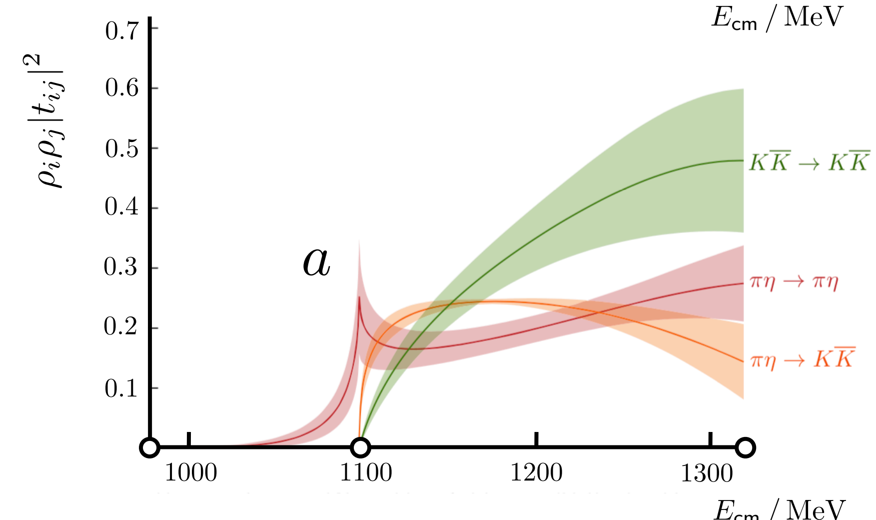
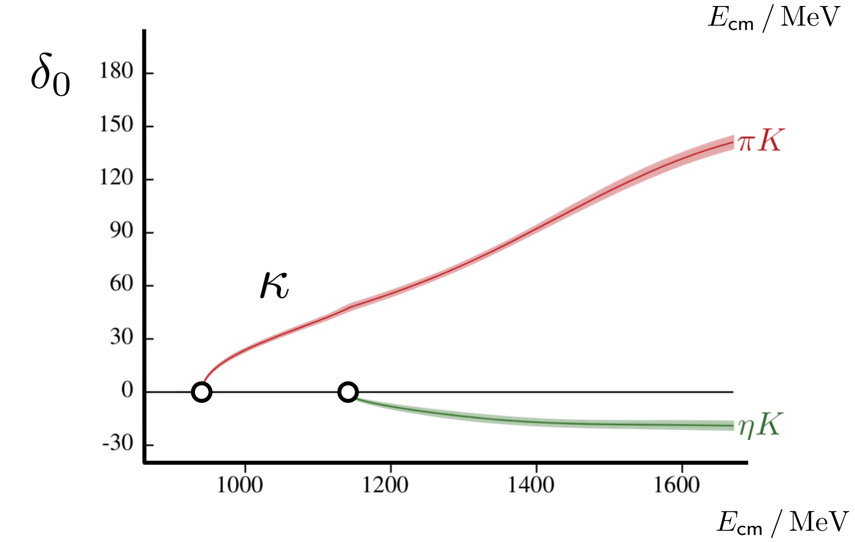


$m_\pi = 391$ MeV









R.A. Briceño et al. [HadSpec]
Phys.Rev. **D97**, (2018) 054513

J.J. Dudek et al. [HadSpec]
Phys.Rev. **D93**, (2016) 094506

J.J. Dudek et al. [HadSpec]
Phys.Rev.Lett. **113**, (2014) 182001

Path to three-body physics from QCD

Follow same path in three-body sector — considerably more challenging

- 8 kinematic variables (vs. 2 in two-body)
- Link between lattice QCD observable to amplitudes involves integral equations

$$\{E_{\mathfrak{n}}\} \longrightarrow \mathcal{K}_3 \longrightarrow \mathcal{T}_3 \longrightarrow m_{\mathrm{res.}}, \Gamma_{\mathrm{res.}}$$

Quantization Condition

Integral Equations

Analytic Continuation

Path to three-body physics from QCD

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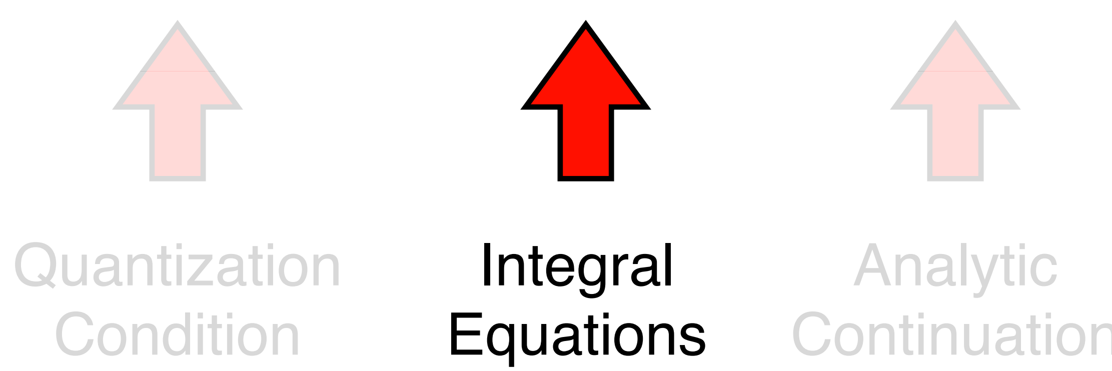
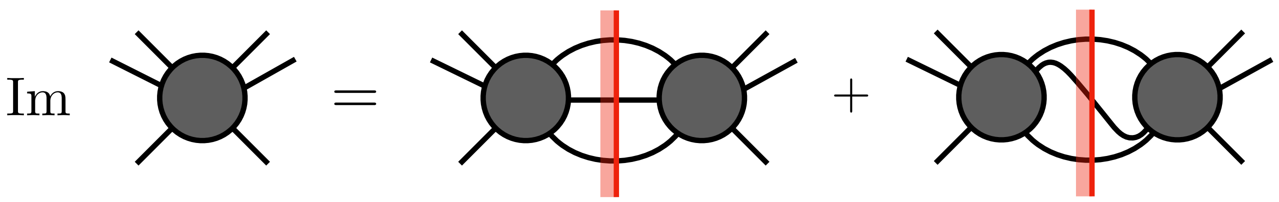


Scattering theory for three-body systems

Using principles of S matrix to constrain amplitude

- — ***Lorentz Invariance***
- — ***Unitarity***
- — ***Analyticity***
- — ***Crossing***



$\sim \mathrm{Im}\,\mathcal{F}$ $\quad\quad$ $\sim \mathrm{Im}\,\mathcal{G}$

Unitarity condition

$$\mathcal{T}_3 = \mathcal{K}_3 - \int \mathcal{K}_3\,\mathcal{F}\,\mathcal{T}_3 - \iint \mathcal{K}_3\,\mathcal{G}\,\mathcal{T}_3$$

On-shell scattering equation

$\mathcal{K}_3$ *Unknown!*
Obtained from Lattice QCD

M. Mai, B. Hu, M. Döring, A. Pilloni, and A. Szczepaniak
Eur. Phys. J. A **53**, 177 (2017)

AJ et al. [JPAC]
Eur. Phys. J. C **79**, no. 1, 56 (2019)

AJ et al. [JPAC]
Phys. Rev. D **100**, 034508 (2019)

AJ, *in preparation*

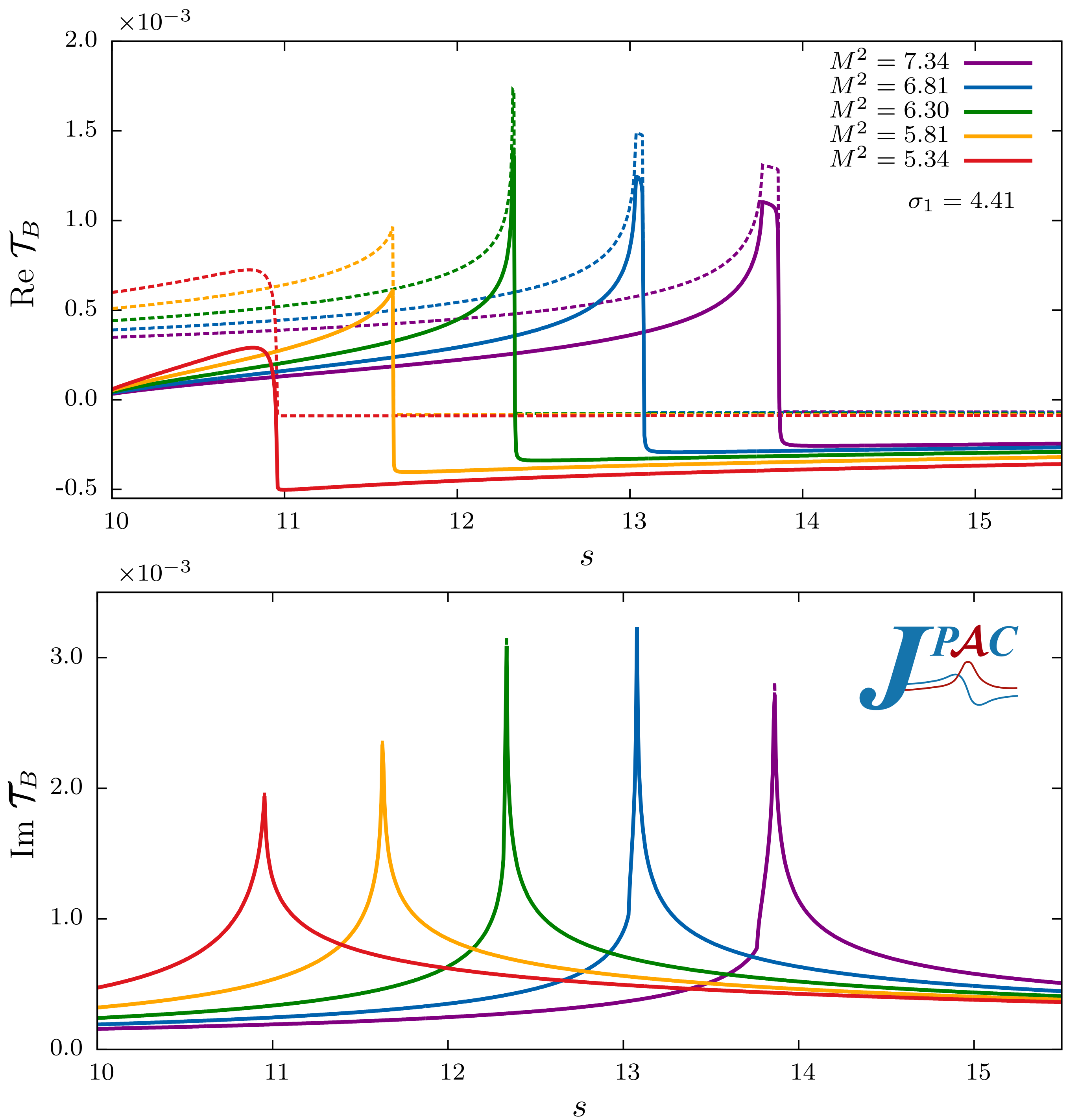
cf. two-body case: $\mathcal{T}_2 = \mathcal{K}_2 - \mathcal{K}_2\,\mathcal{F}\,\mathcal{T}_2$

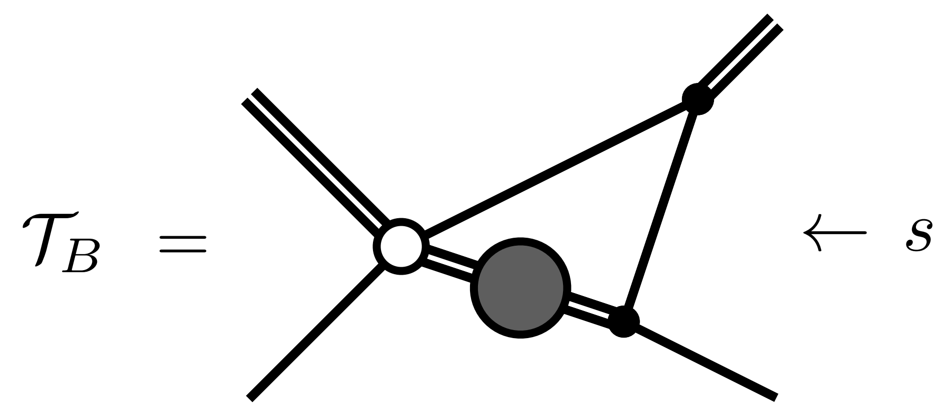
Scattering theory for three-body systems

Using principles of S matrix to constrain amplitude

- Informs the analytic structure — needed to understand resonance phenomena

e.g. effects of triangle singularities





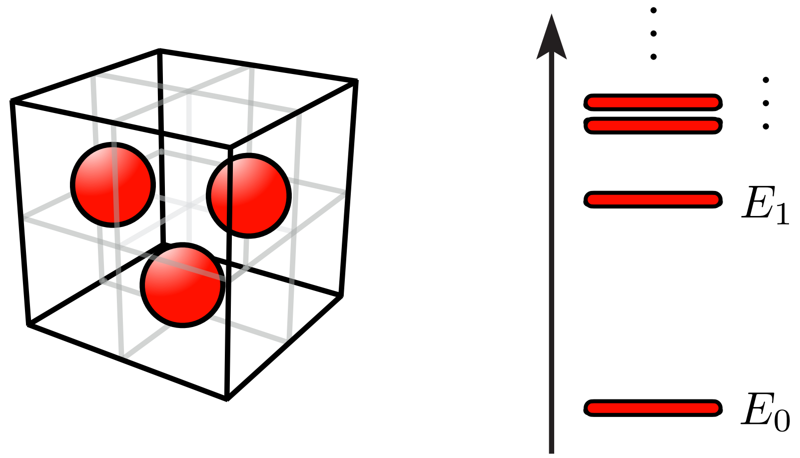
$$\mathcal{T}_B \subset \mathcal{T}_3$$

AJ et al. [JPAC]
Eur. Phys. J. C **79**, no. 1, 56 (2019)

Connecting Lattice QCD to three-body amplitudes

For a given K matrix, can solve integral equations

- K matrices given from lattice QCD



$$\det\left(1 + \mathcal{K}_3 \left(\mathcal{F}_L + \mathcal{G}_L \right) \right)_{E=E_n} = 0$$

Quantization condition

$$\mathcal{T}_3 = \mathcal{K}_3 - \int \mathcal{K}_3 \, \mathcal{F} \, \mathcal{T}_3 - \iint \mathcal{K}_3 \, \mathcal{G} \, \mathcal{T}_3$$

On-shell scattering equation

M. Hansen and S. Sharpe
Phys. Rev. D **90**, 116003 (2014), Phys. Rev. D **95**, 034501 (2017)

M. Mai and M. Döring
Eur. Phys. J. A **53**, 240 (2017), Phys. Rev. Lett. **122**, 062503 (2019)

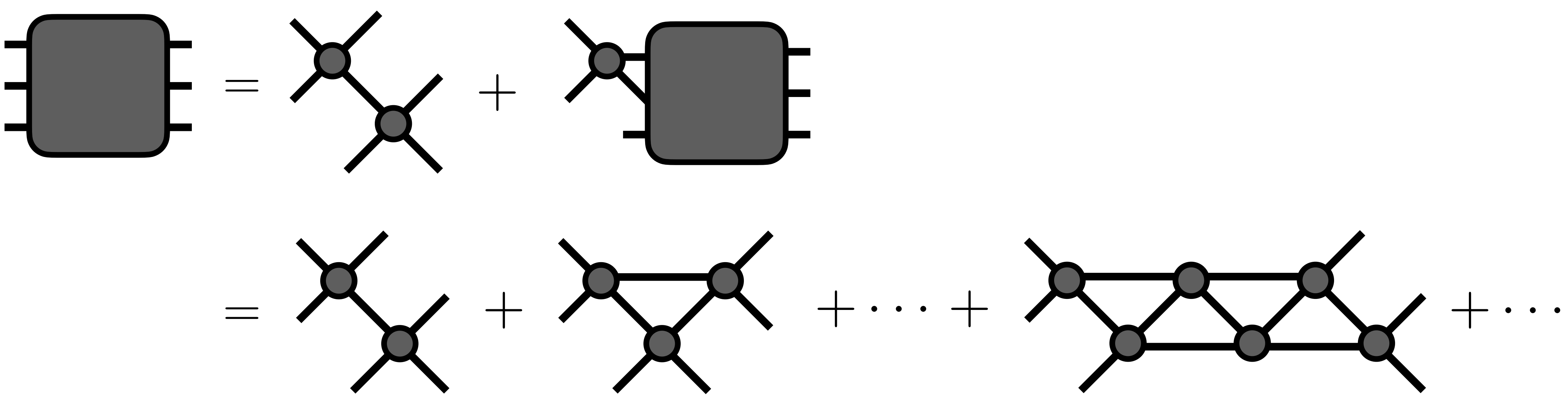
AJ, *in preparation*

Solving the three-body equations

Examine toy-model — $3\varphi \to 3\varphi$

- Assume exchange dominance — ***No short-range three-body forces***
- Scalar system — $J = 0$
- Two-hadron pair forms bound state — $2\varphi \to b$

Toy model version of $3N \to 3N$ *with* $2N \to d$



AJ, R. Briceño, S. Dawid, M.H. Islam, and C. McCarty
Phys. Rev. D **104** 014507 (2021)

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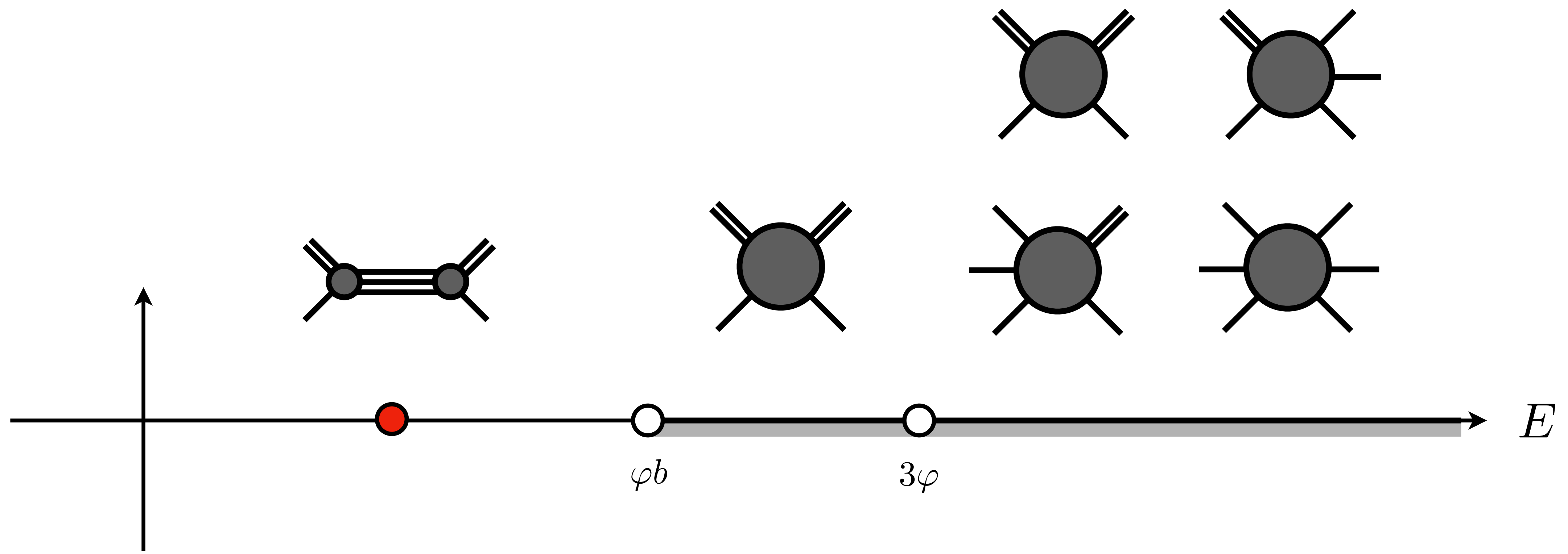

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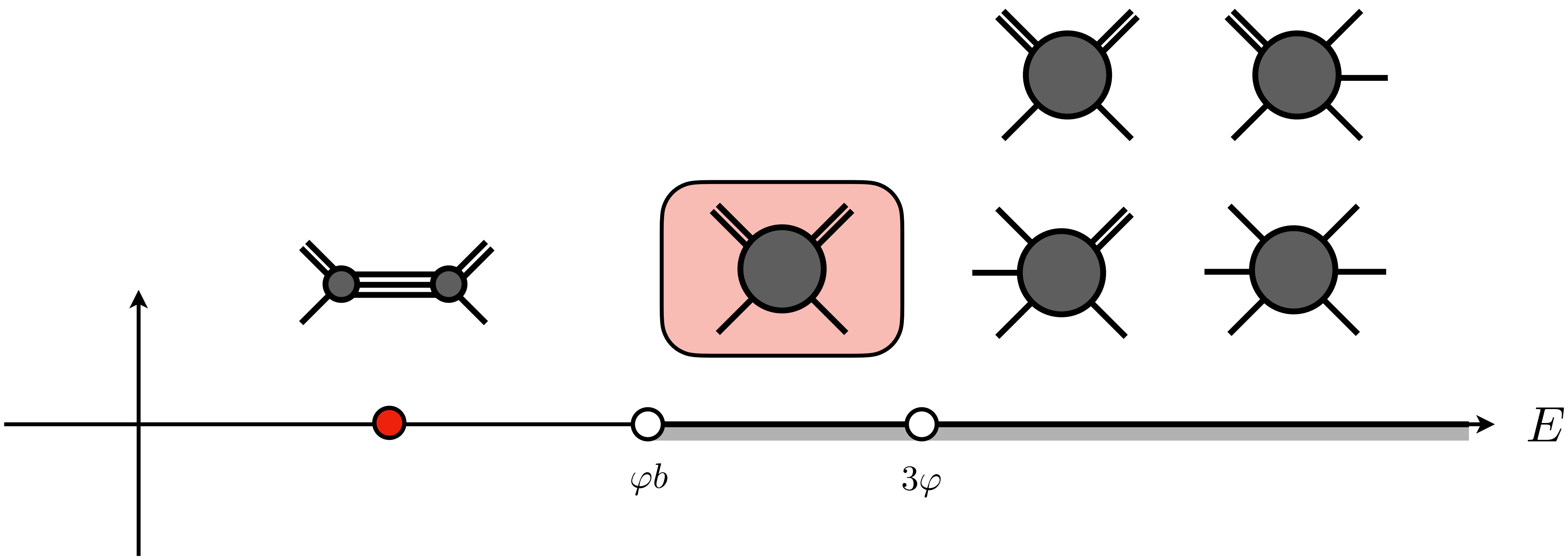

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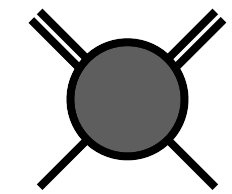

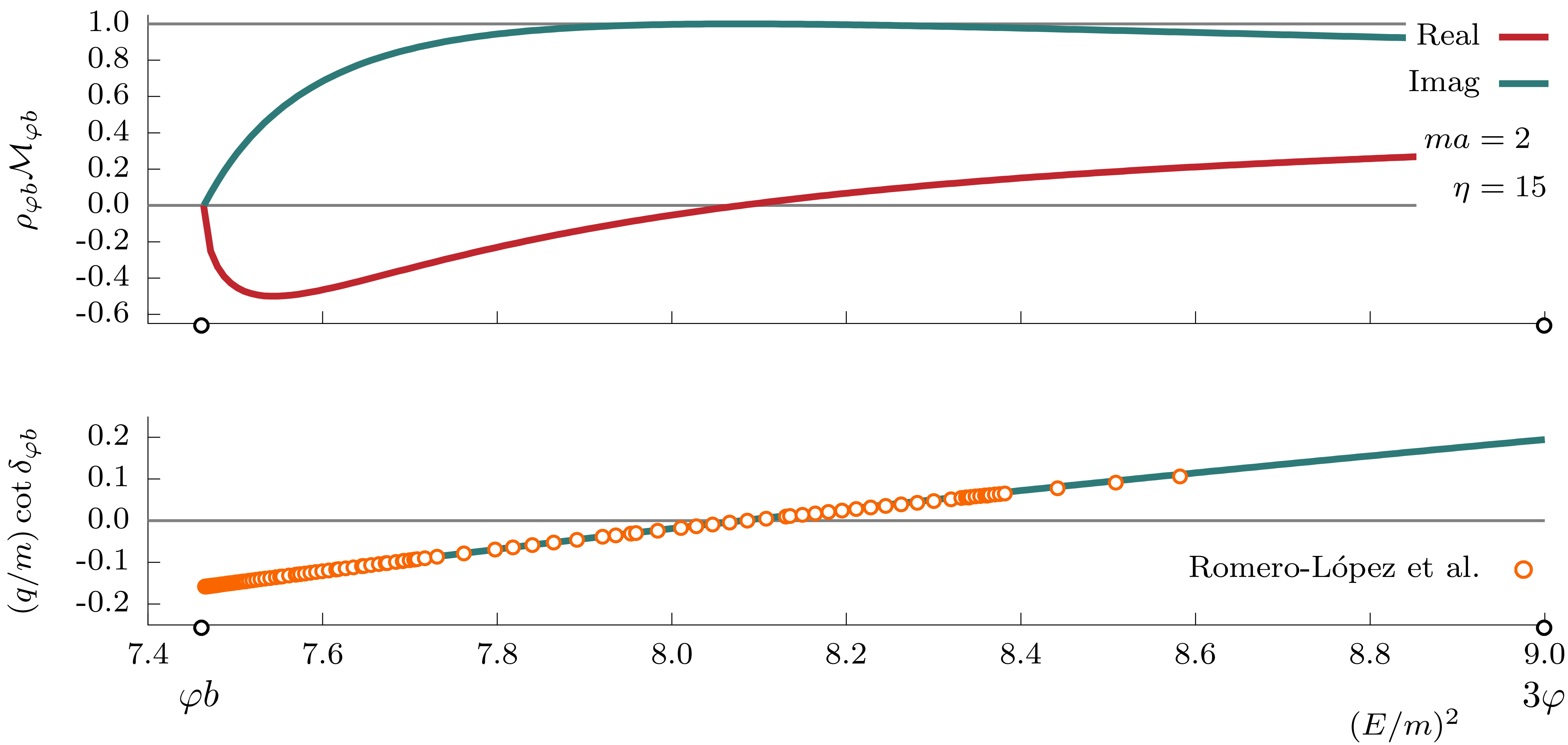

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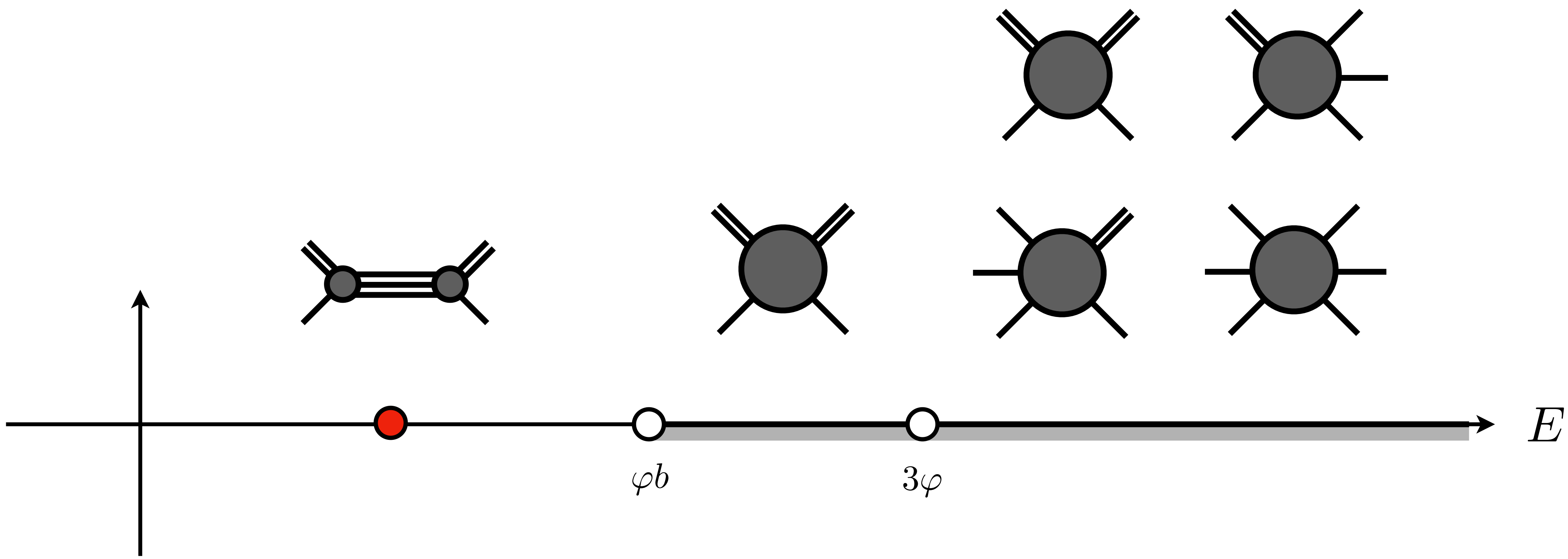
Phys. Rev. D **104** 014507 (2021)

Above the three-body threshold

Methodology not limited to below 3-body threshold

- Allows for calculation of breakup / recombination amplitude



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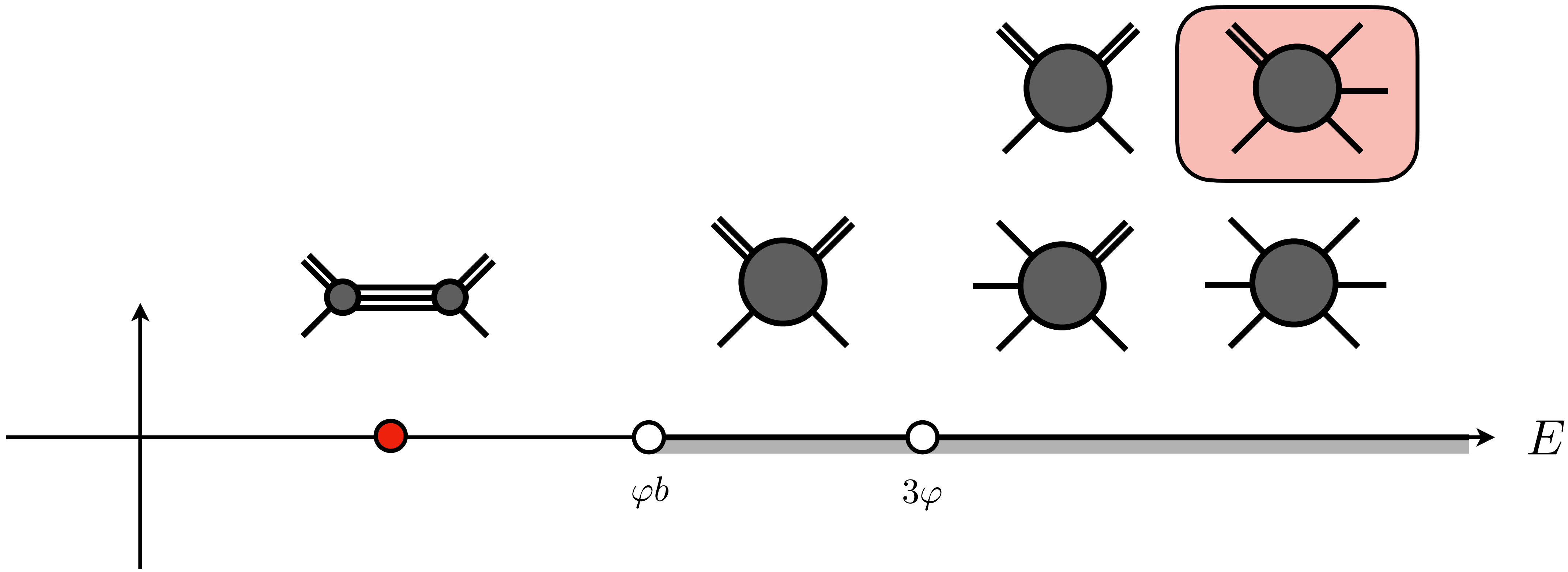

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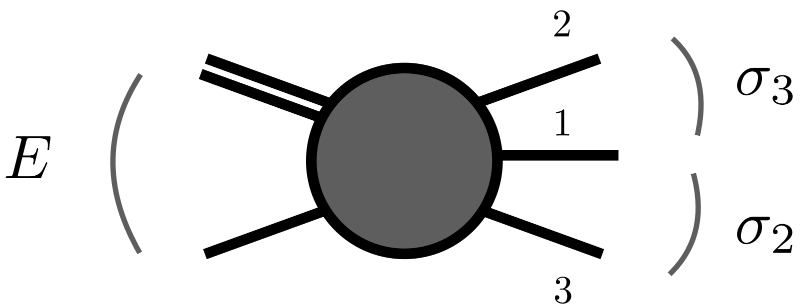

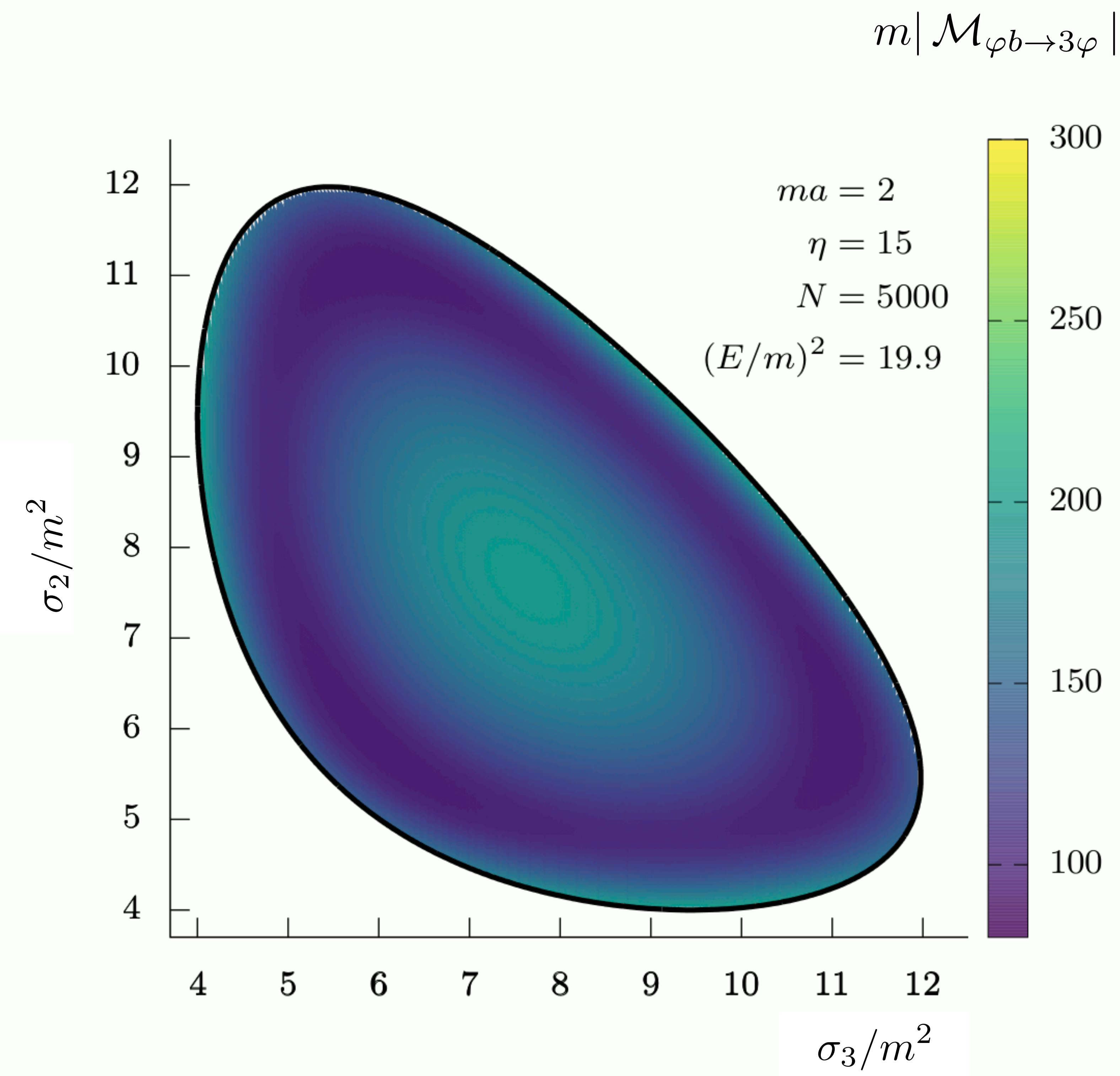

Unfortunately… pretty boring…

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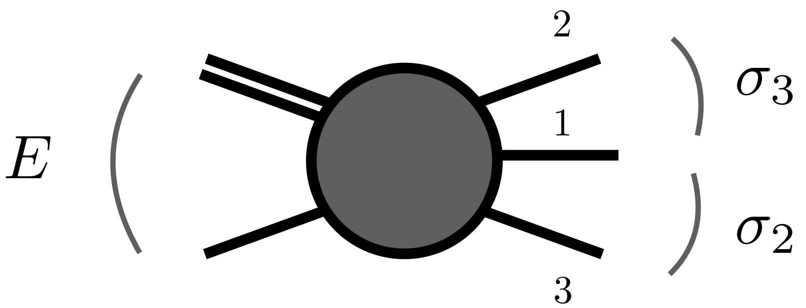

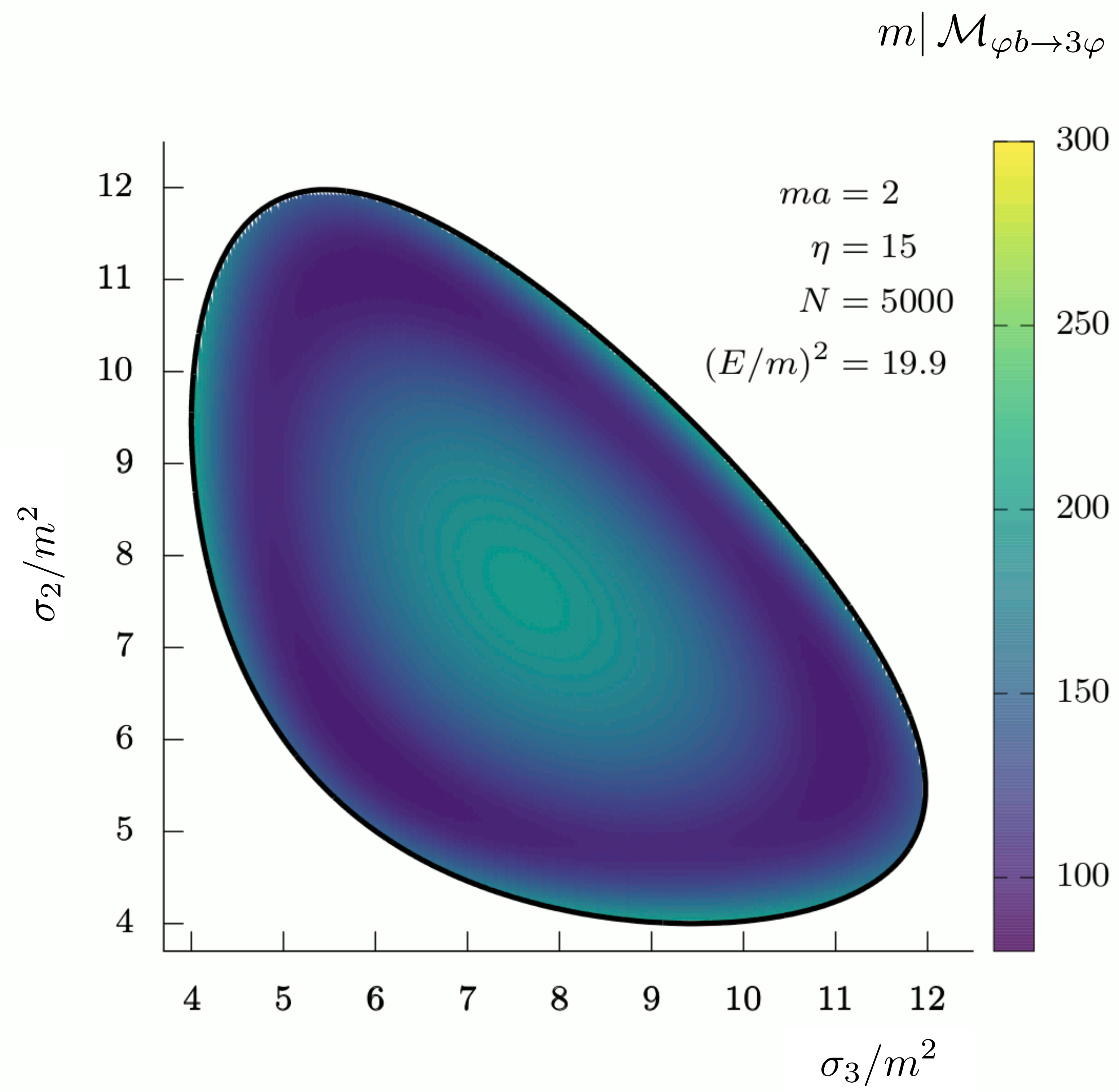

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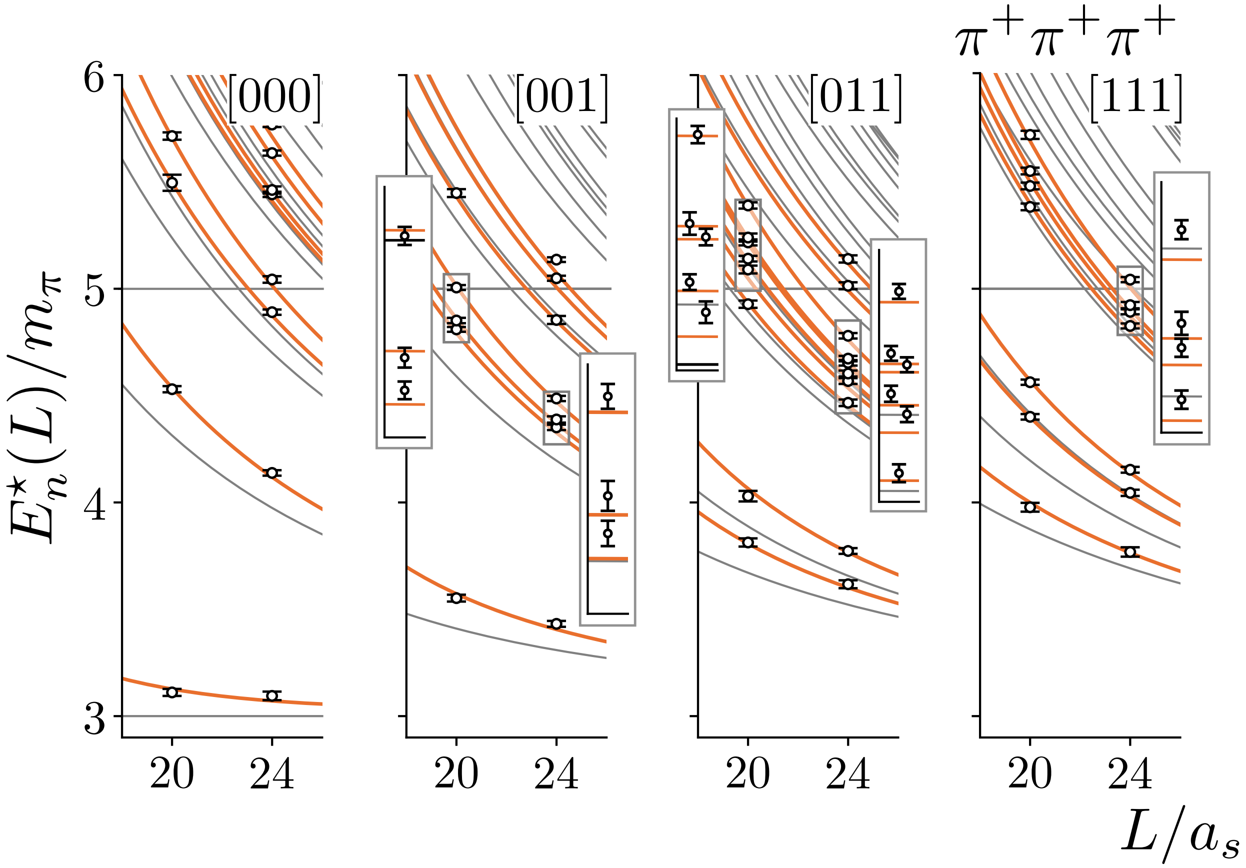
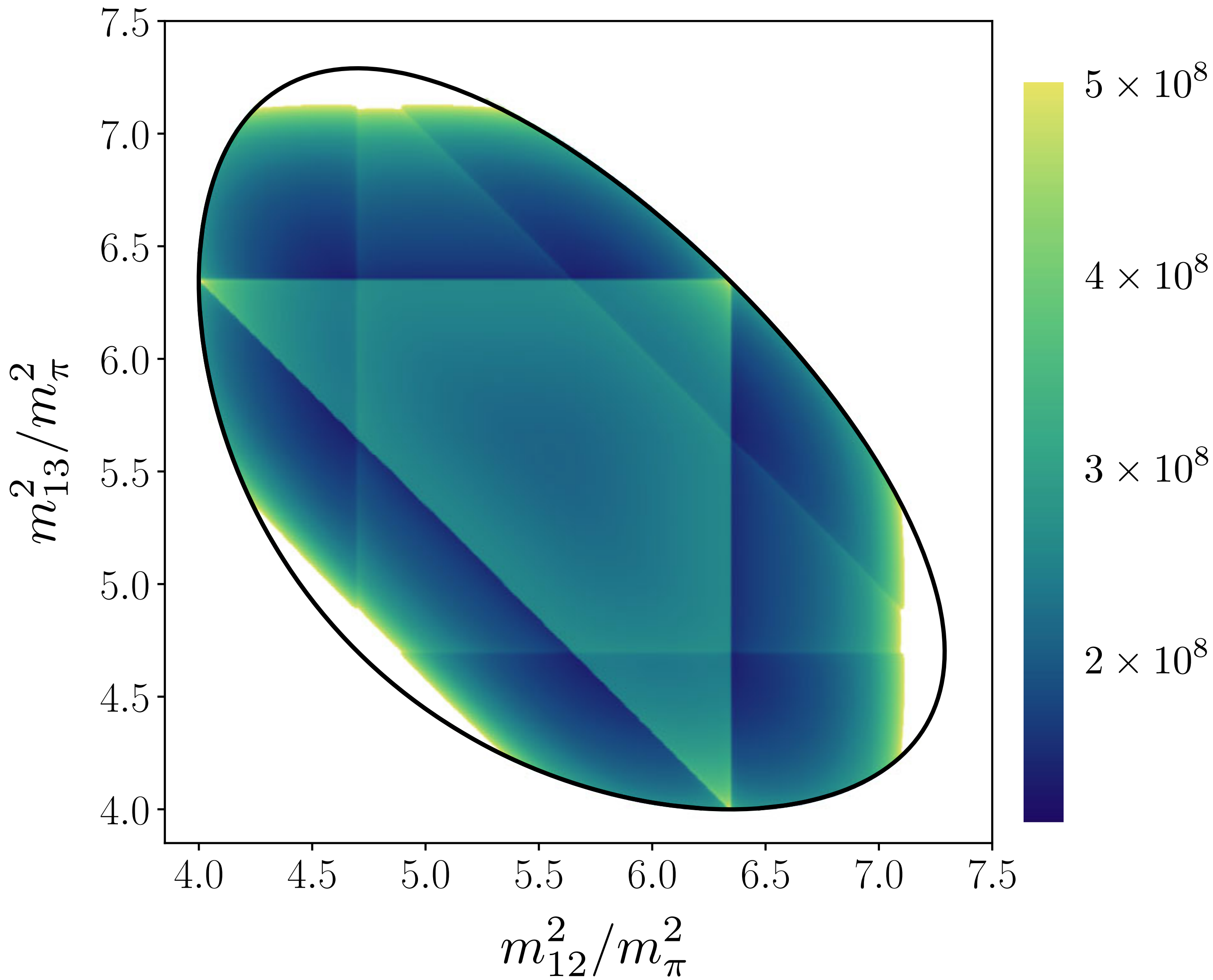
Applications to $3\pi^+$



See M. Hansen,
Meson Spectroscopy 5
28th July, 2021

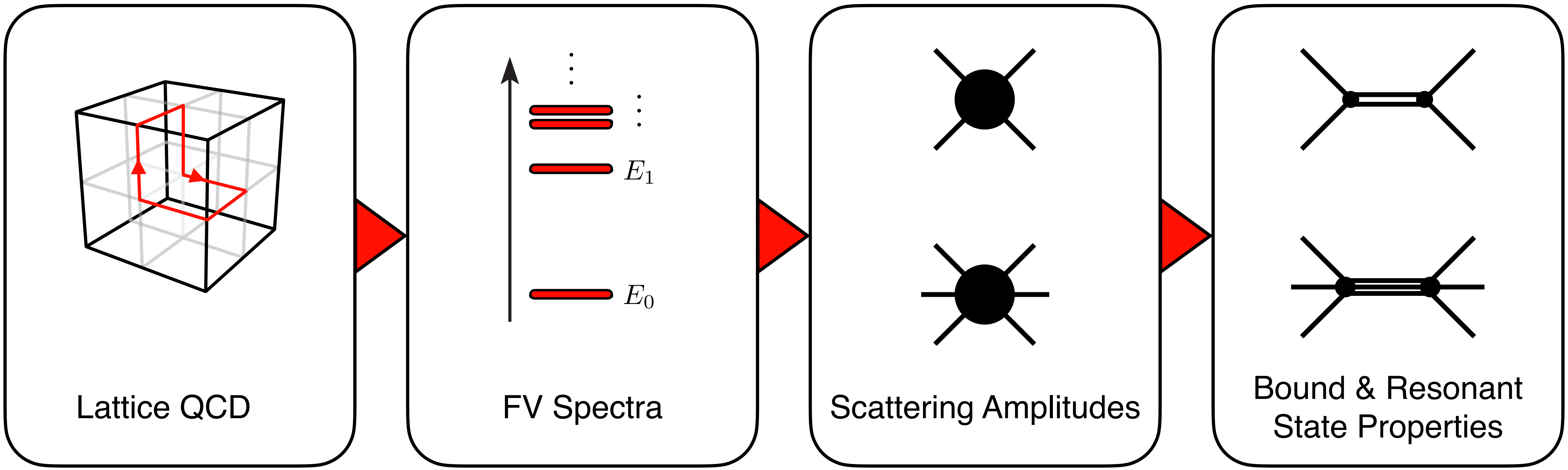


M. Hansen et al. [HadSpec]
Phys. Rev. Lett. **126**, (2021) 012001

Path to three-body physics from QCD

Three-body resonances from QCD are within reach

- Clear systematic path via Lattice QCD
- Scattering theory constraints allow for model-independent analyses



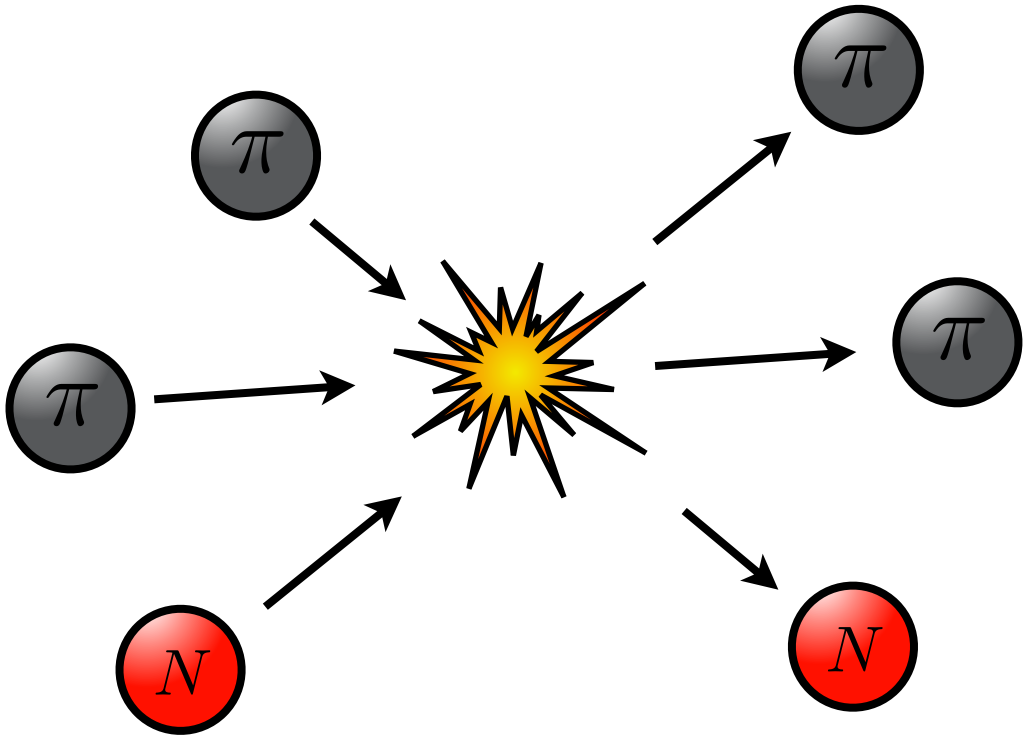
Summary

Three-body interactions play a key role in many outstanding problems in hadron physics

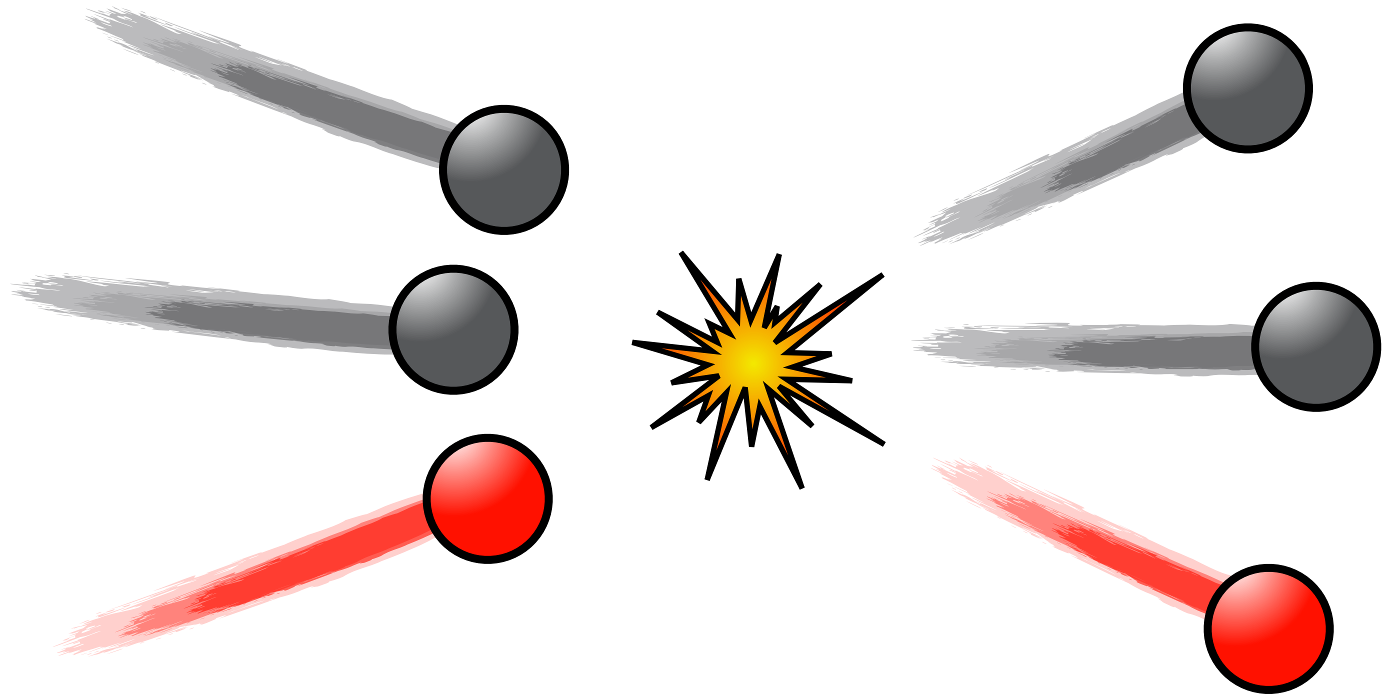
- Most excited states strongly couple to three (or more) particle channels

Framework in place to determine three-body hadron physics from lattice QCD

- Rapid development in formalisms relating lattice QCD observables to amplitudes
- Scattering phenomenology is advancing in tandem
- First applications of these ideas appearing in the literature



Extra Slides



On-shell scattering equations

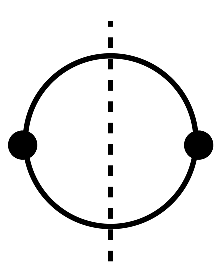
S-matrix unitarity fixes on-shell structure of scattering amplitudes

$2 \rightarrow 2$ scattering

$$\mathcal{M}_2 = \mathcal{K}_2 - \mathcal{K}_2 \mathcal{I} \mathcal{M}_2$$

On-shell two-particle rescattering

$$\mathrm{Im}\,\mathcal{I} = -\rho \ \sim$$



K-matrices

- *Unknown real function characterizing short-distance physics*
- *Parameterize with analytic function and determine from lattice QCD*
- *Scheme dependent (unphysical)*

On-shell scattering equations

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$2 \to 2$ scattering

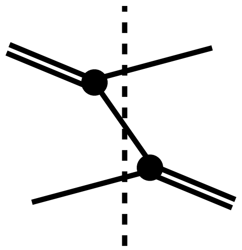
$$\mathcal{M}_2 = \mathcal{K}_2 - \mathcal{K}_2 \mathcal{I} \mathcal{M}_2$$

$3 \to 3$ scattering

$$\mathcal{M}_3 = \mathcal{K}_3 - \mathcal{K}_3 \mathcal{I} \mathcal{M}_2 - \mathcal{K}_2 \mathcal{I} \mathcal{M}_3 - \int \mathcal{K}_3 \mathcal{I} \mathcal{M}_3$$
$$- \mathcal{K}_2 G \mathcal{M}_2 - \int \mathcal{K}_3 G \mathcal{M}_2 - \mathcal{K}_2 \int G \mathcal{M}_3 - \int\int \mathcal{K}_3 G \mathcal{M}_3$$

On-shell exchange

$$\mathrm{Im}\, G = -\Delta \ \sim$$



K-matrices

- *Unknown real function characterizing short-distance physics*
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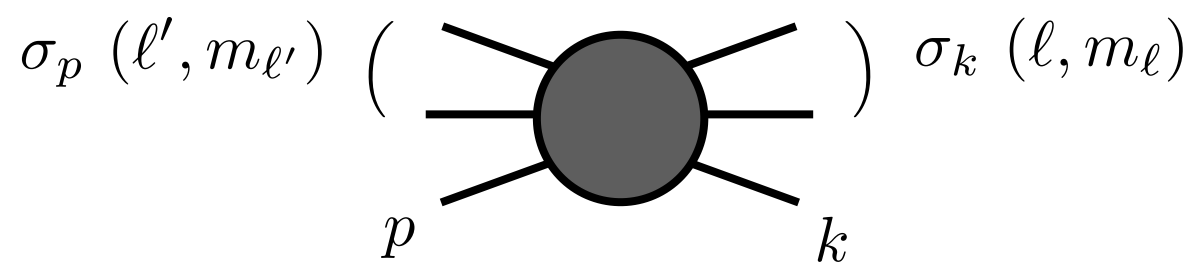
Solving three-body scattering equations

Given $\mathcal{K}_2$, and $\mathcal{K}_3$, solve integral equation for $\mathcal{M}_3$

$$\mathcal{M}_3 = \mathcal{K}_3 - \mathcal{K}_3\mathcal{I}\mathcal{M}_2 - \mathcal{K}_2\mathcal{I}\mathcal{M}_3 - \int \mathcal{K}_3\mathcal{I}\mathcal{M}_3$$
$$- \mathcal{K}_2 G\mathcal{M}_2 - \int \mathcal{K}_3 G\mathcal{M}_2 - \mathcal{K}_2 \int G\mathcal{M}_3 - \int\int \mathcal{K}_3 G\mathcal{M}_3$$



- *Many equivalent forms (Hansen-Sharpe, Blanton-Sharpe, Döring-Mai, JPAC-AJ,Mikhasenko)*
- *Matrix equation in pair angular momenta*
- *Integral equation in spectator momenta*
- *Singular kernels*
- *Scheme dependent K-matrices (unphysical)*

Solving three-body scattering equations

Given $\mathcal{K}_2$, and $\mathcal{K}_3$, solve integral equation for $\mathcal{M}_3$

$$\mathcal{M}_3 = \mathcal{K}_3 - \mathcal{K}_3\mathcal{I}\mathcal{M}_2 - \mathcal{K}_2\mathcal{I}\mathcal{M}_3 - \int \mathcal{K}_3\mathcal{I}\mathcal{M}_3$$
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Consider toy model: $3\varphi \to 3\varphi$ such that $2\varphi \to b$

AJ, R. Briceño, S. Dawid, M.H. Islam, and C. McCarty
arXiv:2010.09820

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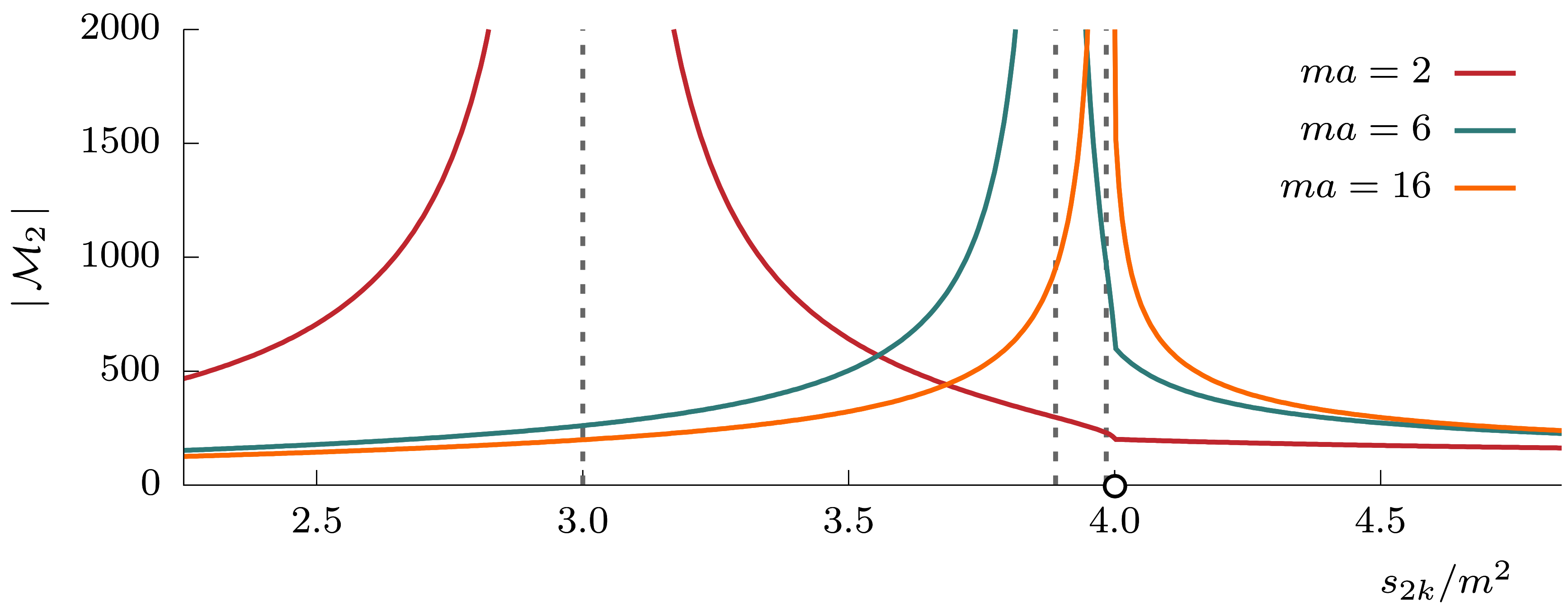
$$\mathcal{K}_2^{-1} \sim -\frac{1}{a}$$



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Assume $\mathcal{K}_3 = 0$

$$\mathcal{M}_3\big|_{\mathcal{K}_3=0} \equiv \mathcal{D} \quad \Longrightarrow \quad \mathcal{D} = -\mathcal{M}_2 G \mathcal{M}_2 - \mathcal{M}_2 \int G \mathcal{D}$$

Solving three-body scattering equations

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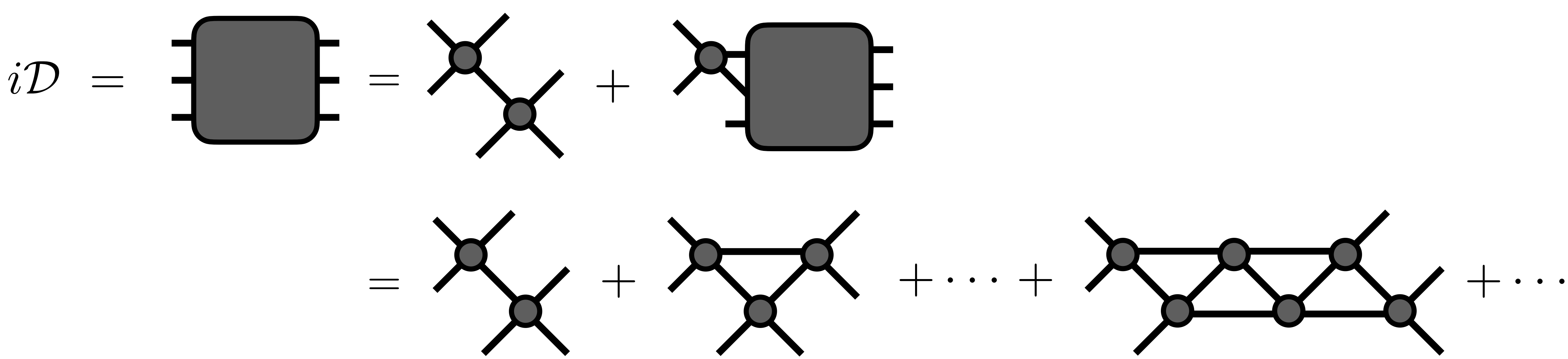
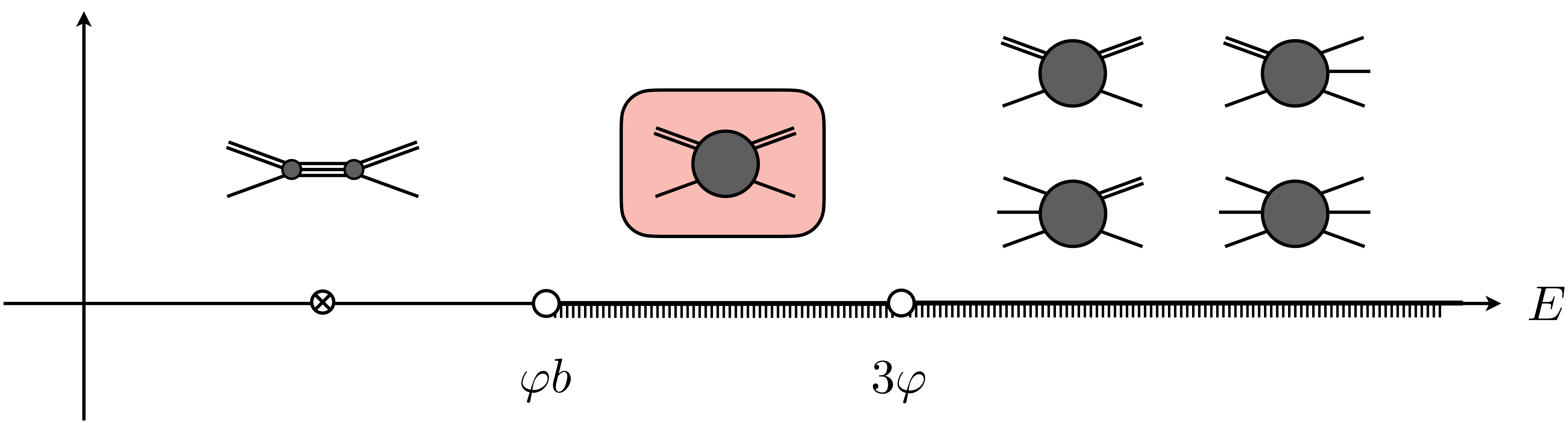


Solving three-body scattering equations

Focus on case where 2-body systems forms bound state

- Consider energies below three particle threshold

$$\lim_{\sigma_p,\sigma_k \to \sigma_b} i\mathcal{M}_3 = i g_b \frac{i}{\sigma_p - \sigma_b} i\mathcal{M}_{\varphi b} \frac{i}{\sigma_k - \sigma_b} i g_b$$



Solving three-body scattering equations

Convert integral equation to linear equation

- Introduce regulators N (matrix size) and ϵ (pole shift)
- Recover amplitude in $N \to \infty, \ \epsilon \to 0^+$ limit

$$\mathcal{M}_{\varphi b} = \lim_{N\to\infty} \lim_{\epsilon\to 0^+} \mathcal{M}_{\varphi b}(N, \epsilon)$$

Several methods

- *Brute-force*
- *Remove bound-state pole explicitly*
- *Splines — Glöckle, Hasberg, Neghabian Z. Phys.* ***A305*** *(1982) 217*

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S-matrix unitarity provides a way to check quality of solutions

- Deviation from unitarity guides quality of solution

$$\mathrm{Im}\,\mathcal{M}_{\varphi b}^{-1}(E) = -\rho_{\varphi b}(E)$$

$$\Delta\rho_{\varphi b}(E;N) \equiv \left| \frac{\mathrm{Im}\left[\mathcal{M}_{\varphi b}^{-1}(E;N)\right] + \rho_{\varphi b}(E)}{\rho_{\varphi b}(E)} \right| \times 100$$

Several methods

- *Brute-force*
- *Remove bound-state pole explicitly*
- *Splines — Glöckle, Hasberg, Neghabian Z. Phys.* ***A305*** *(1982) 217*

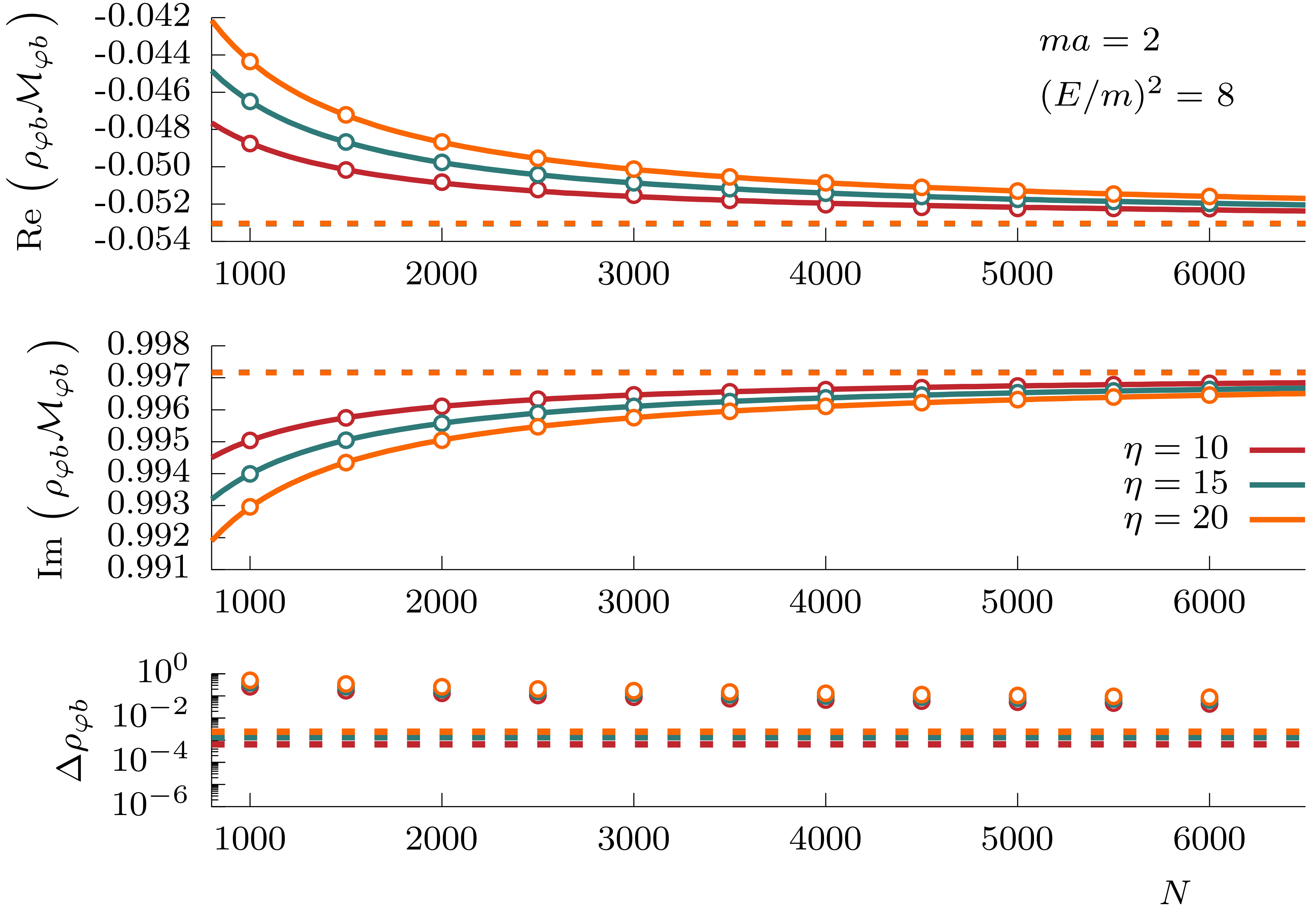
$N \to \infty$ Extrapolations

Compute multiple N solutions — extrapolate to $N \to \infty$ limit

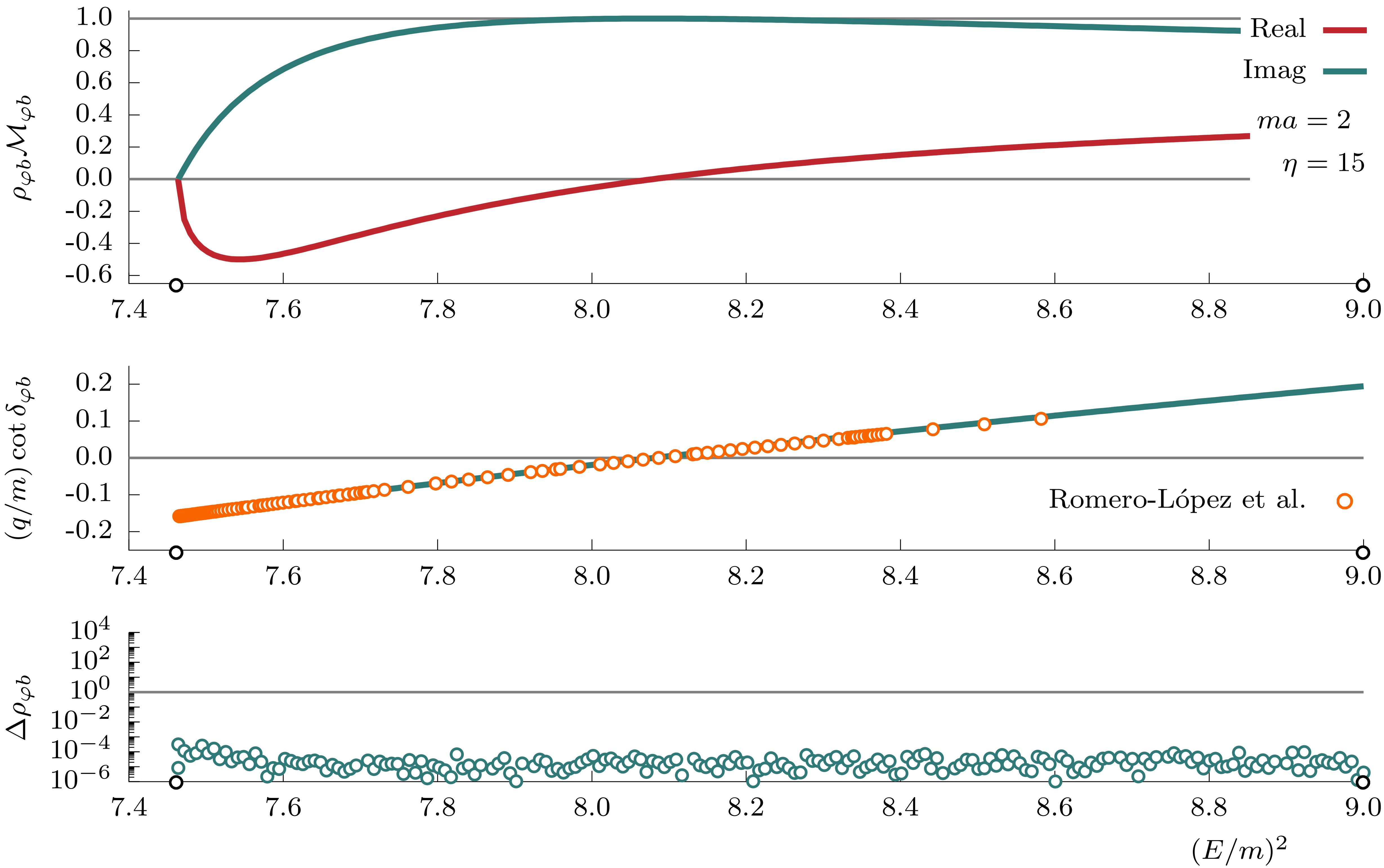
$$\mathcal{M}_{\varphi b}(E; N) \approx \mathcal{M}_{\varphi b}(E) + \frac{\alpha}{N}$$

- Unitarity deviation greatly improved

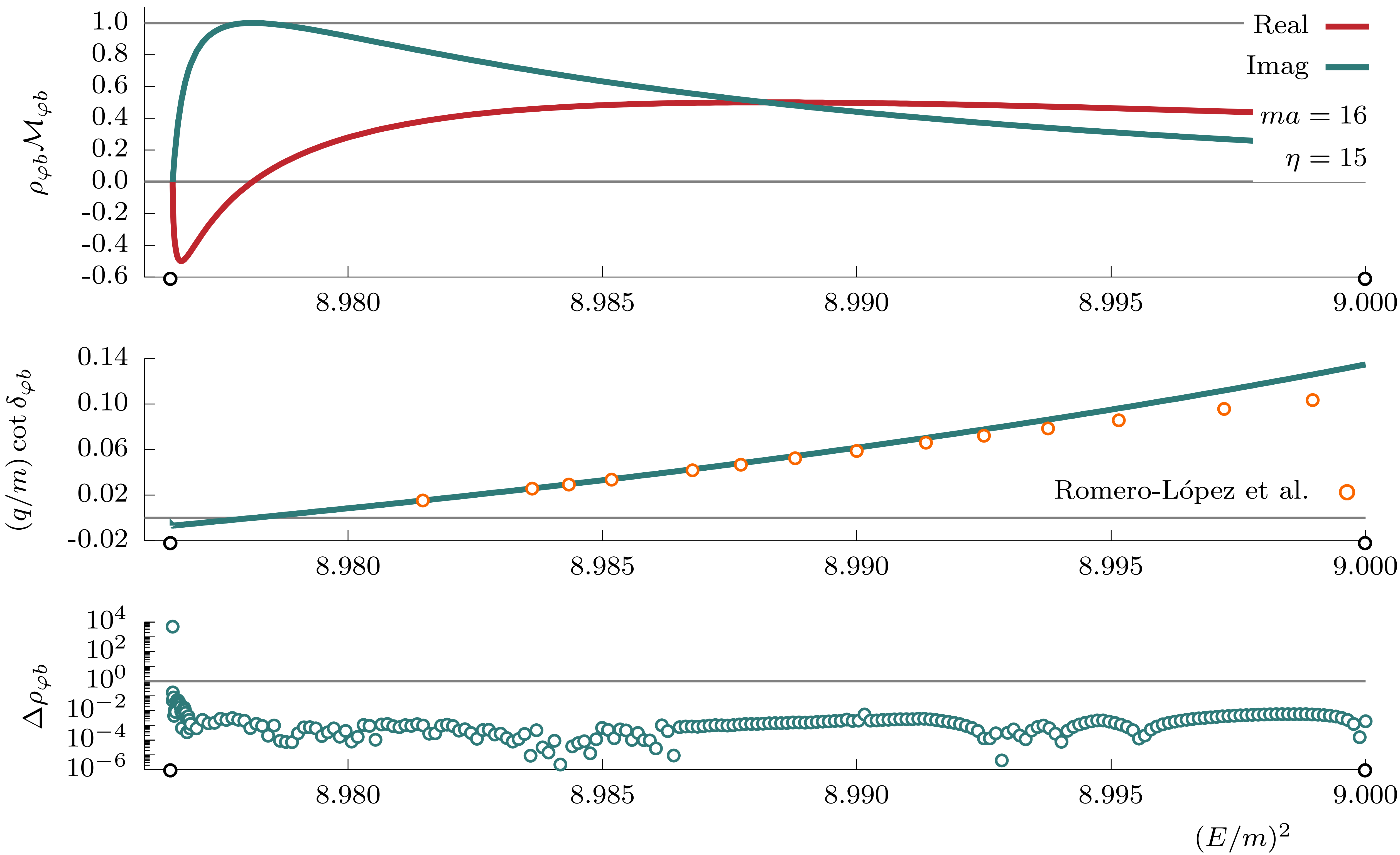
$$\epsilon \propto \frac{\eta}{N}$$



Example of solution



Example of solution

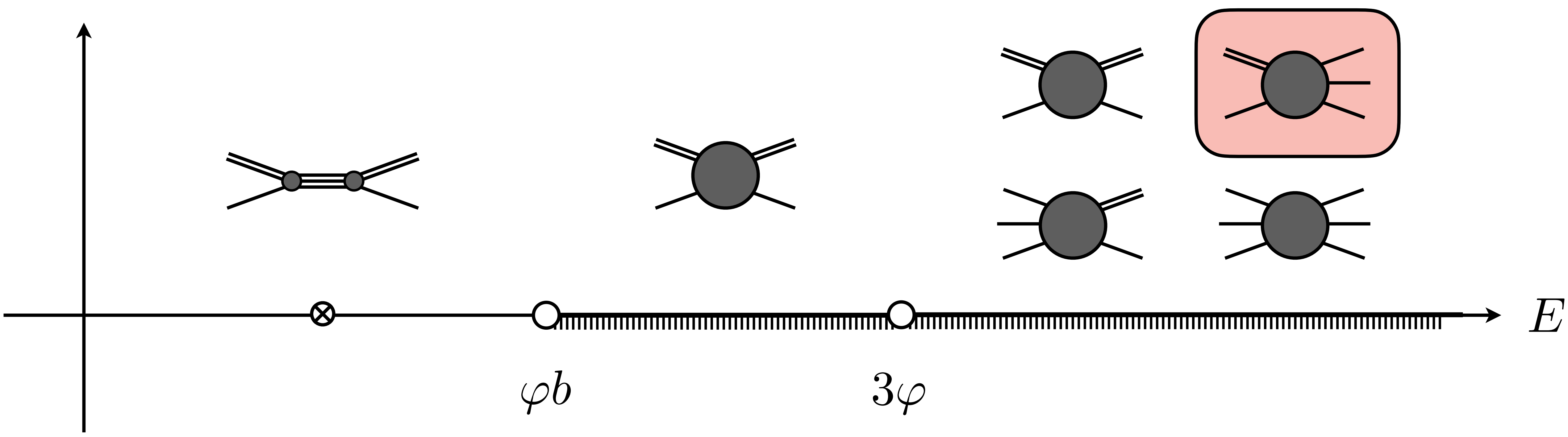


Above the three-body threshold

Methodology not limited to below 3-body threshold

- Allows for calculation of breakup / recombination amplitude

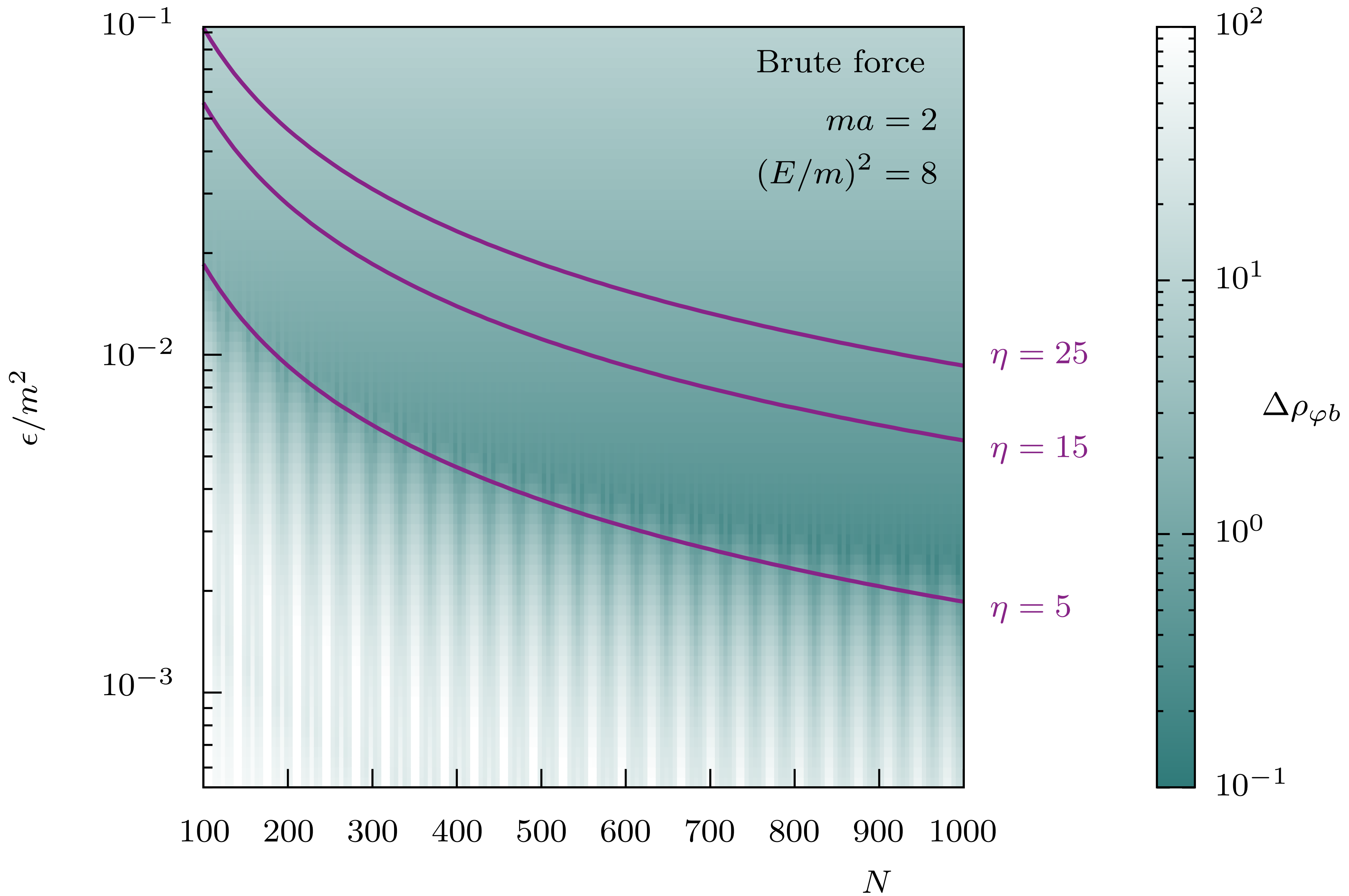
$$\mathcal{M}_{\varphi\to 3\varphi}(\sigma_p) = -\lim_{\sigma_k\to\sigma_b}\frac{\sigma_k-\sigma_b}{g}\mathcal{M}_3(\sigma_p,\sigma_k)$$



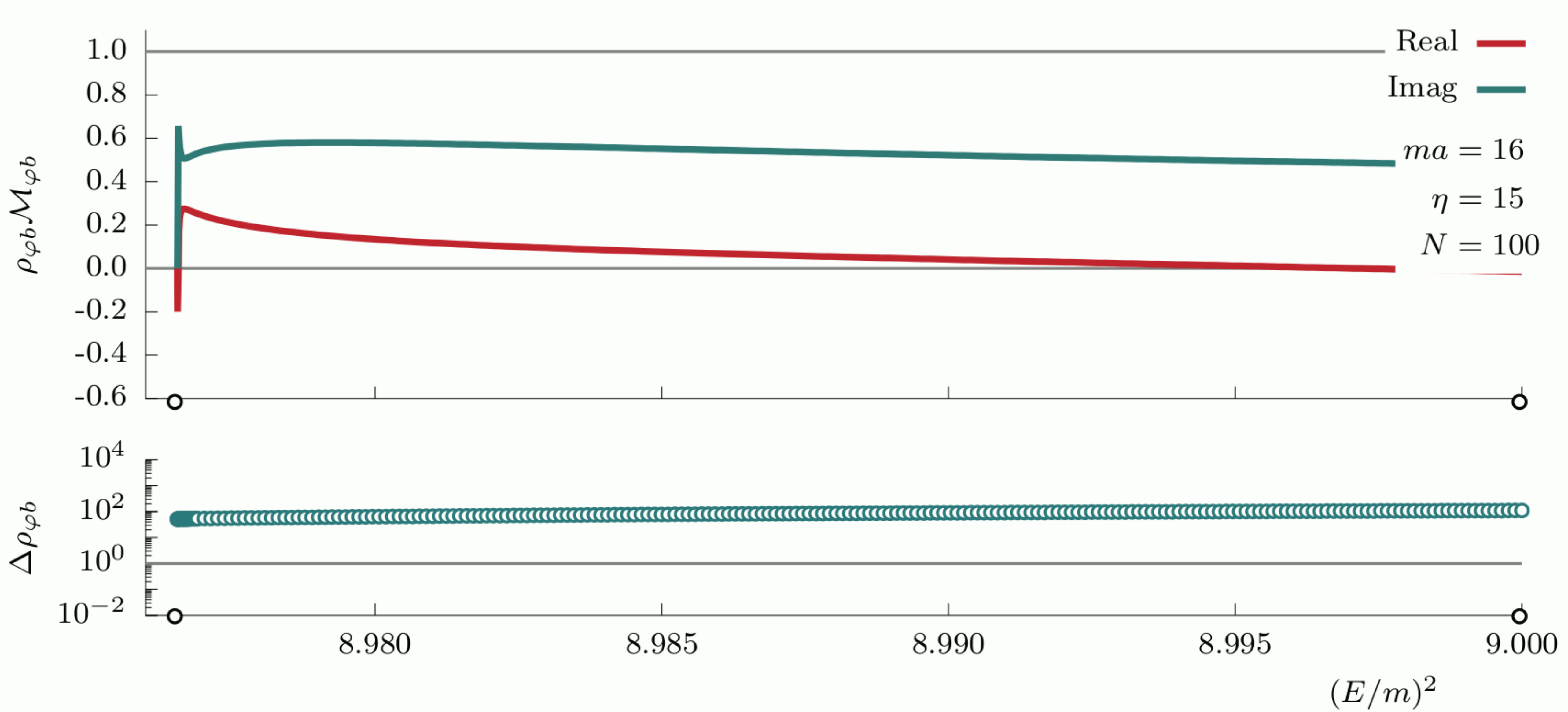
$\epsilon \to 0$ limit

Ensure $\epsilon \to 0$ through $N \to \infty$ limit

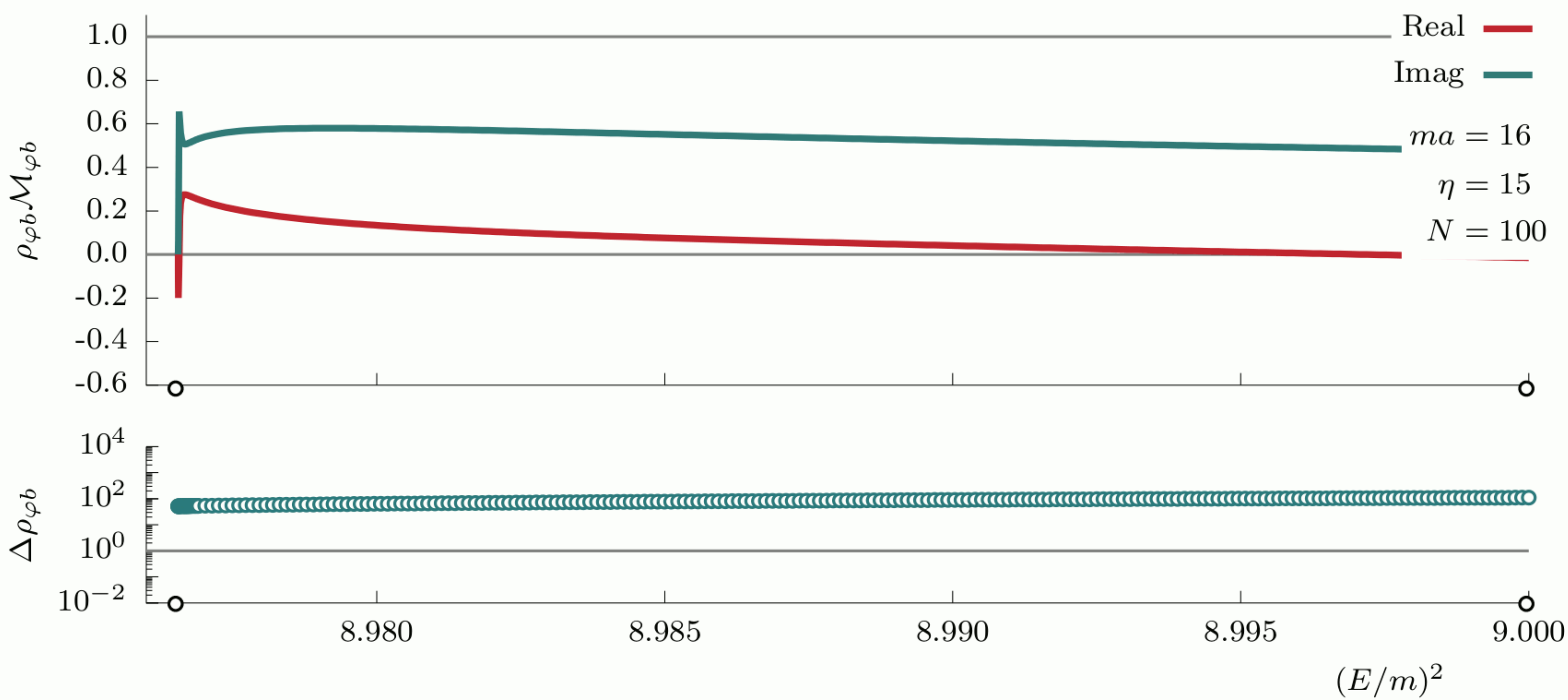
$$\left[\sum_x \Delta x - \int \mathrm{d}x\right] \frac{1}{x^2 - x_0^2 + i\epsilon} \sim e^{-2\pi\epsilon/\Delta x} \quad \Longrightarrow \quad \epsilon \propto \frac{\eta}{N}$$



Evolution of solutions



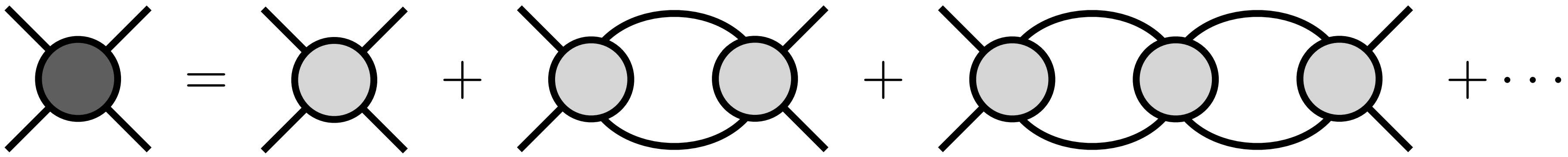
Evolution of solutions



On-shell scattering amplitudes from RFT

Sum to all orders in generic EFT all relevant cuts leading to singularities in physical region

- e.g. $\mathbf{2} \rightarrow \mathbf{2}$



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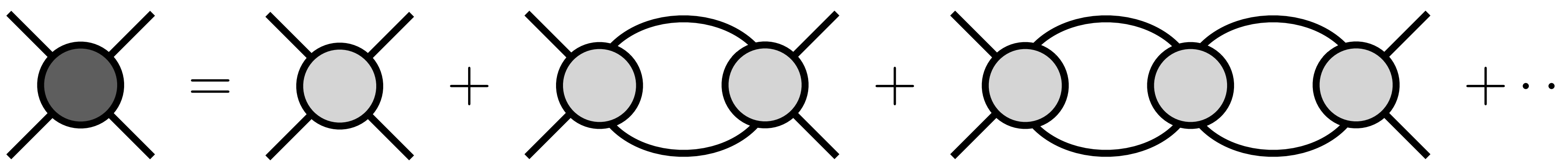

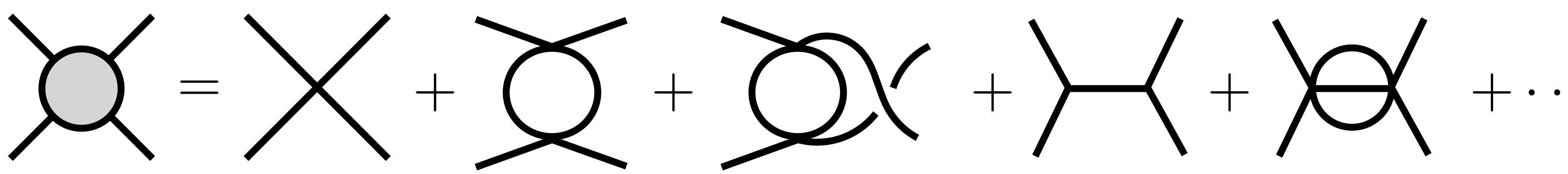

All 2PI diagrams - left hand cuts and higher multi-particle thresholds

On-shell scattering amplitudes from RFT

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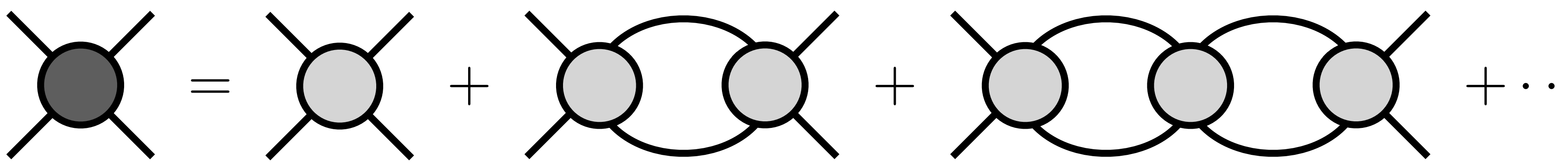

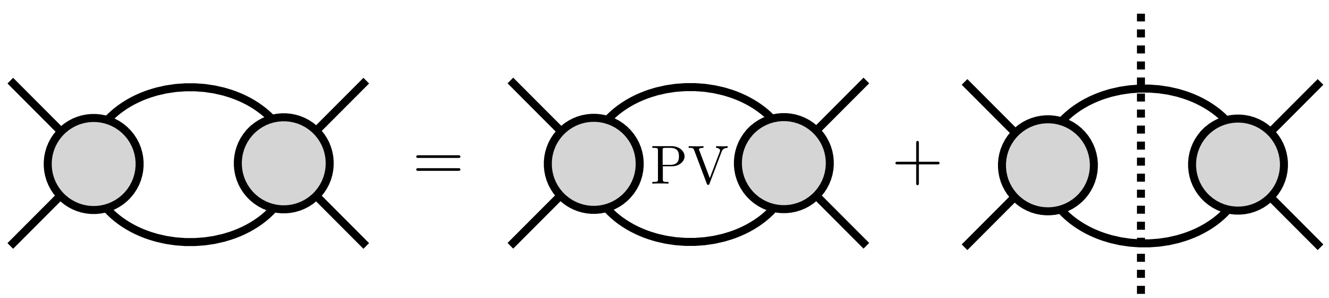

$$\rho = \frac{q}{8\pi E} \sim \sqrt{s - s_{\rm th}}$$

On-shell scattering amplitudes from RFT

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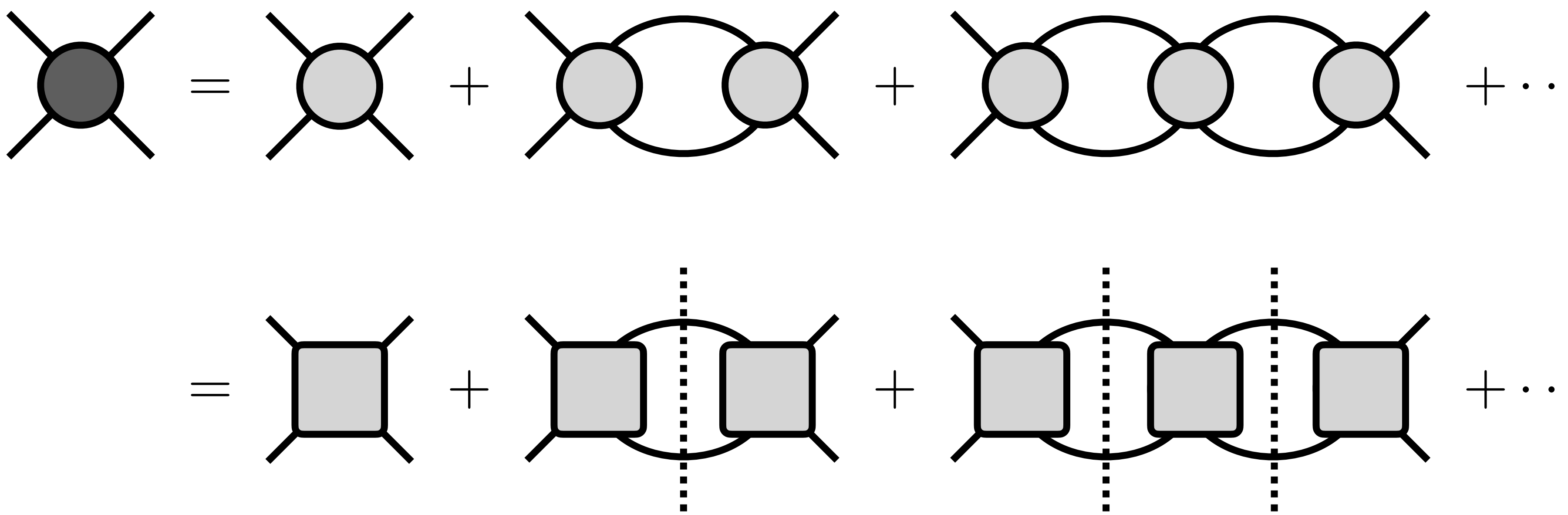
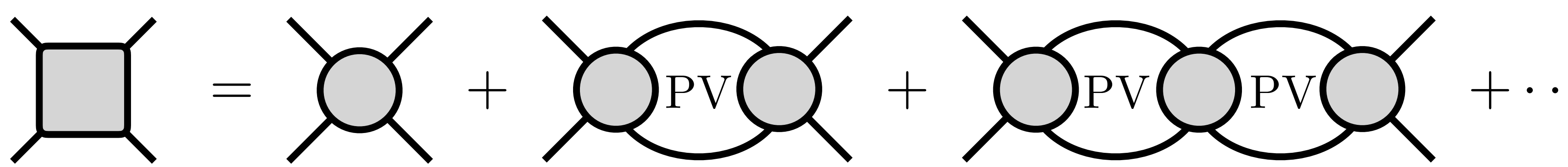
- e.g. $\mathbf{2} \to \mathbf{2}$





K matrix — unknown dynamical function unconstrained by unitarity

On-shell scattering amplitudes from RFT

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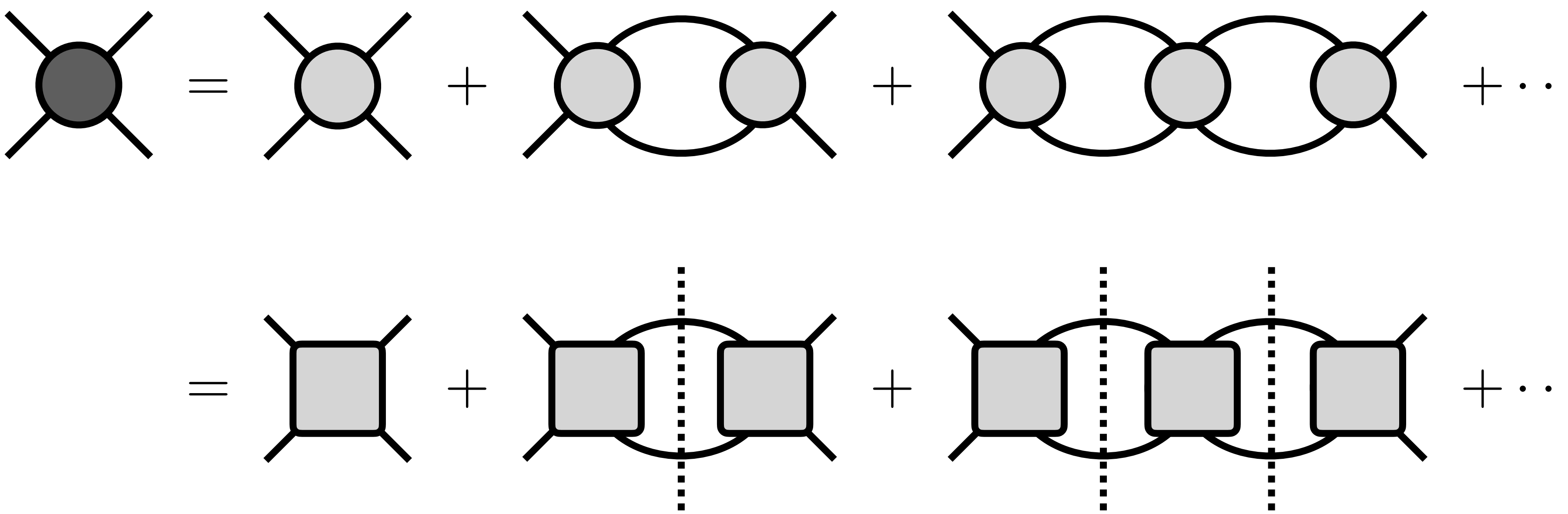
- e.g. $\mathbf{2} \to \mathbf{2}$



$$\mathcal{M}_2 = \mathcal{K}_2 + \mathcal{K}_2 i\rho\mathcal{M}_2$$

For given K matrix, obtain on-shell solution for amplitude

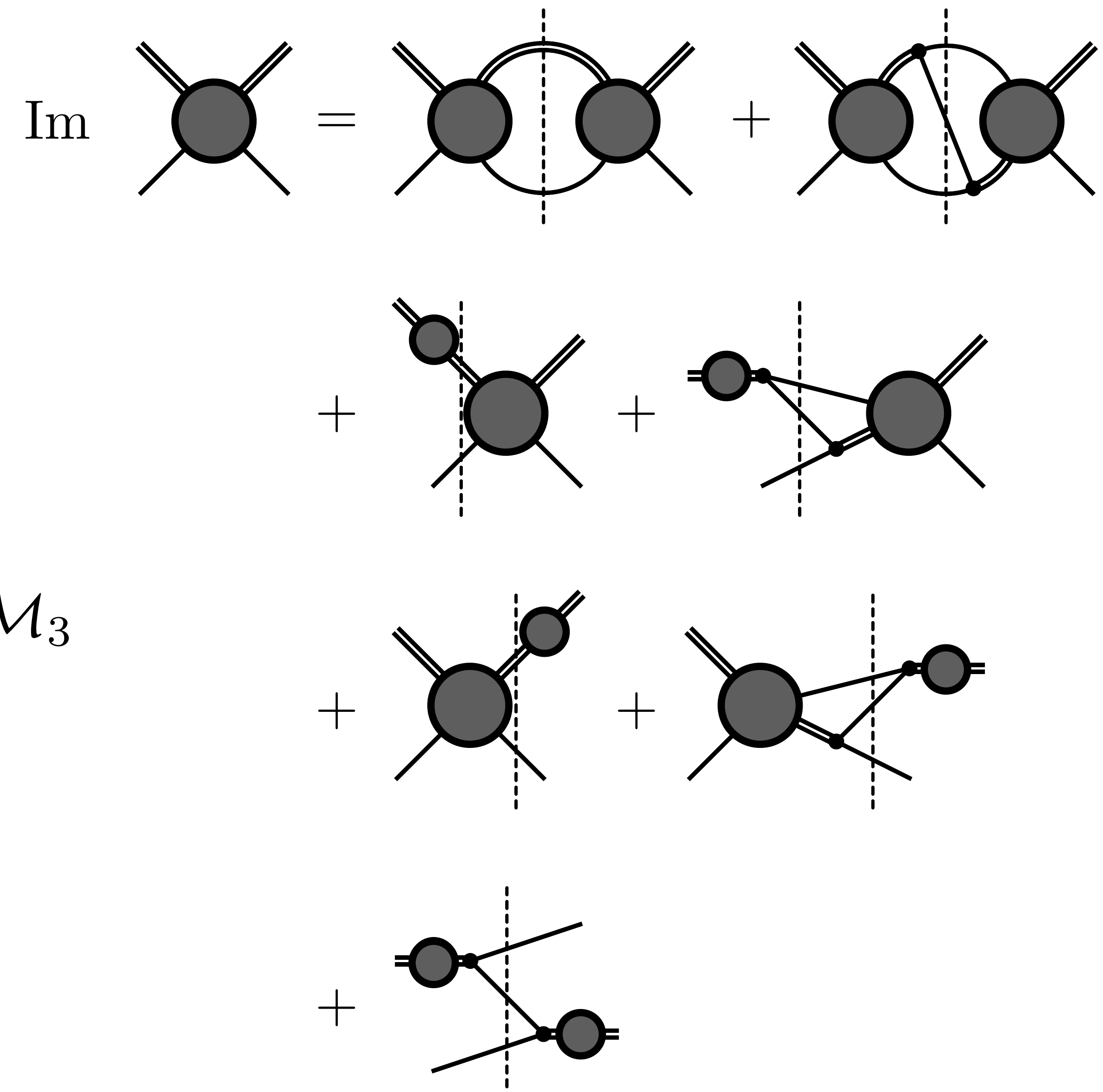
On-shell scattering amplitudes from unitarity

Start with *S* matrix unitarity

- Provides constrain for amplitude on real axis in physical region
- On-shell amplitude satisfies linear equation — check unitarity constraint

$$\mathcal{M}_3 = \mathcal{M}_2(\mathcal{R} - G)\mathcal{M}_2 + \int \mathcal{M}_2(\mathcal{R} - G)\mathcal{M}_3$$

R is a different short-distance function



M. Mai, B. Hu, M. Döring, A. Pilloni, and A. Szczepaniak
Eur. Phys. J. A **53**, 177 (2017)

AJ et al. [JPAC]
Eur. Phys. J. C **79**, no. 1, 56 (2019)

M. Mikhasenko, AJ et al. [JPAC]
Phys. Rev. D **98**, 096021 (2018)

M. Mikhasenko, AJ et al. [JPAC]
JHEP **08**, 080 (2019)

Extension to FV

M. Mai and M. Döring
Eur. Phys. J. A **53**, 240 (2017)

Equivalence of relativistic methods

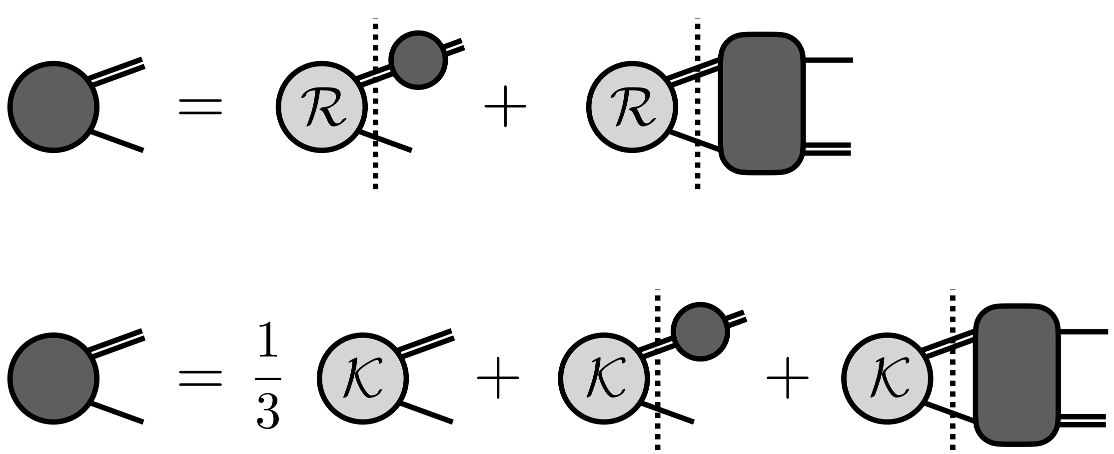
The RFT vs FVU methods can be summarized as “Bottom up” vs “Top down”

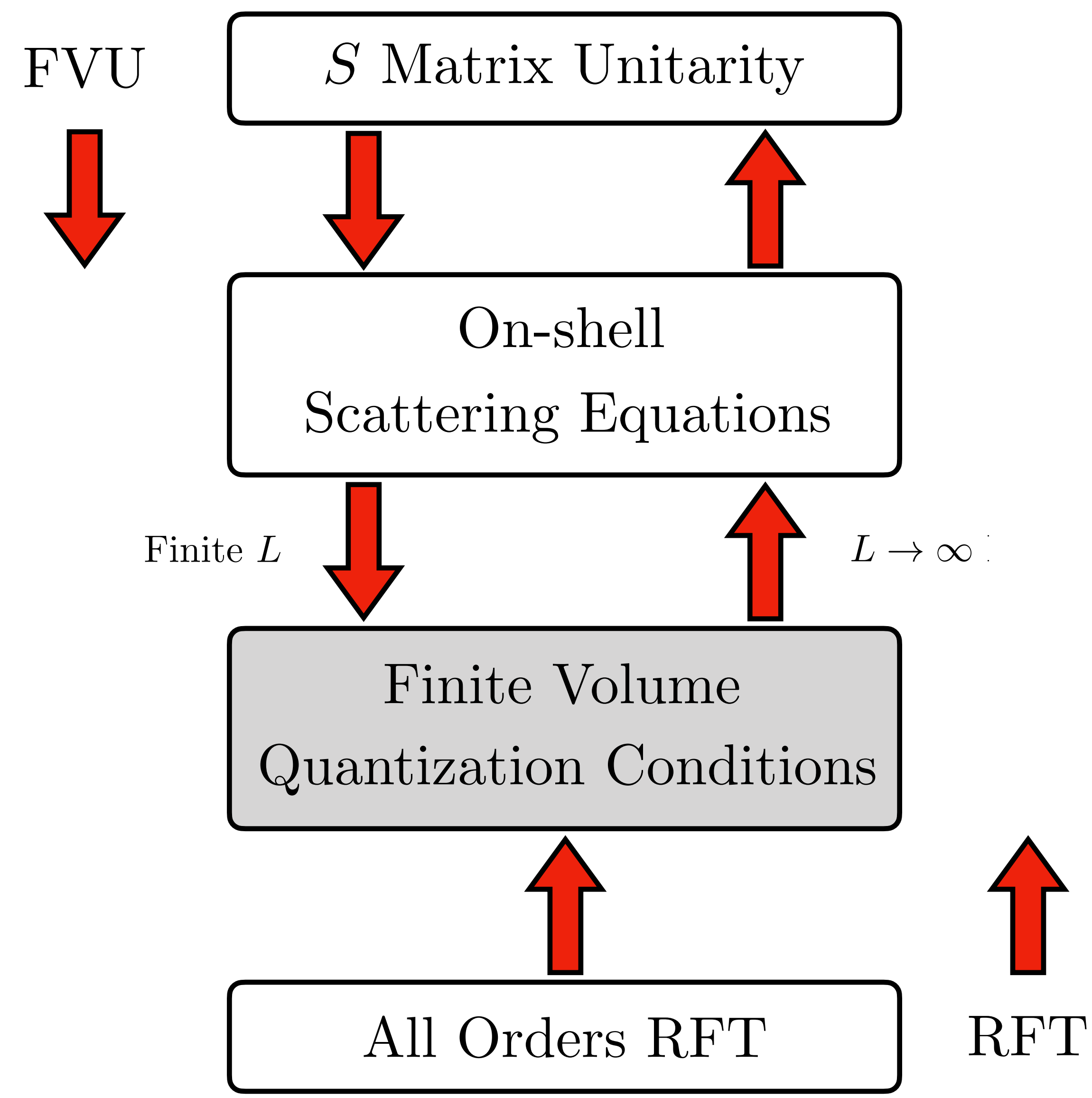
On-shell scattering equations are equivalent

AJ et al.
Phys. Rev. D **100**, 034508 (2019)

Quantization conditions are equivalent

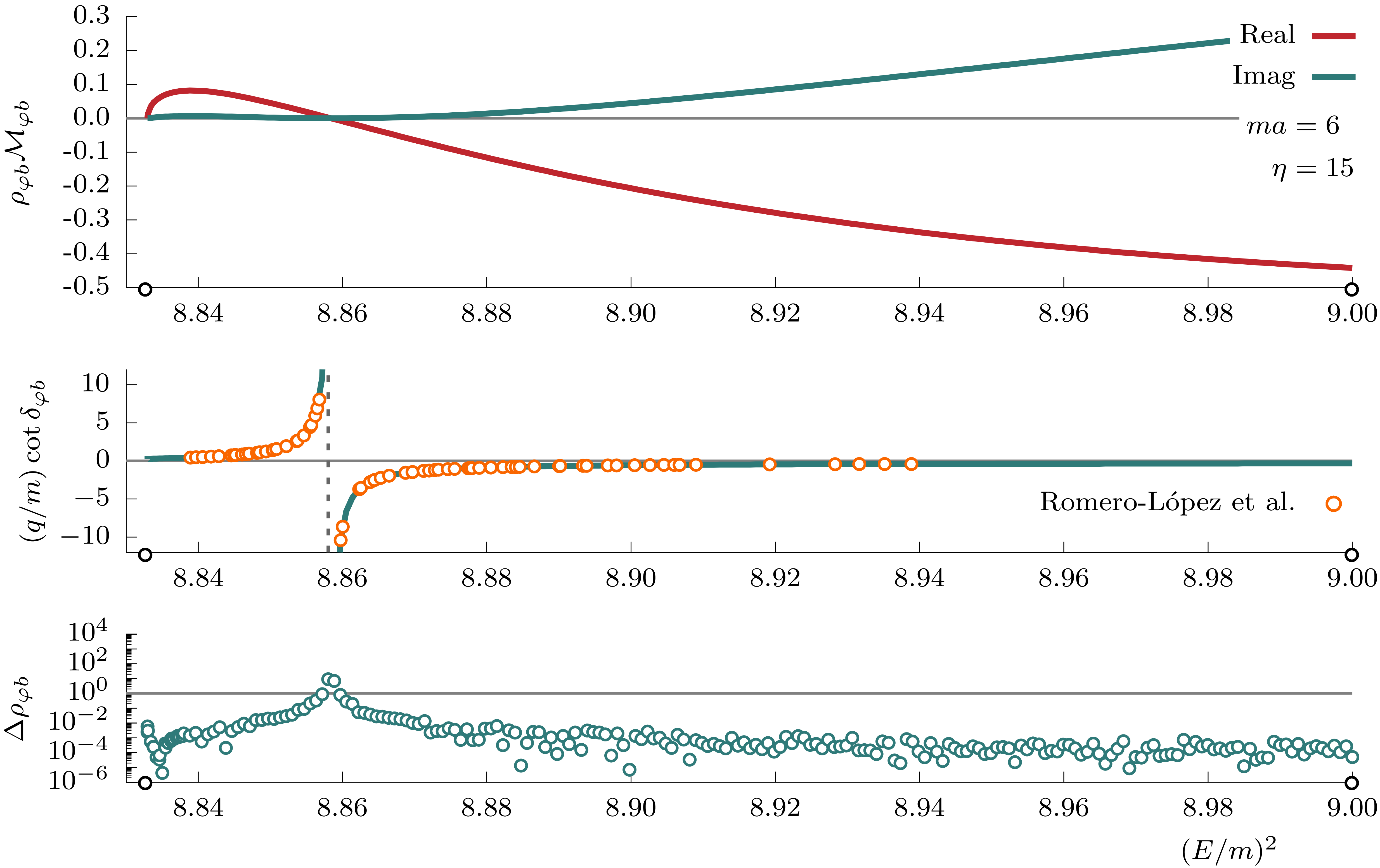
T. Blanton and S. Sharpe
arXiv:2007.16190 (2020)





Small details, e.g. aspects of symmetrization — see literature for information

Example of solution — Extrapolated result



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