γ-ray detectors (II):

(1) Tools & stats; (2) Skymaps; (3) discussion.

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Some tools I use

- Statistics: the basic and the needed. Poisson, gauss, χ^2, likelihood.
- Python: I am not an expert; just a recycled Fortranosaurus.
  - Anaconda installation, spyder editor.
  - Tasks: open, close, read files...
  - Packages: numpy (scipy), matplotlib, healpy.

More than just presenting I would like to discuss some topics.

Poisson distribution

The probability of counting k events given an average μ, given by

$$p_k = \frac{\mu^k}{k!} e^{-\mu}.$$

One proves,

$$\sum_{k=0}^{\infty} p_k = 1, \quad \langle k \rangle = \mu, \quad \sigma(k) = \sqrt{\langle k^2 \rangle - \langle k \rangle^2} = \mu\,.$$

Normal (Gaussian) distribution

Limit of the central limit theorem. Probability density, of mean μ and standard deviation σ,

$$g(x) = \frac{1}{\sqrt{2\pi}\sigma} \exp\left\{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2\right\}, \quad -\infty < x < +\infty\,. \qquad (1)$$

The cumulative distribution is given by the error function,

$$\mathrm{erf}(x) = \frac{2}{\sqrt{\pi}} \int_0^x e^{-t^2}\, dt, \qquad (2)$$

such that

$$G(x) \equiv P(> x) = \frac{1}{2} - \frac{1}{2}\mathrm{erf}\left(\frac{x-\mu}{\sigma\sqrt{2}}\right)\,.$$

Given a probability P (confidence level), the bound on x is,

$$x = \mu + (\sigma/\sqrt{2})\mathrm{erf}^{-1}(2G-1).$$

Gaussian confidence

Beware of the two sides...

s	$\mathrm{erf}(s/\sqrt{2})$	$G(< s)$	$G(> s)$
1	0.6827	0.8413	0.1587
1.645	0.9000	0.9500	0.0500
2	0.9545	0.9772	0.0228
2.326	0.9800	0.9900	0.0100
3	0.9973	0.9987	0.0013
5	1.0000	1.0000	2.867×10^{-7}

Cuadro: Significances and probabilities for $s = (x - \mu)/\sigma$. The function $\mathrm{erf}(s\sqrt{2})$ gives the probability to be between $\pm s$.

Likelihood and maximization

Likelihood is defined as

$$\mathcal{L} = \prod_j p_j \, .$$

Assume gaussian probabilities with constant mean and deviation. Then

$$\ln(\mathcal{L}) = \sum_j -\frac{1}{2}\left(\frac{x_j - \mu}{\sigma}\right)^2 - \ln(\sqrt{2\pi}\sigma) \to \chi^2 \, .$$

Maximizing $\ln(\mathcal{L})$ becomes minimizing χ^2. For a single parameter,

$$\frac{d}{d\mu}\ln(\mathcal{L}) = 0 \;\Rightarrow\; \frac{d}{d\mu}\left\langle (x_j - \mu)^2 \right\rangle = 0 \;\Rightarrow\; \mu = \langle x_j \rangle \, .$$

We also want to determine the uncertainty in μ...

Generic χ^2 minimization

Consider datapoints $\{x_i; y_i\}$ with experimental error δy_i; and $f(x_i, q_k)$ a fitting function of parameters q_k. For gaussian statistics maximum likelihood is χ^2 minimization,

$$\text{mín}\left[\chi^2 = \sum_i \left(\frac{y_i - f(x_i, q_k)}{\delta y_i}\right)^2\right] \quad \Rightarrow \quad \frac{\partial \chi^2}{\partial q_k} = 0 \qquad (3)$$

Contours in fit parameter space are given by the second order expansion around the best parameters, $\{q_k^0\}$,

$$\chi(q_k) = \chi_0 + \frac{1}{2}\sum_{\ell,k}\left(\frac{\partial^2 \chi}{\partial q_\ell \partial q_k}\right)_0 (q_\ell - q_\ell^0)(q_k - q_k^0) + \ldots, \qquad (4)$$

using $(\partial\chi^2/\partial q_k)_0 = 0$.

Linear χ^2 minimization (1)

As a practical example, consider the linear fit, $f(x) = ax + b$, leading to the variance weighted least squares expression,

$$
\begin{aligned}
\frac{\partial \chi^2}{\partial b} = 0 \quad &\Rightarrow \quad \left\langle -\frac{2}{\delta y}\left(\frac{y - ax - b}{\delta y}\right)\right\rangle = 0, \\
&\Rightarrow \quad \left\langle \frac{y}{\delta y^2}\right\rangle = a\left\langle \frac{x}{\delta y^2}\right\rangle + b\left\langle \frac{1}{\delta y^2}\right\rangle; \\
\frac{\partial \chi^2}{\partial a} = 0 \quad &\Rightarrow \quad \left\langle -\frac{2}{\delta y}\left(\frac{y - ax - b}{\delta y}\right)x\right\rangle = 0, \\
&\Rightarrow \quad \left\langle \frac{yx}{\delta y^2}\right\rangle = a\left\langle \frac{x^2}{\delta y^2}\right\rangle + b\left\langle \frac{x}{\delta y^2}\right\rangle .
\end{aligned}
$$

Linear χ^2 minimization (1)

The solution

$$\begin{aligned} a_0 &= \frac{< yx/\delta y^2 > - < y/\delta y^2 >< x/\delta y^2 >}{< x^2/\delta y^2 > - < x/\delta y^2 >^2}, \\ b_0 &= \left\langle y/\delta y^2 \right\rangle - a \left\langle x/\delta y^2 \right\rangle . \end{aligned} \tag{5}$$

Taking the statistical errors in the determination of these two parameters as the contour where the reduced χ^2 increases by one[1], the error ellipse is given by,

$$\left\langle \frac{x^2}{\delta y^2} \right\rangle (a - a_0)^2 + 2 \left\langle \frac{x}{\delta y^2} \right\rangle (a - a_0)(b - b_0) + \left\langle \frac{1}{\delta y^2} \right\rangle (b - b_0)^2 = 1 . \tag{6}$$

The individual errors are given by the horizontal and vertical widths of the ellipse,

$$\Delta a = \left\langle \frac{x^2}{\delta y^2} \right\rangle^{-1/2}, \quad \Delta b = \left\langle \frac{1}{\delta y^2} \right\rangle^{-1/2} .$$

[1] reduced χ^2 is χ^2/n_{dof} with $n_{dot} = n - p$ for n data points and p fit

Likelihood of photon counting data[2]

Likelihood defined as

$$\mathcal{L} = \prod_j p_j \,, \quad \text{with} \quad p_j = \frac{\theta_j^{n_j}}{n_j!}\, e^{-\theta_j} \,,$$

for photon counting data. θ_j is the model at pixel j, n_j are the counts at pixel j. Index j can be multidimensional. For a model θ,

$$\ln(\mathcal{L}) = \sum_j n_j \ln(\theta_j) - \sum_j \theta_j \,,$$

neglecting the last term. A model can be made of a background, background point sources and a test source (a),

$$\theta_j = B + \sum_k c_k \,\mathrm{psf}(\hat{r}_k, j) + c_a \,\mathrm{psf}(\hat{r}_a, j) \,.$$

[2] One reference: Mattox et al. 1996, ApJ 461, 396.

Likelihood of photon counting data (2)

Likelihood is to be maximized relative to a vector of parameters, say $\Lambda = \{B, c_k, c_a\}$, where hypothesis H_0 ignores c_a, while hypothesis H_1 considers c_a. Maximization is now more complex, $D = 0$, with

$$D_i = \frac{\partial \ln \mathcal{L}}{\partial \Lambda_i} = 0\,,$$

and errors in parameters are given by the Hessian,

$$\mathcal{H}_{ij} = -\frac{\partial^2 \ln \mathcal{L}}{\partial \Lambda_i \partial \Lambda_j} \Rightarrow \sigma(\Lambda_i) = \left(\mathcal{H}^{-1}\right)_{ii}^{1/2}\,.$$

A decrease in $\Delta \ln \mathcal{L} = -1/2$ from maximum corresponds to 68 % or 1-σ confidence.

Skymaps and HEALPix

- Celestial maps are generally referred to celestial (North Pole & vernal point) or galactic coordinates (NGP, Galactic center). See arXiv 1010.3773 for transformation between them.
- HEALPix: Hierarchical Equal Area IsoLatitude Pixelization, divides the celestial sphere in a discrete set of equal solid angle elements.
- HAWC uses $N_\theta = 3 \Rightarrow$ two polar regions + one equatorial band; each divided in four $\Rightarrow N_\varphi = 4$.

$$N_{pix} = 12 \times N_{side}^2 ,$$

with one map per f_{hit} bin.

HEALPix: Górski et al. 2005, ApJ 622, 759; healpix.jpl.nasa.gov; healpix.sourceforge.io; healpy@python

HEALPix

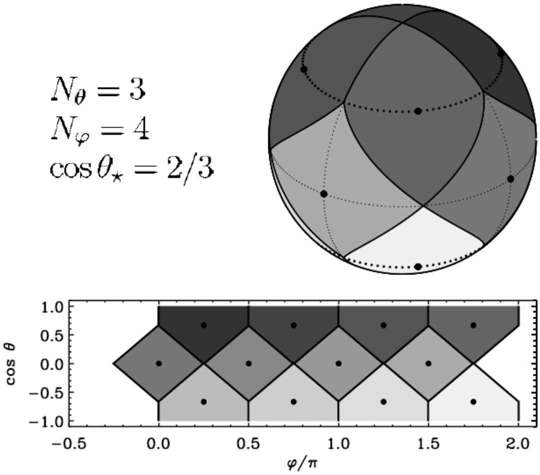

Figura: HAWC scheme is for $N_\theta = 3$, $N_\varphi = 4 \Rightarrow N_{pix} = 12 \times N_{side}^2$.

HEALPix

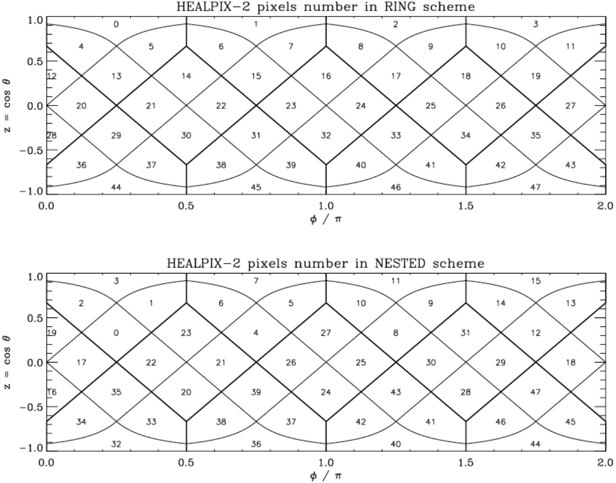

Figura: HAWC uses RING scheme.

HEALPix

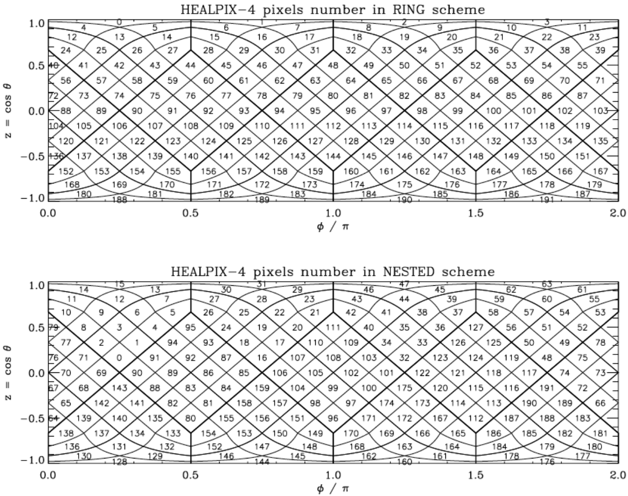

Figura: HAWC uses RING scheme.

How to simulate HAWC data?

- Produce a python script to mimic a skymap, or a region around an injected source?
- How many degrees in the celestial sphere? How many degrees in the 2HWC skymap?
- Rates in HAWC scale exponentially: half in $\mathcal{B} = 0$; $1/4$ in $\mathcal{B} = 1$; ... What is the background per (HEALpix) pixel $(-20^\circ \leq \delta \leq +60^\circ)$ for each f_{hit} bin?
- Background generation in python.
- Dependence with declination? Look at HAWC monitoring plots....
- HAWC response? Cuts, energy intervals, Crab counts ($N(E) \propto E^{-2.63}$).

HAWC monitoring

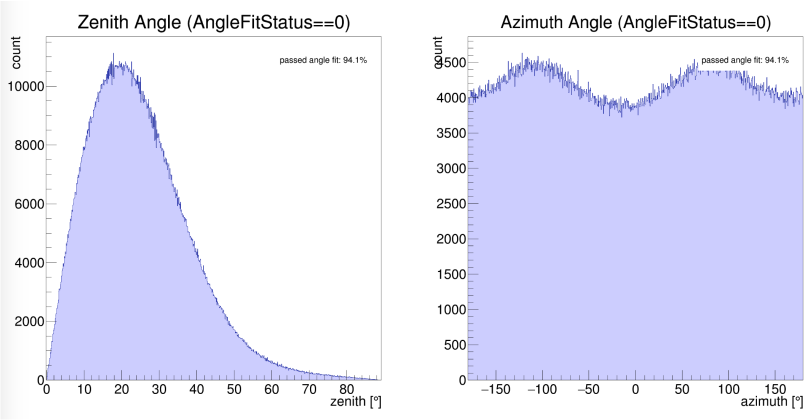

Figura: Rate as function of zenith and azimuth.

HAWC monitoring

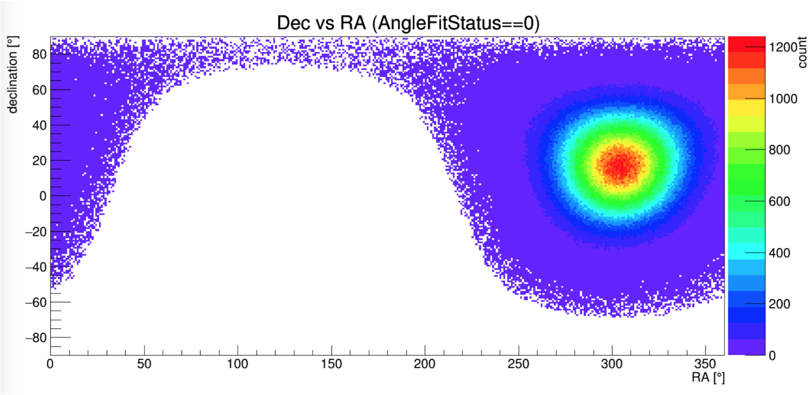

Figura: Rate projected in the sky.